

Hadamard type asymptotics for eigenvalues of the Neumann problem for elliptic operators

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Abstract. This paper considers how the eigenvalues of the Neumann problem for an elliptic operator depend on the domain. The proximity of two domains is measured in terms of the norm of the difference between the two resolvents corresponding to the reference domain and the perturbed domain, and the size of eigenfunctions outside the intersection of the two domains. This construction enables the possibility of comparing both nonsmooth domains and domains with different topology. An abstract framework is presented, where the main result is an asymptotic formula where the remainder is expressed in terms of the proximity quantity described above when this is relatively small. As an application, we develop a theory for the Laplacian in Lipschitz domains. In particular, if the domains are assumed to be $C^{1,\alpha}$ regular, an asymptotic result for the eigenvalues is given together with estimates for the remainder, and we also provide an example which demonstrates the sharpness of our obtained result.

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1. Introduction

The aim of this article is to describe how the eigenvalues of the Neumann problem for an elliptic operator depend on the domain. A large quantity of studies of the corresponding Dirichlet problem exists in the literature; see, for instance, Grinfeld [5], Henrot [7], Kozlov [11, 14], Kozlov and Nazarov [12], and references found therein. However, less has been written about the Neumann problem. In this article, we present a framework for the Neumann problem similar to the one developed for the Dirichlet problem in [11].

Investigations of how eigenvalues change when the domain is perturbed is a classical problem. Rayleigh [17] studied eigenvalues and domain perturbation in connection with acoustics as early as in the nineteenth century. In the early twentieth century, Hadamard [6] studied perturbations of domains with smooth boundary, where the perturbed domain Ω_ε is represented by $x_\nu = \varepsilon h(x')$ where $x' \in \partial\Omega_0$, x_ν is the signed distance to the boundary ($x_\nu < 0$ for $x \in \Omega_0$), h is a smooth function, and ε is a small parameter. Hadamard considered the Dirichlet problem, but a formula of Hadamard-type for the first nonzero eigenvalue of the Neumann-Laplacian is given by

$$\Lambda(\Omega_\varepsilon) = \Lambda(\Omega_0) + \varepsilon \int_{\partial\Omega_0} h(|\nabla\varphi|^2 - \Lambda(\Omega_0)\varphi^2) dS + o(\varepsilon),$$

where dS is the surface measure on $\partial\Omega_0$ and φ is an eigenfunction corresponding to $\Lambda(\Omega_0)$ such that $\|\varphi\|_{L^2(\Omega_0)} = 1$; compare with Grinfeld [5]. A study of asymptotics for singular perturbations can be found in, e.g., Mel'nyk and Nazarov [16], Laurain et al. in [15], Kozlov and Nazarov [13], and references found therein. The problem of domain dependence of eigenvalues is closely related to shape optimization. We refer to Henrot [7], and Sokołowski and Zolésio [18], and references found therein.

Suppose that Ω_1 and Ω_2 are domains in \mathbf{R}^n , $n \geq 2$. This article considers the spectral problems

$$\begin{cases} -\Delta u = \Lambda(\Omega_1)u & \text{in } \Omega_1, \\ \partial_\nu u = 0 & \text{on } \partial\Omega_1 \end{cases} \quad (1.1)$$

and

$$\begin{cases} -\Delta v = \Lambda(\Omega_2)v & \text{in } \Omega_2, \\ \partial_\nu v = 0 & \text{on } \partial\Omega_2, \end{cases} \quad (1.2)$$

where ∂_ν is the normal derivative with respect to the outwards normal and if the boundary is nonsmooth, we consider the corresponding weak formulation of the problem. Our results are, however, applicable to a wider class of partial differential operators. In particular to uniformly elliptic operators of second order.

We start the paper with an abstract setting of the problem in a Hilbert space H . We assume that two subspaces H_1 and H_2 are given together with positive definite operators K_1 and K_2 acting in H_1 and H_2 , respectively. We assume that K_1 is a compact operator. We choose an eigenvalue λ^{-1} of K_1 and denote by $\mathcal{X} \subset H_1$ the linear combination of all eigenvectors corresponding to eigenvalues greater than or equal to λ^{-1} . The proximity of the operators K_1 and K_2 is measured by a constant ε in the inequalities

$$\|\varphi - S\varphi\|^2 \leq \varepsilon\|\varphi\|^2 \quad \text{for every } \varphi \in \mathcal{X}$$

and

$$\|(K_2 - SK_1S^*)w\|^2 \leq \varepsilon\|w\|^2 \quad \text{for every } w \in H_2.$$

Here, $S = S_2$ is the orthogonal projector from H into H_2 and S^* is the adjoint operator of $S: H_1 \rightarrow H_2$. Under these assumption we prove that the operator K_2 has exactly the same number of eigenvalues in a neighborhood of λ^{-1} , independent of ε , as the multiplicity of the eigenvalue λ^{-1} of K_1 . This is a consequence of the continuous dependence of eigenvalues on the domain; see, e.g., Henrot [7]. Moreover, we present an asymptotic formula for these eigenvalues where the remainder term is relatively small compared to the leading term. This asymptotic result improves Theorem 1 in [14] in two ways. First, we consider H_1 and H_2 as subspaces of a fixed Hilbert space and can compare operators acting there with the help of orthogonal projectors, which simplifies the conditions of Theorem 1. Secondly, and perhaps more importantly, the remainder term in our theorem is “smaller” with respect to the leading term, which is not necessarily the case in Theorem 1 from [14].

To characterize how close the two domains are, we will use the Hausdorff distance between the sets Ω_1 and Ω_2 , i.e.,

$$d = \max\left\{ \sup_{x \in \Omega_1} \inf_{y \in \Omega_2} |x - y|, \sup_{y \in \Omega_2} \inf_{x \in \Omega_1} |x - y| \right\}. \quad (1.3)$$

We do not assume that one domain is a subdomain of the other. It should be noted however, that the abstract result presented below permits a more general type of proximity quantity for the two domains; see (2.3) and (2.4) in Section 2.

If Ω_1 is a $C^{1,\alpha}$ domain with $0 < \alpha < 1$ and Ω_2 is a Lipschitz perturbation of Ω_1 in the sense that the perturbed domain Ω_2 can be characterized by a function h defined on the boundary $\partial\Omega_1$ such that every point $(x', x_\nu) \in \partial\Omega_2$ is represented by $x_\nu = h(x')$, where $(x', 0) \in \partial\Omega_1$ and x_ν is the signed distance to $\partial\Omega_1$ as defined above. Moreover, the function h is assumed to be Lipschitz continuous and satisfy $|\nabla h| \leq Cd^\alpha$. Then we obtain the following result; see Corollary 6.17.

Theorem 1.1. *Suppose that Ω_1 is a $C^{1,\alpha}$ -domain with $0 < \alpha < 1$ and Ω_2 is as described above, that the problem in (1.1) has a discrete spectrum, and that m is fixed. Then there exists a constant $d_0 > 0$ such that if $d \leq d_0$, then*

$$\Lambda_k(\Omega_2) - \Lambda_m(\Omega_1) = \kappa_k + O(d^{1+\alpha}) \quad (1.4)$$

for every $k = 1, 2, \dots, J_m$, where J_m is the dimension of the eigenspace corresponding to $\Lambda_m(\Omega_1)$. Here $\kappa = \kappa_k$ is an eigenvalue of the problem

$$\kappa(\varphi, \psi) = \int_{\partial\Omega_1} h(x')(\nabla\varphi \cdot \nabla\psi - \Lambda_m(\Omega_1)\varphi\psi) dS(x') \quad \text{for all } \psi \in X_m, \quad (1.5)$$

where $\varphi \in X_m$. Moreover, $\kappa_1, \kappa_2, \dots, \kappa_{J_m}$ in (1.4) run through all eigenvalues of (1.5) counting their multiplicities.

Observe that (1.5) can be phrased as a spectral problem on the Hilbert space X_m by using the Riesz representation theorem of the operator on the right-hand side.

In Section 6.7, we consider a specific example of a Lipschitz perturbation of a cylinder in two dimensions. We prove that if $\eta : \mathbf{R} \rightarrow \mathbf{R}$ is a periodic non-negative Lipschitz continuous function with period 1, and $\Omega_1 \subset \mathbf{R}^2$ is defined by $0 < x < T$ and $0 < y < R$, where R and T are constants, and the subdomain $\Omega_2 \subset \Omega_1$ is defined by $0 < x < T$ and $\delta\eta(x/\delta) < y < R$ for a small parameter δ , then

$$\Lambda_k(\Omega_2) - \Lambda_m(\Omega_1) = \kappa_k + O(\delta^2) \quad \text{for every } k = 1, 2, \dots, J_m, \quad (1.6)$$

where J_m is the dimension of the eigenspace corresponding to $\Lambda_m(\Omega_1)$. Here, $\kappa = \kappa_k$ is an eigenvalue of the problem

$$\begin{aligned} \kappa(\varphi, \psi) = & \delta\eta_0 \int_0^T (\nabla\varphi(x, 0) \cdot \nabla\psi(x, 0) - \Lambda_m(\Omega_1)\varphi(x, 0)\psi(x, 0)) dx \\ & + \delta\eta_1 \int_0^T \nabla\varphi(x, 0) \cdot \nabla\psi(x, 0) dx \end{aligned} \quad (1.7)$$

for all $\psi \in X_m$, where $\varphi \in X_m$ and

$$\eta_0 = \int_0^1 \eta(X) dX \quad \text{and} \quad \eta_1 = \int_0^1 V(X, \eta(X))\eta'(X) dX.$$

The function V is the solution to $-\Delta V = 0$ for $0 < X < 1$ and $Y > \eta(X)$ with the boundary condition $\partial_\nu V(X, \eta(X)) = \eta'(X)(1 + (\eta'(X))^2)^{-1/2}$ on $Y = \eta(X)$ and periodic boundary conditions on the remaining boundary. The constant η_1 is not zero if η is not identically constant. Observe that the first term in the right-hand side of (1.7) coincides with the right-hand side of (1.5) up to order $O(\delta^2)$. This proves that the condition $\alpha > 0$ is sharp in Theorem 1.1.

In Corollary 6.11, we obtain as a consequence of the methods developed that eigenvalues satisfy the following estimate for (uniformly) Lipschitz perturbations. There exists a constant C , independent of d , such that $|\Lambda_k(\Omega_2) - \Lambda_m(\Omega_1)| \leq Cd$ for every $k = 1, 2, \dots, J_m$; see Corollary 6.11. This estimate can be compared to results presented in, e.g., Burenkov and Davies [2] in the case when $\Omega_2 \subset \Omega_1$.

2. Abstract setting: perturbation of eigenvalues

The fact that zero is an eigenvalue for the problems in (1.1) and (1.2) is trivial, and to avoid technicalities due to this, we will consider the operator $1 - \Delta$. A number λ is an eigenvalue of the operator $1 - \Delta$ if and only if $\lambda - 1$ is an eigenvalue of $-\Delta$. Let $\Lambda_k(\Omega_1) = \lambda_k - 1$, $k = 1, 2, \dots$, be the eigenvalues of (1.1) enumerated according to $0 < \lambda_1 < \lambda_2 < \dots$. We assume here that Ω_1 is connected. Similarly, we let $\Lambda_k(\Omega_2) = \mu - 1$ be the eigenvalues of (1.2). The subset X_k of H_1 is the eigenspace corresponding to eigenvalue $\Lambda_k(\Omega_1)$, with the dimension denoted by $J_k = \dim(X_k)$. Observe that X_k is also the eigenspace for the eigenvalue λ_k of the to (1.1) corresponding problem for $1 - \Delta$.

We proceed by introducing an abstract setting for the problems in (1.1) and (1.2). Suppose that H_1 and H_2 are infinite dimensional subspaces of a Hilbert space H . Let the operators $K_j: H_j \rightarrow H_j$ be positive definite and self-adjoint for $j = 1, 2$. Furthermore, let K_1 be compact. We consider the spectral problems

$$K_1\varphi = \lambda^{-1}\varphi, \quad \varphi \in H_1, \quad (2.1)$$

and

$$K_2U = \mu^{-1}U, \quad U \in H_2, \quad (2.2)$$

and denote by λ_k^{-1} for $k = 1, 2, \dots$ the eigenvalues of K_1 . Let $X_k \subset H_1$ be the eigenspace corresponding to eigenvalue λ_k^{-1} . Moreover, we denote the dimension of X_k by J_k and define $\mathcal{X}_m = X_1 + X_2 + \dots + X_m$, where $m \geq 1$ is any integer. In this article we study eigenvalues of (2.2) located in a neighborhood of λ_m , where m is fixed.

In order to define the proximity of the operators K_1 and K_2 , which are defined on different spaces, we introduce the orthogonal projectors $S_1: H \rightarrow H_1$ and $S_2: H \rightarrow H_2$. To simplify the notation, we also introduce the operator S as the restriction of S_2 to H_1 . Thus S maps H_1 into H_2 and its adjoint operator $S^*: H_2 \rightarrow H_1$ is given by $S^* = S_1S_2$.

