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Global well-posedness and long-time asymptotics for the defocussing Davey–Stewartson II equation in $H^{1,1}(\mathbb{C})$

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(with Appendix A by Michael Christ[2])

Abstract. We use the $\bar{\partial}$-inverse scattering method to obtain global well-posedness and large-time asymptotics for the defocussing Davey–Stewartson II equation. We show that these global solutions are dispersive by computing their leading asymptotic behavior as $t \to \infty$ in terms of an associated linear problem. These results appear to be sharp.

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1. Introduction

In this paper we will use the inverse scattering method to prove global well-posedness for the defocussing Davey–Stewartson II (DS II) equation

$$iu_t + 2(\bar{\partial}^2 + \partial^2)u + (g + \bar{g})u = 0, \tag{1.1a}$$

$$\bar{\partial} g + \partial(|u|^2) = 0, \tag{1.1b}$$

a nonlinear, completely integrable dispersive equation in two space dimensions. Here and in what follows, $z = x_1 + ix_2$ and

$$\bar{\partial} = \frac{1}{2}\Big(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\Big), \quad \partial = \frac{1}{2}\Big(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\Big).$$

The defocussing DS II equation may be regarded as a two-dimensional analogue of the defocussing cubic nonlinear Schrödinger equation in one space dimension: it is one of a multiparameter family of models proposed by Benny and Roskes [12] and Davey and Stewartson [19] to model the propagation of weakly nonlinear surface waves in shallow water (see Ghidaglia and Saut [23] for a physical derivation and extensive local well-posedness results).

We will prove that the Cauchy problem for (1.1) is globally well-posed for initial data in the space $H^{1,1}(\mathbb{C})$. Here and in what follows, $H^{\alpha,\beta}(\mathbb{C})$ denotes the weighted Sobolev space

$$H^{\alpha,\beta}(\mathbb{C}) = \{f \in L^2(\mathbb{C})\colon \langle D\rangle^\alpha f,\ \langle\cdot\rangle^\beta f(\cdot) \in L^2(\mathbb{C})\}.$$

Here $\langle z\rangle = (1 + |z|^2)^{1/2}$ and $\langle D\rangle^\beta$ is the Fourier multiplier with symbol $\langle\xi\rangle^\beta$.

We will also show, under stronger conditions on the initial data, that solutions are asymptotic in L^∞-norm to solutions of the linear problem

$$iv_t + 2(\bar{\partial}^2 + \partial^2)v = 0. \tag{1.2}$$

To state our result precisely, we recall the formulation of (1.1) as a nonlinear integral equation. Denote by $S(t)$ the solution operator for the linear problem (1.2). For $T > 0$, let

$$X_T = C([0,T], L^2(\mathbb{C})) \cap L^4([0,T] \times \mathbb{C}).$$

We say that a function $u \in X_T$ solves the Davey–Stewartson II equation with initial data $u_0 \in L^2(\mathbb{C})$ if $u(t)$ solves the equation

$$u(t) = S(t)u_0 + \Lambda(u)(t), \tag{1.3}$$

for $t \in (0, T)$. Here

$$\Lambda(u)(t) = i \int_0^t S(t-s)(2u(s)\,\mathrm{Re}[\mathcal{S}(|u(s)|^2)])\,ds,$$

where $\mathcal{S} = \partial\bar{\partial}^{-1}$ is the Beurling transform (see Lemma 2.4). Strichartz estimates for $S(t)$ (see [23, §2 and Appendix]) show that the solution operator $S(t)$ and the nonlinear mapping Λ takes X_T to itself, so that (1.3) can be formulated as a fixed-point problem in this space. It is not difficult to see that a classical solution of (1.1) belonging to $C^1([0,T], \mathcal{S}(\mathbb{C}))$ also solves (1.3). Ghidalgia and Saut [23, Theorem 2.1] showed that for initial data $u_0 \in L^2(\mathbb{C})$, problem (1.3) has a solution in X_T for some $T > 0$ depending on the initial data. For $u_0 \in H^{1,1}(\mathbb{C})$ we can globalize this result by the inverse scattering method. We will prove:

Theorem 1.1. *There exists a continuous map*

$$\begin{gathered} H^{1,1}(\mathbb{C}) \times \mathbb{R} \longrightarrow H^{1,1}(\mathbb{C}), \\ (u_0, t) \longmapsto u(t), \end{gathered}$$

so that the function u is a solution of the Davey–Stewartson II *equation* (1.1) *with initial data u_0 in the sense that the integral equation* (1.3) *holds for all t. Moreover, $\|u(t)\|_2$ is conserved.*

Since $H^{1,1}(\mathbb{C}) \subset L^p(\mathbb{C})$ for all $p \in (1, \infty)$ (see (2.2)) it is easy to see that $C([0,T], H^{1,1}(\mathbb{C}))$ is continuously embedded in X_T. Hence, the global solution constructed in Theorem 1.1 coincides with the local Ghidaglia-Saut solution for all T, so that these solutions extend to $T = \infty$ when $u_0 \in H^{1,1}(\mathbb{C})$.

Our proof exploits the completely integrable method for the defocussing DS II equation developed by Fokas [22], Ablowitz and Fokas [1, 2, 3], Beals and Coifman [7, 8, 9], Sung [33], and Brown [13]. For $u_0 \in \mathcal{S}(\mathbb{C})$, the function

$$u(z,t) = \mathcal{I}[e^{4it\,\mathrm{Re}((\cdot)^2)}(\mathcal{R}u_0)(\cdot)](z) \tag{1.4}$$

solves the Cauchy problem for (1.1) with initial data $u_0 \in \mathcal{S}(\mathbb{C})$ (see Appendix B for a self-contained proof and for references to the literature). Here $\mathcal{R}$ and $\mathcal{I}$ are the direct and inverse scattering transforms for the DS II equation which we now describe.

For $z = x_1 + ix_2$ and $k = k_1 + ik_2$, let $e_k(z)$ be the unimodular function

$$e_k(z) = e^{\bar{k}\bar{z} - kz} = \exp(-2i(k_1 x_2 + k_2 x_1)).$$

The *direct scattering map* $\mathcal{R}$ is defined by the $\bar{\partial}$-problem (in the z-variable)

$$\bar{\partial}\mu_1 = \frac{1}{2} e_k u \overline{\mu_2}, \tag{1.5a}$$

$$\bar{\partial}\mu_2 = \frac{1}{2} e_k u \overline{\mu_1}, \tag{1.5b}$$

$$\lim_{|z|\to\infty} (\mu_1(z,k), \mu_2(z,k)) = (1,0), \tag{1.5c}$$

and the representation formula

$$(\mathcal{R}u)(k) = \frac{1}{\pi} \int e_k(z) u(z) \overline{\mu_1(z,k)}\, dA(z) \tag{1.6}$$

(here and in what follows, dA denotes Lebesgue measure on $\mathbb{C}$). Given $u \in \mathcal{S}(\mathbb{C})$ one first solves (1.5) for μ_1, μ_2, and then computes $\mathcal{R}u$ from (1.6). The linearization of the map $\mathcal{R}$ at $u = 0$ is the map

$$(\mathcal{F} f)(k) = \frac{1}{\pi} \int e_k(z) f(z)\, dA(z) \tag{1.7}$$

which is the usual two-dimensional Fourier transform up to a linear change of variables.

The *inverse scattering map* $\mathcal{I}$ is similarly defined by the $\bar{\partial}$-problem

$$\bar{\partial}_k \nu_1 = \frac{1}{2} e_k \bar{r}\, \overline{\nu_2}, \tag{1.8a}$$

$$\bar{\partial}_k \nu_2 = \frac{1}{2} e_k \bar{r}\, \overline{\nu_1}, \tag{1.8b}$$

$$\lim_{|k|\to\infty} (\nu_1(z,k), \nu_2(z,k)) = (1,0), \tag{1.8c}$$

and the representation formula

$$(\mathcal{I}r)(z) = \frac{1}{\pi} \int e_{-k}(z) r(k) \nu_1(z,k)\, dA(k). \tag{1.9}$$

Here $\bar{\partial}_k$ denotes the $\bar{\partial}$-operator acting in the k variable. Given $r \in \mathcal{S}(\mathbb{C})$, one first solves (1.8) for ν_1, ν_2, and then computes $\mathcal{I}r$ from (1.9). The linearization of the map $\mathcal{I}$ at $r = 0$ is the inverse Fourier transform

$$(\mathcal{F}^{-1} g)(z) = \frac{1}{\pi} \int e_{-k}(z) g(k)\, dA(k). \tag{1.10}$$

From the definitions it is formally obvious that

$$\mathcal{I} = C \circ \mathcal{R} \circ C, \tag{1.11}$$

where C is complex conjugation. This fact, proved in Lemma 3.11 of what follows (see also [5, §2]), will allow us to apply our analysis of $\mathcal{R}$ directly to $\mathcal{I}$.

The solution formula (1.4) for (1.1) should be compared to the Fourier transform solution formula

$$v(z,t) = \mathcal{F}^{-1}[e^{4it\,\mathrm{Re}((\,\cdot\,)^2)}(\mathcal{F}v_0)(\,\cdot\,)](z) \tag{1.12}$$

for the linearized problem (1.2). Using the definition (1.8)–(1.9) of the map $\mathcal{I}$, we can recast the solution formula (1.4) for the DS II equation as a $\bar{\partial}$-problem depending on space and time as parameters. Given $u_0 \in \mathcal{S}(\mathbb{C})$, one computes $r_0 = \mathcal{R}u_0$ and solves the $\bar{\partial}$-problem

$$\bar{\partial}_k \nu_1 = \frac{1}{2} e^{-itS} \overline{r_0 \nu_2}\,, \tag{1.13a}$$

$$\bar{\partial}_k \nu_2 = \frac{1}{2} e^{-itS} \overline{r_0 \nu_1}\,, \tag{1.13b}$$

$$\lim_{|k|\to\infty} (\nu_1, \nu_2) = (1, 0). \tag{1.13c}$$

Here

$$S(z,k,t) = \frac{kz - \bar{k}\bar{z}}{it} + 4\,\mathrm{Re}(k^2) \tag{1.14}$$

is a real-valued phase function with a single nondegenerate critical point

$$k_c = iz/4t. \tag{1.15}$$

We then recover the solution from the formula

$$u(z,t) = \frac{1}{\pi} \int e^{itS(z,k,t)} r_0(k) \nu_1(z,k,t)\, dA(k). \tag{1.16}$$

By a careful study of the $\bar{\partial}$-problems (1.5) and (1.8), we will prove:

Theorem 1.2. *The maps $\mathcal{R}$ and $\mathcal{I}$, initially defined on $\mathcal{S}(\mathbb{C})$ by (1.5)–(1.6) and (1.8)–(1.9), extend to locally Lipschitz continuous maps from $H^{1,1}(\mathbb{C})$ to itself. Moreover, $\mathcal{R} \circ \mathcal{I} = \mathcal{I} \circ \mathcal{R} = I$ where I is the identity mapping on $H^{1,1}(\mathbb{C})$. Finally, the Plancherel relations $\|\mathcal{R}u\|_2 = \|r\|_2$ and $\|\mathcal{I}r\|_2 = \|r\|_2$ hold.*

Proof of Theorem 1.1, given Theorem 1.2. First, we show that the map defined by (1.4) has the claimed continuity properties. For u_1 and u_2 in a fixed bounded subset of $H^{1,1}(\mathbb{C})$ and $t, t' > 0$, let

$$U_1(z,t) = \mathcal{I}[e^{4it\,\mathrm{Re}((\,\cdot\,)^2)}\mathcal{R}(u_1)](z),$$
$$U_2(z,t') = \mathcal{I}[e^{4it'\,\mathrm{Re}((\,\cdot\,)^2)}\mathcal{R}(u_2)](z).$$

Then

$$\begin{aligned}\|U_1(\,\cdot\,,t) - U_2(\,\cdot\,,t')\|_{H^{1,1}} &\le C\,\|e^{4it\,\mathrm{Re}((\,\cdot\,)^2)}\mathcal{R}(u_1) - e^{4it'\,\mathrm{Re}((\,\cdot\,)^2)}\mathcal{R}(u_2)\|_{H^{1,1}}\\ &\le C\,\|e^{4it\,\mathrm{Re}((\,\cdot\,)^2)}[\mathcal{R}(u_1) - \mathcal{R}(u_2)]\|_{H^{1,1}}\\ &\quad + C\,\|(e^{4it\,\mathrm{Re}((\,\cdot\,)^2)} - e^{4it'\,\mathrm{Re}((\,\cdot\,)^2)})\mathcal{R}(u_2)\|_{H^{1,1}},\end{aligned}$$

where C is uniform in u_1 and u_2 in a fixed bounded subset of $H^{1,1}(\mathbb{R})$. The continuity now follows from the Lipschitz continuity of $\mathcal{R}$, the estimate

$$\|e^{4it\,\mathrm{Re}((\,\cdot\,)^2)}f\|_{H^{1,1}} \le C(1+|t|)\|f\|_{H^{1,1}}$$

and the fact that

$$\lim_{|t|\to 0}\|[e^{4it\,\mathrm{Re}((\,\cdot\,)^2)} - 1]f\|_{H^{1,1}} = 0$$

for each fixed $f \in H^{1,1}(\mathbb{C})$ by dominated convergence.

Next we prove that the map (1.4) solves the DS II equation (1.3) for initial data $u_0 \in H^{1,1}(\mathbb{R}^2)$. For u_1 and u_2 in a fixed bounded subset B of $H^{1,1}(\mathbb{C})$ and $T > 0$, we have

$$\sup_{t\in[0,T]}\|U_1(\,\cdot\,,t) - U_2(\,\cdot\,,t)\|_{H^{1,1}} \le C\,\|u_1 - u_2\|_{H^{1,1}}, \tag{1.17}$$

where $C = C(T,B)$, by Theorem 1.2. Now let $u_0 \in H^{1,1}(\mathbb{C})$ be given and let $\{u_{n,0}\}_{n=1}^{\infty}$ be a sequence from $\mathcal{S}(\mathbb{C})$ with $u_{n,0} \to u_0$ in $H^{1,1}(\mathbb{C})$. Let

$$U_n(z,t) = \mathcal{I}(e^{4it\,\mathrm{Re}((\,\cdot\,)^2)}\mathcal{R}(u_{n,0}))$$

and

$$U(z,t) = \mathcal{I}(e^{4it\,\mathrm{Re}((\,\cdot\,)^2)}\mathcal{R}(u_0)).$$

By (1.17),

$$\sup_{t\in[0,T]}\|U_n(\,\cdot\,,t) - U(\,\cdot\,,t)\|_{H^{1,1}} \le C\,\|u_{n,0} - u_0\|_{H^{1,1}}$$

so that, in particular, $U_n \to U$ in X_T. Since $u_{n,0} \in \mathcal{S}(\mathbb{C})$, we have $U_n \in C([0,T], \mathcal{S}(\mathbb{C}))$, and each U_n satisfies

$$U_n(t) = S(t)u_{n,0} + \Lambda(U_n)(t). \tag{1.18}$$

Since

$$\|U - U_n\|_{X_T} \leq C(T) \sup_{t\in[0,T]} \|U(\cdot,t) - U_n(\cdot,t)\|_{H^{1,1}(\mathbb{C})}$$

and Λ is a continuous mapping from X_T to itself, it follows that $\Lambda(U_n) \to \Lambda(U)$ in X_T. Taking limits in (1.18) in the X_T-topology, we conclude that

$$U(t) = S(t)u_0 + \Lambda(U)(t)$$

so that U solves the DS II equation (1.3) with initial data u_0. □

Through a careful study of the $\bar{\partial}$-problem (1.13a), we will prove:

Theorem 1.3. *Suppose that $u_0 \in H^{1,1}(\mathbb{C}) \cap L^1(\mathbb{C})$. The solution u of the defocussing DS II equation with Cauchy data u_0 obeys the asymptotic formula*

$$u(z,t) = v(z,t) + o(t^{-1})$$

in L^∞_z-norm, where

$$v(z,t) = \mathcal{F}^{-1}(e^{4it\,\mathrm{Re}((\cdot)^2)}(\mathcal{R}u_0)(\cdot)).$$

Remark 1.4. In an earlier version of this paper, the hypothesis that $u_0 \in L^1(\mathbb{C})$ was erroneously omitted. The condition $u_0 \in L^1(\mathbb{C})$ implies that r_0 is continuous (see Remark 3.7). The additional hypothesis appears to be necessary for the proof: see Lemma 5.9 for the key step where the continuity of r_0 is used.

