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On the spectra of the Sturm–Liouville operator with the fast oscillating potential

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Abstract. A spectral Problem for a Sturm–Liouville operator is considered in the functional Hilbert space on the half axis. The potential is "fast oscillating." The conditions giving a discrete spectra are indicated. The spectral asymptotic is determined.

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1. Introduction. The main result

The standard Sturm–Liouville operator L_q is defined in the Hilbert space $L_2[0, +\infty)$ as a self adjoined extension of a minimal operator acting by

$$L_q y = -y''(x) + q(x)y(x), \quad x > 0,$$

for $y \in C_0^2[0, \infty)$. The function $x \mapsto q(x)$ (potential) is real and continuous. The classical spectral theory says that if $q(x) \to +\infty$ for $x \to +\infty$ then the spectra of L_q is discrete; asymptotical formulas for spectra in this case are known [4]. The situation is more intricate and the results are not as complete in the case $q(x) \to -\infty$ for $x \to +\infty$, see [1] and [2].

In this paper we consider *fast oscillating potentials.* The latter term denotes potentials of the form $q(x) = u(x)\cos v(x)$ where the functions $x \to u(x)$, $x \to v(x)$ are continuous and $u(x) \to +\infty$, $v(x) \to +\infty$ for $x \to +\infty$. It would be of interest to study the spectra of L_q for these potentials and, in particular, to find the cases of discrete spectra and to describe the spectral asymptotic

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in these cases. In this paper we partially treat this problem for potentials of the form

$$q(x) = h(v'(x))^2 \cos v(x), \quad h > 0. \tag{1}$$

The following conditions are assumed:

$$\begin{aligned}&\text{the functions } x \mapsto v(x) \text{ and } x \to v'(x)\\ &\text{are increasing to } +\infty \text{ for } x \to +\infty;\end{aligned} \tag{2}$$

$$\frac{v''(x)}{v'(x)^2} \longrightarrow 0 \quad \text{for } x \to +\infty. \tag{3}$$

Potentials of the form of eq. (1) can be classified as "singular cases" in the Sturm–Liouville theory. Discrete spectra for these cases is somewhat unexpected (as well as asymptotics for spectra as we will see later).

To state the results we need some notations. First, for any Sturm–Liouville operator L_q denote by $N(\alpha, \beta)$ the number of points in the interval (α, β) belonging to the spectra (this number may be infinite as well). Second, take the function $\xi \to w(\xi), \xi \in [v'(0), +\infty)$, inverse to v'. We write $f \asymp g$ for functions f and g if $A < f/g < B$ for some numbers $A > 0$ and $B > 0$. We write $f \simeq g$ if $f/g \to 1$ for $x \to +\infty$.

Theorem. *Let* $1/2 < h < 1$. *Then the spectra of* L_q *is discrete. The following asymptotical formulas hold*:

$$N(0, t) \asymp \sqrt{t}\, w(\sqrt{t})$$

and

$$N(-t, 0) \asymp v(w(\sqrt{t})), \quad \textit{for } t > 1.$$

Examples. 1) If $q(x) = he^{2x} \cos e^x$, then

$$N(0, t) \asymp \sqrt{t} \log t, N(-t, 0) \asymp \sqrt{t}.$$

2) If $q(x) = hb^2 x^{2b-2} \cos x^b, b > 1$, then

$$N(0, t) \asymp t^{b/2(b-1)}, N(-t, 0) \asymp t^{b/2(b-1)}.$$

(In both examples the condition $1/2 < h < 1$ is assumed.)

The latter result improves the results of the authors paper [3] where the case $4hx^2 \cos x^2$ was considered. (Note that the information on h was scarce in [3]: it was only stated the existence of numbers h giving the desired asymptotics.)