We introduce a quantity $\varepsilon > 0$ as a constant in the inequalities

$$\|(K_2 - SK_1S^*)w\|^2 \leq \varepsilon\|w\|^2 \quad \text{for every } w \in H_2 \quad (2.3)$$

and

$$\|\varphi - S\varphi\|^2 \leq \varepsilon\|\varphi\|^2 \quad \text{for every } \varphi \in \mathcal{X}_m. \quad (2.4)$$

The parameter ε is the measure we use to describe the proximity of the spaces H_1 and H_2 and the operators K_1 and K_2 . In the following analysis, an important role is played by the operator

$$B: H_1 \longrightarrow H_2$$

defined as

$$B = K_2S - SK_1.$$

Remark 2.1. A common way to compare the proximity of domains in shape optimization is the parameter σ in

$$\|(K_2S_2 - K_1S_1)w\|^2 \leq \sigma\|w\|^2 \quad \text{for every } w \in H. \quad (2.5)$$

Let us show that ε can be chosen as

$$\varepsilon = \sigma \max \left\{ 1, 4 \sum_{k=1}^m \lambda_k^2 \right\}.$$

The fact that (2.3) holds can be verified directly. To verify that (2.4) holds, let $\varphi \in \mathcal{X}_m$. Then $\varphi = \sum_{k=1}^m c_k \varphi_k$, where $\varphi_k \in X_k$ are orthonormal and c_k are constants. Thus,

$$\begin{aligned} \|\varphi - S\varphi\| &\leq \sum_{k=1}^m |c_k \lambda_k| \|K_1\varphi_k - SK_1\varphi_k\| \\ &\leq \sum_{k=1}^m |c_k \lambda_k| (\|K_1\varphi_k - K_2S_2\varphi_k\| + \|K_2S_2\varphi_k - K_1\varphi_k\|) \\ &\leq 2\sigma^{1/2} \sum_{k=1}^m |c_k \lambda_k|, \end{aligned}$$

which implies that

$$\|\varphi - S\varphi\|^2 \leq 4\sigma \left(\sum_{k=1}^m \lambda_k^2 \right) \|\varphi\|^2.$$

3. Main results

Let P_m be the orthogonal projection of H_1 onto SX_m . We now state results about the stability of eigenvalues and eigenvectors depending on the parameter ε . The first lemma is a consequence of the continuous dependence of eigenvalues on the domain; see, for instance, Kato [9] (Sections IV.3 and V.3) or Henrot [7] and references therein.

Lemma 3.1. *There exists positive constants ε_0 , c , and C , depending on the eigenvalues $\lambda_1, \dots, \lambda_{m+1}$, such that, for $\varepsilon \leq \varepsilon_0$, the following assertions are valid:*

(i) *The operator K_2 has precisely J_m eigenvalues in*

$$(\lambda_{m+1}^{-1} + c\varepsilon^{1/2}, \lambda_{m-1}^{-1} - c\varepsilon^{1/2})$$

and all of them are located in

$$(\lambda_m^{-1} - c\varepsilon^{1/2}, \lambda_m^{-1} + c\varepsilon^{1/2}).$$

(ii) *If μ^{-1} is an eigenvalue of (2.2) from the interval*

$$(\lambda_m^{-1} - c\varepsilon^{1/2}, \lambda_m^{-1} + c\varepsilon^{1/2})$$

and U is a corresponding eigenfunction, then

$$\|U - P_m U\| \leq C\varepsilon^{1/2}\|U\|.$$

We denote by μ_k^{-1} for $k = 1, 2, \dots, J_m$, the eigenvalues of the spectral problem (2.2) located in the interval $(\lambda_m^{-1} - c\varepsilon^{1/2}, \lambda_m^{-1} + c\varepsilon^{1/2})$, where c is the same constant as in Lemma 3.1. The quantity ρ is defined by

$$\rho = \sup_{\varphi \in X_m, \|\varphi\|=1} (\lambda_m \|K_2 B\varphi\|^2 + \varepsilon \lambda_m \|B\varphi\|^2). \quad (3.1)$$

Theorem 3.2. *The following asymptotic formula holds:*

$$\mu_k^{-1} = \lambda_m^{-1} + \tau_k + O(\rho + |\tau_k|\varepsilon) \quad \text{for every } k = 1, 2, \dots, J_m, \quad (3.2)$$

where $\tau = \tau_k$ is an eigenvalue of the problem

$$\tau(S\varphi, S\psi) = \lambda_m(B\varphi, B\psi) + (B\varphi, S\psi) \quad \text{for all } \psi \in X_m, \quad (3.3)$$

where $\varphi \in X_m$. Moreover, $\tau_1, \tau_2, \dots, \tau_{J_m}$ in (3.2) run through all eigenvalues of (3.3) counting their multiplicities.

In applications, the term $\|K_2 B\varphi\|$ is typically significantly smaller than $\max |\tau_k|$; see, e.g., Lemma 6.9. This implies that ρ is small compared to τ_k for every k . Note also that the right-hand side of (3.3) can be expressed more compactly as $\lambda_m(B\varphi, K_2 S\psi)$.

The asymptotic formula in (3.2) has similarities to the one presented in Kozlov [14]. The main difference is how the remainder term is constructed; in Theorem 3.2, ρ is typically small compared to the main terms above. However, the same is not necessarily true in [14].

4. Proof of Lemma 3.1

The following properties hold.

(I) $(1 - \varepsilon)\|\varphi\|^2 \leq \|S\varphi\|^2 \leq \|\varphi\|^2$ for every $\varphi \in \mathcal{X}_m$.

(II) There exists a positive constant C , depending on the operator norm of K_1 , such that

$$\|B\varphi\| \leq C\varepsilon^{1/2}\|\varphi\| \quad \text{for all } \varphi \in \mathcal{X}_m. \quad (4.1)$$

(III) $(K_2 w, w) \leq (K_1 S^* w, S^* w) + \varepsilon^{1/2}\|w\|^2$ for all $w \in H_2$.

The inequality in (I) follows from

$$\|S\varphi\|^2 \geq \|\varphi\|^2 - \|\varphi - S\varphi\|^2 \geq (1 - \varepsilon)\|\varphi\|^2.$$

To prove (II), suppose that $\varphi \in \mathcal{X}_m$. Then

$$\begin{aligned} \|B\varphi\| &\leq \|BS^*S\varphi\| + \|B(\varphi - S^*S\varphi)\| \\ &\leq \|(K_2 - SK_1S^*)S\varphi\| + \|SK_1(\varphi - S^*S\varphi)\| \\ &\leq \varepsilon^{1/2}\|S\varphi\| + C\|\varphi - S^*S\varphi\| \\ &\leq C\varepsilon^{1/2}\|\varphi\|, \end{aligned}$$

where we used (2.3), (2.4), and the fact that $SS^*S\varphi = S\varphi$. The property in (III) follows from the fact that

$$\begin{aligned} (K_2 w, w) - (K_1 S^* w, S^* w) &= ((K_2 - SK_1 S^*)w, w) \\ &\leq \|(K_2 - SK_1 S^*)w\| \|w\| \\ &\leq \varepsilon^{1/2}\|w\|^2. \end{aligned}$$

The arguments in Section 3.2, 3.3, and 3.4 of [II], are now valid with small modifications. Specifically, we substitute S for the operator S_2 in these sections, and replace inequality (32) by (I). Furthermore, the proof of inequality (34) is analogous, inequality (36) is replaced by (II), and finally, inequality (39) is replaced by (III). This completes the proof of Lemma 3.1.

5. Proof of Theorem 3.2

The proof of Theorem 3.2 mirrors that of the corresponding theorem in Kozlov [II]. Equation (5.2) below corresponds to (7) in [II], but in this case we have the explicit solution given in (5.1). In Sections 5.1–5.3, we provide results similar to the ones found in Section 4 of [II].

Let $Q_m = I - P_m$, where I is the identity operator on H_2 , and suppose henceforth that φ and ψ belong to X_m . To simplify the notation, define

$$\Psi_\varphi = -\lambda_m B\varphi \quad \text{for any } \varphi \in X_m. \quad (5.1)$$

Then Ψ_φ solves the equation

$$(\Psi_\varphi, w) = (\varphi, w) - \lambda_m(S\varphi, K_2w) \quad \text{for every } w \in H_2. \quad (5.2)$$

To verify (5.1), suppose that $w \in H_2$. Then

$$\begin{aligned} (B\varphi, w) &= (K_2S\varphi, w) - (SK_1\varphi, w) \\ &= -\lambda_m^{-1}((S\varphi, w) - \lambda_m(S\varphi, K_2w)) \\ &= -\lambda_m^{-1}(\Psi_\varphi, w). \end{aligned}$$

5.1. Representation of $(Q_m B\varphi, B\psi)$. From (5.1) it follows that

$$(Q_m B\varphi, B\psi) = \lambda_m^{-2}((\Psi_\varphi, \Psi_\psi) - (P_m \Psi_\varphi, \Psi_\psi)).$$

Let $\{\Upsilon_k\}_{k=1}^{J_m}$ be an ON-basis in SX_m . Then, for each $k = 1, \dots, J_m$, there exists $\varphi_k \in X_m$ such that $\Upsilon_k = S\varphi_k$. Thus,

$$(P_m \Psi_\varphi, \Psi_\psi) = \sum_{k=1}^{J_m} (\Psi_\varphi, S\varphi_k)(S\varphi_k, \Psi_\psi). \quad (5.3)$$

From (5.1) and (II), it is clear that

$$|(\Psi_\varphi, S\varphi_k)| = \lambda_m |(B\varphi, S\varphi_k)| \leq \lambda_m \|B\varphi\| \|S\varphi_k\| \leq C\varepsilon^{1/2} \|\varphi\|$$

for $k = 1, \dots, J_m$. Moreover, letting $w = S\varphi_k$ in (5.2) proves that

$$\begin{aligned} (\Psi_\varphi, S\varphi_k) &= \lambda_m (\Psi_\varphi, K_2 S\varphi_k) + (\Psi_\varphi, \Psi_{\varphi_k}) \\ &= \lambda_m (K_2 \Psi_\varphi, S\varphi_k) - \lambda_m (\Psi_\varphi, B\varphi_k), \end{aligned} \quad (5.4)$$

from which it follows together with (I) that

$$\begin{aligned} |(\Psi_\varphi, S\varphi_k)| &\leq \lambda_m (\|K_2 \Psi_\varphi\| \|S\varphi_k\| + \|\Psi_\varphi\| \|B\varphi_k\|) \\ &\leq C (\|K_2 B\varphi\| + \varepsilon^{1/2} \|B\varphi\|) \end{aligned} \quad (5.5)$$

Analogously,

$$(\Psi_\psi, S\varphi_k) = \lambda_m (K_2 \Psi_\psi, S\varphi_k) + (\Psi_\psi, \Psi_{\varphi_k}),$$

and thus

$$|(S\varphi_k, \Psi_\psi)| \leq C (\|K_2 B\psi\| + \varepsilon^{1/2} \|B\psi\|). \quad (5.6)$$

Now, the identity in (5.3), and the estimates in (5.5) and (5.6), imply that

$$|(P_m \Psi_\varphi, \Psi_\psi)| = \lambda_m^2 |(P_m B\varphi, B\psi)| \leq C(\rho(\varphi) + \rho(\psi)), \quad (5.7)$$

where

$$\rho(\varphi) = \lambda_m (\|K_2 B\varphi\|^2 + \varepsilon \|B\varphi\|^2), \quad \varphi \in X_m.$$

5.2. Estimate of $(K_2 Q_m B\varphi, Q_m B\psi)$. Since $P_m + Q_m = I$, it is clear that

$$\begin{aligned} (K_2 Q_m B\varphi, Q_m B\psi) &= (K_2 B\varphi, Q_m B\psi) - (K_2 P_m B\varphi, Q_m B\psi) \\ &= (K_2 B\varphi, Q_m B\psi) - (K_2 P_m B\varphi, B\psi) \\ &\quad + (K_2 P_m B\varphi, P_m B\psi). \end{aligned}$$

Now,

$$|(K_2 B\varphi, Q_m B\psi)| \leq \|K_2 B\varphi\| \|Q_m B\psi\| \leq C\varepsilon^{1/2} \|\varphi\| \|K_2 B\psi\|.$$

Similarly,

$$|(K_2 P_m B\varphi, B\psi)| = |(P_m B\varphi, K_2 B\psi)| \leq C\varepsilon^{1/2} \|\varphi\| \|K_2 B\psi\|.$$