Remark 1.5. This result shows that, in contrast to the one-dimensional cubic nonlinear Schrödinger equation, there is no "logarithmic phase shift" in the solution due to the nonlinear term. See Deift-Zhou [20] for an analysis of this phenomenon and for references to the literature.

Remark 1.6. Suppose that r_0 is continuous and that $\mathcal{F}^{-1}r_0 \in L^1(\mathbb{C})$. This assumption holds, for example, when $u_0 \in \mathcal{S}(\mathbb{C})$, so that $r_0 \in \mathcal{S}(\mathbb{C})$ by Sung's work [33, Paper II, §4] on the scattering transform. The function $v(z,t)$ is given by

$$v(z,t) = \int \Gamma_t(z-z')(\mathcal{F}^{-1}r_0)(z')\,dA(z'),$$

where

$$\Gamma_t(z) = \frac{e^{i(z^2+\bar{z}^2)/8t}}{4t}.$$

From this formula, we obtain

$$\lim_{t\to\infty} \Gamma_t(z)^{-1} v(z,t) = \int (\mathcal{F}^{-1} r_0)(z')\, dA(z') = \pi r_0(0)$$

which shows that the remainder $o(t^{-1})$ is indeed of lower order provided $r_0(0) \neq 0$.

The results of Theorem 1.3 were first obtained by Kiselev [24] (see also [25, Theorem 7]). On the one hand, Kiselev's result treats both the focusing and defocussing DS II equations; on the other, he imposes a "small data" restriction and more stringent integrability and regularity assumptions. Kiselev's analysis relies in part on separate asymptotic expansions of the solution $\nu_1(z,k,t)$ in the 'exterior region' $|k - k_c| \geq t^{-1/4}$ and in the 'interior region' $|k - k_c| < 2t^{1/4}$ with matching in the transition region.

In our proof, we remove Kiselev's small data restriction in the defocussing case and replace the asymptotic expansions with a finer analysis of the integral operator M (see (5.3)) used to solve (1.13a). Our analysis rests on scaling arguments and on the simple integration by parts formula (2.9) previously used by Bukhgeim [17] in his analysis of the inverse conductivity problem.

Inverse scattering for the defocussing Davey–Stewartson II equation was studied by Fokas [22], Ablowitz and Fokas [1, 2, 3], Beals and Coifman [7, 8, 9], Sung [33], and Brown [13]. Beals and Coifman construct global solutions for the defocussing DS II equation with initial data in $\mathcal{S}(\mathbb{C})$ by inverse scattering methods, while Sung constructs solutions for the same case if $u \in L^1 \cap L^\infty$ and the Fourier transform of u lies in $L^1 \cap L^\infty$ (see paper III of [33]). Sung [34] obtained the leading t^{-1} decay rate (but not the asymptotic formula) for solutions of the DS II equation with Schwarz class initial data, using his earlier work [33] on the inverse scattering method for DS II. Sung and Brown construct solutions to the focusing DS II equations with small initial data; the small data hypothesis avoids soliton solutions (see [4] and Section 10.5 of [21] for an exposition and additional references) and the blow-up phenomena discussed below. Brown actually shows Lipschitz continuity of the solution map for (1.2) for small Cauchy data in L^2 for either the focusing or defocussing DS II equation. More recently, Astala, Faraco, and Rogers [5] proved Lipschitz continuity of the scattering map $\mathcal{R}$ from $H^{s,s}$ to L^2 for $s \in (0,1)$ and proved a Plancherel identity for $\mathcal{R}$.

The same analysis used here can also be applied to the focusing DS II equation with small initial data, which differs from (1.1) in that the second equation reads

$$\bar{\partial} g - \bar{\partial}(|u|^2) = 0,$$

changing the sign of the nonlinear term. The "small data" condition is used to replace the Fredholm argument in Lemma 3.1. Details will be given in [30].

Ozawa [28] constructed a solution to the focusing DS II equation with the following properties: (1) the initial data $u_0 \in L^2$, but $|\nabla u_0(z)|,\ |zu(z)| \geq C(1+|z|)^{-1}$ for a positive constant C, (2) the measure $|u(z,t)|^2 dA(z)$ concentrates to a δ-function in finite time (see also C. Sulem and P. Sulem [32], pp. 229-230). Since ∇u_0 and $(\,\cdot\,)u_0(\,\cdot\,)$ lie in weak-L^2 but not L^2, Ozawa's results suggest that $H^{1,1}(\mathbb{C})$ is a natural limit for the inverse scattering method.

In [29], we use the results of this paper and previous work of Lassas, Mueller, and Siltanen [26] and Lassas, Mueller, Siltanen, and Stahel [27] to find global solutions of the Novikov–Veselov equation with initial data of conductivity type by the inverse scattering method. In [14], co-authored with Russell Brown, Katharine Ott, and Nathan Serpico, we show that the maps $\mathcal{R}$ and $\mathcal{I}$ have mapping properties between weighted Sobolev spaces which parallel those of the Fourier transform. Our analysis in [14] relies in part on a key estimate of Astala, Faraco, and Rogers [5] that generalizes our Lemma 3.2. These authors prove a Plancherel formula for the map $\mathcal{R}$ under less restrictive hypotheses than ours.

We close by sketching the contents of this paper. In §2, we fix notation, recall basic facts about integral operators associated to the $\bar{\partial}$-problem, recall key Brascamp–Lieb inequalities, and prove an important lemma on integration by parts. In §3, we study the $\bar{\partial}$-problem (1.5) in depth. We apply these results in §4 to prove Theorem 1.2. Finally, we prove Theorem 1.3 in §5. Appendix A, written by Michael Christ, proves Brown's multilinear estimate (Proposition 2.5 and [13, Lemma 3]) by the methods of Bennett, Carbery, Christ, and Tao [10, 11]. In Appendix B we present a concise proof that the inverse scattering formula (1.4) gives a classical solution of the DS II equation for initial data in $\mathcal{S}(\mathbb{C})$. Appendix C computes large-z asymptotic expansions for solutions of (1.8) that are used in Appendix B.

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2. Preliminaries

Notation, function spaces. We denote by $C^0(\mathbb{C})$ the bounded continuous functions on $\mathbb{C}$ equipped with the sup norm, and by $C_0(\mathbb{C})$ the continuous functions that vanish at infinity. The spaces $L^p(\mathbb{C})$ are the usual Lebesgue spaces and p' the Hölder conjugate exponent. We sometimes write $L^p_z(\mathbb{C})$ or $L^p_k(\mathbb{C})$ to clarify the choice of integration variable z or k. The space $L^{2,1}(\mathbb{C})$ consists of complex-valued measurable functions $f \in L^2(\mathbb{C})$ with $\langle z \rangle f \in L^2(\mathbb{C})$. We denote by $\langle f, g \rangle$ the dual pairing

$$\langle f, g \rangle = \frac{1}{\pi} \int \overline{f(z)} g(z) \, dA(z). \tag{2.1}$$

To quantify regularity of solutions for (1.5) and (1.8), we use the usual Hölder spaces. For $\alpha \in (0,1)$, let C^α denote the bounded, Hölder continuous functions of order α on $\mathbb{C}$ equipped with the norm

$$\|f\|_{C^\alpha} = \|f\|_\infty + \sup_{z \neq z'} \frac{|f(z) - f(z')|}{|z - z'|^\alpha}.$$

If X and Y are Banach spaces, $\mathcal{B}(X, Y)$ (resp. $\overline{\mathcal{B}}(X, Y)$) is the Banach space of linear (resp. antilinear) operators from X to Y. We write $\mathcal{B}(X)$ for $\mathcal{B}(X, X)$, and similarly for $\overline{\mathcal{B}}(X)$.

The space $H^{1,1}(\mathbb{C})$ is continuously embedded in $L^p(\mathbb{C})$ for any $p \in (1, \infty)$. Thus, for any $s \in (1, \infty)$,

$$\|u\|_s \leq C_s \|u\|_{H^{1,1}}, \tag{2.2}$$

where C_s depends only on s. We also have the following standard compact embedding result. We give a proof since we have not found a reference although the result is well-known.

Lemma 2.1. *The space $H^{1,1}(\mathbb{C})$ is compactly embedded in $L^p(\mathbb{C})$ for any $p \in (1, \infty)$.*

Proof. Observe that $\mathcal{F}$ preserves $H^{1,1}(\mathbb{C})$ and maps $L^p(\mathbb{C})$ continuously to $L^{p'}(\mathbb{C})$ for $p \in [1, 2]$. Hence, if we show that $H^{1,1}(\mathbb{C})$ is compactly embedded in $L^p(\mathbb{C})$ for $p \in (1, 2]$, the same fact for $p \in [2, \infty)$ is follows by composing with the continuous map $\mathcal{F}$.

To show that $H^{1,1}(\mathbb{C})$ is compactly embedded in $L^p(\mathbb{C})$ for $p \in (1, 2]$, let $\eta \in C_0^\infty(\mathbb{C})$ with $\eta(w) = 1$ for $|w| \leq 1$ and $\eta(w) = 0$ for $|w| \geq 2$, and let $\eta_R(z) = \eta(w/R)$. Let $T_R f = \eta_R \cdot f$. Since T_R is a bounded map from $H^{1,1}(\mathbb{C})$ to $W^{1,p}(\mathbb{C})$ for any $p \in [1, 2]$, it follows from the Rellich-Kondrakov Theorem that T_R is a compact mapping from $H^{1,1}(\mathbb{C})$ to $L^q(\mathbb{C})$ for any $q \in [1, \infty)$. For $q \in (1, 2]$

we have by Hölder's inequality that $\|(I - T_R) f\|_q \le C_q R^{\frac{2(1-q)}{q}} \|f\|_{L^{2,1}}$, so that $\|(I - T_R)\|_{\mathcal{B}(L^q)}$ vanishes as $R \to \infty$. The compact embedding now follows from norm-closure of the compact operators. □

Estimates and vanishing theorem for the $\bar{\partial}$-problem. The solid Cauchy transform is given by

$$(Pf)(z) = \frac{1}{\pi} \int \frac{1}{z - \zeta} f(\zeta)\, dA(\zeta)$$

and is an inverse for the $\bar{\partial}$-operator in the sense that, for $f \in C_0^\infty(\mathbb{C})$,

$$P(\bar{\partial} f) = \bar{\partial}(Pf) = f. \tag{2.3}$$

Results analogous to those described below also hold for the operator

$$(\bar{P} f)(z) = \frac{1}{\pi} \int \frac{1}{\bar{z} - \bar{\zeta}} f(\zeta)\, dA(\zeta)$$

which is an inverse for the ∂-operator.

The following estimates extend P to a larger domain. They are proved, for example, in Vekua [35, Chapter I.6] or Astala, Iwaniec, and Martin [6, §4.3].

(1) Fractional integration and Hölder estimates. If $q \in (1, 2)$ then $\tilde{q}$ denotes the Sobolev conjugate $(\tilde{q})^{-1} = q^{-1} - 1/2$. It follows from the Hardy-Littlewood-Sobolev inequality that

$$\|Pf\|_{\tilde{q}} \le C_q \|f\|_q. \tag{2.4}$$

We usually take $\tilde{q} = p$ and $q = 2p/(p + 2)$ for $p \in (2, \infty)$. From this inequality and Hölder's inequality we see that for $p \in (2, \infty)$, $v \in L^2(\mathbb{C})$ and $u \in L^p(\mathbb{C})$

$$\|P(vf)\|_p \le C_p \|v\|_2 \|f\|_p. \tag{2.5}$$

It follows from Hölder's inequality that for any q, r with $1 < q < 2 < r < \infty$,

$$\|Pf\|_\infty \le C_{q,r}(\|f\|_q + \|f\|_r). \tag{2.6}$$

(2) Hölder continuity and asymptotic behavior. For any $p > 2$ and $f \in L^p(\mathbb{C}) \cap L^{p'}(\mathbb{C})$,

$$|(Pf)(z) - (Pf)(z')| \le C_p |z - z'|^{1-2/p} \|f\|_p. \tag{2.7}$$

If $p \in (2, \infty)$ and $f \in L^p \cap L^{p'}$ then

$$\lim_{|z| \to \infty} (Pf)(z) = 0. \tag{2.8}$$

By (2.6) and a density argument, it is enough to show that (2.8) holds for $f \in C_0^\infty(\mathbb{C})$. This is a straightforward computation.

The following lemma will allow us to recast (1.5), (1.8), and (1.13a) as integral equations.

Lemma 2.2. *Suppose $f \in L^q(\mathbb{C})$ for $q \in (1,2)$. A function $u \in L^{\tilde{q}}(\mathbb{C})$ solves $\bar{\partial} u = f$ in distribution sense if and only if $u = Pf$.*

Proof. For any $f \in L^q(\mathbb{C})$, it follows from (2.3) and (2.4) that $u = Pf$ solves $\bar{\partial} u = f$ in distribution sense.

Suppose, on the other hand, that $f \in L^q(\mathbb{C})$, that $u \in L^{\tilde{q}}(\mathbb{C})$, and that $\bar{\partial} u = f$ in distribution sense. Let $v = u - Pf$. It follows that $\partial\bar{\partial} v = 0$ in distribution sense, so that $v \in C^\infty$ by Weyl's lemma. Thus, v is a holomorphic function belonging to $L^p(\mathbb{C})$, so v vanishes identically by Liouville's Theorem. □

The following vanishing theorem is a special case of Brown and Uhlmann [16, Corollary 3.11] that will suffice for our purpose.

Lemma 2.3. *Suppose that $w \in L^p(\mathbb{C}) \cap L^2_{\mathrm{loc}}(\mathbb{C})$ for some $p \in (1,\infty)$, that $a \in L^2(\mathbb{C})$, and that $\bar{\partial} w = a\bar{w}$ in distribution sense. Then $w = 0$.*

Basic estimates on the Beurling transform. The Beurling transform $\mathcal{S}$ is defined on $C_0^\infty(\mathbb{C})$ by

$$(\mathcal{S} f)(z) = -\frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_{|w-z|>\varepsilon} \frac{1}{(z-w)^2} f(w)\, dA(w)$$

and obeys the relation $\bar{\partial}(\mathcal{S} f) = \partial f$. We refer the reader to [6, §4.3] for discussion and proofs.

Lemma 2.4. *The operator $\mathcal{S}$ extends to a bounded operator from $L^p(\mathbb{C})$ to itself for any $p \in (1,\infty)$, unitary if $p = 2$. Moreover, if $\nabla\varphi$ belongs to $L^q(\mathbb{C})$ for some $q \in (1,\infty)$, then $\mathcal{S}(\bar{\partial}\varphi) = \partial\varphi$.*

Thus, if $u \in L^p(\mathbb{C})$ for some $p \in (1,\infty)$ and $\nabla u \in L^q(\mathbb{C})$ for $q \in (1,\infty)$, the norms $\|\bar{\partial} u\|_q$, $\|\partial u\|_q$, and $\|\nabla u\|_q$ are mutually equivalent.

We will also use the analogous results for the transform

$$(\mathcal{S}^* f)(z) = -\frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_{|w-z|>\varepsilon} \frac{1}{(\bar{z}-\bar{w})^2} f(w)\, dA(w)$$

which satisfies $\partial(\mathcal{S}^* f) = \bar{\partial} f$ on $C_0^\infty(\mathbb{C})$.

Integration by parts. If $\varphi(z,\theta)$ is a smooth, real-valued function with isolated critical points in the integration variable z, and if $f \in C_0^\infty$ with support away from the critical points of φ, then

$$P[e^{i\varphi} f](z) = \frac{e^{i\varphi}}{i\varphi_{\bar z}} f(z) - \frac{1}{i} P[e^{i\varphi} \bar\partial_z(\varphi_{\bar z}^{-1} f)](z). \tag{2.9}$$

In case $i\varphi(z,k) = \bar k \bar z - kz$ for $k \neq 0$, the phase function φ has no critical points. Hence, for any $f \in C_0^\infty(\mathbb{C})$, the identity

$$P[e_k f] = \frac{e_k}{\bar k} f - \frac{1}{\bar k} P[e_k(\bar\partial_z f)] \tag{2.10}$$

holds.