2. Proof of the theorem

The core of the Proof is the following fact proved in [3]. Consider an arbitrary Sturm–Liouville operator L_q in $L_2[0,+\infty)$. For any interval

$$\Delta = [a,b] \subset [0,+\infty)$$

let

$$\lambda_1(\Delta) = \inf(L_q y, y), y \in C_0^\infty(\Delta), \quad (y,y) = 1.$$

This is the minimal eigenvalue of the spectral problem

$$L_q y = \lambda y, \quad y(a) = y(b) = 0.$$

Suppose we are given two sequences $0 = a_0 < a_1, \dots$ and $0 = b_0 < b_1, \dots$ with $a_k \to +\infty$ and $b_k \to +\infty$ for $k \to +\infty$. Take the *distribution functions* $A(x) = \max\{k : a_k < x\}$ and $B(x) = \max\{k : b_k < x\}$. Take numbers $\alpha < \beta$.

Proposition. 1) *Let* $\lambda_1[a_{k-1}, a_k] \le \alpha$, *and* $\lambda_1[b_{k-1}, b_k] \ge \beta$ *for all* $k \ge 1$. *Then*

$$N(\alpha,\beta) \le \underline{\lim}(B(x) - A(x)) + 1, \quad x \to +\infty.$$

2) *Let* $\lambda_1[a_{k-1}, a_k] \ge \alpha$, *and* $\lambda_1[b_{k-1}, b_k] \le \beta$ *for all* $k \ge 1$. *Then*

$$N(\alpha,\beta) \ge \overline{\lim}(B(x) - A(x)) - 1, \quad x \to +\infty.$$

(For readers convenience we sketch the proof of this proposition in appendix.)

Proof of the theorem. We divide the proof in six steps.

Step 1. Denote by $\gamma_1(h,s)$ the minimal eigenvalue of the spectral problem

$$L_{h,s} z = \gamma z, \quad z(0) = z(2\pi) = 0$$

where

$$L_{h,s} z = -z'' + h\cos(t+s) z(t).$$

(In what follows we will need only the values $s = 0$ and $s = \pi$.)

Lemma 2.1. *If* $1/2 < h < 1$ *then*

$$\gamma_1(h, 0) < 0, \quad \gamma_1(h, \pi) > 0.$$

Proof of Lemma 2.1. 1) Let $s = 0$. Take a function $z(t) = \sin(t/2)$. We have then $z(0) = z(2\pi) = 0, (L_{h,s}z, z) = \pi(1/4 - h/2)$. Thus $(L_{h,s}z, z) \leq 0$ and consequently $\gamma_1(h, 0) < 0$ if $h > 1/2$.

2) Let $s = \pi$. For any function $t \to z(t)$ with $z(0) = z(2\pi) = 0$ we have

$$\int_0^{\pi/2} (|z'|^2 - h \cos t |z(t)^2|) dt \geq \int_0^{\pi/2} (|z'|^2 - h|z(t)|^2) dt$$
$$\geq (1 - h) \int_0^{\pi/2} |z|^2 dt.$$

A similar inequality holds also for the interval $[3\pi/2, 2\pi]$. Thus $\gamma_1(h, \pi) > 0$ if $h < 1$. Lemma 2.1 is proved. □

STEP 2. For any $s \in [0, 2\pi)$ consider a sequence $c_k(s), k = 0, 1, \ldots$ defined by equations

$$v(c_k(s)) = s + 2\pi k.$$

Consider intervals $\Delta_k(s) = [c_{k-1}(s), c_k(s)]$; put

$$|\Delta_k(s)| = c_k(s) - c_{k-1}(s).$$

Lemma 2.2. *We have*

$$v'(x) = \frac{2\pi}{|\Delta_k(s)|} + o\Big(\frac{1}{|\Delta_k(s)|}\Big) \tag{4}$$

and

$$v(x) = s + 2\pi k + \frac{2\pi(x - c_{k-1}(s))}{|\Delta_k(s)|} + o(1) \tag{5}$$

for $x \in \Delta_k(s)$ *and* $k \to +\infty$.

It follows that

$$|\Delta_k(s)| \simeq |\Delta_{k-1}(s)| \simeq 2\pi / v'(c_k(s)).$$

Proof of Lemma 2.2. We begin with proving the following property of the function v. Fix a number $l > 0$; let the numbers x and y vary in such a way that $x \to +\infty$ and $y \to +\infty$ with $|v(x) - v(y)| \le l$; we have then $v'(x) \simeq v'(y)$. To prove this fact apply the Cauchy–L'Hopital formula to the functions $\ln v'$ and v. We obtain then

$$\frac{\ln v'(x) - \ln v'(y)}{v(x) - v(y)} = \frac{(\ln v'(z))'}{v'(z)} = \frac{v''(z)}{(v'(z))^2}$$

for some z lying between x and y. Using now property (3) and the condition $|v(x) - v(y)| \le l$ we see that $\ln v'(x) - \ln v'(y) \to 0$ for x, y described above; thus $v'(x) \simeq v'(y)$. So the desired property is proved.

