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A note on the resonance counting function for surfaces with cusps

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Abstract. We prove sharp upper bounds for the number of resonances in boxes of size 1 at high frequency for the Laplacian on finite volume surfaces with hyperbolic cusps. As a corollary, we obtain a Weyl asymptotic for the number of resonances in balls of size $T \to \infty$ with remainder $O(T^{3/2})$.

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In this short note, we prove sharp bounds on resonance-counting functions for the Laplacian on finite volume surfaces with hyperbolic cusps. Let M be a complete non-compact surface, equipped with a Riemannian metric g. We assume that (M, g) can be decomposed as the union of a compact manifold with boundary and a finite number κ of hyperbolic cusps, each one being isometric to

$$(a, +\infty)_y \times \mathbb{S}^1_\theta \text{ with metric } \frac{dy^2 + d\theta^2}{y^2}$$

for some $a > 0$. The spectral properties of the Laplacian Δ were first studied by Selberg [9] and Lax-Phillips [5] in constant negative curvature, and by Colin-de-Verdière [2], Müller [6], Parnovski [7] in the non-constant curvature setting.

On such surfaces, the resolvent $R(s) = (\Delta - s(1-s))^{-1}$ of the Laplacian admits a meromorphic extension from $\{\Re s > 1/2\}$ to $\mathbb{C}$ as an operator mapping L^2_{comp} to L^2_{loc} and the natural discrete spectral set for Δ is the set of poles denoted by

$$\mathcal{R} \subset \{s \in \mathbb{C} \mid \Re s \leq 1/2\} \cup (1/2, 1].$$

The poles are called *resonances* and are counted with multiplicity $m(s)$ (the multiplicity $m(s)$ is defined below and corresponds, for all but finitely many resonances, to the rank of the residue of the resolvent at s). We shall recall in the next section how the set of resonances is built. To study their distribution in the complex plane, we define two counting functions:

$$N_{\mathcal{R}}(T) := \sum_{s\in\mathcal{R}, |s-1/2|\leq T} m(s), \tag{1}$$

$$N_{\mathcal{R}}(T,\delta) := \sum_{s\in\mathcal{R}, |s-1/2-iT|\leq \delta T} m(s). \tag{2}$$

The first result on the resonance counting function was proved by Selberg [9, p. 25] for the special case of hyperbolic surfaces with finite volume: the following Weyl type asymptotic expansion holds as $T \to \infty$

$$N_{\mathcal{R}}(T) = \frac{\mathrm{Vol}(M)}{2\pi}T^2 - \frac{2\kappa}{\pi}T\log(T) + \frac{2\kappa(1-\log 2)}{\pi}T + O\Big(\frac{T}{\log(T)}\Big). \tag{3}$$

In variable curvature, Müller gives a Weyl asymptotic [6, Theorem 1.3.a] of the form

$$N_{\mathcal{R}}(T) = \frac{\mathrm{Vol}(M)}{2\pi}T^2 + o(T^2),$$

and this was improved by Parnovski [7] who showed that for all $\epsilon > 0$

$$N_{\mathcal{R}}(T) = \frac{\mathrm{Vol}(M)}{2\pi}T^2 + O(T^{3/2+\epsilon}). \tag{4}$$

Parnovski's proof relies on a Weyl type asymptotic expansion involving the scattering phase $\mathcal{S}(T)$ (see next section for a precise definition):

$$2\pi N_d(T) + \mathcal{S}(T) = \frac{\mathrm{Vol}(M)}{2}T^2 - 2\kappa T\ln T + O(T), \tag{5}$$

where κ is the number of cusps, and N_d is the counting function for the L^2 eigenvalues of Δ embedded in the continuous spectrum.