Let $q \in (1,2)$. Approximating $f \in W^{1,q}(\mathbb{C})$ by a sequence from $C_0^\infty(\mathbb{C})$, we can conclude that for $f \in W^{1,q}(\mathbb{C})$, the equality (2.10) holds in $L^{\tilde q}$. Note that, if $f \in W^{1,q}(\mathbb{C})$, Sobolev embedding implies that $f \in L^{\tilde q}$ so the statement makes sense. From this we obtain the estimate

$$\|P[e_k f]\|_{\tilde q} \leq C_q \langle k\rangle^{-1} \|f\|_{W^{1,q}}. \tag{2.11}$$

Brascamp–Lieb type estimates. The following multilinear estimate, due to Russell Brown [13, Lemma 3] (see also Nie-Brown [15]), plays a crucial role in the analysis of solutions to (1.5) and (1.8). See Appendix A for a proof of the estimate by the methods of Bennett, Carbery, Christ and Tao [10, 11]. Define

$$\Lambda_n(\rho, u_0, u_1, \ldots, u_{2n}) = \int_{\mathbb{C}^{2n+1}} \frac{|\rho(\zeta)||u_0(z_0)| \ldots |u_{2n}(z_{2n})|}{\prod_{j=1}^{2n} |z_{j-1} - z_j|} dA(z),$$

where $dA(z)$ is product measure on $\mathbb{C}^{2n+1}$ and

$$\zeta = \sum_{j=0}^{2n} (-1)^j z_j.$$

Proposition 2.5. [13] *For any functions $\rho, u_0, u_1, \ldots, u_{2n} \in L^2(\mathbb{C})$, the estimate*

$$|\Lambda_n(\rho, u_0, u_1, \ldots, u_{2n})| \leq C_n \|\rho\|_2 \prod_{j=0}^{2n} \|u_j\|_2 \tag{2.12}$$

holds.

Remark 2.6. Let $T^{(j)}\psi = Pe_k u_j \bar{\psi}$ where $u_j \in L^2(\mathbb{C})$. Consider the form

$$\langle 1, e_k u_0 T^{(1)} \dots T^{(2n)} 1 \rangle \tag{2.13}$$

which defines a function of k. Integrating (2.13) against a test function $\widehat{\rho}$ in the k-variable and applying (2.12) shows that (2.13) defines an L^2 function of k with

$$\| \langle 1, e_k u_0 T^{(1)} \dots T^{(2n)} 1 \rangle \|_2 \leq C_n \prod_{j=0}^{2n} \|u_j\|_2.$$

For details we refer the reader to the proof of Theorem 2 in [13] where a very similar estimate is proved.

3. An oscillatory $\bar{\partial}$-problem

In this section we study the $\bar{\partial}$ problem (1.5). The main results used in §4 are Lemmas 3.6 and 3.12. Because the problem (1.8) has a nearly identical structure, the results of this section apply to the problem (1.8) with typographical changes. Fix $p \in (2, \infty)$, $k \in \mathbb{C}$ and $u \in H^{1,1}(\mathbb{C})$. It follows from Lemma 2.2 that a pair of functions (μ_1, μ_2) with $\mu_1 - 1, \mu_2 \in L^p(\mathbb{C})$ solves (1.5a)–(1.5b) if and only if

$$\begin{aligned} \mu_1 - 1 &= T_k \mu_2, \\ \mu_2 &= T_k \mu_1, \end{aligned} \tag{3.1}$$

where T_k is the antilinear operator

$$[T_k \psi](z) = \frac{1}{2} P[e_k(\cdot) u(\cdot) \overline{\psi(\cdot)}](z).$$

We sometimes write $T_{k,u}$ for T_k to emphasize its dependence on u. We will solve these integral equations and then check that (1.5c) holds for the solutions so constructed (see (3.15)).

Formally, $\mu_1 = (I - T_k^2)^{-1} 1$. To prove and analyze this solution formula, we will need the following estimates which are easily deduced from (2.4)–(2.7). Here C_p (resp. $C_{p,q}$) represent numerical constants depending only on p (resp. p, q).

$$\|T_k\|_{\overline{\mathcal{B}}(L^p)} \leq C_p \|u\|_2, \tag{3.2}$$

$$\|T_{k,u} - T_{k,u'}\|_{\overline{\mathcal{B}}(L^p)} \leq C_p \|u - u'\|_2, \tag{3.3}$$

$$\|T_k\|_{\overline{\mathcal{B}}(L^p, L^\infty)} \leq C_p (\|u\|_{2p/(p-1)} + \|u\|_{p(p+2)/(p-2)}), \tag{3.4}$$

$$|(T_k\psi)(z) - (T_k\psi)(z')| \le C_{p_0,p}|z-z'|^{1-2/p_0}\|u\|_{H^{1,1}}\|\psi\|_p, \quad 2 < p_0 < p, \tag{3.5}$$

$$\|T_{k,u} - T_{k',u'}\|_{\mathcal{B}(L^p)} \le C_p(\|(e_{k-k'}(\cdot)-1)u\|_2 + \|u-u'\|_2), \tag{3.6}$$

$$\|T_{k,u} - T_{k,u'}\|_{\mathcal{B}(L^p,L^\infty)} \le C_p(\|u-u'\|_{2p/(p-1)} + \|u-u'\|_{p(p+2)/(p-2)}), \tag{3.7}$$

$$\begin{aligned}\|T_{k,u} - T_{k',u}\|_{\mathcal{B}(L^p,L^\infty)} \le C_p(&\|(e_{k-k'}-1)u\|_{2p/(p-1)} \\ &+ \|(e_{k-k'}-1)u\|_{p(p+2)/(p-2)}).\end{aligned} \tag{3.8}$$

In (3.4), we used (2.6) with $q = 2p/(p+1)$ and $r = (p+2)/2$. In (3.5), we used (2.7) together with $\|u\psi\|_{p_0} \le \|u\|_s\|\psi\|_p$ for $s^{-1} = p_0^{-1} - p^{-1}$. Estimate (3.7) follows from (3.4) and the linear dependence of $T_{k,u}$ on u. Estimate (3.8) follows from (3.4) and the linear dependence of $T_{k,u}$ on $e_k u$.

We also have from (2.2) and (2.4) that

$$\|T_k 1\|_p \le C_p\|u\|_{H^{1,1}}, \tag{3.9}$$

$$\|T_{k,u}1 - T_{k,u'}1\|_p \le C_p\|u-u'\|_{H^{1,1}}. \tag{3.10}$$

Using the inequality $|e^{i\theta}-1| \le 2^{1-\alpha}|\theta|^\alpha$ for any $\alpha \in [0,1]$, (2.4), and Hölder's inequality, we have

$$\|T_k 1 - T_{k'}1\|_p \le C_p|k-k'|^\alpha\|u\|_{L^{2,1}}, \quad \alpha \in [0, 1-2/p), \tag{3.11}$$

while from (2.6) with $q = 2p/(p+1)$, $r = (p+2)/2$, we have

$$\|T_k 1 - T_{k'}1\|_\infty \le C_p(\|(e_{k-k'}-1)u\|_{2p/(p+1)} + \|(e_{k-k'}-1)u\|_{(p+2)/2}). \tag{3.12}$$

In what follows, it will be important to track uniformity of estimates for u in bounded subsets of $H^{1,1}(\mathbb{C})$. For given $M_0 > 0$, we denote

$$B_0 = \{u \in H^{1,1}(\mathbb{C}) \colon \|u\|_{H^{1,1}} \le M_0\}.$$

We denote by $C(M_0)$ a constant depending only on M_0.

We first construct the resolvent $(I - T_{k,u}^2)^{-1}$ for $(k,u) \in \mathbb{C} \times L^2(\mathbb{C})$ using Fredholm theory.

Lemma 3.1. *For any $(k,u) \in \mathbb{C} \times L^2(\mathbb{C})$ and $p \in (2,\infty)$, $(I - T_{k,u}^2)^{-1}$ exists as a bounded operator on L^p and the map*

$$(k,u) \longmapsto (I - T_{k,u}^2)^{-1}$$

is continuous from $\mathbb{C} \times L^2$ into $\mathcal{B}(L^p)$.

Proof. First, we show that T_k is a compact operator on L^p. By the norm-closure of compact operators, the estimate (3.3), and the density of $C_0^\infty(\mathbb{C})$ in $L^2(\mathbb{C})$, it suffices to show that T_k is compact for $u \in C_0^\infty(\mathbb{C})$. Let $p' \in (1,2)$ be the conjugate exponent to p. It suffices to show that the Banach space adjoint $T_k' = -\frac{1}{2}e_k u P$ is compact from $L^{p'}(\mathbb{C})$ to itself. Let $\Omega \subset \mathbb{C}$ be a bounded set with smooth boundary containing the support of u. If $f \in L^{p'}(\mathbb{C})$ then $Pf \in L^{2p/(p-2)}(\mathbb{C})$ by (2.4) while $\nabla Pf \in L^{p'}(\mathbb{C})$ by Lemma 2.4. Thus

$$\|uPf\|_{W^{1,p'}} \leq C(1+|\Omega|^{1/2})\|f\|_{p'}$$

and compactness follows from the Rellich-Kondrakov Theorem.

Next, we recall the standard argument (see, for example, [8, §7]) that $\ker(I - T_k^2)$ is trivial. Suppose that $\psi \in L^p(\mathbb{C})$ with $\psi = T_k^2\psi$. Then by Lemma 2.2, the pair $(\psi, T\psi)$ is a weak solution of the system (1.5a)–(1.5b). It follows that $\phi_+ = \psi + T\psi$ and $\phi_- = \psi - T\psi$ each solve the scalar problem $\bar{\partial} w = a\bar{w}$ with $a = \pm\frac{1}{2}e_k u \in L^2(\mathbb{C})$ and $w \in L^p(\mathbb{C})$. We now conclude from Lemma 2.3 that $\phi_+ = \phi_- = 0$ so $\psi = 0$.

It now follows from the Fredholm alternative that $(I - T_{k,u}^2)^{-1}$ exists. To prove that the resolvent is continuous in $(k,u) \in \mathbb{C} \times L^2(\mathbb{C})$, we appeal to (3.2), (3.6), the Dominated Convergence Theorem, and the second resolvent formula. $\square$

For $u \in H^{1,1}(\mathbb{C})$, the operator T_k^2 has small norm for large $|k|$.

Lemma 3.2. *Fix $p \in (2,\infty)$. For $u \in H^{1,1}(\mathbb{C})$ and $|k| \geq 1$, the estimate*

$$\|T_{k,u}^2\|_{\mathcal{B}(L^p)} \leq C_p \|u\|_{H^{1,1}}^2 \langle k \rangle^{-1}$$

holds. Moreover,

$$\|T_{k,u}^2 1\|_p \leq C_p \|u\|_{H^{1,1}}^2 \langle k \rangle^{-1}.$$

Proof. From (2.11) with $f = u\bar{\psi}$ and $q = 2p/(p+2)$ we have the estimate

$$\|T_k\psi\|_p \leq C_p \langle k \rangle^{-1}(\|u\|_2\|\psi\|_p + \|\bar{\partial}u\|_2\|\psi\|_p + \|u\|_p\|\bar{\partial}\bar{\psi}\|_2) \tag{3.13}$$

so that

$$\|T_k^2\psi\|_p \leq C_p\langle k \rangle^{-1}(\|u\|_2\|T_k\psi\|_p + \|\bar{\partial}u\|_2\|u\|_2\|\psi\|_p + \|u\|_p\|u\|_{2p/(p-2)}\|\psi\|_p).$$

In the second term we used (3.2), and in the third term, we used $\|\bar{\partial}\,\overline{T_k\psi}\|_2 = \|\partial\overline{T_k\psi}\|_2 = \|u\psi\|_2$ (the second step follows from the unitarity of the Beurling transform; see Lemma 2.4). Using (2.2) and (3.2), we obtain the first estimate.

To obtain the second estimate, we use (3.13) with $\psi = T_k 1$ together with (2.2) and (2.4). $\square$

From Lemma 3.1 and Lemma 3.2, we obtain the following uniform estimate on the resolvent.

Lemma 3.3. *Fix $M_0 > 0$ and $p > 2$. The estimate*

$$\sup\{\|(I - T_{k,u}^2)^{-1}\|_{\mathcal{B}(L^p)} : k \in \mathbb{C}, u \in B_0\} \le C(M_0, p)$$

holds.

Proof. By Lemma 3.2, given M_0, we can find R_0 so that $\|(I - T_{k,u}^2)^{-1}\|_{\mathcal{B}(L^p)} \le 2$ for all (k, u) with $|k| > R_0$ and $\|u\|_{H^{1,1}} \le M_0$. On the other hand, the set

$$\{(k, u) : |k| \le R_0, \|u\|_{H^{1,1}} \le M_0\}$$

is bounded in $\mathbb{C} \times H^{1,1}$, hence precompact in $\mathbb{C} \times L^2$ by Lemma 2.1. The image of this set under the map $(k, u) \mapsto (I - T_{k,u}^2)^{-1}$ is therefore a bounded set in $\mathcal{B}(L^p)$ by the continuity asserted in Lemma 3.1. □

Lemma 3.4. *Let $u \in H^{1,1}(\mathbb{C})$. For each $k \in \mathbb{C}$ and any $p \in (2, \infty)$, the functions*

$$\mu_1 := 1 + (I - T_k^2)^{-1} T_k^2 1, \quad \mu_2 := T_k \mu_1 \tag{3.14}$$

are the unique solutions of (3.1) with $\mu_1 - 1, \mu_2 \in C^\alpha \cap L^p$. For these solutions,

$$\lim_{|z| \to \infty} (\mu_1, \mu_2) = (1, 0). \tag{3.15}$$

Moreover, for any $p \in (2, \infty)$, any $\alpha \in [0, 1 - 1/(2p))$, any $M_0 > 0$ and any $u \in B_0$, the following estimates hold:

$$\sup_{k \in \mathbb{C}} (\|\mu_1 - 1\|_p + \|\mu_2\|_p) \le C(M_0, p), \tag{3.16}$$

$$\sup_{k \in \mathbb{C}} (\|\mu_1\|_{C^\alpha(\mathbb{C})} + \|\mu_2\|_{C^\alpha(\mathbb{C})}) \le C(M_0, \alpha). \tag{3.17}$$

Proof. Let $p > p_0 > 2$. It follows from (3.4) and (3.5) that for $u \in H^{1,1}$, T_k maps $L^p(\mathbb{C})$ into $C^{1-2/p_0}(\mathbb{C})$ with bound uniform in $k \in \mathbb{C}$ and u in bounded subsets of $H^{1,1}(\mathbb{C})$. From (3.9), we have $T_k 1 \in L^p$ with norm bounded uniformly in $k \in \mathbb{C}$ and u with $\|u\|_{H^{1,1}} \le M_0$. It follows from this fact and Lemma 3.1 that $(I - T_k^2)^{-1} T_k^2 1 = T_k (I - T_k^2)^{-1} T_k 1 \in L^p(\mathbb{C}) \cap C^{1-2/p_0}(\mathbb{C})$ for $p_0 > p > 2$. Hence, the functions given by (3.14) solve (3.1) with $\mu_1 - 1 \in L^p \cap C^{1-2/p_0}$. The assertion about limiting behavior follows from (2.8).

Estimate (3.16) follows from (3.2), (3.9), and Lemma 3.3. Estimate (3.17) follows from the uniform estimate on $\|T_k\|_{\mathcal{B}(L^p, C^{1-2/p_0})}$. □

Next, we study the k-dependence of the solutions (3.14). For brevity we write $\mu = (\mu_1, \mu_2)$.

Lemma 3.5. *Fix $p > 2$, $M_0 > 0$ and $u \in B_0$. Then*

$$\|\mu(\cdot, k) - \mu(\cdot, k')\|_{L^p} \leq C(M_0, \alpha)|k - k'|^\alpha, \tag{3.18}$$

for any $\alpha \in [0, 1 - 2/p)$, and

$$\sup_{z \in \mathbb{C}} |\mu(z, k) - \mu(z, k')| \leq C(M_0, \alpha) F(|k - k'|), \tag{3.19}$$

where $F(0) = 0$, F is continuous, and F depends only on u and p. Finally, for each fixed $z \in \mathbb{C}$,

$$\lim_{|k| \to \infty} (\mu_1(z, k), \mu_2(z, k)) = (1, 0). \tag{3.20}$$

Proof. From (3.6) and the inequality $|e_k(z) - 1| \leq 2^{1-\alpha}|k|^\alpha |z|^\alpha$, we easily see that for any $p \in (2, \infty)$ and $\alpha \in [0, 1]$, the estimate

$$\|T_k - T_{k'}\|_{\mathcal{B}(L^p)} \leq C_{p,\alpha} \|u\|_{H^{1,1}} |k - k'|^\alpha \tag{3.21}$$

holds. From this estimate, Lemma 3.3, and the second resolvent formula, we conclude that $\|(I - T_k^2)^{-1} - (I - T_{k'}^2)^{-1}\|_{\mathcal{B}(L^p)} \leq C(M_0, p, \alpha)|k - k'|^\alpha$ for any $\alpha \in [0, 1]$. From (3.11), (3.14), and (3.21) again, we conclude that

$$\|\mu_1(\cdot, k) - \mu_1(\cdot, k')\|_{L^p} \leq C(M_0, p, \alpha)|k - k'|^\alpha, \quad \alpha \in [0, 1 - 2/p).$$

A similar estimate holds for μ_2 by the formula $\mu_2 = T_k \mu_1$, estimates (3.2) and (3.11), the continuity estimate on μ_1, and (3.21). This proves (3.18).