Now proceed with proving Lemma 2.2. Using the definition of values $c_k(s)$ we conclude that $v'(\xi_k)\Delta_k(s) = 2\pi$ for some $\xi_k \in \Delta_k(s)$. Applying now the above proved property we obtain formula (4).

We have further

$$v(x) = v(c_{k-1}(s)) + (x - c_{k-1}(s))v'(\nu_k)$$

for some $\nu_k \in \Delta_k(s)$. Applying now the formula (4) we come to formula (5). Lemma 2.2 is proved. □

We consider now the eigenvalues $\lambda_1(\Delta_k(s))$. (They depend also on positive number h.)

Lemma 2.3. *Let* $1/2 < h < 1$. *Then*

$$A_1|\Delta_k(\pi)|^{-2} < \lambda_1(\Delta_k(\pi)) < B_1|\Delta_k(\pi)|^{-2}, \tag{6}$$

$$-A_2|\Delta_k(0)|^{-2} < \lambda_1(\Delta_k(0)) < -B_2|\Delta_k(0)|^{-2}, \tag{7}$$

$$-A_2|\Delta_k(s)|^{-2} < \lambda_1(0, c_k(s)) < -B_2|\Delta_k(s)|^{-2} \tag{8}$$

for $k > 1$ *where the numbers* A_i, B_i *depend on* h *and* $A_i > 0, B_i > 0$.

Proof of Lemma 2.3. Using Lemma 2.2 we can easily transform the equation

$$L_q y = \lambda y$$

to

$$-y''(x) + \Big(h\Big(\frac{2\pi}{|\Delta_k(s)|}\Big)^2 \cos\Big(s + \frac{2\pi(x - c_{k-1})}{|\Delta_k(s)|}\Big) + o\Big(\frac{1}{|\Delta_k(s)|^2}\Big)\Big)y(x) = \lambda y(x). \tag{9}$$

Consider a mapping

$$\Delta_k(s) \longrightarrow [0, 2\pi]$$

given by

$$x \longrightarrow t, \quad t = 2\pi \frac{(x - c_{k-1}(s))}{|\Delta_k(s)|}$$

and transfer the latter spectral problem for L_q in $\Delta_k(s)$ to the interval $[0, 2\pi]$ by using the substitution $y(x) = z(t)$ where x and t are related by the previous mapping $\Delta_k(s) \to [0, 2\pi]$. Equation (9) leads to

$$-z''(t) + h\cos(s+t)z(t) = \Big(\lambda\Big(\frac{|\Delta_k(s)|}{2\pi}\Big)^2 + o(1)\Big)z(t).$$

Taking the minimal eigenvalue of this spectral problem we come to

$$\lambda_1(\Delta_k(s)) = (2\pi/|\Delta_k(s)|)^2(\gamma_1(h,s) + o(1)).$$

Taking here the values $s = 0$ and $s = \pi$ and applying Lemma 2.1 we come to the first and second assertions of Lemma 2.3. The third assertion follows from inequalities $\lambda_1(0, c_k(s)) \le \lambda_1(\Delta_k(s))$ and $q(x) = h(v'(x))^2 \cos v(x) \ge -A_2|\Delta_k(s)|^{-2}$ for some $A_2 > 0$. Lemma 2.3 is proved. □

We add to Lemma 2.3 the following fact. Divide some interval $\Delta_k(s)$ in m_k intervals $\Delta_{kj}(s)$ of the same length $|\Delta_k(s)|/m_k$. Then obviously

$$\lambda_1(\Delta_{kj}(\pi)) \ge A_1 m_k^2 |\Delta_k(\pi)|^{-2} \tag{10}$$

and

$$\lambda_1(\Delta_{kj}(0)) \le B_1 m_k^2 |\Delta_k(0)|^{-2} \tag{11}$$

for any $j \le m_k$ where A_1, B_1 depend on h, s and $A_1 > 0, B_1 > 0$. (To simplify notations we have used here the same constants A_1 and B_1 as in (6).)