Using a Poisson formula proved by Müller [6] and estimate (5), we are able to improve the results of Parnovski:

Theorem 1. *For $T > 1$, and $0 \leq \delta \leq 1/2$, the following estimates hold*

$$N_{\mathcal{R}}(T,\delta) = O(T^2\delta + T), \tag{6}$$

$$N_{\mathcal{R}}(T) = \frac{\mathrm{Vol}(M)}{2\pi}T^2 + O(T^{3/2}). \tag{7}$$

In the first estimate with $\delta = 1/T$, the exponent in T is sharp in general, as can be seen from Selberg's result (3) and the additionnal estimate also from [9]

$$\sum_{s\in\mathcal{R},0\leq\Im s\leq T} \Re s - 1/2 = \frac{\kappa}{4\pi} T \log\frac{T}{\pi} - \frac{1}{2\pi}\Big(\frac{\kappa}{2} + \log|c|\Big)T + O(\log T), \quad (8)$$

where c is a constant depending on the surface, introduced by Selberg. Together these formulae imply that as $T \to \infty$

$$N_{\mathcal{R}}(T, 1/T) = \frac{\mathrm{vol}(M)}{2\pi} T + O\Big(\frac{T}{\log T}\Big). \quad (9)$$

In n-dimensional Euclidan scattering, upper bounds $O(T^{n-1})$ on the number of resonances in boxes of fixed size at frequency T were obtained by Petkov and Zworski [8] using the Breit–Wigner approximation and the scattering phase; our scheme of proof is inspired from their approach. Their result was extended to the case of non-compact perturbations of the Laplacian by Bony [1]. In general, it is expected that the number of resonances in such boxes is controlled by the (fractal) dimension of the trapped set (see for example Zworski [11], Guillopé, Lin, and Zworski [4], Sjöstrand and Zworski [10], and Datchev and Dyatlov [3]).

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1. Preliminaries

We start by recalling well-known facts on scattering theory on surfaces with cusps, and we refer to the article of Müller [6] for details. Let (M, g) be a complete Riemannian surface that can be decomposed as follows:

$$M = M_0 \cup Z_1 \cup \cdots \cup Z_k,$$

where M_0 is a compact surface with smooth boundary, and Z_j are hyperbolic cusps

$$Z_j \simeq (a_j, +\infty) \times \mathbb{S}^1, \quad j = 1 \ldots k,$$

with $a_j > 0$ and the metric on Z_j in coordinates $(y, \theta) \in (a_j, +\infty) \times \mathbb{S}^1$ is

$$ds^2 = \frac{\mathrm{d}y^2 + \mathrm{d}\theta^2}{y^2}.$$

Notice that the surface has finite volume when equipped with this metric.

The non-negative Laplacian Δ acting on $C_0^\infty(M)$ functions has a unique self-adjoint extension to $L^2(M)$ and its spectrum consists of

(1) absolutely continuous spectrum

$$\sigma_{\mathrm{ac}} = [1/4, +\infty)$$

with multiplicity κ (the number of cusps);

(2) discrete spectrum

$$\sigma_{\mathrm{d}} = \{\lambda_0 = 0 < \lambda_1 \leq \cdots \leq \lambda_i \leq \dots\},$$

possibly finite, and which may contain embedded eigenvalues in the continuous spectrum. To $\lambda \in \sigma_{\mathrm{d}}$, we associate a family of orthogonal eigenfunctions that generate its eigenspace $(u_\lambda^i)_{i=1\dots d_\lambda} \in L^2(M) \cap C^\infty(M)$.

The generalized eigenfunctions associated to the absolutely continuous spectrum are the Eisenstein functions, $(E_j(x,s))_{i=1\dots\kappa}$. Each E_j is a meromorphic family (in s) of smooth functions on M. Its poles are contained in the open half-plane $\{\Re s < 1/2\}$ or in $(1/2, 1]$. The Eisenstein functions are characterized by two properties:

(1) $\Delta E_j(., s) = s(1-s)E_j(., s)$;

(2) in the cusp Z_i, $i = 1\dots\kappa$, the zeroth Fourier coefficient of E_j in the θ variable equals $\delta_{ij} y_i^s + \phi_{ij}(s) y_i^{1-s}$ where y_i denotes the y coordinate in the cusp Z_i and $\phi_{ij}(s)$ is a meromorphic function of s.