Using (3.18), estimates (3.2), (3.4),(3.6), (3.8), (3.12), (3.16), and the identity

$$\mu_1(z, k) - \mu_1(z, k') = (T_k^2 \mu_1)(z, k) - (T_{k'}^2 \mu_1)(z, k'),$$

we conclude that (3.19) holds for μ_1 with

$$\begin{aligned} F(k - k') = {} & \|(e_{k-k'} - 1)u\|_{2p/(p-1)} + \|(e_{k-k'} - 1)u\|_{(p+2)/p} \\ & + \|(e_{k-k'} - 1)u\|_{2p/(p-1)} + \|(e_{k-k'} - 1)u\|_{p(p+2)/(p-2)}. \end{aligned}$$

To estimate μ_2 we write $\mu_2 = T_k \mu_1$ and use (3.4), (3.8), (3.12), (3.16), and (3.18) to obtain an estimate with the same F as above.

From (3.17) and (3.19), it follows that the solutions (3.14) are jointly continuous in (z, k) so that, in particular, point evaluations make sense. If we can show that $\lim_{|k| \to \infty} T_k 1 = 0$ in $L^p(\mathbb{C})$, (3.20) will follow from (3.14), Lemma 3.3, and the uniform estimate (3.4). By (3.10) and a density argument, it suffices to show that $\lim_{|k| \to \infty} T_k 1 = 0$ for $u \in C_0^\infty(\mathbb{C})$. This is an immediate consequence of (2.10). □

Next, we prove Lipschitz continuity of μ as a function of u. We write $\mu(z, k, u)$ to emphasize the dependence of μ on u.

Lemma 3.6. *Fix $M_0 > 0$ and $p \in (2, \infty)$, and suppose that $u, u' \in B_0$. Then*

$$\sup_{k \in \mathbb{C}} \|\mu(\,\cdot\,, k, u) - \mu(\,\cdot\,, k, u')\|_p \leq C(M_0)\|u - u'\|_{H^{1,1}}, \tag{3.22}$$

$$\sup_{(z,k) \in \mathbb{C} \times \mathbb{C}} |\mu(z, k, u) - \mu(z, k, u')| \leq C(M_0)\|u - u'\|_{H^{1,1}}. \tag{3.23}$$

Proof. If (3.22) holds, we can use (3.2), (3.3), (3.4), (3.7), and the identity $(\mu_1 - 1, \mu_2) = (T_k^2 \mu_1, T_k \mu_1)$ to conclude that (3.23) holds.

If (3.22) holds with μ replaced by μ_1, then the same estimate for μ_2 follows from the formula $\mu_2 = T_k 1 + T_k(\mu_1 - 1)$ and (3.2), (3.3), and (3.10).

It remains to prove (3.22) for μ replaced by μ_1. By the second resolvent formula, (3.2), (3.3), and Lemma 3.3, for any u, u' in B,

$$\|(I - T_{k,u}^2)^{-1} - (I - T_{k,u'}^2)^{-1}\|_{\mathcal{B}(L^p)} \leq C(M_0, p)\|u - u'\|_2. \tag{3.24}$$

Using the identity

$$\mu_1(\,\cdot\,, k, u) - \mu_1(\,\cdot\,, k, u') = [T_{k,u}(I - T_{k,u}^2)^{-1} T_{k,u} 1 - T_{k,u'}(I - T_{k,u'}^2)^{-1} T_{k,u'} 1],$$

estimates (3.2), (3.3), (3.9), (3.10), the uniform estimate from Lemma 3.3, and the Lipschitz estimate (3.24), we conclude that μ_1 satisfies the L^p Lipschitz estimate. $\square$

We now turn to the scattering map (1.6). If $u \in H^{1,1}(\mathbb{C})$ we may define $r = \mathcal{R}u$ by

$$r(k) = \frac{1}{\pi} \int e_k(z) u(z)\, dA(z) + \frac{1}{\pi} \int e_k(z) u(z) (\overline{\mu_1(z, k)} - 1)\, dA(z). \tag{3.25}$$

The first term is a Fourier transform and is well-defined as an element of $H^{1,1}(\mathbb{C})$. The second integral defines a bounded continuous function of k by (3.18) since $u \in L^{p'}$. It follows from (3.25) that if $r = \mathcal{R}u$, $r' = \mathcal{R}u'$, and $\|u\|_{H^{1,1}}, \|u'\|_{H^{1,1}} \leq M_0$, then

$$\begin{aligned} r - r' &= \frac{1}{\pi} \int e_k(u - u') \\ &\quad + \frac{1}{\pi} \int e_k (u[\overline{\mu_1(z, k; u)} - 1] - u'[\overline{\mu_1(z, k; u')} - 1]) \\ &= I_1(k) + I_2(k), \end{aligned}$$

where for any $p \in (2, \infty)$

$$\|I_1\|_p \leq C_p \|u - u'\|_{p'}, \quad \|I_2\|_\infty \leq C(M_0)\|u - u'\|_{H^{1,1}}. \tag{3.26}$$

The first estimate follows from the Hausdorff-Young inequality. In the second estimate, we used (3.22) and (3.16).

Remark 3.7. If $u \in H^{1,1}(\mathbb{C}) \cap L^1(\mathbb{C})$ and $r = \mathcal{R}u$, we may compute

$$r(k) = \int e_k(z) u(z) \mu_1(z, k)\, dA(z),$$

where the integral is absolutely convergent by (3.17). By the Dominated Convergence Theorem and (3.18), this expression defines a continuous function of k.

We claim that, moreover, $r \in C_0(\mathbb{C})$, the continuous functions vanishing at infinity. To see this we use (3.25) and note that the first term vanishes as $|k| \to \infty$ by the Riemann–Lebesgue lemma, while the second term vanishes as $|k| \to \infty$ by (3.20) and dominated convergence.

We now give a self-contained proof of the standard result (see [8, §3.2], the formal argument in [33, I, §1], and justification in [33, II, §4]) that the functions (μ_1, μ_2) are determined by the $\bar{\partial}$-data r. More precisely, we will show that the functions

$$\nu_1 = \mu_1, \quad \nu_2 = e_k \overline{\mu_2} \tag{3.27}$$

solve the $\bar{\partial}_k$ problem (1.8). We will prove this by direct differentiation of the solution formulas (3.14) in k.

To do so, we will need the following well-known lemma which shows that the 'analyticity defect' of the operator T_k^2 is a rank-one operator. We give a proof for completeness.

Lemma 3.8. *The identity*

$$(\bar{\partial}_k T_k^2) = \frac{1}{2}(T_k 1)\mathcal{F}^{-1}(\bar{u}\,\cdot) \tag{3.28}$$

holds as a derivative in $\mathcal{B}(L^p)$ operator norm, where $\mathcal{F}^{-1}$ is given by (1.10).

Proof. This identity is formally obvious but we need an explicit estimate to prove differentiability in operator norm. Write $z = x_1 + ix_2$ and $k = k_1 + ik_2$ so that $e_k(z) = \exp(-2i(k_1x_2 + k_2x_1))$ and $\bar{\partial}_k = (1/2)(\partial_{k_1} + i\partial_{k_2})$. We claim that

$$\left(\frac{\partial}{\partial k_1}T_k^2 f\right)(z) = -\frac{i}{2\pi^2}\int \frac{1}{z-z'}e_k(z'-z'')\frac{x_2'-x_2''}{\bar{z}'-\bar{z}''}u(z')u(z'')f(z'')\,dA(z'')\,dA(z'), \tag{3.29}$$

$$\left(\frac{\partial}{\partial k_2}T_k^2 f\right)(z) = -\frac{i}{2\pi^2}\int \frac{1}{z-z'}e_k(z'-z'')\frac{x_1'-x_1''}{\bar{z}'-\bar{z}''}u(z')u(z'')f(z'')\,dA(z'')\,dA(z'), \tag{3.30}$$

from which (3.28) follows. We will prove (3.29) since the proof of (3.30) is similar. Using the estimate

$$|e^{iht} - 1 - iht| \le 2^{1-\theta}|h|^{1+\theta}|t|^{1+\theta}$$

(for $\theta \in (0,1)$ to be chosen), denoting $T_k^2 f$ by $F(k_1)$, and denoting the right-hand side of (3.29) by $F'(k_1)$, we can estimate $|h^{-1}(F(k_1+h) - F(k_1)) - F'(k_1)|$ by $|h|^\theta$ times

$$\int \frac{|x_2'-x_2''|^\theta}{|z-z'|}\,|u(z')|\,|u(z'')|\,|f(z'')|\,dA(z'')\,dA(z'). \tag{3.31}$$

To prove norm differentiability, it suffices to bound (3.31) as an L^p function of z uniformly in f with $\|f\|_p \le 1$. The L^p norm of the expression (3.31) is bounded by $2^{-\theta}$ times the sum of the L^p norms of the functions

$$I_1(z) = \int \frac{1}{|z-z'|}|z'|^\theta\,|u(z')|\,|u(z'')|\,|f(z'')|\,dA(z'')\,dA(z'),$$

$$I_2(z) = \int \frac{1}{|z-z'|}|z''|^\theta\,|u(z')|\,|u(z'')|\,|f(z'')|\,dA(z'')\,dA(z').$$

By Hölder's inequality and (2.4),

$$\|I_1\|_p \le \|u\|_{p'}\|(\cdot)^\theta u(\cdot)\|_{2p/(p+2)},$$

$$\|I_2\|_p \le \|u\|_{2p/(p+2)}\|(\cdot)^\theta u(\cdot)\|_{p'},$$

so it suffices to choose θ so the weighted norms of u are bounded for $u \in H^{1,1}$. A short calculation shows that the norm $\|(\cdot)^{\theta}u(\cdot)\|_s$ is bounded by constants times $\|\langle\cdot\rangle u(\cdot)\|_2$ provided $0 < \theta < 2 - 2/s$. Choosing any θ with $0 < \theta < \min(1 - 2/p, 2/p)$ gives the desired bound. $\square$

First, we consider $u \in \mathcal{S}(\mathbb{C})$.

Lemma 3.9. *Let $u \in \mathcal{S}(\mathbb{C})$, and let (μ_1, μ_2) be given by (3.14). Then, for each $z \in \mathbb{C}$, $\nu_1(z, \cdot), \nu_2(z, \cdot)$ defined by (3.27) are strong solutions of the system (1.8).*

Proof. The asymptotic condition (1.8c) is an immediate consequence of (3.20) and the definition of (ν_1, ν_2). To show that (ν_1, ν_2) satisfy (1.8a)–(1.8b), it suffices to show that

$$\bar{\partial}_k \mu_1 = \frac{1}{2}\bar{r}\mu_2, \quad (\partial_k + z)\mu_2 = \frac{1}{2}r\mu_1. \tag{3.32}$$

We will prove these identities by differentiating the solution formulas (3.14) with respect to k and using (3.28).

For $u \in \mathcal{S}(\mathbb{C})$ it is easy to see that

$$\bar{\partial}_k T_k^2 1 = (\mathcal{F}^{-1}\bar{u})(k)(T_k 1),$$

where the Fourier transform defines a continuous function of k since $u \in \mathcal{S}(\mathbb{C})$. Using the operator identity

$$\bar{\partial}_k (I - T_k^2)^{-1} = (I - T_k^2)^{-1}(\bar{\partial}_k T_k^2)(I - T_k^2)^{-1}$$

together with the formula $\mu_1 - 1 = (I - T_k^2)^{-1} T_k^2 1$ and (3.28), we compute

$$\begin{aligned}
\bar{\partial}_k \mu_1 &= \bar{\partial}_k ((I - T_k^2)^{-1} T_k^2 1) \\
&= [(I - T_k^2)^{-1} T_k 1](\mathcal{F}^{-1}(\bar{u}(\mu_1 - 1)) + [(I - T_k^2)^{-1} T_k 1]\mathcal{F}^{-1}(\bar{u}) \\
&= \mu_2 \mathcal{F}^{-1}(\bar{u}\mu_1) \\
&= \bar{r}\mu_2.
\end{aligned}$$

To compute $(\partial + k)\mu_2$ we will use the identity $(\partial_k + z)T_k f = \mathcal{F}(u\bar{f}) + T_k(\bar{\partial}_k f)$, which holds pointwise if $u \in \mathcal{S}(\mathbb{C})$, $f(\cdot, k) \in C(\mathbb{C})$, and $(\bar{\partial}_k f)(\cdot, k) \in C(\mathbb{C})$ with bounds uniform in k. We then compute

$$\begin{aligned}(\partial_k + z)\mu_2 &= (\partial_k + z)T_k\mu_1\\ &= \mathcal{F}(u\overline{\mu_1}) + T_k(\partial_k\mu_1)\\ &= r + rT_k\mu_2\\ &= r\mu_1,\end{aligned}$$

where in the last step we used $T_k\mu_2 = T_k^2\mu_1 = \mu_1 - 1$. □

Next, we use Lemma 3.6 to extend the result to $u \in H^{1,1}$.

Lemma 3.10. *Let $u \in H^{1,1}(\mathbb{C})$ and let (μ_1, μ_2) be given by (3.14). Then, for each $z \in \mathbb{C}$, $\nu_1(z,\cdot), \nu_2(z,\cdot)$ defined by (3.27) are weak solutions of the system (1.8).*

Proof. It suffices to show that for each $\varphi \in C_0^\infty(\mathbb{C})$ and each fixed $z \in \mathbb{C}$,

$$\int (-\bar{\partial}_k\varphi)\mu_1(z,k)\,dA(k) = \frac{1}{2}\int \varphi(k)\overline{r(k)}\mu_2(z,k)\,dA(k), \tag{3.33}$$

$$\int (-\partial + z)\varphi(k)\mu_2(z,k)\,dA(k) = \frac{1}{2}\int \varphi(k)r\mu_1(z,k)\,dA(k). \tag{3.34}$$

Let $\{u_n\}_{n=1}^\infty$ be a sequence from $C_0^\infty(\mathbb{C})$ converging in $H^{1,1}(\mathbb{C})$ to u, and denote by $\mu_{1,n}, \mu_{2,n}$ the corresponding solutions given by (3.14). Finally, let

$$r_n = \pi^{-1}\int e_k u_n\overline{\mu_{1,n}}.$$

By Lemma 3.9 and an integration by parts, the identities

$$\int (-\bar{\partial}_k\varphi)\mu_{1,n}(z,k)\,dA(k) = \frac{1}{2}\int \varphi(k)\overline{r_n(k)}\mu_{2,n}(z,k)\,dA(k), \tag{3.35}$$

$$\int ((-\partial + z)\varphi(k))\mu_{2,n}(z,k)\,dA(k) = \frac{1}{2}\int \varphi(k)r_n\mu_{1,n}(z,k)\,dA(k) \tag{3.36}$$

hold. We will prove (3.33)–(3.34) by taking limits in (3.35)–(3.36) as $n \to \infty$. We give the proof for (3.33) since the other is similar. The left-hand side of (3.35) converges to the left-hand side of (3.33) as $n \to \infty$ by (3.23). To show convergence of the right-hand side, we estimate

$$\left|\int \varphi(k)(\overline{r_n(k)}\,\mu_{2,n} - \overline{r(k)}\,\mu_2(z,k))\,dA(k)\right| \le C(M_0)\|u_n - u\|_{H^{1,1}},$$

where we used uniform bounds (3.16) and (3.17) together with Lipschitz estimates (3.22), (3.23), and (3.26). □

We now briefly discuss the $\bar{\partial}$-problem (1.8) and prove:

Lemma 3.11. *For any $r \in \mathcal{S}(\mathbb{C})$, the relation* (1.11) *holds.*

Proof. Let S_z be the antilinear operator

$$[S_z\psi](k) = \frac{1}{2}P_k[e_{(\cdot)}(z)\overline{r(\cdot)\psi(\cdot)}](k),$$

where P_k is the Cauchy transform acting on the k variable. Write $S_z = S_{z,r}$ to emphasize the dependence of S_z on r. Observe that, as operators on $L^p(\mathbb{C})$, we have

$$[S_{z,r}f](k) = [T_{z,\bar{r}}f](k). \tag{3.37}$$

Formally, (1.8) is solved by

$$\nu_1 := 1 + (I - S_z^2)^{-1}S_z^2 1, \quad \nu_2 := S_z\nu_1 \tag{3.38}$$

(compare (3.14)). Tracing through the proofs of Lemmas 3.1–3.4 one can easily prove that these formulas give the unique solution to (1.8). The uniqueness of solutions to the $\bar{\partial}$-problems for μ and ν and the identity $e_k(z) = e_z(k)$ easily imply the identity

$$\nu_1(z,k,r) = \mu_1(k,z,\bar{r}). \tag{3.39}$$

One may then compute, for $r \in \mathcal{S}(\mathbb{C})$,

$$\begin{aligned}
(C \circ \mathcal{R} \circ C)(r) &= C \circ \mathcal{R}(\bar{r}) \\
&= C\Big(\frac{1}{\pi}\int e_z(k)\overline{r(k)\mu_1(k,z;\bar{r})}\,dA(k)\Big) \\
&= \frac{1}{z}\int e_{-k}(z)r(k)\nu_1(z,k,r)\,dA(k) \\
&= \mathcal{I}(r),
\end{aligned}$$

where we used $e_k(z) = e_z(k)$ and, in the third line, we used (3.39). $\square$

Finally, we obtain expansions for the solution μ which will facilitate a finer analysis of the scattering map.