In what follows we explore the numbers A_i, B_i from inequalities (6)–(11).

STEP 3. Evaluate the value $N(0,t)$ from above for $t > 0$. For this sake apply the proposition (Part 1) taking for $\{a_k\}$ the sequence $\{c_k(0)\}$. Recall that

$$\lambda_1(\Delta_k(0)) \le 0 \quad \text{for } k \ge 1;$$

hence the sequence $\{a_k\}$ satisfies the condition indicated in proposition (Part 1). To construct the sequence $\{b_k\}$ indicated in this proposition use the inequalities (6) and (10). Let

$$p = \min\{k; A_1|\Delta_k(\pi)|^{-2} \ge t\}.$$

For any $k \le p$ take the minimal positive integer m_k such that

$$A_1 m_k^2 |\Delta_k(\pi)|^{-2} \ge t.$$

Divide any interval $\Delta_k(\pi)$ in m_k intervals $\Delta_{kj}(\pi)$ of the same length $|\Delta_k(\pi)|/m_k$. Clearly

$$m_k \le A_3 |\Delta_k(\pi)| \sqrt{t}$$

for some A_3 and (by Part 1 of the proposition)

$$N(0,t) \le \sum m_k \le A_3 \sum |\Delta_k(\pi)| \sqrt{t} = A_3 c_p(\pi) \sqrt{t}.$$

Further,

$$v'(c_p(\pi)) \asymp 1/|\Delta_p(\pi)| \asymp \sqrt{t},$$

and hence

$$c_p(\pi) \asymp w(\sqrt{t}).$$

Thus

$$N(0,t) \le A_4 w(\sqrt{t}) \sqrt{t}$$

for some A_4.

STEP 4. Evaluate the value $N(0,t)$ from below. The arguments are similar to previous ones. Apply Part 2 of the proposition taking for $\{a_k\}$ the sequence $\{c_k(\pi)\}$. To construct the sequence $\{b_k\}$ indicated in the proposition above apply inequality (11) and choose maximal positive integers m_k such that the

$$B_1 m_k^2 |\Delta_k(0)|^{-2} \le t.$$

Divide any interval $\Delta_k(0)$ in m_k intervals $\Delta_{kj}(0)$ of the same length $|\Delta_k(0)|/m_k$. (The interval $\Delta_k(0)$ remains undivided for sufficiently big k.) Clearly

$$m_k \ge A_6 |\Delta_k(0)| \sqrt{t}$$

for some A_6 and

$$N(0,t) \ge \sum m_k \ge A_6 \sum |\Delta_k(0)| \sqrt{t} = A_6 c_p(0) \sqrt{t}.$$

Repeating the same arguments as in Step 3 we obtain

$$N(0,t) \ge A_7 w(\sqrt{t}) \sqrt{t}$$

for some A_7.

STEP 5. Evaluate the value $N(-t,0)$ from above. Apply Part 1 of the proposition. Use inequality (8) and take for $\{a_k\}$ the sequence

$$0, \quad c_{p+1}(0), \quad c_{p+2}(0), \quad \dots$$

where

$$p = \min\{k\colon\, B_2|\Delta_k(0)|^{-2} > t\}.$$

$$\lambda_1(0, c_p(0)) \le -t \quad \text{and} \quad \lambda_1(\Delta_k(0)) \le -t \quad \text{for } k \ge p.$$

Thus $\lambda_1(a_{k-1}, a_k) \le -t$ for all $k \ge 1$. Take for $\{b_k\}$ the sequence $c_k(\pi)$. Then $\lambda_1(b_{k-1}, b_k)(\pi) > 0$ and so (by Part 2 of the proposition above) $N(-t,0) \le p$.