We can collect the *scattering coefficients* ϕ_{ij} in a meromorphic family of matrices, $\phi(s) = (\phi_{ij})_{ij}$ called *scattering matrix*. We denote its determinant by $\varphi(s) = \det \phi(s)$. Then the following identities hold

$$\phi(s)\phi(1-s) = \mathrm{Id}, \quad \overline{\phi(s)} = \phi(\overline{s}), \quad \phi(s)^* = \phi(\overline{s}).$$

The line $\Re s = 1/2$ corresponds to the continuous spectrum. On that line, $\phi(s)$ is unitary, $\varphi(s)$ has modulus 1. We also define the scattering phase

$$\mathcal{S}(T) = -\int_0^T \frac{\varphi'}{\varphi}\Big(\frac{1}{2} + it\Big)\mathrm{d}t. \tag{10}$$

The set of poles of φ, ϕ and $(E_j)_{j=1\dots\kappa}$ is the same, we call them them *scattering poles* and we shall denote Λ this set. It is contained in $\{\Re s < 1/2\} \cup (1/2, 1]$. The union of this set with the set of $s \in \mathbb{C}$ such that $s(1-s)$ is an L^2 eigenvalue, is called the *resonance set,* and denoted $\mathcal{R}$. Following [6, p. 287], the multiplicities $m(s)$ are defined as follows:

(1) if $\Re s \geq 1/2$, $s \neq 1/2$, $m(s)$ is the dimension of $\ker_{L^2}(\Delta - s(1-s))$;

(2) if $\Re s < 1/2$, $m(s)$ is the dimension of $\ker_{L^2}(\Delta - s(1-s))$ minus the order of φ at s;

(3) $m(1/2)$ equals $(\mathrm{Tr}(\phi(1/2)) + \kappa)/2$ plus twice the dimension of $\ker_{L^2}(\Delta - 1/4)$.

For convenience, we define two counting functions for the discrete spectrum and the poles of φ:

$$N_d(T) := \sum_{|s_i - 1/2| \leq T} m(s_i), \tag{11}$$

$$N_\Lambda(T) := \sum_{s \in \Lambda, |s-1/2| \leq T} m(s), \tag{12}$$

so that

$$N_{\mathcal{R}}(T) := \sum_{s \in \mathcal{R}, |s-1/2| \leq T} m(s) = 2N_d(T) + N_\Lambda(T).$$

2. Main observation

In this section, we obtain estimate for $N_{\mathcal{R}}(T)$ in boxes at high frequency.

From the asymptotic expansion (5), we deduce that for $0 \leq \delta \leq 1/2$,

$$\begin{aligned} &2\pi[N_d(T+T\delta) - N_d(T-T\delta)] + \mathcal{S}(T+T\delta) - \mathcal{S}(T-T\delta) \\ &\quad = 2\mathrm{Vol}(M)T^2\delta - 4\kappa T\delta \ln T + O(T). \end{aligned} \tag{13}$$

Next, we recall the Poisson formula for resonances proved by Müller [6, Th. 3.32]

$$\mathcal{S}'(T) = \log \frac{1}{q} + \sum_{\rho \in \Lambda} \frac{1 - 2\Re\rho}{(\Re\rho - 1/2)^2 + (\Im\rho - T)^2}. \tag{14}$$

where q is some positive constant (not necessarily < 1). Let $C > 1$, $0 < \epsilon < 1$ and

$$\Omega_{T,\delta} := \{s \in \mathbb{C};\ |s - 1/2 - iT| \leq T\delta/C \text{ and } 0 \leq 1/2 - \Re s \leq \epsilon T\delta\}.$$

Then, for $s \in \Omega_{T,\delta}$,

$$\int_{[T-T\delta,T+T\delta]} \frac{1-2\Re s}{(\Re s-1/2)^2+(t-\Im s)^2}\mathrm{d}t = 2\Big[\arctan\frac{t-\Im s}{1/2-\Re s}\Big]_{T-T\delta}^{T+T\delta}.$$