Lemma 3.12. *Fix $M_0 > 0$ and suppose that $u \in B_0$. Then for any positive integer N,*

$$\mu_1(z,k) = 1 + \sum_{j=1}^{N} T^{2j} 1 + R_{1,N}(z,k;u),$$

$$\mu_2(z,k) = \sum_{j=0}^{N} T^{2j+1} 1 + R_{2,N}(z,k;u),$$

where for any $p \in (2,\infty)$

$$\|R_{1,N}(\,\cdot\,,z;u)\|_p \le C(p,M_0)\langle k\rangle^{-N-1}, \tag{3.40}$$

$$\|R_{2,N}(\,\cdot\,,z;u)\|_p \le C(p,M_0)\langle k\rangle^{-N-1}. \tag{3.41}$$

Moreover for any $p \in (2,\infty)$ and $u, u' \in B_0$, the estimates

$$\sup_{k\in\mathbb{C}} \langle k\rangle^N \|R_{1,N}(\,\cdot\,,k,u) - R_{1,N}(\,\cdot\,,k,u')\|_p \le C(M_0,p)\|u-u'\|_{H^{1,1}}, \tag{3.42}$$

$$\sup_{k\in\mathbb{C}} \langle k\rangle^N \|R_{2,N}(\,\cdot\,,k,u) - R_{2,N}(\,\cdot\,,k,u')\|_p \le C(M_0,p)\|u-u'\|_{H^{1,1}} \tag{3.43}$$

hold.

Proof. By iterating the integral equations (3.1), we see that

$$\mu_1 = 1 + \sum_{j=1}^{N} T_k^{2j} 1 + T_k^{2N+2}\mu_1,$$

$$\mu_2 = \sum_{j=0}^{N} T_k^{2j+1} 1 + T_k^{2N+3}\mu_2.$$

Thus

$$R_{1,N} = T_k^{2N+2}\mu_1 \quad \text{and} \quad R_{2,N} = T_l^{2N+3}\mu_2.$$

The norm estimates (3.40)–(3.41) follow from (3.2), (3.9), Lemma 3.2, and (3.16). The Lipschitz estimates (3.42)–(3.43) follow from Lemma 3.2 and (3.22). □

4. Direct and inverse scattering transforms

In this section we study the direct and inverse maps $\mathcal{R}$ and $\mathcal{I}$ defined respectively by (1.5)–(1.6) and (1.8)–(1.9). As in §3, for given $M_0 > 0$, B_0 denotes the ball of radius M_0 in $H^{1,1}(\mathbb{C})$.

First, we will prove:

Proposition 4.1. *The map $\mathcal{R}$ defined initially on $\mathcal{S}(\mathbb{C})$ by (1.5) and (1.6) extends to $H^{1,1}(\mathbb{C})$. Moreover, for any $M_0 > 0$, $u, u' \in B_0$, we have $\mathcal{R}u, \mathcal{R}u' \in H^{1,1}(\mathbb{C})$ and*

$$\|\mathcal{R}u - \mathcal{R}u'\|_{H^{1,1}} \leq C(M_0)\|u - u'\|_{H^{1,1}}.$$

Remark 4.2. The proof of Proposition 4.1 shows that for $u \in H^{1,1}(\mathbb{C})$, the scattering transform can be computed as

$$(\mathcal{R}u)(k) = \mathcal{F}(u)(k) + \frac{1}{\pi}\int e_k(\zeta)u(\zeta)(\overline{\mu_1(\zeta,k)} - 1)\, dA(\zeta),$$

where the second right-hand term is an absolutely convergent integral for each k.

We prove Proposition 4.1 in several steps.

Lemma 4.3. *Fix $M_0 > 0$. For $u, u' \in B_0$, $\mathcal{R}u$ and $\mathcal{R}u'$ belong to $L^2(\mathbb{C})$ and the estimate*

$$\|\mathcal{R}u - \mathcal{R}u'\|_2 \leq C(M_0)\|u - u'\|_{H^{1,1}}$$

holds.

Proof. We use Lemma 3.12 with $N = 2$. Substituting the expansion for μ_1 into the integral formula (3.25) we see that

$$\begin{aligned}\mathcal{R}u = {} & \frac{1}{\pi}\int e_k u(z)\, dA(z) + \frac{1}{\pi}\int e_k(z)u(z)(T_k^2 1 + T_k^4 1)\, dA(z) \\ & + \frac{1}{\pi}\int e_k u(z) R_{1,2}(z,k)\, dA(z).\end{aligned}$$

The first term is a Fourier transform which is Lipschitz continuous as a map from $H^{1,1}$ to L^2. The second two terms are multilinear forms in u and define L^2 functions of k by Remark 2.6. Lipschitz continuity follows from multilinearity. Since $R_{1,2}(\cdot,k)$ has L^p norm of order $\langle k\rangle^{-2}$, it follows from Hölder's inequality and (3.40) that the last right-hand term defines a function in L^2, Lipschitz continuous in u by (3.42). $\square$

Now we extend the Lipschitz estimates to the weighted space $L^{2,1}(\mathbb{C})$. Initially we compute for $u \in \mathcal{S}(\mathbb{C})$ to justify the integrations by parts that occur.

Lemma 4.4. *Fix $M_0 > 0$. For $u, u' \in B_0$, $\mathcal{R}u$ and $\mathcal{R}u'$ belong to $L^{2,1}(\mathbb{C})$ and*

$$\|\mathcal{R}u - \mathcal{R}u'\|_{L^{2,1}} \leq C(M_0)\|u - u'\|_{H^{1,1}}.$$

Proof. By Lemma 4.3, it suffices to show that the map $u \mapsto (\cdot)r(\cdot)$ is well-defined and Lipschitz continuous from $H^{1,1}$ to L^2. We will prove Lipschitz continuity on the dense subset $\mathcal{S}(\mathbb{C})$ and extend by continuity to $H^{1,1}(\mathbb{C})$.

Using the trivial identity $\partial_z(e_k) = -ke_k$ and integrating by parts in (3.25), we see that $kr(k) = \mathcal{F}(\partial_z u) + I_1 + I_2$ where

$$I_1 = \frac{1}{\pi} \int e_k(\zeta)(\partial_\zeta u)(\zeta)(\overline{\mu_1(\zeta,k)} - 1)\, dA(\zeta),$$

$$I_2 = \frac{1}{2\pi} \int |u(\zeta)|^2 \mu_2(\zeta,k)\, dA(\zeta),$$

where in the second line we used (1.5a).

To analyze I_1, let $\eta \in C_0^\infty(\mathbb{C})$ with $\eta(z) = 1$ for $|z| \leq 1$ and $\eta(z) = 0$ for $|z| \geq 2$. Then $I_1 = I_{11} + I_{12}$ where

$$I_{11} = \frac{1}{\pi} \int e_k(\zeta)\eta(\zeta)\,(\partial_\zeta u)(\zeta)\,[\overline{\mu_1(\zeta,k)} - 1]\, dA(\zeta),$$

$$I_{12} = \frac{1}{\pi} \int e_k(\zeta)(1-\eta(\zeta))\,(\partial_\zeta u)(\zeta)\,[\overline{\mu_1(\zeta,k)} - 1]\, dA(\zeta).$$

In I_{11}, the function $\eta\partial_\zeta u$ belongs to $L^{p'}$ for any $p \in (2,\infty)$, so we can show that I_{11} has the required continuity properties by mimicking the proof of Lemma 4.3 with u replaced by $\eta\partial_\zeta u$. In I_{12}, substitute

$$\begin{aligned}\mu_1(\zeta,k) - 1 &= \frac{1}{2\pi\zeta} \int e_k(\zeta')u(\zeta')\,\overline{\mu_2(\zeta',k)}\, dA(\zeta') \\ &\quad + \frac{1}{2\pi\zeta} \int \frac{e_k(\zeta')}{\zeta - \zeta'}\,\zeta' u(\zeta')\,\overline{\mu_2(\zeta',k)}\, dA(\zeta').\end{aligned} \tag{4.1}$$

Inserting the second right-hand term of (4.1) in I_{12} leads to an integral that can be analyzed along the same lines as I_1 since $(1-\eta(\zeta))\bar{\zeta}^{-1}(\partial_\zeta u)(\zeta)$ belongs to $L^{p'}$ for $p \in (2,\infty)$ while $\zeta u(\zeta)$ belongs to L^2. Inserting the first right-hand term of (4.1) into I_{12} gives the product of $\mathcal{F}(\bar{\zeta}^{-1}(1-\eta)\partial_\zeta u)$ and $\int e_{-k}\bar{u}\overline{\mu_2}$. The first factor is the Fourier transform of an L^2 function and Lipschitz continuous from $H^{1,1}$ into L^2. Thus, it suffices to show that the second factor is a Lipschitz continuous map from $H^{1,1}$ into L^∞. This follows from Hölder's inequality, (2.2) with $s = p'$, (3.16), and (3.22).

To analyze I_2, we use Lemma 3.12 with $N = 2$. The term corresponding to $R_{2,N}$ belongs to L^p with appropriate Lipschitz continuity by (3.41) and (3.43) together with the fact that $\||u|^2\|_{p'}$ is bounded for any $p \in (2, \infty)$ using (2.2) with $s = 2p'$. The remaining terms take the form

$$\langle |u|^2, T^{2j+1}1\rangle = \frac{1}{2}\langle |u|^2, P(e_k u(\overline{T^{2j}1}))\rangle = \langle e_{-k}w, \overline{T^{2j}1}\rangle, \tag{4.2}$$

where $w = \bar{u}\bar{P}(|u|^2)$ satisfies $\|w\|_2 \leq C\|u\|^3_{H^{1,1}}$ owing to (2.2) and (2.4). By Remark 2.6, the form (4.2) defines a multilinear map from $H^{1,1}$ to L^2. □

To finish the proof that $\mathcal{R}$ is Lipschitz continuous from $H^{1,1}$ to itself, we consider the derivatives $\partial_k r$ and $\bar{\partial}_k r$.

Lemma 4.5. *Fix $M_0 > 0$. For any $u, u' \in B_0$, $\nabla(\mathcal{R}u)$ and $\nabla(\mathcal{R}u')$ belong to L^2 and the estimate*

$$\|\nabla(\mathcal{R}u) - \nabla(\mathcal{R}u')\|_2 \leq C(M_0)\|u - u'\|_{H^{1,1}}$$

holds.

Proof. By Lemma 2.4, to show Lipschitz continuity of $u \mapsto \nabla(\mathcal{R}u)$, it suffices to study the map $u \mapsto \partial_k r$. As usual, we check Lipschitz continuity on $\mathcal{S}(\mathbb{C})$ and extend by density.

For $u \in \mathcal{S}(\mathbb{C})$ we compute $\partial_k r = -\mathcal{F}((\,\cdot\,)u(\,\cdot\,)) + I_1 + I_2$ where

$$\begin{aligned} I_1 &= -\frac{1}{\pi}\int e_k(\zeta)\,\zeta u(\zeta)\,[\overline{\mu_1(\zeta,k)} - 1]\,dA(\zeta), \\ I_2 &= \frac{1}{2\pi} r(k)\int e_k(\zeta)u(\zeta)\overline{\mu_2(\zeta,k)}\,dA(\zeta), \end{aligned}$$

where we used the first equation in (3.32). To see that $u \mapsto I_1$ is Lipschitz continuous, we may mimic the analysis of I_1 in the proof of Lemma 4.4. The map $u \mapsto I_2$ defines a Lipschitz continuous map since $u \mapsto r$ is Lipschitz continuous as a map from $H^{1,1}$ to L^2 by Lemma 4.3, $u \in L^{p'}$ by (2.2), and $u \mapsto \mu_2$ is Lipschitz continuous from $H^{1,1}$ into L^p by Lemma 3.6. □

Proof of Proposition 4.1. An immediate consequence of Lemmas 4.3-4.5. □

The following result is an immediate consequence of Lemma 3.11 and Proposition 4.1.

Proposition 4.6. *The map $\mathcal{I}$, initially defined on $\mathcal{S}(\mathbb{C})$ by (1.8) and (1.9), extends to $H^{1,1}(\mathbb{C})$. Moreover, for any $M_0 > 0$, and $r, r' \in B_0$, we have $\mathcal{I}r, \mathcal{I}r' \in H^{1,1}(\mathbb{C})$ and*

$$\|\mathcal{I}r - \mathcal{I}r'\|_{H^{1,1}} \leq C(M_0)\|r - r'\|_{H^{1,1}}.$$

Remark 4.7. In analogy to Remark 4.2, the extension of $\mathcal{I}$ to $H^{1,1}(\mathbb{C})$ can be computed as

$$(\mathcal{I}r)(z) = \mathcal{F}^{-1}(r)(z) + \frac{1}{\pi}\int e_{-k}(z)r(k)(\nu_1(z,k) - 1)\,dA(k),$$

where the second right-hand integral is absolutely convergent.

Next, we show that $\mathcal{R}$ and $\mathcal{I}$ are mutual inverses.

Lemma 4.8. *Suppose that $u \in H^{1,1}(\mathbb{C})$ and that $r = \mathcal{R}u$. Let (ν_1, ν_2) solve the system (1.8) with $r = \mathcal{R}u$. Then*

$$u(z) = \frac{1}{\pi}\int e_{-k}(z)r(k)\nu_1(z,k)\,dA(k).$$

That is, $(\mathcal{I} \circ \mathcal{R})u = u$ for all $u \in H^{1,1}(\mathbb{C})$. Similarly, $(\mathcal{R} \circ \mathcal{I})r = r$ for all $r \in H^{1,1}(\mathbb{C})$.

Proof. If $\mathcal{I} \circ \mathcal{R}$ is the identity map I on $H^{1,1}(\mathbb{C})$, the relation $\mathcal{R} \circ \mathcal{I} = I$ is an immediate consequence of (1.11). Hence, it suffices to show that $\mathcal{I} \circ \mathcal{R}$ is the identity map.

The analysis of §3 applies with no essential changes to (1.8) and shows that this equation has a unique solution for each fixed $z \in \mathbb{C}$ and given $r \in H^{1,1}(\mathbb{C})$. By this uniqueness, the functions (ν_1, ν_2) obtained by setting $(\nu_1, \nu_2) = (\mu_1, e_k\overline{\mu_2})$ coincide with the functions (ν_1, ν_2) obtained by solving the $\bar{\partial}$-problem (1.8) with $r = \mathcal{R}u$. We will first show that

$$\lim_{|k|\to\infty} (\partial_z + k)\nu_2 = \frac{1}{2}\bar{u},$$

where $\nu_2 = e_k\overline{\mu_2}$, u is the given $u \in H^{1,1}(\mathbb{C})$, and the limit is taken in the $L^s(\mathbb{C})$ topology for some $s \in (2, \infty)$. We will then show that, if ν_2 is the solution to (1.8), the relation

$$\lim_{|k|\to\infty} (\partial_z + k)\nu_2 = \frac{1}{2\pi}\int e_k(z)\,\overline{r(k)\nu_1(z,k)}\,dA(k) = \frac{1}{2}\overline{\mathcal{I}r}$$

also holds. This proves that $u = \mathcal{I}r$ in $L^s(\mathbb{C})$. Since u and $\mathcal{I}r$ belong to $H^{1,1}(\mathbb{C})$, it follows that the equality holds in $H^{1,1}(\mathbb{C})$.