As in Section 5.1, let $\{\Upsilon_k\}_{k=1}^{J_m}$ be an ON-basis in SX_m . Then there exists eigenfunctions $\varphi_k \in X_m$ such that $\Upsilon_k = S\varphi_k$ for every $k = 1, \dots, J_m$. Thus,

$$(K_2 P_m B\varphi, P_m B\psi) = \sum_{k=1}^{J_m} (P_m B\varphi, S\varphi_k)(K_2 S\varphi_k, P_m B\psi). \quad (5.8)$$

Using (5.1) and (5.5), it is clear that

$$|(P_m B\varphi, S\varphi_k)| \leq |(B\varphi, S\varphi_k)| \leq C(\|K_2 B\varphi\| + \varepsilon^{1/2}\|B\varphi\|).$$

Furthermore, (5.1) and (5.5), with φ_l in the place of φ_k and ψ replaced by φ , proves that

$$\begin{aligned} |(K_2 S\varphi_k, P_m B\psi)| &= \left| \sum_{l=1}^{J_m} (K_2 S\varphi_k, S\varphi_l)(P_m B\psi, S\varphi_l) \right| \\ &\leq C \sum_{l=1}^{J_m} |(P_m B\psi, S\varphi_l)| \\ &\leq C(\|K_2 B\psi\| + \varepsilon^{1/2}\|B\psi\|). \end{aligned}$$

Thus,

$$(\|K_2 B\varphi\| + \varepsilon^{1/2}\|B\varphi\|)(\|K_2 B\psi\| + \varepsilon^{1/2}\|B\psi\|) \leq C(\rho(\varphi) + \rho(\psi)).$$

Finally, we obtain that

$$|(K_2 Q_m B\varphi, Q_m B\psi)| \leq C(\rho(\varphi) + \rho(\psi)). \quad (5.9)$$

5.3. Proof of Theorem 3.2. Analogously with the argument used in Kozlov [11], it is possible to reduce the spectral problem (2.2) to a finite dimensional situation using the projectors P_m and Q_m :

$$(\mu^{-1} - K_2)(S\varphi + w) = 0, \quad (5.10)$$

where $\varphi \in X_m$ and $w \in Q_m H_2$. Indeed, proceeding accordingly with Section 4.1 in [11], we obtain that

$$\hat{\tau}(S\varphi, S\psi) - (B\varphi, S\psi) - \mu(Q_m B\varphi, B\psi) - (L(\mu)B\varphi, B\psi) = 0, \quad (5.11)$$

where

$$L(\mu) = \mu Q_m K_2 Q_m (\mu^{-1} - Q_m K_2 Q_m)^{-1} Q_m$$

and

$$\hat{\tau} = \mu^{-1} - \lambda_m^{-1}.$$

We assume that $|\hat{\tau}| \leq \varepsilon^{1/2}$. Moreover, the operator $(\mu^{-1} - Q_m K_2 Q_m)^{-1}$ is bounded from $Q_m H_2$ into $Q_m H_2$:

$$\|(\mu^{-1} - Q_m K_2 Q_m)^{-1} w\|_{Q_m H_2} \leq C \|w\|_{Q_m H_2} \quad \text{for every } w \in H_2.$$

Hence,

$$|(L(\mu)B\varphi, B\psi)| \leq C(K_2 Q_m B\psi, Q_m B\psi).$$

It follows from the identity $\mu^{-1} = \lambda_m^{-1} + \hat{\tau}$ that

$$\mu(Q_m B\varphi, B\psi) = \lambda_m(B\varphi, B\psi) - b_2(\varphi, \psi),$$

where

$$b_2(\varphi, \psi) = \frac{\lambda_m \hat{\tau}}{\hat{\tau} + \lambda_m^{-1}}(Q_m B\varphi, B\psi) + \mu(P_m B\varphi, B\psi).$$

Then

$$|b_2(\varphi, \psi)| \leq C|\hat{\tau}|\varepsilon + C(\rho(\varphi) + \rho(\psi)). \quad (5.12)$$

Put $b(\varphi, \psi) = (L(\mu)B\varphi, B\psi) + b_2(\varphi, \psi)$. Then

$$\hat{\tau}(S\varphi, S\psi) = \lambda_m(B\varphi, K_2 S\psi) + b(\varphi, \psi), \quad (5.13)$$

where $b(\varphi, \psi)$ satisfies

$$|b(\varphi, \psi)| \leq C(\rho(\varphi) + \rho(\psi) + |\hat{\tau}|\varepsilon) \quad (5.14)$$

according to (5.12) and (5.9).

Suppose that $j = 1, \dots, J_m$. Let $U_j \in H_2$ be an eigenfunction of K_2 corresponding to the eigenvalue μ_j^{-1} . Then there exists $V_j \in X_m$ satisfying $P_m U_j = S V_j$. By $\hat{\tau}_j$ we denote an eigenvalue of (5.13) with eigenfunction $\varphi = V_j$. Suppose also that τ_k is an eigenvalue of (3.3) and $\Phi_j \in X_m$ a corresponding eigenfunction. Analogously with Section 4.5 in Kozlov [11], we may assume that there exists a constant $c_* > 0$ such that

$$(S V_j, S \Phi_j) \geq c_* \quad (5.15)$$

after possible rearrangement of the eigenfunctions Φ_j spanning X_m .

Choosing $\varphi = \Phi_j$ and $\psi = V_j$ in equation (3.3), and $\varphi = V_j$ and $\psi = \Phi_j$ in equation (5.13), and then subtracting (3.3) from (5.13), we obtain that

$$(\hat{\tau}_j - \tau_j)(S V_j, S \Phi_j) = \lambda_m((B V_j, K_2 S \Phi_j) - (B \Phi_j, K_2 S V_j)) + b(V_j, \Phi_j).$$

The fact that K_2 is self-adjoint, that Φ_j and V_j belong to X_m , and the definition of B , imply that

$$\begin{aligned} (BV_j, K_2S\Phi_j) - (B\Phi_j, K_2SV_j) &= \lambda_m^{-1}((S\Phi_j, K_2SV_j) - (SV_j, K_2S\Phi_j)) \\ &= 0. \end{aligned}$$

Hence,

$$(\hat{\tau}_j - \tau_j)(SV_j, S\Phi_j) = b(V_j, \Psi_j),$$

from which it follows from (5.14) and (5.15) that

$$|\hat{\tau}_j - \tau_j| \leq C(\rho(V_j) + \rho(\Psi_j) + |\hat{\tau}_j|\varepsilon).$$

Taking the supremum over V_j and Ψ_j in X_m with $\|V_j\| = \|\Psi_j\| = 1$, we obtain that

$$|\hat{\tau}_j - \tau_j| \leq C(\rho + |\hat{\tau}_j|\varepsilon),$$

where

$$\rho = \sup_{\substack{\varphi \in X_m \\ \|\varphi\|=1}} \rho(\varphi) = \lambda_m \sup_{\substack{\varphi \in X_m \\ \|\varphi\|=1}} (\|K_2B\varphi\|^2 + \varepsilon\|B\varphi\|^2).$$

This also implies that

$$|\hat{\tau}_j - \tau_j| \leq C(\rho + |\tau_j|\varepsilon),$$

6. Applications

In this section we consider the Neumann problem for the operator $1 - \Delta$ in different domains. Let Ω_1 and Ω_2 be two domains in \mathbf{R}^n with nonempty intersection. We put $H = L^2(\mathbf{R}^n)$ and $H_j = L^2(\Omega_j)$ for $j = 1, 2$. Functions in H_j are extended to \mathbf{R}^n by zero outside of Ω_j . Observe that we do not require that one subdomain Ω_j is a subset of the other. For $f \in L^2(\Omega_j)$, the weak solution to the Neumann problem $(1 - \Delta)W_j = f$ in Ω_j and $\partial_\nu W_j = 0$ on $\partial\Omega_j$ for $j = 1, 2$ satisfies

$$\int_{\Omega_j} (\nabla W_j \nabla v + W_j v) dx = \int_{\Omega_j} f v dx \quad \text{for every } v \in H^1(\Omega_j). \quad (6.1)$$

It follows from (6.1) with $v = W_j$ and the Cauchy–Schwarz inequality that

$$\|\nabla W_j\|_{L^2(\Omega_j)} + \|W_j\|_{L^2(\Omega_j)} \leq \|f\|_{L^2(\Omega_j)}.$$

We let K_j for $j = 1, 2$ be defined on $L^2(\Omega_j)$ as the solution operators corresponding to the domains Ω_j , i.e., $K_j f = W_j$. Then K_j maps $L^2(\Omega_j)$ into the Sobolev space $H^1(\Omega_j)$, and

$$\|K_j u\|_{H^1(\Omega_j)} \leq C \|u\|_{L^2(\Omega_j)}.$$

Moreover, $(1 - \Delta)K_j u = u$ and $\partial_\nu K_j u = 0$ on $\partial\Omega_j$ in the weak sense. The operators K_j are also self-adjoint and positive definite, and if Ω_j are, e.g., Lipschitz, also compact.

To characterize how close the two domains are, we will use the Hausdorff distance d between the sets Ω_1 and Ω_2 given in (1.3).

6.1. Perturbations of Lipschitz- and $C^{1,\alpha}$ -domains. We now consider two cases of regularity of the boundaries $\partial\Omega_j$, namely $C^{1,\alpha}$ and Lipschitz boundaries. Let us first consider the Lipschitz case. Let Ω_1 be the reference domain which will be fixed throughout. Then there exists a positive constant M such that the boundary $\partial\Omega_1$ can be covered by a finite number of balls B_k , $k = 1, 2, \dots, N$, where there exists orthogonal coordinate systems in which

$$B_k \cap \Omega_1 = B_k \cap \{y = (y', y_n) : y_n > h_k^{(1)}(y')\}$$

where the center of B_k is at the origin and $h_k^{(1)}$ are Lipschitz functions, i.e.,

$$|h_k^{(1)}(y') - h_k^{(1)}(x')| \leq M|y' - x'|,$$

such that $h_k^{(1)}(0) = 0$. We assume that Ω_2 belongs to the class of domains where Ω_2 is close to Ω_1 in the sense that Ω_2 can be described by

$$B_k \cap \Omega_2 = B_k \cap \{y = (y', y_n) : y_n > h_k^{(2)}(y')\},$$

where $h_k^{(2)}$ are also Lipschitz continuous with Lipschitz constant M . Clearly all such domains belong to a ball D of sufficiently large radius depending only on M and B_1, B_2, \dots, B_N . Note also that $\Omega_1 \cap \Omega_2$ is a Lipschitz domain of this type and that we can use the same covering and Lipschitz constant.

Remark 6.1. Observe that d is comparable to

$$\hat{d} = \max_{k=1,2,\dots,N} \sup\{|h_k^{(1)}(y') - h_k^{(2)}(y')| : y = (y', y_n) \in B_k \cap \partial\Omega_1\}$$

in the sense that there exists positive constants c_1 and c_2 depending only on M and B_k , $k = 1, 2, \dots, N$, such that $c_1 \hat{d} \leq d \leq c_2 \hat{d}$.

The case of a $C^{1,\alpha}$ domain is defined in the same manner, with the additional assumptions that $h_k^{(1)}$ are $C^{1,\alpha}$ -functions such that

$$h_k^{(1)}(0) = \partial_{x_i} h_k^{(1)} = 0, \quad i = 1, 2, \dots, n-1.$$

Moreover, we suppose that

$$|\nabla(h_k^{(1)} - h_k^{(2)})| \leq Cd^\alpha. \quad (6.2)$$

Note that $h_k^{(2)}$ are only assumed to be Lipschitz continuous and satisfy (6.2). It is also worth noting that these domains constitute a subset of the class of Lipschitz domains used in Section 6.2. Thus, results that hold for Lipschitz domains are also valid for this class of domains.

6.2. Lipschitz domains. Solutions to elliptic partial differential equations in Lipschitz domains often belong to Hardy-type spaces. Let Ω be a Lipschitz domain. The truncated cones $\Gamma(x')$ at $x' \in \partial\Omega$ are given by, e.g.,

$$\Gamma(x') = \{x \in \Omega : |x - x'| < 2\text{dist}(x, \partial\Omega)\}$$

and the non-tangential maximal function is defined on the boundary $\partial\Omega$ by

$$N(u)(x') = \max_{k=1,2,\dots,N} \sup\{|u(x)| : x \in \Gamma(x') \cap B_k\}.$$

The non-tangential convergence of $u(x)$ to some number $u(x')$ is defined as

$$\lim_{\Gamma(x') \ni x \rightarrow x'} u(x) = u(x'), \quad x' \in \partial\Omega,$$

provided that the limit exists. Thus only approaches inside the cone $\Gamma(x')$ are considered. Let $n(x')$ denote the normal vector at x' and furthermore, if T is any tangential vector of Ω at x' , the tangential gradient $\nabla_T u$ with respect to T is defined as $\nabla u \cdot T$. We refer to Kenig [10] for further details. The next two lemmas consists of known results which we prove for completeness sake.