We have $v'(c_p) \asymp 1/|\Delta_p(0)| \asymp \sqrt{t}$ and hence $c_p \asymp w(\sqrt{t})$. Further, we have $v(c_p) \asymp p$. Hence

$$N(-t,0) \le p \asymp v(c_p) \asymp v(w(\sqrt{t})).$$

STEP 6. Evaluate the value $N(-t,0)$ from below. Apply Part 2 of the proposition above. Use inequality (8) and take for $\{a_k\}$ the sequence

$$0, \quad c_p(\pi), \quad c_{p+1}(\pi), \quad \dots$$

where

$$p = \max\{k\colon\, A_2|\Delta_k(\pi)|^{-2} < t\}.$$

Then it follows from (8) that $\lambda_1(0, c_p(\pi) \ge -t$; we have also $\lambda_1((\Delta_k(\pi)) > 0 > -t$ for $k \ge p$. Thus $\lambda_1(a_{k-1}, a_k) \ge -t$ for $k \ge 1$. Take for $\{b_k\}$ the sequence $c_k(0)$; so $\lambda_1(b_{k-1}, b_k) \le 0$. Thus, using Part 1 of the proposition and the same arguments as in Step 5 we obtain $N(-t,0) \ge p \asymp v(c_p(\pi)) \asymp v(w(\sqrt{t}))$. □

3. Appendix

We sketch the proof of proposition above. For any interval $\Delta = [a,b] \subset [0,+\infty)$ denote by $\lambda_1(\Delta) < \cdots < \lambda_m(\Delta) < \cdots$ the eigenvalues of the spectral Problem

$$L_q y = \lambda y, \quad y(a) = y(b) = 0$$

and by $N_{[a,b]}(\alpha,\beta)$ the number of these eigenvalues lying in (α,β). We need the following lemmata.

Lemma 3.1. *Divide the interval $[a,b]$ in intervals δ_k, $1 \le k \le m$. Then*

$$\min \lambda_1(\delta_k) \le \lambda_m([a,b]) \le \max \lambda_1(\delta_k).$$

In the next Lemma we denote by $N_{[0,c]}(\alpha,\beta)$ the number of eigenvalues of our Sturm–Liouville operator in the interval $[0,c]$ with the boundary conditions $y(0) = y(c) = 0$.

Lemma 3.2. *The following inequalities hold*

$$\overline{\lim}_{c\to+\infty} N_{[0,c]}(\alpha,\beta) \le N(\alpha,\beta) \le \underline{\lim}_{c\to+\infty} N_{[0,c]}(\alpha,\beta) + 1.$$

These lemmata follow from elementary Sturm–Liouville Theory (mini-max principle, the properties of solutions of Sturm–Liouville equation).

Prove now Part 1. Take an arbitrary $c > 0$ and put $A(c) = q, B(c) = p$. Lemma 3.1 implies that $\lambda_q \le \alpha, \lambda_p \ge \beta,$. Thus $N_{[0,c]}(\alpha,\beta) \le p - q$. Lemma 3.2 gives $N(\alpha,\beta) \le p - q + 1$. Part 2 can be proved similarly.

References

[1] F. V. Atkinson and C. T. Fulton, Some limit circle eigenvalue problems and asymptotic formulae for eigenvalues. In W. N. Everitt and B. D. Sleeman (eds.), *Ordinary and partial differential equations.* Proceedings of the Seventh Conference held at the University of Dundee, Dundee, March 29–April 2, 1982. Lecture Notes in Mathematics, 964. Springer, Berlin etc., 1982, 28–55. MR 0693100 MR 0693098 (collection) Zbl 0514.34018 Zbl 0488.00008 (collection)

[2] Pt. Heywood, On the asymptotic distribution of eigenvalues. *Proc. London Math. Soc.* (3) **4** (1954). 456–470. MR 0067291 Zbl 0058.07301

[3] R. S. Ismagilov, Spectrum of the Sturm–Liouville equation with an oscillating potential. *Mat. Zametki* **37** (1985), no. 6, 869–879, 942. In Russian. English translation, *Math. Notes* **37** (1985), no. 5-6, 476–482. MR 0802430 Zbl 0597.34017

[4] E. C. Titchmarsh, *Eigenfunction expansions associated with second-order differential equations.* Oxford, at the Clarendon Press, 1946. MR 0019765 Zbl 0061.13505

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