First, we may compute for each $k \in \mathbb{C}$ that

$$(\partial_z + k)\nu_2 = e_k\overline{(\bar\partial_z \mu_2)} = \frac{1}{2}\bar u\,\mu_1 = \frac{1}{2}\bar u\,\nu_1$$

as vectors in $L^p(\mathbb{C})$ where we used (1.5b). On the other hand,

$$\nu_1 - 1 = \frac{1}{2}P_k(e_k \bar r\,\overline{\nu_2})$$

by (1.8a) and Lemma 2.2, so that

$$(\partial_z + k)\nu_2 - \frac{1}{2}\bar u = \frac{1}{4}\bar u\,\overline{P_k(e_k \bar r\,\overline{\nu_2})}.$$

For each k and any $s \in (2,\infty)$ we may therefore estimate

$$\|(\partial_z + k)\nu_2 - \frac{1}{2}\bar u\,\|_s \le \frac{1}{4}\|u\|_s\|\nu_2\|_{C^0(\mathbb{C}\times\mathbb{C})}\,|P_k(|r|)(k)|.$$

The second right-hand factor is bounded owing to (3.17) since $\nu_2 = e_k\overline{\mu_2}$. The third right-hand factor is a bounded function that vanishes as $|k| \to \infty$ by (2.8).

Second, from the formula $\nu_2 = \frac{1}{2}P_k[e_k\bar r\overline{\nu_1}]$, the fact that $\nu_1 = \mu_1$, and (1.5a), we may compute

$$\begin{aligned}(\partial_z + k)\nu_2 - \overline{\mathcal{I}r} &= (\partial_z + k)\nu_2 - \frac{1}{2\pi}\int e_k\overline{r(k)\nu_1(z,k)}\,dA(k)\\ &= \frac{1}{2}P_k[e_{-k}\bar r\overline{\bar\partial_z\nu_1}]\\ &= \frac{1}{4}\bar u P_k[\bar r\mu_2].\end{aligned}$$

We may then estimate, for each $k \in \mathbb{C}$,

$$\|\bar u P_k[\bar r\mu_2]\|_s \le \|u\|_s\|\mu_2\|_{C^0(\mathbb{C}\times\mathbb{C})}|P_k(|r|)|$$

and conclude as before that $\|\bar u P_k[\bar r\mu_2]\|_s$ vanishes as $k \to \infty$. $\square$

Next, we prove Plancherel-type identities for $\mathcal{R}$ and $\mathcal{I}$.

Lemma 4.9. *For u and r belonging to $H^{1,1}(\mathbb{C})$, the identities*

$$\|\mathcal{R}u\|_2 = \|r\|_2, \quad \|\mathcal{I}r\|_2 = \|u\|_2$$

hold.

Proof. We prove the first since the second then follows from (1.11). By Lipschitz continuity it suffices to prove the result for $u \in C_0^\infty(\mathbb{C})$. Letting $r = \mathcal{R}u$ we may compute

$$\begin{aligned}\int |r(k)|^2 \, dA(k) &= \lim_{R\to\infty} \frac{1}{\pi} \int_{|k|\le R} \overline{r(k)} \left(\int e_k(\zeta) u(\zeta) \overline{\mu_1(\zeta,k)} \, dA(\zeta) \right) dA(k) \\ &= \lim_{R\to\infty} \frac{1}{\pi} \int u(\zeta) \left(\int_{|k|\le R} e_k(\zeta) \overline{r(k)\nu_1(\zeta,k)} \, dA(k) \right) dA(\zeta) \\ &= \lim_{R\to\infty} (I_1(R) + I_2(R)),\end{aligned}$$

where

$$\begin{aligned} I_1(R) &= \frac{1}{\pi} \int u(\zeta) \left(\int_{|k|\le R} e_k(\zeta) \overline{r(k)} \, dA(k) \right) dA(\zeta), \\ I_2(R) &= \frac{1}{\pi} \int u(\zeta) \left(\int_{|k|\le R} e_k(\zeta) \overline{r(k)} [\overline{\nu_1(\zeta,k)} - 1] \, dA(k) \right) dA(\zeta). \end{aligned}$$

Since $\frac{1}{\pi} \int_{|k|\le R} e_k(\zeta) \overline{r(k)} \, dA(k)$ converges in L^2 to $\mathcal{F}(\bar{r})$ we have

$$\lim_{R\to\infty} I_1(R) = \int u(\zeta) (\mathcal{F}\bar{r})(\zeta) \, dA(\zeta).$$

The analogue of Lemma 3.4 for (1.8) guarantees that $\nu_1(\zeta, \cdot) - 1 \in L^p(\mathbb{C})$ uniformly in $\zeta \in \mathbb{C}$, so that

$$\lim_{R\to\infty} I_2(R) = \frac{1}{\pi} \int u(\zeta) \left(\int e_k(\zeta) \overline{r(k)} [\overline{\nu_1(\zeta,k)} - 1] \, dA(k) \right) dA(\zeta).$$

The Plancherel identity now follows from Remark 4.7 and the identity $\mathcal{F}\bar{r} = \overline{\mathcal{F}^{-1} r}$. □

Proof of Theorem 1.2. An immediate consequence of Propositions 4.1 and 4.6 together with Lemmas 4.8 and 4.9. □

5. Large-time asymptotics

In this section, we prove Theorem 1.3 using the formulation (1.16) of the inverse scattering method. For $u_0 \in H^{1,1}(\mathbb{C}) \cap L^1(\mathbb{C})$, we have $r_0 \in H^{1,1}(\mathbb{C}) \cap C_0(\mathbb{C})$ by Remark 3.7. In this section we will assume that $r_0 \in H^{1,1}(\mathbb{C}) \cap C_0(\mathbb{C})$ and set

$$\gamma = \|r_0\|_{H^{1,1}} + \|r_0\|_{C^0(\mathbb{C})}.$$

Observe that the solution formula (1.12) for initial data $v_0 = \mathcal{F}^{-1} r_0$ may be written

$$v(z,t) = \frac{1}{\pi} \int e^{itS(z,k,t)} r_0(k)\, dA(k), \tag{5.1}$$

where the real-valued phase function S is given by (1.14).

By (5.1), to prove Theorem 1.3, we need to show that

$$u(z,t) - v(z,t) = \frac{1}{\pi} \int e^{itS(z,k,t)} r_0(k)[\nu_1(z,k,t) - 1]\, dA(k) = o(t^{-1}) \tag{5.2}$$

in L^∞_z-norm, where ν_1 is determined by (1.13a).

To solve the $\bar{\partial}$-problem (1.13a) and obtain the estimates on $\nu_1 - 1$ needed to prove (5.2), we introduce the integral operator

$$M\psi = \frac{1}{2} P_k(e^{-itS} \overline{r_0 \psi}) \tag{5.3}$$

which depends parametrically on z and t through the phase function S. It follows from the theory of §3 that M is a compact operator on $L^p_k(\mathbb{C})$ for each fixed z, t and any $p \in (2, \infty)$, that the resolvent $(I - M^2)^{-1}$ is a bounded operator on $L^p_k(\mathbb{C})$, and

$$\nu_1 - 1 = (I - M^2)^{-1} M^2 1 \tag{5.4}$$

as vectors in $L^p_k(\mathbb{C})$. In Lemma 5.5 we reduce the proof of estimate (5.2) to the estimate

$$\frac{1}{\pi} \int e^{itS(z,k,t)} r_0(k) (M^2 1)(z,k,t)\, dA(k) = o(t^{-1}) \tag{5.5}$$

in $L^\infty_z(\mathbb{C})$ norm. We prove estimate (5.5) in Lemmas 5.6–5.9. For the proofs of Lemmas 5.6–5.8, it suffices to assume that $r \in H^{1,1}(\mathbb{C}) \cap C^0(\mathbb{C})$. For Lemma 5.9, we need to assume that $r \in C_0(\mathbb{C})$.

We begin with stationary phase estimates on the operator M. Recalling (1.14) and (1.15) we may write

$$S(z,k,t) = 4\,\mathrm{Re}((k - k_c)^2) + S_0, \quad S_0 = \frac{1}{4} \mathrm{Re}(z^2/t^2).$$

Hence

$$S_{\bar{k}}(z,k,t) = 4(\bar{k} - \bar{k}_c).$$

Since S has a single stationary point at $k = k_c$, we introduce a cutoff function

$$\chi(k) = \eta(t^{1/4}(k - k_c)),$$

where $\eta \in C^\infty_0(\mathbb{C})$ with $\eta(w) = 1$ for $|w| \leq 1$ and $\eta(w) = 0$ for $|w| \geq 2$. Note that, for any $\sigma \in [1, \infty]$,

$$\|\chi\|_\sigma \leq C_\sigma t^{-1/(2\sigma)}. \tag{5.6}$$

We will estimate M by splitting $M = M\chi + M(1-\chi)$, use the small support of χ to estimate the first term, and the oscillations of the factor $\exp(itS)$ to estimate the second term.

Lemma 5.1. *Suppose that* $p \in (2,\infty)$, *that* $\psi \in W^{1,p}(\mathbb{C})$, *and that* $r_0 \in H^{1,1}(\mathbb{C})$. *Then, as vectors in* $L^p(\mathbb{C})$,

$$M[(1-\chi)\psi] = -\frac{e^{-itS}}{2itS_{\bar{k}}}(1-\chi)\overline{r_0\psi} + \frac{1}{2it}P_k[e^{-itS}\bar{\partial}_k(S_{\bar{k}}^{-1}(1-\chi)\overline{r_0}\bar{\psi})].$$

Proof. For $\psi, r_0 \in C_0^\infty(\mathbb{C})$ this is a direct consequence of the integration by parts formula (2.9) with φ replaced by $-tS$. Now let $\psi \in W^{1,p}(\mathbb{C})$ and $r_0 \in H^{1,1}(\mathbb{C})$. If $\{\psi_n\}$ and $\{r_n\}$ are sequences from $C_0^\infty(\mathbb{C})$ with $\psi_n \to \psi$ in $W^{1,p}$ and $r_n \to r_0$ in $H^{1,1}(\mathbb{C})$, we have $\psi_n \to \psi$ in sup norm so $r_n\psi_n \to r_0\psi$ in $L^p \cap L^{2p/(p+2)}$ and $\bar{\partial}_\zeta(r_n\psi_n) \to \bar{\partial}(r_0\psi)$ in $L^{2p/(p+2)}$. Using (2.5), we conclude that the identity holds in L^p-sense for $\psi \in W^{1,p}$ and $r_0 \in H^{1,1}(\mathbb{C})$. □

We'll use the following estimates on singular factors $S_{\bar{k}}^{-1}$ and $S_{\bar{k}}^{-2}$ that occur in the integrations by parts. We omit the elementary proofs.

Lemma 5.2. *For any* $\sigma \in (2,\infty]$,

$$\|S_{\bar{k}}^{-1}(1-\chi)\|_\sigma \le C_\sigma t^{1/4-1/(2\sigma)}. \tag{5.7}$$

For any $\sigma \in (1,\infty]$,

$$\|S_{\bar{k}}^{-2}(1-\chi)\|_\sigma \le C_\sigma t^{1/2-1/(2\sigma)}, \tag{5.8}$$

$$\|S_{\bar{k}}^{-1}(\bar{\partial}_k\chi)\|_\sigma \le C_\sigma t^{1/2-1/(2\sigma)}. \tag{5.9}$$

Using the estimates above we can now estimate M away from points of stationary phase.

Lemma 5.3. *Suppose that* $r_0 \in H^{1,1}(\mathbb{C}) \cap C^0(\mathbb{C})$. *For any* $p \in (2,\infty)$, *the estimate*

$$\|M(1-\chi)\psi\|_p \le C_p\gamma t^{-3/4}(\|\psi\|_p + \|\partial\psi\|_p) \tag{5.10}$$

holds.

Proof. We will use Lemma 5.1. We compute

$$M(1-\chi)\psi = \frac{e^{-itS}}{itS_{\bar{k}}}(1-\chi)\overline{r_0\psi} + \frac{1}{\pi i t}\int \frac{e^{-itS}}{k-\zeta}\bar{\partial}_\zeta\left(S_{\bar{k}}^{-1}(1-\chi)\overline{r_0\psi}\right)\,dA(\zeta) \tag{5.11}$$

so that

$$\|M(1-\chi)\psi\|_p \le C_p t^{-1}\Big(\sum_{j=0}^{4}\|J_j\|_p\Big). \tag{5.12}$$

Here, J_0 is t times the first right-hand term in (5.11). The terms J_1, J_2, J_3, J_4 are t times the integrals that arise in the second right-hand term of (5.11) by applying the product rule to

$$\begin{aligned}\bar{\partial}_\zeta(S_{\bar{k}}^{-1}(1-\chi)\overline{r_0}\,\bar{\psi}) = -4S_{\bar{k}}^{-2}(1-\chi)\overline{r_0}\,\bar{\psi} - S_{\bar{k}}^{-1}(\bar{\partial}\chi)\overline{r_0}\,\bar{\psi}\\ + S_{\bar{k}}^{-1}(1-\chi)\bar{\partial}\,\overline{r_0}\,\bar{\psi} + S_{\bar{k}}^{-1}(1-\chi)\overline{r_0}\,\overline{\partial\psi}.\end{aligned} \tag{5.13}$$

By (2.4), to estimate $\|J_i\|_p$ for $i = 1,2,3,4$, we must estimate the $L^{2p/(p+2)}$ norms of each of the four right-hand terms in (5.13).

J_0: Using Hölder's inequality and (5.7), we estimate

$$\|J_0\|_p \le \|S_{\bar{k}}^{-1}(1-\chi)\overline{r_0}\,\bar{\psi}\|_p \le C_p t^{\frac{1}{4}}\gamma\|\psi\|_p$$

which shows that J_0 is estimated by a constant times $t^{1/4}$.

J_1, J_2: We estimate J_1 since the estimate for J_2 is similar. Using (5.8), we have

$$\|J_1\|_p \le C_p\|4S_{\bar{k}}^{-2}(1-\chi)\overline{r_0}\,\bar{\psi}\|_{2p/(p+2)} \le C_{p,\sigma}t^{\frac{1}{4}}\gamma\|\psi\|_p,$$

where in the last step we used Hölder's inequality, (5.8) with $\sigma = 2$, and the bound $\|r_0\|_\infty \le \gamma$. In the estimate for J_2, we replace (5.8) by (5.9).

J_3, J_4: To estimate J_3, we use (5.7) and Hölder's inequality to conclude that

$$\begin{aligned}\|J_3\|_p &\le C_p\|S_{\bar{k}}^{-1}(1-\chi)\overline{\partial r_0}\,\bar{\psi}\|_{2p/(p+2)}\\ &\le C_p\|S_{\bar{k}}^{-1}(1-\chi)\|_{\sigma_1}\,\|\partial r_0\|_2\,\|\psi\|_{\sigma_2}\\ &\le C_p\gamma t^{1/4-1/(2\sigma_1)}\|\psi\|_{\sigma_2}.\end{aligned}$$

Here $\sigma_1^{-1} + \sigma_2^{-1} = p^{-1}$. If $\sigma_2 = p$ we may take $\sigma_1 = \infty$. Hence, we can estimate J_3 in all cases by a constant times $t^{1/4}$. The estimate for J_4 is similar, with $\|\partial r_0\|_2$ replaced by $\|r_0\|_2$ in the estimates. Recalling (5.12) and combining these estimates leads to (5.10). $\square$

We will make use of the following estimates on M.

Lemma 5.4. *Suppose that* $r_0 \in H^{1,1}(\mathbb{C}) \cap C^0(\mathbb{C})$. *For any* p *with* $p > 2$, *the following estimates hold*:

$$\|M\|_{\overline{\mathcal{B}}(L^p)} \leq C_p \gamma, \tag{5.14}$$

$$\|M\chi\|_{\overline{\mathcal{B}}(L^p)} \leq C_p \gamma t^{-1/4}, \tag{5.15}$$

$$\|M^2\|_{\mathcal{B}(L^p)} \leq C_p t^{-1/4} \gamma^2, \tag{5.16}$$

$$\|M\chi 1\|_p \leq C_p \gamma t^{-1/4-1/(2p)}, \tag{5.17}$$

$$\|M(1-\chi)1\|_p \leq C_p \gamma t^{-3/4}, \tag{5.18}$$

$$\|M^4 1\|_p \leq C_p \gamma^4 t^{-1-1/(2p)}. \tag{5.19}$$

Proof. Estimate (5.14) is an immediate consequence of (2.5).