Lemma 6.2. *If $g \in L^2(\partial\Omega)$, where $\Omega \subset D$ is a Lipschitz domain, then there exists a unique function $u \in H^1(\Omega)$ such that $(1 - \Delta)u = 0$ in Ω and $\partial_\nu u = g$ on $\partial\Omega$ in the sense that $n \cdot \nabla u \rightarrow g$ nontangentially at almost every point on $\partial\Omega$, where n is the outwards normal. Moreover,*

$$\|N(u)\|_{L^2(\partial\Omega)} + \|N(\nabla u)\|_{L^2(\partial\Omega)} \leq C \|g\|_{L^2(\partial\Omega)},$$

where the constant C depends only on M and B_1, B_2, \dots, B_N and the tangential gradient $\nabla_T u$ exists in $L^2(\partial\Omega)$ in the sense of a weak limit in L^2 of mean value integrals $(\nabla_T u)_r$ (see Section 1.8 of Kenig [10]).

Proof. The problem $(1 - \Delta)u = 0$ in Ω and $\partial_\nu w = g$ on $\partial\Omega$ has a weak solution $w \in H^1(\Omega)$ for every $g \in L^2(\partial\Omega)$ such that

$$\|u\|_{H^1(\Omega)} \leq C \|g\|_{L^2(\partial\Omega)},$$

where C is independent of g and u . Let us extend u to a function $\tilde{u} \in H^1(D)$ with compact support such that $\|\tilde{u}\|_{H^1(D)} \leq C \|u\|_{H^1(\Omega)}$. Put $u = u_0 + u_1$, where $\Delta u_0 = \tilde{u}$ on D and $u_0 = 0$ on ∂D . Then $u_0 \in H^2(D)$ and

$$\|u_0\|_{H^2(D)} \leq C \|g\|_{L^2(\partial\Omega)}.$$

We also obtain that $\Delta u_1 = 0$ in Ω and $\partial_\nu u_1 = h$ with $h = \partial_\nu u - \partial_\nu u_0$ satisfying $\|h\|_{L^2(\partial\Omega)} \leq C \|g\|_{L^2(\partial\Omega)}$.

Suppose that $U = 1$. Then $\Delta U = 0$ and $U = 1$ on $\partial\Omega$, and by Green's formula,

$$\begin{aligned} \int_{\partial\Omega} (\partial_\nu u - \partial_\nu u_0) U dS &= \int_{\Omega} (\nabla(u - u_0) \cdot \nabla U + \Delta(u - u_0) U) dx \\ &= \int_{\Omega} (u - \tilde{u}) dx = 0. \end{aligned}$$

The homogeneous Neumann problem $\Delta u_1 = 0$ in Ω with $\partial_\nu u_1 = h$ on $\partial\Omega$ has a unique solution $u_1 \in H^1(\Omega)$ such that

$$\|N(u_1)\|_{L^2(\partial\Omega)} + \|N(\nabla u_1)\|_{L^2(\partial\Omega)} \leq C \|g\|_{L^2(\partial\Omega)}, \quad (6.3)$$

where N is the non-tangential maximal function; see Jerison and Kenig [8]. Now, (6.3) and the fact that $u = u_0 + u_1$ imply that

$$\|N(u)\|_{L^2(\partial\Omega)} + \|N(\nabla u)\|_{L^2(\partial\Omega)} \leq C \|g\|_{L^2(\partial\Omega)}.$$

For the convergence of the tangential gradient, see Kenig [10]. □

Lemma 6.3. *If $f \in L^2(\Omega)$, where $\Omega \subset D$ is a Lipschitz domain, then there exists a unique function $u \in H^1(\Omega)$ such that $(1 - \Delta)u = f$ in Ω , and $\partial_\nu u = 0$ on $\partial\Omega$ in the nontangential sense. Moreover,*

$$\|N(u)\|_{L^2(\partial\Omega)} + \|N(\nabla u)\|_{L^2(\partial\Omega)} \leq C \|f\|_{L^2(\Omega)}, \quad (6.4)$$

where the constant C depends only on M and B_1, B_2, \dots, B_N .

Proof. Extend $f \in L^2(\Omega)$ by zero to a function $\tilde{f} \in L^2(D)$. Let $v \in H^2(D)$ be the solution to $(1 - \Delta)v = \tilde{f}$ and $v = 0$ on ∂D such that

$$\|v\|_{H^2(D)} \leq C \|f\|_{L^2(\Omega)}, \quad (6.5)$$

and put $u = v + w$. It follows that $(1 - \Delta)w = 0$ in Ω and $\partial_\nu w = -\partial_\nu v$ on $\partial\Omega$. Since $\nabla v \in H^1(\mathbf{R}^n)$ and (6.5) holds, the trace $\partial_\nu v \in L^2(\partial\Omega)$ satisfies

$$\|\partial_\nu v\|_{L^2(\partial\Omega)} \leq C \|v\|_{H^1(\mathbf{R}^n)} \leq C \|f\|_{L^2(\mathbf{R}^n)}. \quad (6.6)$$

Applying Lemma 6.2 with $g = -\partial_\nu v$, we obtain the unique $w \in H^1(\Omega)$ such that $(1 - \Delta)w = 0$, $\partial_\nu w = g$, and

$$\|N(w)\|_{L^2(\partial\Omega)} + \|N(\nabla w)\|_{L^2(\partial\Omega)} \leq C \|f\|_{L^2(\Omega)},$$

where we used (6.6). Since $u = v + w$, we have now proved the statements in the lemma. \square

Notice that Lemmas 6.2 and 6.3 imply that

$$\|N(K_j u)\|_{L^2(\partial\Omega_j)} + \|N(\nabla K_j u)\|_{L^2(\partial\Omega_j)} \leq C \|u\|_{L^2(\Omega_j)}, \quad j = 1, 2. \quad (6.7)$$

6.3. Extension operators. It will be necessary for our purposes to extend functions from either Lipschitz- or $C^{1,\alpha}$ -domains. The following result provides the possibility to accomplish this.

Lemma 6.4. (i) *Suppose that $f \in H^1(\partial\Omega)$ and $g \in L^2(\partial\Omega)$, where Ω is a Lipschitz domain. Then there exists a function $u \in H^1(\Omega^c)$ such that $u \rightarrow f$ and $n \cdot \nabla u \rightarrow g$ nontangentially at almost every point on $\partial\Omega$, where n is the outwards normal of Ω , and there exists a constant C such that*

$$\|N(u)\|_{L^2(\partial\Omega)} + \|N(\nabla u)\|_{L^2(\partial\Omega)} \leq C(\|f\|_{H^1(\partial\Omega)} + \|g\|_{L^2(\partial\Omega)}),$$

where C depends on M and B_1, B_2, \dots, B_N .

(ii) *Suppose that $f \in C^{1,\alpha}(\partial\Omega)$ and $g \in C^{0,\alpha}(\partial\Omega)$, where Ω is a $C^{1,\alpha}$ domain. Then there exists a function $u \in C^{1,\alpha}(\Omega^c)$ such that $u = f$ and $\partial_\nu u = g$ on $\partial\Omega$, and there exists a constant C such that*

$$\|u\|_{C^{1,\alpha}(\Omega^c)} \leq C(\|f\|_{C^{1,\alpha}(\partial\Omega)} + \|g\|_{C^{0,\alpha}(\partial\Omega)}). \quad (6.8)$$

Proof. Let B_k be given as in Section 6.2. Choose $\eta_k \in C_c^\infty(B_k)$, $k = 1, 2, \dots, N$, such that $\eta_1 + \eta_2 + \dots + \eta_N = 1$ in an open neighborhood containing $\partial\Omega$. For each k , define

$$f_k = \eta_k f \quad \text{and} \quad g_k = \eta_k g \quad \text{on } B_k \cap \partial\Omega,$$

and let

$$f_k = g_k = 0 \quad \text{on } \partial B \cap \Omega^c.$$

Let D_k be the bounded domain with boundary $(\partial\Omega \cap B) \cup (\partial B \cap \Omega^c)$. Then D_k is a Lipschitz domain with connected boundary, $f_k \in H^1(\partial D_k)$, and $g_k \in L^2(\partial D_k)$. According to, e.g., Dahlberg et al. [3], there exists a solution u to $\Delta^2 u = 0$ in D_k such that $u \rightarrow f_k$ and $n \cdot \nabla u \rightarrow g_k$ nontangentially at almost every point on ∂D_k , where $-n$ is the outwards normal at ∂D_k . Moreover,

$$\begin{aligned} \|N(u)\|_{L^2(\partial D_k)} + \|N(\nabla u)\|_{L^2(\partial D_k)} &\leq C(\|f_k\|_{H^1(\partial D_k)} + \|g_k\|_{L^2(\partial D_k)}) \\ &\leq C(\|f\|_{H^1(\partial\Omega)} + \|g\|_{L^2(\partial\Omega)}), \end{aligned} \quad (6.9)$$

where C is independent of u , f , and g , but depends on the Lipschitz constant of D_k . Carrying out the same argument for all of the balls B_k in Section 6.2, which is a finite number, we obtain $u \in H^1(D)$, where $D = D_1 \cup D_2 \cup \dots \cup D_m$. We may extend u to all of Ω^c by letting $u = 0$ outside D and obtain $u \in H^1(\Omega^c)$ which satisfies the statement in 6.4(i).

The proof of Lemma 6.4(ii) can be carried out analogously with the Lipschitz case. However, the result is well known for $C^{1,\alpha}$ -domains and the proof is omitted. \square

We will commonly denote the extension for, e.g., a function u , obtained from this Lemma by \tilde{u} .

6.4. Determination of the quantity ε . We now proceed by determining a quantity ε suitable for our purpose. Let us investigate the assertions in (2.3) and (2.4). The assumption in (2.4) is in our case

$$\int_{\Omega_1 \setminus \Omega_2} |\varphi|^2 dx \leq \varepsilon \|\varphi\|_1^2 \quad \text{for every } \varphi \in \mathcal{X}_m. \quad (6.10)$$

There exists a constant C , depending on the domain Ω_1 and λ , such that for every weak solution to the elliptic problem $(1 - \Delta)\varphi = \lambda\varphi$ in Ω_1 with $\partial_\nu \varphi = 0$ on $\partial\Omega_1$,

$$\|\varphi\|_{L^\infty(\Omega_1)} \leq C \|\varphi\|_{L^2(\Omega_1)};$$

see, e.g, Theorem 8.15 in Gilbarg and Trudinger [4]. This enables us to estimate the left-hand side of (6.10) by

$$\int_{\Omega_1 \setminus \Omega_2} |\varphi|^2 dx \leq \|\varphi\|_{L^\infty(\Omega_1 \setminus \Omega_2)}^2 |\Omega_1 \setminus \Omega_2| \leq Cd \|\varphi\|_1^2, \quad (6.11)$$

where d is the Hausdorff distance between Ω_1 and Ω_2 and $|\Omega_1 \setminus \Omega_2|$ is the Lebesgue measure of $\Omega_1 \setminus \Omega_2$.

To prove the assertion in (2.3), we use the following lemmas.