The addition formula for arctan, with $x, y > 0$ and $xy > 1$ is given by

$$\arctan x + \arctan y = \pi + \arctan\frac{x+y}{1-xy}.$$

Thus

$$\begin{aligned}\int_{[T-T\delta,T+T\delta]} &\frac{1-2\Re s}{(\Re s-1/2)^2+(t-\Im s)^2}\mathrm{d}t\\ &= 2\pi - 2\arctan\frac{2T\delta(1/2-\Re s)}{T^2\delta^2-|s-1/2-iT|^2}\\ &\geq 2\pi - 2\arctan\widetilde{C}\epsilon,\end{aligned}$$

where $\widetilde{C}$ is set to be $2/(1-1/C^2)$. For ϵ small enough, this is bigger than, say, π.

Since all but a finite number of terms in (14) are positive, we have

$$\mathcal{S}(T+T\delta)-\mathcal{S}(T-T\delta) \geq O(T\delta) + \sum_{\rho\in\Lambda\cap\Omega_{T,\delta}} \pi.$$

Combining with (13), we deduce that

$$N_d(T+T\delta)-N_d(T-T\delta)+\#\Lambda\cap\Omega_{T,\delta} = O(T^2\delta)+O(T)+O(T\delta).$$

This is the content of (6) in our main theorem.

3. Consequence

Now, we prove the second part of Theorem 1. We will follow the method of Müller [6, p. 282], which is a global and quantitative version of the argument used in the previous section. Integrating the Poisson formula over $[-T, T]$, we relate the scattering phase asymptotics to the poles of ϕ. Using the arctan addition formula, we are left with the sum of $N_\Lambda(T)$ and an expression with arctan's (equation (4.9) in [6]):

$$\begin{aligned}\frac{1}{2\pi}&\mathcal{S}(T)\\ &= \frac{1}{2}N_\Lambda(T) + \frac{1}{2\pi}\sum_{\rho\in\Lambda,\Re\rho<1/2}\arctan\Big[\frac{1-2\Re\rho}{|\rho-1/2|^2}T\Big(1-\frac{T^2}{|\rho-1/2|^2}\Big)^{-1}\Big] + O(T).\end{aligned} \tag{15}$$

The sum is then split between {1} the poles in $\{|T - |\rho - 1/2|| > T^{1/2}\}$, and {2}, the others. Müller proved that the sum {1} is $O(T^{3/2})$. The sum {2} can be bounded by

$$\frac{1}{4}(N_\Lambda(T + \sqrt{T}) - N_\Lambda(T - \sqrt{T})).$$

From [6, Corollary 3.29], we also recall that

$$\sum_{\eta \in \Lambda, \eta \neq 1/2} m(\eta) \frac{1 - 2\Re\eta}{|\eta - 1/2|^2} < \infty.$$

Consider the set $\widetilde{\Lambda} = \{\eta \in \Lambda;\ (2\Re\eta - 1)^2 > \Im\eta,\ |\eta| > 1\}$. On $\widetilde{\Lambda}$, we have that $|\eta - 1/2|^{1/2} \leq 1 - 2\Re\eta$, thus

$$\sum_{\eta \in \widetilde{\Lambda}, \eta \neq 1/2} m(\eta) \frac{1}{|\eta - 1/2|^{3/2}} < \infty.$$

If $\tilde{n}(T)$ is the counting function for $\widetilde{\Lambda}$, we deduce that

$$\sum_{k=1}^{\infty} \tilde{n}(k) \Big[\frac{1}{k^{3/2}} - \frac{1}{(k+1)^{3/2}} \Big] < \infty.$$

Since $\tilde{n}$ is non-decreasing, $\tilde{n}(k) = o(k^{3/2})$. Now,

$$N_\Lambda(T - \sqrt{T}) - N_\Lambda(T + \sqrt{T}) \leq \tilde{n}(T) + N_{\mathfrak{R}}(T, \sqrt{T}^{-1}) + N_{\mathfrak{R}}(-T, \sqrt{T}^{-1}).$$

This concludes the proof.

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