To prove (5.15), we use (2.5) to estimate

$$\|M\chi\psi\|_p \leq C_p \|r_0 \chi\|_2 \|\psi\|_p \leq C_p \gamma \|\chi\|_2 \|\psi\|_p$$

and use (5.6) with $\sigma = 2$.

To prove (5.16), we use (5.10) and (5.15) to estimate

$$\|M\varphi\|_p \leq C_p \gamma (t^{-1/4} \|\varphi\|_p + t^{-3/4} (\|\varphi\|_p + \|\bar{\partial}\varphi\|_p)),$$

where in the last term we used $\|\partial\varphi\|_p \leq C_p \|\bar{\partial}\varphi\|_p$ owing to Lemma 2.4. Setting $\varphi = M\psi$ and using the estimate above, (5.14), and the trivial estimate $\|\bar{\partial} M\psi\|_p \leq \gamma \|\psi\|_p$ we obtain (5.16).

To prove (5.17), we use (2.4) to estimate $\|M\chi 1\|_p \leq C_p \gamma \|\chi\|_{2p/(p+2)}$ and apply (5.6).

To prove (5.18), we trace through the proof of (5.10) with $\psi = 1$.

To prove (5.19), we first note that

$$\|M1\|_p \leq C_p \gamma t^{-1/4-1/(2p)} \tag{5.20}$$

by (5.17) and (5.18). Next, from (5.10) and (5.15), the estimate

$$\|M\psi\|_p \leq C_p \gamma t^{-1/4} (\|\psi\|_p + t^{-1/2} \|\bar{\partial}\psi\|_p) \tag{5.21}$$

holds, where in the last term we used Lemma 2.4. Starting with (5.20) and iterating with (5.21) we see that

$$\|M^j 1\|_p \leq C_p \gamma^j t^{-j/4-1/(2p)}.$$

The case $j = 4$ gives (5.19). $\square$

From (5.16) it follows that for each $p \in (2, \infty)$, there is a $T = T(\gamma, p) > 0$ so that

$$\sup_{t > T(\gamma,p)} \|(I - M^2)^{-1}\|_{\mathcal{B}(L^p)} \leq 2. \tag{5.22}$$

Lemma 5.5. *Suppose that $r_0 \in H^{1,1}(\mathbb{C}) \cap C^0(\mathbb{C})$. Then, the estimate*

$$\sup_{z \in \mathbb{C}} \left| u(z,t) - v(z,t) - \frac{1}{\pi} \int e^{itS} r_0 M^2 1 \right| \leq C(p,\gamma) t^{-1-1/(2p)}$$

holds for any $p \in (2, \infty)$ and all $t > T(\gamma, p)$.

Proof. From the first equality in (5.2), (5.4), and the identity

$$(I - M^2)^{-1} - I - M^2 = (I - M^2)^{-1} M^4$$

we conclude that

$$u(z,t) - v(z,t) - \frac{1}{\pi} \int e^{itS} r_0 M^2 1 = \frac{1}{\pi} \int e^{itS} r_0 (I - M^2)^{-1} M^4 1.$$

The result now follows from Hölder's inequality, the fact that $r_0 \in L^{p'}(\mathbb{C})$ for any $p \in (2, \infty)$, (5.19), and (5.22). □

It remains to estimate

$$\int e^{itS} r M^2 1 = I_1 + I_2 + I_3 + I_4, \tag{5.23}$$

where

$$\begin{aligned}
I_1 &= \int e^{itS} r_0 M[(1 - \chi) M(\chi)], \\
I_2 &= \int e^{itS} r M[\chi M(1 - \chi)], \\
I_3 &= \int e^{itS} r M[(1 - \chi) M(1 - \chi)], \\
I_4 &= \int e^{itS} r M[\chi M(\chi)].
\end{aligned}$$

First, we analyze the mixed terms I_1 and I_2. In each integral we will split $e^{itS} r_0 = e^{itS} r_0 \chi + e^{itS} r_0 (1 - \chi)$ and bound each of the terms separately. To control the first type of term, we will use the estimate

$$\left| \int e^{itS} r_0 \chi \psi \right| \leq C \gamma t^{1/(2p) - 1/2} \|\psi\|_p \tag{5.24}$$

true for any $p \in (2, \infty)$. To control the second type of term, we will use the integration by parts formula

$$\int e^{itS} r_0(1-\chi)\psi = -\frac{1}{it}\int e^{itS}\bar{\partial}_k(S_{\bar{k}}^{-1} r_0(1-\chi)\psi).$$

Expanding the $\bar{\partial}_k$-derivative into four terms, using Hölder's inequality, and applying the inequalities (5.7)–(5.9), we conclude that for any $p \in (2, \infty)$,

$$\begin{aligned}\left|\int e^{itS} r_0(1-\chi)\psi\right| &\le C_p\gamma t^{1/(2p)-1}\|\psi\|_p + C_p\gamma t^{1/(2p)-1}\|\psi\|_p \\ &\quad + C_p\gamma t^{1/(2p)-1}\|\psi\|_p + C_p\gamma t^{1/(2p)-1}\|\bar{\partial}\psi\|_p \\ &\le C_p\gamma t^{1/(2p)-1}(\|\psi\|_p + \|\bar{\partial}\psi\|_p).\end{aligned} \tag{5.25}$$

First, we consider I_1.

Lemma 5.6. *Suppose that* $r_0 \in H^{1,1}(\mathbb{C}) \cap C^0(\mathbb{C})$, $p \in (2, \infty)$, *and* $t > 1$. *Then*

$$\left|\int e^{itS} rM[(1-\chi)M\chi]\right| \le C_p\gamma^3 t^{1/(2p)-5/4}. \tag{5.26}$$

Proof. We split $I_1 = J_1 + J_2$ where

$$\begin{aligned}J_1 &= \int e^{itS}\chi rM[(1-\chi)M\chi], \\ J_2 &= \int e^{itS}(1-\chi)rM[(1-\chi)M\chi].\end{aligned}$$

To bound J_1, we use (5.24) with $\psi = M(1-\chi)M\chi$ and (5.10) with $s = p$ to estimate

$$\begin{aligned}|J_1| &\le C_p\gamma t^{1/(2p)-1/2}\|M(1-\chi)M\chi\|_p \\ &\le C_p\gamma^2 t^{1/(2p)-5/4}(\|M\chi\|_p + \|\bar{\partial}(M\chi)\|_p) \\ &\le C_p\gamma^3 t^{-5/4},\end{aligned}$$

where in the last step we used $\|\bar{\partial}(M\chi)\|_p = \|r_0\chi\|_p$ and (5.6).

To bound J_2, we (5.25) with $\psi = M(1-\chi)M\chi$, the operator bound (5.14), and (5.17) to estimate

$$\begin{aligned}|J_2| &\le C_p\gamma t^{1/(2p)-1}(\|M(1-\chi)M\chi\|_p + \|\bar{\partial}M(1-\chi)M\chi\|_p) \\ &\le C_p\gamma^3 t^{1/(2p)-5/4}.\end{aligned}$$

Combining these two estimates completes the proof. □

Next, we consider I_2.

Lemma 5.7. *Suppose that $r_0 \in H^{1,1}(\mathbb{C}) \cap L^\infty(\mathbb{C})$, $t > 1$, and $p \in (2, \infty)$. Then*

$$\left| \int e^{itS} r M[\chi M(1-\chi)] \right| \leq C_p \gamma^3 t^{1/(2p)-5/4}. \tag{5.27}$$

Proof. As before we write $I_2 = J_1 + J_2$ where

$$\begin{aligned} J_1 &= \int e^{itS} r_0 \chi M[\chi M(1-\chi)], \\ J_2 &= \int e^{itS} r_0 (1-\chi) M[\chi M(1-\chi)]. \end{aligned}$$

To estimate J_1, we use (5.24) with $\psi = M\chi M(1-\chi)$, (5.14), and (5.18) to estimate

$$\begin{aligned} |J_1| &\leq C_p \gamma^2 t^{1/(2p)-1/2} \|\chi M(1-\chi)\|_p \\ &\leq C_p \gamma^3 t^{1/(2p)-5/4}. \end{aligned}$$

To estimate J_2, we use (5.25) with $\psi = M\chi M(1-\chi)$, (5.14), and (5.18) to estimate

$$\begin{aligned} |J_2| &\leq C_p \gamma t^{1/(2p)-1} \|M\chi M(1-\chi)\|_p + \|\bar{\partial} M\chi M(1-\chi)\|_p \\ &\leq C_p \gamma^2 t^{1/(2p)-1} \|M(1-\chi)\|_p \\ &\leq C_p \gamma^3 t^{1/(2p)-7/4}. \end{aligned}$$

Combining these estimates completes the proof. □

Next, we bound I_3.

Lemma 5.8. *Suppose $r_0 \in H^{1,1}(\mathbb{C}) \cap C^0(\mathbb{C})$, $t > 1$, and $p \in (2, \infty)$. Then*

$$\left| \int e^{itS} r M(1-\chi) M[(1-\chi)] \right| \leq C_p \gamma^3 t^{1/(2p)-5/4}. \tag{5.28}$$

Proof. First, we insert $1 = \chi + (1-\chi)$ and write the integral to be estimated as $J_1 + J_2$ where

$$J_1 = \int e^{itS} \chi r_0 M[(1-\chi) M[(1-\chi)]] \, dA,$$

$$J_2 = \int e^{itS} (1-\chi) r_0 M[(1-\chi) M[(1-\chi)]] \, dA.$$

We estimate, using (5.24) with $\psi = M(1-\chi)[M(1-\chi)]$,

$$\begin{aligned}|J_1| &\leq C_p \gamma t^{1/(2p)-1/2} \|M(1-\chi)M(1-\chi)\|_p \\ &\leq C_p \gamma^3 t^{1/(2p)-5/4},\end{aligned}$$

where in the last step we used (5.18) and (5.14).

To estimate J_2, we use (5.25) with $\psi = M(1-\chi)M(1-\chi)$, (5.14), and (5.18) to conclude that

$$\begin{aligned}|J_2| &\leq C_p \gamma t^{1/(2p)-1} (\|M(1-\chi)M(1-\chi)\|_p + \gamma \|M(1-\chi)\|_p) \\ &\leq C_p \gamma^3 t^{1/(2p)-7/4}.\end{aligned}$$

Combining these two estimates completes the proof. □

Finally, we show that I_4 is $o(t^{-1})$. Recall that, for $u_0 \in L^1(\mathbb{C})$, $r_0 \in C_0(\mathbb{C})$ by Remark 3.7.

Lemma 5.9. *Suppose $r_0 \in H^{1,1}(\mathbb{C}) \cap C_0(\mathbb{C})$. Then*

$$\lim_{t\to+\infty} t \int e^{itS} r_0 \, M(\chi(M(\chi 1))) = 0. \tag{5.29}$$

Proof. We first write

$$t \int e^{itS} r_0 M(\chi(M(\chi 1))) = J_1 + J_2,$$

where

$$J_1 = t \int e^{itS} \chi r_0 \psi_M, \quad J_2 = t \int e^{itS} (1-\chi) r_0 \psi_M,$$

and

$$\psi_M = M(\chi(M(\chi 1))).$$

We first show that $J_2 \to 0$ as $t \to \infty$ using estimate (5.25) with $\psi = \psi_M$. We obtain, for any $p \in (2, \infty)$,

$$\begin{aligned}|J_2| &\leq C_p \gamma t^{1/(2p)} (\|\psi_M\|_p + \|\bar{\partial}\psi_M\|_p) \\ &\leq C_p \gamma t^{1/(2p)} (\|\psi_M\|_p + \gamma \|\chi M(\chi)\|_p) \\ &\leq C_p \gamma^3 t^{-1/4}.\end{aligned}$$

In the second step, we used (5.14), and in the last step, we used (5.17). This shows that $J_2 \to 0$ as $t \to \infty$.

We now turn to J_1. Let us write $\tilde{k}$ for $k - k_c$. We may compute

$$J_1 = \frac{t e^{itS_0}}{4\pi^2} \int_{\mathbb{C}^3} \frac{e^{4it \operatorname{Re}(\tilde{k}^2 - (\tilde{k}')^2 + (\tilde{k}'')^2)} G(k, k', k'')}{(k - k')(\overline{k'} - \overline{k''})} \, dA(k, k', k''),$$

where

$$G(k, k', k'') = r_0(k) \overline{r_0(k')} r_0(k'') \chi(k) \chi(k') \chi(k'').$$

Define $\zeta = t^{1/4}(k - k_c)$ and similarly for ζ' and ζ''. We see that the expression J_1 is given by

$$I(z,t) = \frac{e^{itS_0}}{4\pi^2} \int \frac{e^{4it^{1/2} \operatorname{Re}(\zeta^2 - \zeta'^2 + \zeta''^2)} H(\zeta, \zeta', \zeta'')}{(\zeta - \zeta')(\overline{\zeta'} - \overline{\zeta''})} dA(\zeta, \zeta', \zeta''),$$

where

$$H(\zeta, \zeta', \zeta'') = \eta(\zeta)\eta(\zeta')\eta(\zeta'') \times r_0(k_c + \zeta/\sqrt[4]{t}) \, \overline{r_0(k_c + \zeta'/\sqrt[4]{t})} \, r_0(k_c + \zeta''/\sqrt[4]{t}).$$

Clearly, $|I(z,t)| \leq C \|r_0\|^3_{C_0(\mathbb{C})}$, where C is bounded uniformly in z and t, and $I(z,t)$ is a continuous multilinear function of $r_0 \in C_0(\mathbb{C})$. Thus, to show that $\lim_{t \to \infty} I(z,t) = 0$, it suffices to check for $r_0 \in C_0^\infty(\mathbb{C})$ since such r_0 are dense in $C_0(\mathbb{C})$. For such r_0, we have

$$\begin{aligned} I(z,t) = e^{itS_0} |r_0(k_c)|^2 r_0(k_c) \int \frac{e^{4it^{1/2} \operatorname{Re}(\zeta^2 - \zeta'^2 + \zeta''^2)} \eta(\zeta)\eta(\zeta')\eta(\zeta'')}{(\zeta - \zeta')(\overline{\zeta'} - \overline{\zeta''})} dA(\zeta, \zeta', \zeta'') \\ + \mathcal{O}(t^{-1/4}). \end{aligned}$$

Consider now the integral

$$J(t) = \int \frac{e^{4it^{1/2} \operatorname{Re}(\zeta^2 - \zeta'^2 + \zeta''^2)} \eta(\zeta)\eta(\zeta')\eta(\zeta'')}{(\zeta - \zeta')(\overline{\zeta'} - \overline{\zeta''})} \, dA(\zeta, \zeta', \zeta'').$$

The integrand is an $L^1(\mathbb{C}^3)$ function owing to the compact support of η. The integral $J(t)$ is thus a special case of the integral

$$J(t; f) = \int_{\mathbb{C}^3} e^{4it^{1/2} \operatorname{Re}(\zeta^2 - \zeta'^2 + \zeta''^2)} f(\zeta, \zeta', \zeta'') \, dA(\zeta, \zeta', \zeta'').$$

It suffices to show that $J(t, f) \to 0$ as $t \to \infty$. Owing to the trivial bound $|J(t, f)| \leq \|f\|_1$, it suffices to do so for a dense set of $f \in L^1(\mathbb{C}^3)$. We first observe that finite linear combinations of compactly supported product functions of the form $g_1(\zeta) g_2(\zeta') g_3(\zeta'')$ are dense in $L^1(\mathbb{C}^3)$, so it suffices to show that

$$\lim_{t \to \infty} \int e^{4it^{1/2} \operatorname{Re} \zeta^2} g(\zeta) \, dA(\zeta) = 0.$$

Now write $\zeta = \zeta_1 + i\zeta_2$ and note that, by a further density argument, we may take $g(\zeta) = h_1(\zeta_1)h_2(\zeta_2)$. As $\mathrm{Re}(\zeta^2) = \zeta_1^2 - \zeta_2^2$ it now suffices to show that

$$\lim_{t\to\infty} \int e^{\pm 4it^{1/2}s^2} h(s)\, ds = 0$$

for bounded and compactly supported h. This is now an easy consequence of the Riemann–Lebesgue lemma and a simple change of variables.

□

Proof of Theorem 1.3. An immediate consequence of Lemma 5.5, (5.23), and estimates (5.26), (5.27), (5.28), and (5.29). □

Appendices

A. Multilinear estimates

Michael Christ

In this appendix we establish a rather general multilinear inequality in terms of weak type Lebesgue spaces, then specialize it to deduce the inequality of Brown [13] stated in Proposition 2.5.