Lemma 6.5. *Suppose that $v = K_2 S_2 w - S_2 K_1 S_1 w$, where $w \in L^2(D)$. Then v satisfies $(1 - \Delta)v = 0$ in $\Omega_1 \cap \Omega_2$ and $v \in H^1(\partial(\Omega_1 \cap \Omega_2))$. Moreover, there exists a positive constant C , depending only on M and B_1, B_2, \dots, B_N , such that*

(i) *if $w \in L^2(\Omega_1 \cap \Omega_2)$, then*

$$\|v\|_{H^1(\Omega_1 \cap \Omega_2)}^2 \leq Cd \|g\|_{L^2(\partial(\Omega_1 \cap \Omega_2))} \|w\|_{L^2(\Omega_1 \cap \Omega_2)},$$

(ii) *if $w \in L^2(\Omega_1 \setminus \Omega_2)$, then*

$$\|v\|_{H^1(\Omega_1 \cap \Omega_2)}^2 \leq Cd^{1/2} \|g\|_{L^2(\partial(\Omega_1 \cap \Omega_2))} \|w\|_{L^2(\Omega_1 \setminus \Omega_2)},$$

(iii) *and if $w \in L^2(\Omega_2 \setminus \Omega_1)$, then*

$$\|v\|_{H^1(\Omega_1 \cap \Omega_2)}^2 \leq Cd^{1/2} \|g\|_{L^2(\partial(\Omega_1 \cap \Omega_2))} \|w\|_{L^2(\Omega_2 \setminus \Omega_1)},$$

where w is extended by zero outside the respective domains, and

$$g = \partial_\nu v \quad \text{on } \partial(\Omega_1 \cap \Omega_2).$$

Proof. Since $v \in H^1(\Omega_1 \cap \Omega_2)$ satisfies $(1 - \Delta)v = 0$ in $\Omega_1 \cap \Omega_2$ and $\partial_\nu v$ belongs to $L^2(\partial(\Omega_1 \cap \Omega_2))$, Lemma 6.2 implies that

$$\|N(v)\|_{L^2(\partial(\Omega_1 \cap \Omega_2))} + \|N(\nabla v)\|_{L^2(\partial(\Omega_1 \cap \Omega_2))} \leq C \|g\|_{L^2(\partial(\Omega_1 \cap \Omega_2))} \quad (6.12)$$

and that $v \in H^1(\partial(\Omega_1 \cap \Omega_2))$. Moreover, Lemma 6.4(i) ensures the existence of an extension $\tilde{v} \in H^1(\mathbf{R}^n)$ such that

$$\|N(\tilde{v})\|_{L^2(\partial(\Omega_1 \cap \Omega_2))} + \|N(\nabla \tilde{v})\|_{L^2(\partial(\Omega_1 \cap \Omega_2))} \leq C \|g\|_{L^2(\partial(\Omega_1 \cap \Omega_2))}. \quad (6.13)$$

Now,

$$\begin{aligned}
& \int_{\Omega_1 \cap \Omega_2} (v^2 + |\nabla v|^2) dx \\
&= \int_{\partial(\Omega_1 \cap \Omega_2)} v \partial_\nu v dS \\
&= \int_{\partial\Omega_1 \cap \Omega_2} v \partial_\nu K_2 S_2 w dS - \int_{\Omega_1 \cap \partial\Omega_2} v \partial_\nu K_1 S_1 w dS \\
&= - \int_{\partial(\Omega_2 \setminus \Omega_1)} \tilde{v} \partial_\nu K_2 S_2 w dS + \int_{\partial(\Omega_1 \setminus \Omega_2)} \tilde{v} \partial_\nu K_1 S_1 w dS,
\end{aligned}$$

where we used the fact that $\partial_\nu K_2 S_2 w = 0$ on $\partial\Omega_2$ and $\partial_\nu K_1 S_1 w = 0$ on $\partial\Omega_1$.

Since $(1 - \Delta)K_2 S_2 w = S_2 w$ in $\Omega_2 \setminus \Omega_1$, we obtain that

$$- \int_{\partial(\Omega_2 \setminus \Omega_1)} \tilde{v} \partial_\nu K_2 S_2 w dS = \int_{\Omega_2 \setminus \Omega_1} (\tilde{v} S_2 w - \tilde{v} K_2 S_2 w - \nabla \tilde{v} \cdot \nabla K_2 S_2 w) dx. \tag{6.14}$$

If $w \in L^2(\Omega_1 \cap \Omega_2)$, then $S_2 w = 0$ and the right-hand side of (6.14) is bounded by

$$Cd \|g\|_{L^2(\partial(\Omega_1 \cap \Omega_2))} \|w\|_{L^2(\Omega_1 \cap \Omega_2)}. \tag{6.15}$$

This follows from the Cauchy–Schwarz inequality, (6.13), and (6.7), since, e.g.,

$$\begin{aligned}
& \int_{\Omega_2 \setminus \Omega_1} |\tilde{v} K_2 S_2 w| dx, \\
& \leq \left(\int_{\Omega_2 \setminus \Omega_1} \tilde{v}^2 dx \right)^{1/2} \left(\int_{\Omega_2 \setminus \Omega_1} (K_2 S_2 w)^2 dx \right)^{1/2} \\
& \leq Cd \left(\int_{\partial(\Omega_1 \cap \Omega_2)} N(\tilde{v})^2 dx' \right)^{1/2} \left(\int_{\partial(\Omega_1 \cap \Omega_2)} N(K_2 S_2 w)^2 dx' \right)^{1/2}.
\end{aligned}$$

If $w \in L^2(\Omega_1 \setminus \Omega_2)$, then $S_2 w = 0$, and analogously with (6.15), the expression in (6.14) is bounded by $Cd \|g\|_{L^2(\partial(\Omega_1 \cap \Omega_2))} \|w\|_{L^2(\Omega_1 \setminus \Omega_2)}$. If $w \in L^2(\Omega_2 \setminus \Omega_1)$, then $S_2 w = w$. Since

$$\int_{\Omega_2 \setminus \Omega_1} |\tilde{v} w| dx \leq Cd^{1/2} \|g\|_{L^2(\partial(\Omega_1 \cap \Omega_2))} \|w\|_{L^2(\Omega_2 \setminus \Omega_1)},$$

we obtain that (6.14) is bounded by $Cd^{1/2} \|g\|_{L^2(\partial(\Omega_1 \cap \Omega_2))} \|w\|_{L^2(\Omega_2 \setminus \Omega_1)}$.

Analogously, the expression

$$\int_{\partial(\Omega_1 \setminus \Omega_2)} \tilde{v} \partial_\nu K_1 S_1 w \, dS = \int_{\Omega_1 \setminus \Omega_2} (\tilde{v} K_1 S_1 w + \nabla \tilde{v} \cdot \nabla K_1 S_1 w - \tilde{v} S_1 w) \, dx$$

is bounded by

$$Cd \|g\|_{L^2(\partial(\Omega_1 \cap \Omega_2))} \|w\|_{L^2(\Omega_1 \cap \Omega_2)} \quad \text{if } w \in L^2(\Omega_1 \cap \Omega_2),$$

$$Cd \|g\|_{L^2(\partial(\Omega_1 \cap \Omega_2))} \|w\|_{L^2(\Omega_2 \setminus \Omega_1)} \quad \text{if } w \in L^2(\Omega_2 \setminus \Omega_1),$$

$$Cd^{1/2} \|g\|_{L^2(\partial(\Omega_1 \cap \Omega_2))} \|w\|_{L^2(\Omega_1 \setminus \Omega_2)} \quad \text{if } w \in L^2(\Omega_1 \setminus \Omega_2),$$

respectively. \square

Lemma 6.6. *There exists a constant $C > 0$ such that*

$$\|K_2 w - SK_1 S^* w\|^2 \leq Cd^{1/2} \|w\|^2 \quad \text{for every } w \in L^2(\Omega_2) \quad (6.16)$$

and

$$\|B\varphi\|^2 \leq Cd \|\varphi\|^2 \quad \text{for every } \varphi \in \mathcal{X}_m. \quad (6.17)$$

Proof. Put $v = K_2 w - SK_1 S^* w$. We split the domain Ω_2 in two disjoint subdomains: $\Omega_1 \cap \Omega_2$ and $\Omega_2 \setminus \Omega_1$. For the subdomain $\Omega_2 \setminus \Omega_1$, it is clear from (6.7) that

$$\int_{\Omega_2 \setminus \Omega_1} v^2 \, dx = \int_{\Omega_2 \setminus \Omega_1} (K_2 w)^2 \, dx \leq Cd \|w\|_{L^2(\Omega_2)}^2. \quad (6.18)$$

Lemma 6.5 now implies the inequality in (6.16) since

$$\|g\|_{L^2(\partial(\Omega_1 \cap \Omega_2))} \leq C \|w\|_{L^2(\Omega_2)}. \quad (6.19)$$

To prove (6.17), observe first that (6.11) holds. Thus, by letting $w = \varphi$, we can apply Lemma 6.5 with $v = B\varphi$ and obtain that

$$\int_{\Omega_1 \cap \Omega_2} (B\varphi)^2 \, dx \leq Cd \|g\|_{L^2(\partial(\Omega_1 \cap \Omega_2))} \|\varphi\|_{L^2(\Omega_1)}.$$

Since also $B\varphi = v$ on $\Omega_2 \setminus \Omega_1$, inequalities (6.18) and (6.19) are applicable, which concludes the proof of (6.17). \square

Thus, by (6.16) and (6.11), it is clear that we can choose $\varepsilon = Cd^{1/2}$. Furthermore, if Ω_2 is a subdomain of Ω_1 , we obtain a bound depending on d instead of $d^{1/2}$ for a general function $w \in L^2(\Omega_2)$; this is a consequence of that fact that the term $\|w\|_{L^2(\Omega_2 \setminus \Omega_1)}$ vanishes in Lemma 6.5 when $\Omega_2 \subset \Omega_1$.

Remark 6.7. If $\Omega_2 \subset \Omega_1$, then

$$\|K_2 w - SK_1 S^* w\|^2 \leq Cd \|w\|^2 \quad \text{for every } w \in L^2(\Omega_2). \quad (6.20)$$

6.5. Main results for Lipschitz domains. We now derive an expression for the right-hand side of (3.3) and prove that in comparison, the remainder is small. We will then use Theorem 3.2 to obtain a result for eigenvalues of K_2 near λ_m^{-1} .

Lemma 6.8. *If $w \in L^2(\Omega_2)$, then*

$$\begin{aligned} \lambda_m \int_{\Omega_2} B\varphi K_2 w \, dx &= \int_{\Omega_1 \setminus \Omega_2} ((1 - \lambda_m)W\varphi + \nabla W \cdot \nabla \varphi) \, dx \\ &\quad - \int_{\Omega_2 \setminus \Omega_1} ((W - K_2 w)\tilde{\varphi} + \nabla W \cdot \nabla \tilde{\varphi}) \, dx, \end{aligned} \quad (6.21)$$

where $W \in H^1(\mathbf{R}^n)$ is an extension of $K_2^2 w \in H^1(\Omega_2)$.

Proof. We proceed similarly with the proof of Lemma 6.6. Since $(1 - \Delta)B\varphi = 0$ in $\Omega_1 \cap \Omega_2$, we obtain using Green's formula that

$$\begin{aligned} \int_{\Omega_1 \cap \Omega_2} B\varphi K_2 w \, dx &= \int_{\Omega_1 \cap \Omega_2} B\varphi(1 - \Delta)K_2^2 w \, dx \\ &= \int_{\partial(\Omega_1 \cap \Omega_2)} (K_2^2 w \partial_\nu B\varphi - B\varphi \partial_\nu K_2^2 w) \, dS \\ &= \int_{\partial\Omega_1 \cap \Omega_2} (K_2^2 w \partial_\nu K_2 S\psi - B\varphi \partial_\nu K_2^2 w) \, dS \\ &\quad - \int_{\Omega_1 \cap \partial\Omega_2} K_2^2 w \partial_\nu K_1 \varphi \, dS. \end{aligned}$$

Furthermore, $(1 - \Delta)K_2 w = w$ in $\Omega_2 \setminus \Omega_1$ and $\partial_\nu K_2 S\varphi = 0$ on $\partial\Omega_2$. Thus,

$$\begin{aligned} \int_{\partial\Omega_1 \cap \Omega_2} K_2^2 w \partial_\nu K_2 S\varphi \, dS &= - \int_{\partial(\Omega_2 \setminus \Omega_1)} K_2^2 w \partial_\nu K_2 S\varphi \, dS \\ &= - \int_{\Omega_2 \setminus \Omega_1} (K_2^2 w K_2 S\varphi + \nabla K_2^2 w \cdot \nabla K_2 S\varphi) \, dx \\ &\quad - \int_{\Omega_2 \setminus \Omega_1} K_2 w K_2 S\varphi \, dx, \end{aligned}$$

and analogously,

$$\begin{aligned} - \int_{\partial\Omega_1 \cap \Omega_2} B\varphi \partial_\nu K_2^2 w \, dS &= \int_{\partial(\Omega_2 \setminus \Omega_1)} \tilde{B}\varphi \partial_\nu K_2^2 w \, dS \\ &= \int_{\Omega_2 \setminus \Omega_1} (\tilde{B}\varphi K_2^2 w + \nabla \tilde{B}\varphi \cdot \nabla K_2^2 w) \, dx \\ &\quad - \int_{\Omega_2 \setminus \Omega_1} \tilde{B}\varphi K_2 w \, dx, \end{aligned}$$

where $\tilde{B}u = K_2 S u - S \widetilde{K_1} u$ for $u \in L^2(\Omega_1)$ satisfies

$$\begin{aligned} \|\tilde{B}u - Bu\|_{L^2(\Omega_2)}^2 &= \int_{\Omega_2 \setminus \Omega_1} |\widetilde{K_1} u|^2 \, dx \\ &\leq Cd \int_{\partial\Omega_1} |N(K_1 u)|^2 \, dx' \\ &\leq Cd \|u\|_{L^2(\Omega_1)}^2 \end{aligned}$$

by Lemma 6.4(i) and inequality (6.7),

Similar to the treatment of the previous boundary integrals, it follows from the facts that $(1 - \Delta)K_1\varphi = \varphi$ in $\Omega_1 \setminus \Omega_2$ and $\partial_\nu K_1\varphi = 0$ on $\partial\Omega_1$, that

$$\begin{aligned} - \int_{\Omega_1 \cap \partial\Omega_2} K_2^2 w \partial_\nu K_1\varphi \, dS &= \int_{\partial(\Omega_2 \setminus \Omega_1)} W \partial_\nu K_1\varphi \, dS \\ &= \int_{\Omega_1 \setminus \Omega_2} (WK_1\varphi + \nabla W \cdot \nabla K_1\varphi - W\varphi) \, dx. \end{aligned}$$

We have now proved that

$$\begin{aligned} \int_{\Omega_2} B\varphi K_2 w \, dx &= \int_{\Omega_1 \cap \Omega_2} B\varphi K_2 w \, dx + \int_{\Omega_2 \setminus \Omega_1} B\varphi K_2 w \, dx \\ &= \lambda_m^{-1} \int_{\Omega_1 \setminus \Omega_2} ((1 - \lambda_m)W\varphi + \nabla W \cdot \nabla\varphi) \, dx \\ &\quad - \lambda_m^{-1} \int_{\Omega_2 \setminus \Omega_1} ((K_2^2 w - K_2 w)\tilde{\varphi} + \nabla K_2^2 w \cdot \nabla\tilde{\varphi}) \, dx. \end{aligned}$$

This is the equality in (6.21). □

Lemma 6.9. *There exists a constant $C > 0$ such that*

$$\|K_2 B\varphi\|_{L^2(\Omega_2)}^2 \leq Cd^{3/2} \|\varphi\|_{L^2(\Omega_1)}^2 \quad \text{for every } \varphi \in X_m. \quad (6.22)$$

Proof. Since $K_2 B\varphi$ is a solution to $(1 - \Delta)K_2 B\varphi = B\varphi$ with $\partial_\nu K_2 B\varphi = 0$ on $\partial\Omega_2$, we obtain that

$$\int_{\Omega_2} ((K_2 B\varphi)^2 + |\nabla K_2 B\varphi|^2) dx = \int_{\Omega_2} B\varphi K_2 B\varphi dx.$$

Let W be given as in Lemma 6.8 with $w = B\varphi$. Then

$$\|N(W)\|_{L^2(\partial\Omega_2)} + \|N(\nabla W)\|_{L^2(\partial\Omega_2)} \leq C \|B\varphi\|_{L^2(\Omega_2)}$$

and Lemma 6.8 implies that

$$\int_{\Omega_2} B\varphi K_2 B\varphi dx \leq Cd \|B\varphi\|_{L^2(\Omega_2)} \|\varphi\|_{L^2(\Omega_1)}.$$

Since $\|B\varphi\|_{L^2(\Omega_2)} \leq Cd^{1/2} \|\varphi\|_{L^2(\Omega_1)}$ according to Lemma 6.6, we obtain the inequality in (6.22). \square

We now have all the tools available to prove our main result for Lipschitz domains, i.e., expressing the difference between eigenvalues λ_m^{-1} and μ_k^{-1} in known terms.

Proposition 6.10. *Suppose that Ω_1 and Ω_2 are Lipschitz domains in the sense of Section 6.1. Then*

$$\lambda_m^{-1} - \mu_k^{-1} = \tau_k + O(d^{3/2}) \quad \text{for } k = 1, 2, \dots, J_m. \quad (6.23)$$

Here, $\tau = \tau_k$ is an eigenvalue of

$$\begin{aligned} \tau(\varphi, \psi) = & \lambda_m^{-1} \int_{\Omega_1 \setminus \Omega_2} ((1 - \lambda_m) \widetilde{K_2 S\varphi} \psi + \nabla \widetilde{K_2 S\varphi} \cdot \nabla \psi) dx \\ & - \lambda_m^{-1} \int_{\Omega_2 \setminus \Omega_1} ((1 - \lambda_m) (K_2 S\varphi) \tilde{\psi} + \nabla K_2 S\varphi \cdot \nabla \tilde{\psi}) dx \end{aligned} \quad (6.24)$$

for all $\psi \in X_m$, where $\varphi \in X_m$. Moreover, $\tau_1, \tau_2, \dots, \tau_{J_m}$ in (6.23) run through all eigenvalues of (6.24) counting their multiplicities.

Proof. We express $K_2^2 S\psi$ in terms of the operator B :

$$K_2^2 S\psi = K_2(B\psi + SK_1\psi) = K_2 B\psi + BK_1\psi + SK_1^2\psi.$$

If $\psi \in X_m$, then

$$K_2^2 S\psi = K_2 B\psi + \lambda_m^{-1} K_2 S\psi.$$

Put

$$W = \widetilde{K_2 B \psi} + \lambda_m^{-1} \widetilde{K_2 S \psi},$$

where $\widetilde{K_2 B \psi}$ and $\widetilde{K_2 S \psi}$ are the extensions of $K_2 B \psi$ and $K_2 S \psi$, respectively, given by Lemma 6.4(i). Then $W \in H^1(\mathbf{R}^n)$ and

$$\begin{aligned} \|N(W)\|_{L^2(\partial\Omega_2)} + \|N(\nabla W)\|_{L^2(\partial\Omega_2)} &\leq C(\|B\psi\|_{L^2(\Omega_2)} + \|S\psi\|_{L^2(\Omega_2)}) \\ &\leq C\|\psi\|_{L^2(\Omega_1)}. \end{aligned} \quad (6.25)$$

Lemma 6.8 and (6.25) proves that

$$\begin{aligned} \lambda_m^2 \int_{\Omega_2} B\varphi K_2 S \psi \, dx &= \int_{\Omega_1 \setminus \Omega_2} ((1 - \lambda_m) \widetilde{K_2 S \psi} \varphi + \nabla \widetilde{K_2 S \psi} \cdot \nabla \varphi) \, dx \\ &\quad - \int_{\Omega_2 \setminus \Omega_1} ((1 - \lambda_m)(K_2 S \psi) \tilde{\varphi} + \nabla K_2 S \psi \cdot \nabla \tilde{\varphi}) \, dx \\ &\quad + O(d^{3/2})\|\varphi\|_{L^2(\Omega_1)}\|\psi\|_{L^2(\Omega_1)}. \end{aligned} \quad (6.26)$$

Observe also that (6.26) implies that

$$|(B\varphi, K_2 S \psi)| \leq Cd\|\varphi\|_{L^2(\Omega_1)}\|\psi\|_{L^2(\Omega_1)}. \quad (6.27)$$

Lemmas 6.6 and 6.9 imply that

$$\rho = \lambda_m^{-1} \sup_{\|\varphi\|=1} (\|K_2 B \varphi\|^2 + \varepsilon\|B\varphi\|^2) = O(d^{3/2}).$$

Thus, Theorem 3.2 proves that we obtain

$$\mu_k^{-1} - \lambda_m^{-1} = \tau_k + O(\rho + |\tau_k|d^{1/2}) = \tau_k + O(d^{3/2}) \quad (6.28)$$

since

$$\tau_k(\varphi, \psi) = \tau_k(S\varphi, S\psi) + O(|\tau_k|d) = \lambda_m(B\varphi, K_2 S \psi) + O(d^2). \quad (6.29)$$

Now, equations (6.28), (6.29), and (6.26), imply (6.23). \square

From (6.27) we obtain the following corollary.

Corollary 6.11. *With the same assumptions as in Proposition 6.10, there exists a constant C , independent of d , such that*

$$|\lambda_m^{-1} - \mu_k^{-1}| \leq Cd$$

for $k = 1, 2, \dots, J_m$.

If $\Omega_2 \subset \Omega_1$, the solution $v_\varphi = B\varphi$ to $(1 - \Delta)v_\varphi = 0$ and $\partial_\nu v_\varphi = -\lambda_m^{-1} \partial_\nu \varphi$ for $\varphi \in X_m$ can be used to formulate the results above in terms of this solution. This can be an advantage since in many cases these type of partial differential equations are well studied and explicit solutions or estimates for solutions are known. Moreover, we also present an example in Section 6.7 based on this proposition, proving that the condition $\alpha > 0$ is sharp for our result in the $C^{1,\alpha}$ -case.

Proposition 6.12. *Suppose that Ω_1 and Ω_2 are Lipschitz domains in the sense of Section 6.1 and that $\Omega_2 \subset \Omega_1$. Then*

$$\lambda_m^{-1} - \mu_k^{-1} = \tau_k + O(d^{3/2}) \quad \text{for } k = 1, 2, \dots, J_m. \tag{6.30}$$

Here, $\tau = \tau_k$ is an eigenvalue of

$$\tau(\varphi, \psi) = \int_{\Omega_2} (\lambda_m v_\varphi v_\psi + v_\varphi \psi) dx \tag{6.31}$$

for all $\psi \in X_m$, where $\varphi \in X_m$. Moreover, $\tau_1, \tau_2, \dots, \tau_{J_m}$ in (6.30) run through all eigenvalues of (6.31) counting their multiplicities.

6.6. The case of a $C^{1,\alpha}$ domain. We now consider the case when Ω_1 and Ω_2 are $C^{1,\alpha}$ domains, where $0 < \alpha < 1$.

Lemma 6.13. *If Ω_1 is a $C^{1,\alpha}$ -domain, then for every $u \in L^\infty(\mathbf{R}^n)$, $K_1 S_1 u$ belongs to $C^{1,\alpha}(\Omega_1)$.*

Proof. This follows from the results in Section 9 of Agmon et al. [1]. □

Lemma 6.14. *There exists a constant $C > 0$ such that*

$$\sup_{x' \in \partial(\Omega_1 \cap \Omega_2)} |\partial_\nu K_j S_j w(x')| \leq C \|w\|_{L^2(\Omega_1 \cap \Omega_2)} d^\alpha, \quad j = 1, 2, \tag{6.32}$$

for every $w \in L^2(\Omega_1 \cap \Omega_2)$.

Proof. Let n_j be the outwards normal on $\partial\Omega_j$ for $j = 1, 2$. On the boundary $\partial\Omega_1$, $\partial_\nu K_1 S_1 w = 0$, and on $\partial\Omega_2$, $\partial_\nu K_2 S_2 w = 0$. We prove (6.32) for $j = 2$. The proof when $j = 1$ is analogous. Thus,

$$\partial_\nu K_2 S_2 w = n_1 \cdot \nabla K_2 S_2 w = (n_1 - n_2) \cdot \nabla K_2 S_2 w + n_2 \cdot \nabla K_2 S_2 w,$$

and since it is clear that $n_2 \cdot \nabla K_2 S_2 w = 0$ on $\partial\Omega_2$,

$$\sup_{\partial(\Omega_1 \cap \Omega_2)} |\partial_{n_1} K_2 S_2 w| \leq C d^\alpha \|\nabla K_2 S_2 w\|_{L^2(\Omega_2)} \leq C d^\alpha \|w\|_{L^2(\Omega_1 \cap \Omega_2)}.$$

Here, we also used the fact that there exists a constant C , independent of $K_2 S_2 w$, such that $\|\nabla K_2 S_2 w\|_{L^\infty(\Omega_2)} \leq C \|K_2 S_2 w\|_{H^1(\Omega_2)}$. Moreover,

$$|n_1 - n_2| \leq |\nabla(h_k^{(1)} - h_k^{(2)})| \leq C d^\alpha,$$

so we obtain (6.32) for $j = 2$. \square

We can use Lemma 6.14 to refine the estimates provided in Lemma 6.6.

Lemma 6.15. *There exists a constant $C > 0$ such that*

$$\|K_2 w - S K_1 S^* w\|^2 \leq C d^{1/2+\alpha} \|w\|^2 \quad \text{for every } w \in L^2(\Omega_2) \quad (6.33)$$

and

$$\|B\varphi\|^2 \leq C d^{1+\alpha} \|\varphi\|^2 \quad \text{for every } \varphi \in \mathcal{X}_m. \quad (6.34)$$

Proof. Proceeding as in Lemma 6.6, we obtain the inequality in (6.33) and also that

$$\|B\varphi\|_{L^2(\Omega_1 \cap \Omega_2)}^2 \leq C d^{1+\alpha} \|\varphi\|_{L^2(\Omega_1)}^2$$

since (6.32) implies that

$$\|g\|_{L^2(\partial(\Omega_1 \cap \Omega_2))} \leq C d^\alpha \|w\|_{L^2(\Omega_2)}, \quad (6.35)$$

where g is as in Lemma 6.5.

In $\Omega_2 \setminus \Omega_1$, $B\varphi = K_2 S\varphi$. Thus, $B\varphi$ is a solution to $(1 - \Delta)B\varphi = S\varphi = 0$ in $\Omega_2 \setminus \Omega_1$ such that $\partial_\nu B\varphi = 0$ on $\partial\Omega_2 \cap \Omega_1^c$ and $\partial_\nu B\varphi = \partial_\nu K_2 S\varphi$ on $\Omega_2^c \cap \partial\Omega_1$. Lemma 6.14 with $w = S\varphi$ now implies that

$$\|\partial_\nu B\varphi\|_{L^2(\partial(\Omega_2 \setminus \Omega_1))} \leq C d^\alpha \|\varphi\|_{L^2(\Omega_1)}$$

and thus, Lemma 6.2 proves that

$$\|N(B\varphi)\|_{L^2(\partial(\Omega_2 \setminus \Omega_1))} + \|N(\nabla B\varphi)\|_{L^2(\partial(\Omega_2 \setminus \Omega_1))} \leq C d^\alpha \|\varphi\|_{L^2(\Omega_1)}.$$

Hence,

$$\int_{\Omega_2 \setminus \Omega_1} (B\varphi)^2 dx \leq C d^{1+2\alpha} \|\varphi\|_{L^2(\Omega_1)}^2. \quad \square$$

Since a $C^{1,\alpha}$ domain can be considered a Lipschitz domain, we know that the results from the previous section hold for $\varepsilon = C d^{1/2}$. However, in this case we may choose $\varepsilon = C d^{\alpha+1/2}$ if $\alpha \leq 1/2$. This is clear from Lemma 6.15 and inequality (6.11). If $\alpha > 1/2$, we may choose $\varepsilon = C d$. Inequality (6.11) is the reason for the restriction on α .

Similarly to the Lipschitz case, we shall employ Lemma 6.8 to obtain information about the difference $\mu_k^{-1} - \lambda_m^{-1}$. However, we wish to express the extension W in more explicit terms that depend directly on the eigenfunction ψ .

Proposition 6.16. *Suppose that Ω_1 is a $C^{1,\alpha}$ domain and Ω_2 is a perturbation in the sense of Section 6.1 which satisfies (6.2). Then*

$$\lambda_m^{-1} - \mu_k^{-1} = \tau_k + O(d^{1+\alpha}) \quad \text{for } k = 1, 2, \dots, J_m. \quad (6.36)$$

Here, $\tau = \tau_k$ is an eigenvalue of

$$\begin{aligned} \tau(\varphi, \psi) = \lambda_m^{-2} \left(\int_{\Omega_1 \setminus \Omega_2} ((1 - \lambda_m)\varphi\psi + \nabla\varphi \cdot \nabla\psi) dx \right. \\ \left. - \int_{\Omega_2 \setminus \Omega_1} ((1 - \lambda_m)\tilde{\varphi}\tilde{\psi} + \nabla\tilde{\varphi} \cdot \nabla\tilde{\psi}) dx \right) \end{aligned} \quad (6.37)$$

for all $\psi \in X_m$, where $\varphi \in X_m$. Moreover, $\tau_1, \tau_2, \dots, \tau_{J_m}$ in (6.36) run through all eigenvalues of (6.37) counting their multiplicities.

Proof. We express $K_2^2 S\psi$ in terms of the operator B :

$$K_2^2 S\psi = K_2(B\psi + SK_1\psi) = K_2 B\psi + BK_1\psi + SK_1^2\psi.$$

If $\psi \in X_m$, then

$$K_2^2 S\psi = K_2 B\psi + \lambda_m^{-1} K_2 S\psi.$$

Put

$$W = \widetilde{K_2 B\psi} + \lambda_m^{-1} \widetilde{K_2 S\psi},$$

where $\widetilde{K_2 B\psi}$ and $\widetilde{K_2 S\psi}$ are the extensions of $K_2 B\psi$ and $K_2 S\psi$, respectively, given by Lemma 6.4(i). Then $W \in H^1(\mathbf{R}^n)$ and

$$\begin{aligned} \|N(W)\|_{L^2(\partial\Omega_2)} + \|N(\nabla W)\|_{L^2(\partial\Omega_2)} &\leq C(\|B\psi\|_{L^2(\Omega_2)} + \|S\psi\|_{L^2(\Omega_2)}) \\ &\leq C\|\psi\|_{L^2(\Omega_1)}. \end{aligned} \quad (6.38)$$

Moreover, $W = \lambda_m^{-2}\tilde{\psi} + \widetilde{K_2 B\psi} + \lambda_m^{-1}r$ on $\Omega_1 \setminus \Omega_2$ and $\Omega_2 \setminus \Omega_1$, where r is defined as follows. On $\Omega_1 \cap \Omega_2$, we let $r = B\psi$. Then we extend r to \mathbf{R}^n such that $r = \widetilde{K_2 S\psi} - \lambda_m^{-1}\psi$ in $\Omega_1 \setminus \Omega_2$ and $r = K_2 S\psi - \lambda_m^{-1}\tilde{\psi}$ in $\Omega_2 \setminus \Omega_1$, where the extensions $\widetilde{K_2 S\psi}$ and $\tilde{\psi}$ are given by Lemma 6.4(i) and Lemma 6.4(ii), respectively. It is now possible to use the same argument employed in the proof of Lemma 6.15 to obtain that

$$\|N(r)\|_{L^2(\partial(\Omega_1 \cap \Omega_2))} + \|N(\nabla r)\|_{L^2(\partial(\Omega_1 \cap \Omega_2))} \leq Cd^\alpha \|\psi\|_{L^2(\Omega_1)}.$$

Now, this fact and the Cauchy–Schwarz inequality proves that

$$\int_{\Omega_2 \setminus \Omega_1} (|r\tilde{\varphi}| + |\nabla r \cdot \nabla \tilde{\varphi}|) dx \leq Cd^{1+\alpha} \|\psi\|_{L^2(\Omega_1)} \|\varphi\|_{L^2(\Omega_1)}.$$

Similarly, we can bound the corresponding integral over the domain $\Omega_1 \setminus \Omega_2$. Using (6.34), we can also refine the estimate given in (6.9):

$$\|K_2 B \psi\|_{L^2(\Omega_2)}^2 \leq Cd^{3/2+\alpha/2} \|\psi\|_{L^2(\Omega_1)}^2 \leq Cd^{1+\alpha} \|\psi\|_{L^2(\Omega_1)}^2. \quad (6.39)$$

Thus, we obtain from Lemma 6.8 and inequality (6.38) that

$$\begin{aligned} \lambda_m \int_{\Omega_2} B \varphi K_2 S \psi dx &= \lambda_m^{-2} \int_{\Omega_1 \setminus \Omega_2} ((1 - \lambda_m) \varphi \psi + \nabla \varphi \cdot \nabla \psi) dx \\ &\quad - \lambda_m^{-2} \int_{\Omega_2 \setminus \Omega_1} ((1 - \lambda_m) \tilde{\varphi} \tilde{\psi} + \nabla \tilde{\varphi} \cdot \nabla \tilde{\psi}) dx \\ &\quad + O(d^{1+\alpha}) \|\varphi\|_{L^2(\Omega_1)} \|\psi\|_{L^2(\Omega_1)}. \end{aligned} \quad (6.40)$$

Inequalities (6.39) and (6.34) imply that

$$\rho = \lambda_m^{-1} \sup_{\|\varphi\|=1} (\|K_2 B \varphi\|^2 + \varepsilon \|B \varphi\|^2) = O(d^{(3+\alpha)/2}) \quad (6.41)$$

and thus, Theorem 3.2 proves that

$$\mu_k^{-1} - \lambda_m^{-1} = \tau_k + O(\rho + |\tau_k| \varepsilon) = \tau_k + O(d^{1+\alpha})$$

since we can choose

$$\varepsilon = Cd^{\alpha+1/2} \quad \text{if } \alpha \leq 1/2$$

and

$$\varepsilon = Cd \quad \text{if } \alpha > 1/2,$$

and

$$\tau_k(\varphi, \psi) = \tau_k(S\varphi, S\psi) + O(|\tau_k|d) = \lambda_m(B\varphi, K_2 S\psi) + O(d^2). \quad (6.42)$$

Now, equations (6.41), (6.42), and (6.40), imply (6.36). \square

Suppose that it is possible to characterize the perturbed domain Ω_2 by a function h defined on the boundary $\partial\Omega_1$ such that $(x', x_\nu) \in \partial\Omega_2$ is represented by $x_\nu = h(x')$, where $(x', 0) \in \partial\Omega_1$ and x_ν is the signed distance to the boundary $\partial\Omega_1$ (with $x_\nu < 0$ when $x \in \Omega_1$). The function h is assumed to be Lipschitz and satisfy $|\nabla h| \leq Cd^\alpha$. Thus, we obtain the following variation of Proposition 6.16.

Corollary 6.17. *Suppose that in addition to the assumptions of Proposition 6.16, the domain Ω_2 can be characterized by the function h as above. Then*

$$\lambda_m^{-1} - \mu_k^{-1} = \tau_k + O(d^{1+\alpha}) \quad (6.43)$$

for $k = 1, 2, \dots, J_m$. Here, $\tau = \tau_k$ is an eigenvalue of

$$\tau(\varphi, \psi) = \lambda_m^{-2} \int_{\partial\Omega_1} h(x')((1 - \lambda_m)\varphi\psi + \nabla\varphi \cdot \nabla\psi) dS(x') \quad \text{for all } \psi \in X_m, \quad (6.44)$$

where $\varphi \in X_m$. Moreover, $\tau_1, \tau_2, \dots, \tau_{J_m}$ in (6.43) run through all eigenvalues of (6.44) counting their multiplicities.

Proof. We first prove that

$$\begin{cases} \sup_{(x', x_\nu) \in \Omega_1 \setminus \Omega_2} |\varphi(x', x_\nu) - \varphi(x', 0)| \leq Cd^{1+\alpha} \|\varphi\|_{L^2(\Omega_1)}, \\ \sup_{(x', x_\nu) \in \Omega_1 \setminus \Omega_2} |\nabla\varphi(x', x_\nu) - \nabla\varphi(x', 0)| \leq Cd^\alpha \|\varphi\|_{L^2(\Omega_1)}, \end{cases} \quad (6.45)$$

where the corresponding estimates hold for $\tilde{\varphi}$ on $\Omega_2 \setminus \Omega_1$. Since $\varphi \in C^{1,\alpha}(\Omega_1)$, it is clear that for $x = (x', x_\nu) \in \Omega_1 \setminus \Omega_2$,

$$\varphi(x', x_\nu) = \varphi(x', 0) + x_\nu \partial_\nu \varphi(x', 0) + O(d^{1+\alpha}),$$

where the remainder is bounded by $Cd^{1+\alpha} \|\varphi\|_{L^2(\Omega_1)}$. This shows that the first inequality in (6.45) is true. Similarly, the second inequality in (6.45) is also valid. Thus,

$$\int_{\Omega_1 \setminus \Omega_2} (|\varphi(x) - \varphi(x', 0)|^2 + |\nabla\varphi(x) - \nabla\varphi(x', 0)|^2) dx \leq Cd^{1+\alpha} \|\varphi\|_{L^2(\Omega_1)}^2,$$

with the corresponding estimate for $\tilde{\varphi}$ on $\Omega_2 \setminus \Omega_1$. Hence, Proposition 6.16 implies that $\lambda_m^{-1} - \mu_k^{-1}$ is given by

$$\begin{aligned} & \lambda_m^{-2} \left(\int_{\partial\Omega_1 \cap \Omega_2^c} \int_0^{h(x')} ((1 - \lambda_m)\varphi(x', 0)^2 + |\nabla\varphi(x', 0)|^2) dx_\nu dS(x') \right. \\ & \quad \left. - \int_{\partial\Omega_1 \cap \Omega_2} \int_0^{-h(x')} ((1 - \lambda_m)\tilde{\varphi}(x', 0)^2 + |\nabla\tilde{\varphi}(x', 0)|^2) dx_\nu dS(x') \right) \\ & \quad + O(d^{1+\alpha}). \end{aligned}$$

The desired conclusion follows from this statement. \square

6.7. Sharpness of the requirement $\alpha > 0$ in Theorem 1.1. We now employ Proposition 6.12 to a specific Lipschitz perturbation of a two dimensional cylinder. The aim here is also to show that Theorem 1.1 is sharp in the sense that $\alpha > 0$ is necessary.

Suppose that $\eta : \mathbf{R} \rightarrow \mathbf{R}$ is a periodic nonnegative Lipschitz continuous function such that $\eta(t + 1) = \eta(t)$ for all $t \in \mathbf{R}$. Let the rectangle Ω_1 in \mathbf{R}^2 be defined by $0 < x < T$ and $0 < y < R$, where R and T are constants, and the subdomain $\Omega_2 \subset \Omega_1$ be defined by $0 < x < T$ and $\delta\eta(x/\delta) < y < R$, where $\delta = T/N$ for some large integer N . We will consider boundary conditions periodic in x with Neumann data given on the curves $y = \delta\eta(x/\delta)$ and $y = R$.

Proposition 6.18. *For the domains Ω_1 and Ω_2 defined above,*

$$\lambda_m^{-1} - \mu_k^{-1} = \tau_k + O(\delta^2) \quad \text{for } k = 1, 2, \dots, J_m. \tag{6.46}$$

Here, $\tau = \tau_k$ is an eigenvalue of

$$\begin{aligned} \tau(\varphi, \psi) = & \frac{\delta}{\lambda_m^2}(\eta_0 + \eta_1) \int_0^T \nabla\varphi(x, 0) \cdot \nabla\psi(x, 0) dx \\ & + \frac{\delta(1 - \lambda_m)}{\lambda_m^2} \eta_0 \int_0^T \varphi(x, 0)\psi(x, 0) dx \end{aligned} \tag{6.47}$$

for all $\psi \in X_m$, where $\varphi \in X_m$ and

$$\eta_0 = \int_0^1 \eta(X) dX \quad \text{and} \quad \eta_1 = \int_0^1 V(X, \eta(X))\eta'(X) dX.$$

The function V is the solution to $-\Delta_{X,Y}V = 0$ specified in (6.59) below and η_1 is not zero if η is not identically constant. Moreover, $\tau_1, \tau_2, \dots, \tau_{J_m}$ in (6.46) run through all eigenvalues of (6.47) counting their multiplicities.

To prove Proposition 6.18, we will use Proposition 6.12. To this end, we will find the solution v to the problem

$$(1 - \Delta)v = 0 \text{ in } \Omega_2, \quad \partial_\nu v = -\lambda_m^{-1}\partial_\nu\varphi \text{ on } \gamma_\eta, \quad \partial_\nu v = 0 \text{ on } \gamma_R, \tag{6.48}$$

and v is periodic in the first argument with period T , that is,

$$v(0, y) = v(T, y) \quad \text{and} \quad v'_x(0, y) = v'_x(T, y) \quad \text{for all } y \in (0, R). \tag{6.49}$$

By γ_η we denote the part of the boundary of Ω_2 where $y = \delta\eta(x/\delta)$, and by γ_R the part where $y = R$. Similarly, γ_0 is the part of Ω_1 where $y = 0$. The ansatz for the asymptotic expansion of v has the following form:

$$v(x, y) = \delta w_0(x, y) + \delta V_0(X, Y; x) + \delta^2 V_1(X, Y; x) + \dots, \tag{6.50}$$

where w_0 , V_0 , and V_1 are solutions to two model problems, and the remainder consists of higher order terms. Since the construction of the asymptotic expansion of the solution to problem (6.48) is quite standard, we confine ourselves to only finding the leading terms of this expansion. We have also introduced the new coordinates $X = x/\delta$ and $Y = y/\delta$. Substituting (6.50) into (6.48), we obtain

$$\begin{aligned} 0 = & -\delta^{-1} \Delta_{X,Y} V_0(X, Y; x) \\ & - \Delta_{X,Y} V_1(X, Y; x) - 2\partial_X \partial_x V_0(X, Y; x) \\ & + \delta((1 - \Delta_{x,y})w_0(x, y) + V_0(X, Y; x) \\ & - \partial_x^2 V_0(X, Y; x) - 2\partial_X \partial_x V_1(X, Y; x)) + O(\delta^2) \end{aligned} \quad (6.51)$$

with the boundary condition

$$\begin{aligned} -\lambda_m^{-1} \hat{n} \cdot \nabla_{x,y} \varphi(x, y) = & \hat{n} \cdot \nabla_{X,Y} V_0(X, Y; x) \\ & + \delta \hat{n} \cdot (\nabla_{X,Y} V_1(X, Y; x) + \nabla_{x,y} w_0(x, y)) \\ & + \delta \hat{n} \cdot (\partial_x V_0(X, Y; x), 0) + O(\delta^2) \end{aligned} \quad (6.52)$$

on γ_η , where

$$\hat{n}(X) = \frac{(\eta'(X), -1)}{\sqrt{1 + (\eta'(X))^2}} \quad (6.53)$$

is the outwards normal on γ_η .

The function w_0 is the solution with periodic boundary conditions in the sense of (6.49) to

$$(1 - \Delta)w_0 = 0 \text{ in } \Omega_1 \quad \text{and} \quad \partial_\nu w_0 = g_{w_0} \in L^2(\gamma_0) \text{ and } \partial_\nu w_0 = 0 \text{ on } \gamma_R,$$

where the Neumann data g_{w_0} will be specified below.

Let $\tilde{\Omega}_2$ be defined by $0 < X < 1$ and $\eta(X) < Y$. We denote by Γ_η the curve $Y = \eta(X)$ for $0 < X < 1$. The functions V_j , $j = 0, 1$, will be solutions to the following model problem for right-hand sides specified below:

$$-\Delta_{X,Y} W = F \text{ in } \tilde{\Omega}_2 \quad \text{and} \quad \partial_\nu W = G \text{ on } \Gamma_\eta, \quad (6.54)$$

and W is periodic in X :

$$W(0, Y) = W(1, Y) \quad \text{and} \quad W'_X(0, Y) = W'_X(1, Y) \text{ for all } Y.$$

The functions $G \in L^2(\Gamma_\eta)$ and F satisfy

$$\int_{\tilde{\Omega}_2} F dXdY + \int_{\Gamma_\eta} G dS = 0 \quad (6.55)$$

and

$$|F(X, Y)| \leq Ce^{-bY} \quad \text{for some } b > 0.$$

The orthogonality condition above implies that this solution decays exponentially as $Y \rightarrow \infty$:

$$|W(X, Y)| \leq Ce^{-aY} \quad \text{for some } a > 0. \quad (6.56)$$

We now specify the boundary data for the model problems. Since φ satisfies $\varphi'_y(x, 0) = 0$ for all x , a Taylor expansion yields

$$\varphi'_x(x, y) = \varphi'_x(x, 0) + O(y^2) = \varphi'_x(x, 0) + O(\delta^2)$$

if $y = \delta\eta(x/\delta)$. Similarly,

$$\varphi'_y(x, y) = \delta\eta(X)\varphi''_{yy}(x, 0) + O(\delta^2).$$

Thus,

$$\lambda_m^{-1} \partial_\nu \varphi(x, y) = \frac{\eta'(X)\varphi'_x(x, 0)}{\lambda_m \sqrt{1 + (\eta'(X))^2}} - \delta \frac{\eta(X)\varphi''_{yy}(x, 0)}{\lambda_m \sqrt{1 + (\eta'(X))^2}} + O(\delta^2) \quad (6.57)$$

for $(x, y) \in \gamma_\eta$.

We now consider the variables X , Y , and x as independent. Equations (6.51) and (6.52) together with (6.57) imply that

$$-\Delta_{X,Y} V_0 = 0 \text{ in } \tilde{\Omega}_2$$

and

$$\hat{n} \cdot \nabla_{XY} V_0 = -\frac{\eta'(X)\varphi'_x(x, 0)}{\lambda_m \sqrt{1 + (\eta'(X))^2}} \text{ on } \Gamma_\eta. \quad (6.58)$$

It is clear that

$$\int_{\Gamma_\eta} \frac{\eta'(X)\varphi'_x(x, 0)}{\lambda_m \sqrt{1 + (\eta'(X))^2}} dS(X) = \frac{\varphi'_x(x, 0)}{\lambda_m} \int_0^1 \eta'(X) dX = 0.$$

Thus equation (6.54) with $F = 0$ and G equal to the right-hand side of (6.58) has a solution V_0 decaying exponentially as $Y \rightarrow \infty$. The dependence on x can be described as follows. Let V be the periodic (with respect to X) solution to (6.54) with $F = 0$ and

$$G = \frac{\eta'(X)}{1 + (\eta'(X))^2} \text{ on } \Gamma_\eta. \quad (6.59)$$

Then

$$V_0(X, Y; x) = -\lambda_m^{-1} V(X, Y) \varphi'_x(x, 0)$$

and

$$\partial_x V_0(X, Y; x) = -\lambda_m^{-1} V(X, Y) \varphi''_{xx}(x, 0).$$

Similarly with above, equations (6.51), (6.52), and (6.57) also imply that

$$-\Delta_{X,Y} V_1 = 2\partial_X \partial_x V_0(X, Y; x) \quad \text{in } \tilde{\Omega}_2 \quad (6.60)$$

and

$$\hat{n} \cdot \nabla_{XY} V_1 = \frac{\eta(X) \varphi''_{yy}(x, 0) - \eta'(X) \partial_x V_0(X, Y; x)}{\lambda_m \sqrt{1 + (\eta'(X))^2}} \quad \text{on } \Gamma_\eta. \quad (6.61)$$

Put F equal to the right-hand side of (6.60) and G equal to the right-hand side of (6.61). We require the orthogonality condition in (6.55), so

$$0 = \int_{\tilde{\Omega}_2} 2\partial_X \partial_x V_0(X, Y; x) dXdY + \int_{\Gamma_\eta} G dS.$$

Furthermore, since $\Delta V = 0$,

$$\begin{aligned} \int_{\tilde{\Omega}_2} \partial_X V(X, Y) dXdY &= \int_{\Gamma_\eta} V(X, \eta(X)) \frac{\eta'(X)}{\sqrt{1 + (\eta'(X))^2}} dS \\ &= \int_{\tilde{\Omega}_2} \Delta V(X, Y) V(X, Y) dXdY \\ &\quad + \int_{\tilde{\Omega}_2} \nabla V(X, Y) \cdot \nabla V(X, Y) dXdY \\ &= \int_{\tilde{\Omega}_2} \nabla V(X, Y) \cdot \nabla V(X, Y) dXdY. \end{aligned} \quad (6.62)$$

Thus

$$\begin{aligned} 0 &= \int_{\tilde{\Omega}_2} 2\partial_X \partial_x V_0(X, Y; x) dXdY + \int_{\Gamma_\eta} G dS \\ &= \lambda_m^{-1} (\eta_1 \varphi''_{xx}(x, 0) - \eta_0 \varphi''_{yy}(x, 0)) - w_0'(x, 0), \end{aligned}$$

where

$$\eta_0 = \int_0^1 \eta(X) dX \quad \text{and} \quad \eta_1 = \int_0^1 V(X, \eta(X)) \eta'(X) dX.$$

We note here that $\eta_1 > 0$ if V is nonzero (due to (6.62)), or equivalently, if η is not identically constant. For the validity of (6.55), it is sufficient that

$$w_0'(x, 0) = \lambda_m^{-1}(\eta_1 \varphi''_{xx}(x, 0) - \eta_0 \varphi''_{yy}(x, 0)) \quad (6.63)$$

and we have therefore obtained that the function g_{w_0} is given by the right-hand side of (6.63) since $\partial_\nu w_0 = -w_0'$ on the curve γ_0 .

Now, from (6.31) it is clear that we wish to simplify the expression

$$\int_{\Omega_2} (\lambda_m v_\varphi v_\psi + v_\varphi \psi) dx.$$

With the current notation, $v_\varphi = v$ and v_ψ is the corresponding solution to (6.48) with ψ instead of φ . Hence,

$$\int_{\Omega_2} \lambda_m v_\varphi v_\psi dx = O(\delta^2).$$

Thus, we consider

$$\int_{\Omega_2} v \psi dx = \delta \int_{\Omega_2} w_0 \psi dx + \delta \int_{\Omega_2} V_0 \psi dx + \delta^2 \int_{\Omega_2} V_1 \psi dx + \dots$$

From (6.56) it follows that the integrals involving V_0 and V_1 are of order $O(\delta^2)$. For the first term,

$$\int_{\Omega_2} w_0 \psi dx = \int_{\Omega_1} w_0 \psi dx - \int_{\Omega_1 \setminus \Omega_2} w_0 \psi dx,$$

where the second term is of order $O(\delta)$. The first term can be expressed as

$$\begin{aligned} \int_{\Omega_1} w_0 \psi dx &= \int_{\partial\Omega_1} \partial_\nu w_0 \psi dS - \int_{\Omega_1} \nabla w_0 \cdot \nabla \psi dx \\ &= \int_{\partial\Omega_1} \partial_\nu w_0 \psi dS + \int_{\Omega_1} w_0 \Delta \psi dx, \end{aligned}$$

where we used the fact that $\partial_\nu \varphi = 0$ on $\partial\Omega_1$. Moreover, $\partial_\nu w_0 = 0$ at γ_R . This implies that

$$\begin{aligned} \lambda_m \int_{\Omega_1} w_0 \psi dx &= \int_{\gamma_n} \partial_\nu w_0 \psi dS \\ &= -\lambda_m^{-1} \int_0^T (\eta_1 \varphi''_{xx}(x, 0) - \eta_1 \varphi''_{yy}(x, 0)) dx \\ &= \lambda_m^{-1} (\eta_0 + \eta_1) \int_0^T \varphi'_x(x, 0) \psi'_x(x, 0) dx \\ &\quad + \lambda_m^{-1} (1 - \lambda_m) \eta_0 \int_0^T \varphi(x, 0) \psi(x, 0) dx, \end{aligned}$$

where we used the fact that $\varphi''_{xx} + \varphi''_{yy} = (1 - \lambda_m)\varphi$ and integration by parts.

Thus, we have obtained an expression for the right-hand side of (6.31). Moreover, the representation of v in (6.50) implies that $\|K_2 B\varphi\|^2 = O(\delta^2)$. This proves that Proposition 6.18 holds.

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