Let $\mathbb{F}$ be one of the two fields $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, equipped with Lebesgue measure in either case. Consider $\mathbb{C}$–valued multilinear functionals

$$\Lambda(f_1, f_2, \dots, f_m) = \int_{\mathbb{F}^N} \prod_{j=1}^{m} f_j(\ell_j(y))\, dy, \tag{A.1}$$

where each $\ell_j : \mathbb{F}^N \to \mathbb{F}^{N_j}$ is a surjective $\mathbb{F}$-linear transformation, $f_j : \mathbb{F}^{N_j} \to \mathbb{C}$, and dy denotes Lebesgue measure on $\mathbb{F}^N$. A complete characterization of those exponents $(p_1, \dots, p_m) \in [1,\infty]^m$ for which there are inequalities of the form

$$\left|\Lambda(f_1, f_2, \dots, f_m)\right| \le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}} \tag{A.2}$$

has been obtained in [11]. Such an inequality implicitly includes the assertion that the integral (A.2) converges absolutely whenever each f_j belongs to L^{p_j}. To review this result, we first recall key definitions from [10] and [11].

Denote by $\dim_{\mathbb{F}}(V)$ the dimension of a vector space V over $\mathbb{F}$. Throughout the discussion, $\mathbb{F}$ should be considered as fixed; vector spaces, subspaces, and linear mappings are defined with respect to $\mathbb{F}$.

Definition A.1. Relative to a set of exponents $\{p_j\}$, a subspace $V \subset \mathbb{F}^N$ is said to be *critical* if

$$\dim_{\mathbb{F}}(V) = \sum_j p_j^{-1} \dim_{\mathbb{F}}(\ell_j(V)), \tag{A.3}$$

to be *supercritical* if the right-hand side is strictly less than $\dim_{\mathbb{F}}(V)$, and to be *subcritical* if the right-hand side is strictly greater than $\dim_{\mathbb{F}}(V)$.

Throughout the discussion, the reciprocal of any infinite exponent is interpreted as 0. The subspace $\{0\}$ is always critical.

Theorem A.2 ([11]). *Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Let $N \geq 1$ and $N_j \geq 1$ for all $j \in \{1, 2, \dots, m\}$. For each index $j \in \{1, 2, \dots, m\}$ let $\ell_j : \mathbb{F}^N \to \mathbb{F}^{N_j}$ be an $\mathbb{F}$–linear surjective mapping. Let $p_j \in [1, \infty]$. Then* (A.2) *holds if and only if $\mathbb{F}^N$ is critical relative to $\{p_j\}$ and no proper subspace of $\mathbb{F}^N$ is supercritical relative to $\{p_j\}$.*

This theorem was stated in [11] only for $\mathbb{F} = \mathbb{R}$, but the proof given in [11] applies equally well to $\mathbb{F} = \mathbb{C}$. See also [10] for a different proof and more thorough analysis for the case $\mathbb{F} = \mathbb{R}$.

In order to extend this theorem to include Brown's inequality (2.12), we will utilize the Lorentz spaces $L^{p,r}$ as defined for instance in [31]. These spaces are defined for $(p, r) \in [1, \infty) \times [1, \infty]$, and are Banach spaces except in the exceptional case $(p, r) = (1, 1)$. Throughout the following discussion, we assume that (p, r) is not equal to $(1, 1)$. The facts needed about the Lorentz spaces for our discussion are these.

(i) $L^{p,p}$ equals the Lebesgue space L^p.

(ii) $L^{p,\infty}$ equals weak L^p. That is, $f \in L^{p,\infty}(\mathbb{F}^n)$ if and only if there exists $C_f < \infty$ such that for every $\alpha \in (0, \infty)$, $\left|\{x \in \mathbb{F}^n : |f(x)| > \alpha\}\right| \leq C_f^p \alpha^{-p}$. Here $|E|$ denotes the Lebesgue measure of a subset E of $\mathbb{F}^n$. The infimum of all such C_f is denoted by $\|f\|_{L^{p,\infty}}$. This quantity is not in general a norm, but is equivalent to one unless $(p, r) = (1, 1)$; see [31].

(iii) In particular, the functions $|x|^{-d/p}$ and $|z|^{-2d/p}$ belong to $L^{p,\infty}(\mathbb{R}^d)$ and to $L^{p,\infty}(\mathbb{C}^d)$, respectively.

(iv) If $r \geq p$ then $\|f\|_{L^{p,r}} \leq C \|f\|_{L^p}$ for all functions f, where $C < \infty$ depends only on p, r.

The next result extends Theorem A.2 to Lorentz spaces, although perhaps not in the most definitive manner.

Theorem A.3. *Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Let $N \geq 1$ and $N_j \geq 1$ for all $j \in \{1, 2, \ldots, m\}$. For each index $j \in \{1, 2, \ldots, m\}$ let $\ell_j \colon \mathbb{F}^N \to \mathbb{F}^{N_j}$ be an $\mathbb{F}$–linear surjective mapping. Let each exponent p_j belong to the open interval $(1, \infty)$.*

Suppose that with respect to $\{p_j\}$, the total space $\mathbb{F}^N$ is critical, and every nonzero proper subspace of $\mathbb{F}^N$ is subcritical. Then for all exponents $r_j \in [1, \infty]$ satisfying

$$\sum_j r_j^{-1} = 1 \tag{A.4}$$

and, for all functions $f_j \in L^{p_j, r_j}(\mathbb{F}^{N_j})$, $\prod_{j=1}^m f_j \circ \ell_j$ belongs to $L^1(\mathbb{F}^N)$. Moreover, there exists $C < \infty$ independent of $\{f_j\}$ such that

$$\left|\Lambda(f_1, f_2, \ldots, f_m)\right| \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j, r_j}}. \tag{A.5}$$

The proof will utilize the following crude multilinear interpolation theorem, established in [18].

Proposition A.4. *Let $a_j \in [0, \infty)$, and suppose that at least one of these numbers is nonzero. Let $\Omega = \left\{(t_1, \ldots, t_j) \in (0, 1)^m \colon \sum_j a_j t_j = 1\right\}$, equipped with the topology induced by its embedding in $(0, 1)^m$. Let $(X, \mathcal{A}, \mu)$ be any measure space. Let $\Lambda = \Lambda(f_1, \ldots, f_m)$ be a complex-valued multilinear form defined for all m-tuples of measurable simple functions $f_j \colon X \to \mathbb{C}$.*

Let $\mathcal{O}$ be a nonempty open subset of Ω. Suppose that for each $t = (t_1, \ldots, t_m) \in \mathcal{O}$ there exists $C_t < \infty$ such that

$$|\Lambda(f_1, \ldots, f_m)| \leq C_t \prod_j \|f_j\|_{L^{p_j, 1}}, \quad \textit{where } p_j = t_j^{-1}, \tag{A.6}$$

for all m-tuples of simple functions f_j. Then for any relatively compact subset $\mathcal{O}' \subset \mathcal{O}$ there exists $C < \infty$ such that for all $t \in \mathcal{O}'$ and all exponents r_j satisfying $\sum_{j=1}^m r_j^{-1} = 1$, for all m-tuples of measurable simple functions,

$$|\Lambda(f_1, \ldots, f_m)| \leq C \prod_j \|f_j\|_{L^{p_j, r_j}}, \quad \textit{where } p_j = t_j^{-1}. \tag{A.7}$$

Proof of Theorem A.3. It suffices to apply Theorem A.2 and Proposition A.4 in combination. Indeed, if an m-tuple $p = \{p_j : 1 \le j \le m\}$ satisfies the hypotheses of Theorem A.3, then so does any m-tuple $q = \{q_j : 1 \le j \le m\}$ satisfying the equation $\sum_j \frac{N_j}{N} q_j^{-1} = 1$ such that each q_j^{-1} is sufficiently close to p_j^{-1}. Indeed, as V varies over all nonzero proper subspaces of $\mathbb{F}^N$, the numbers $\sum_j p_j^{-1} \frac{\dim_{\mathbb{F}}(\ell_j(V))}{\dim_{\mathbb{F}}(V)}$ take on finitely many values, and are all strictly greater than one by the subcriticality hypothesis. Therefore these strict inequalities continue to hold whenever q is sufficiently close to p. The hypotheses of Proposition A.4 are thus satisfied. Applying that Proposition yields inequality (A.5). $\square$

Consider now the multilinear inequality of Brown [13]. Let

$$\Lambda_n(\rho, q_0, q_1, \ldots, q_{2n}) = \int_{\mathbb{C}^{2n+1}} \frac{|\rho(\zeta)||q_0(z_0)| \ldots |q(z_{2n})|}{\prod_{j=1}^{2n} |z_{j-1} - z_j|} d\mu(z),$$

where $d\mu(z)$ is product measure on $\mathbb{C}^{2n+1}$ and $\zeta = \sum_{j=0}^{2n} (-1)^j z_j$. The inequality states that

$$|\Lambda_n(\rho, q_0, q_1, \ldots, q_{2n})| \le C \|\rho\|_2 \prod_{j=0}^{2n} \|q_j\|_2. \tag{A.8}$$

Note that since Λ_n is multilinear, it follows directly from this statement that the map

$$(\rho, q_0, q_1, \ldots, q_{2n}) \longmapsto \Lambda_n(\rho, q_0, q_1, \ldots, q_{2n})$$

is Lipschitz continuous from any bounded subset of $(L^2(\mathbb{C}))^{2n+1}$ to $\mathbb{C}$.

To deduce (A.8) from Theorem A.3, set $\mathbb{F} = \mathbb{C}$, $N = 2n+1$, and $m = 4n+2$. Let the index j range over $[0, 4n+1]$, set $N_j = 1$ for all $j \in \{0, \ldots, 4n+1\}$, write $z = (z_0, \ldots, z_{2n})$, and consider the linear functionals $l_j : \mathbb{C}^{2n+1} \to \mathbb{C}^1$ defined by

$$l_j(z) = \begin{cases} z_j & \text{for } 0 \le j \le 2n, \\ z_{j-2n} - z_{j-2n-1} & \text{for } 2n < j \le 4n, \\ \sum_{i=0}^{2n} (-1)^i z_i & \text{for } j = 4n+1. \end{cases} \tag{A.9}$$

The following linear algebraic fact will be proved below.

Lemma A.5. *The* $(4n+2)$*-tuple of exponents* $p = (p_j) = (2, 2, \ldots, 2)$ *satisfies the hypotheses of Theorem* A.3.

To apply the lemma to inequality (A.8), set $r_j = (2n+2)^{-1}$ for $0 \le j \le 2n$ and for $j = 4n+1$, and $r_j = \infty$ for $2n < j \le 4n$. For each $j \in (2n, 4n]$, define $f_j : \mathbb{C}^1 \to \mathbb{R}^+$ by $f_j(w) = |w|^{-1}$. Each of these functions belongs to $L^{2,\infty}(\mathbb{C}^1)$. The factors $|z_j - z_{j-1}|^{-1}$ appearing in (A.8) are then $|z_j - z_{j-1}|^{-1} = f_j(\ell_j(z))$, for $j \in (2n, 4n]$. Setting $f_j = q_j$ for all $j \in [0, 2n]$ and $f_{4n+1} = \rho$, $\Lambda_n(\rho, q_0, \ldots, q_{2n})$ equals $\Lambda(f_0, \ldots, f_{4n+1}) = \int_{\mathbb{C}^N} \prod_{j=0}^{4n+1} f_j(\ell_j(z))\, dz$. By Theorem A.3 in conjunction with Lemma A.5,

$$|\Lambda_n(\rho, q_0, \ldots, q_{2n})| \le C \|\rho\|_{2,r} \prod_{j=0}^{2n} \|q_j\|_{L^{2,r}},$$

where $r = 2n+2$. Since $2n+2 \ge 2$, the $L^{2,r}$ norm is majorized by a constant multiple of the L^2 norm. □

This reasoning yields various refinements of (A.8). For instance, any one of the functions ρ, q_j may be taken to be in $L^{2,\infty}(\mathbb{C})$ rather than in L^2.

Proof of Lemma A.5. Firstly, $N = 2n+1$, while

$$\sum_{j=0}^{4n+1} p_j^{-1} \dim_{\mathbb{C}}(\ell_j(\mathbb{C}^N)) = \sum_{j=0}^{4n+1} p_j^{-1} N_j = \sum_{j=0}^{4n+1} 2^{-1} \cdot 1 = 2^{-1} \cdot (4n+2) = N.$$

Thus $\mathbb{F}^N$ is critical relative to $(2, 2, \ldots, 2)$.

It remains to show that any nonzero proper complex subspace V of $\mathbb{C}^N$ is subcritical. For any index j, since ℓ_j is a linear mapping from $\mathbb{C}^N$ to $\mathbb{C}^1$, either $\dim_{\mathbb{C}}(\ell_j(V)) = 1$, or ℓ_j vanishes identically on V. Let S be the set of all $j \in [0, \ldots, 2n]$ such that $z_j \equiv 0$ for all $z = (z_0, \ldots, z_{2n}) \in V$, and let T be the set of all $j \in [1, 2n]$ such that $z_j - z_{j-1} \equiv 0$ for all $z \in V$, but neither j nor $j-1$ belongs to S.

The mapping $l_{j+2n} : V \to \mathbb{C}$ is surjective if $j \in [1, 2n]$ and $j \notin T \cup S$. For if not, then it vanishes identically; $z_j - z_{j-1} = 0$ for all $z \in V$. Since $j \notin T$, the definition of T forces at least one of the indices $j, j-1$ to belong to S, that is, at least one of the functions $z \mapsto z_j$ and $z \mapsto z_{j-1}$ vanishes identically on S. The equation $z_j - z_{j-1} \equiv 0$ then forces both of these functions to vanish identically. Therefore both indices $j, j-1$ belong to S, contradicting the hypothesis that $j \notin T \cup S$.

A further consequence is that the number of $j \in (0, 2n]$ such that $j \notin T$, but $z_j - z_{j-1} \equiv 0$ for all $z \in V$, is at most $|S| - 1$. Equality occurs if and only if $S = [k, k-1+|S|]$ for some $k \in [0, 2n]$.

The set of mappings $\{\ell_j : j \in S\} \cup \{\ell_{j+2n} : j \in T\}$ is linearly independent, and V is contained in the intersection of their nullspaces, so

$$\dim_{\mathbb{C}}(V) \leq 2n + 1 - |S| - |T|.$$

On the other hand,

$$\begin{aligned}
&\sum_{j=0}^{4n+1} 2^{-1} \dim_{\mathbb{C}}(\ell_j(V)) \\
&\quad = \sum_{j=0}^{2n} 2^{-1} \dim_{\mathbb{C}}(\ell_j(V)) + \sum_{j=2n+1}^{4n} 2^{-1} \dim_{\mathbb{C}}(\ell_j(V)) + 2^{-1} \dim_{\mathbb{C}}(\ell_{4n+1}(V)) \\
&\quad \geq 2^{-1}(2n + 1 - |S|) + 2^{-1}(2n - |T| - (|S| - 1)) + 2^{-1} \dim_{\mathbb{C}}(\ell_{4n+1}(V)) \\
&\quad = \big(2n + 1 - |S| - |T|\big) + 2^{-1}|T| + 2^{-1} \dim_{\mathbb{C}}(\ell_{4n+1}(V)) \\
&\quad \geq \dim_{\mathbb{C}}(V) + 2^{-1}|T| + 2^{-1} \dim_{\mathbb{C}}(\ell_{4n+1}(V)).
\end{aligned}$$

This is strictly greater than $\dim_{\mathbb{C}}(V)$ unless $T = \emptyset$, V is contained in the nullspace of ℓ_{4n+1}, $\dim_{\mathbb{C}}(V) = 2n + 1 - |S|$, and $S = [k, k - 1 + |S|]$ for some $k \in [0, 2n]$ with $k - 1 + |S| \leq 2n$.

Suppose that $T = \emptyset$, and that V is contained in the nullspace of ℓ_{4n+1}. S cannot be all of $[0, 2n]$, for this would force $V = \{0\}$, contrary to hypothesis. Therefore the equation $\ell_{4n+1}|_V \equiv 0$ is not forced by the equations $\ell_j|_V \equiv 0$ for all $j \in S$, so $\dim_{\mathbb{C}}(V)$ must be strictly less than $2n + 1 - |S|$. Therefore $\sum_{j=0}^{4n+1} 2^{-1} \dim_{\mathbb{C}}(\ell_j(V))$ is strictly greater than $\dim_{\mathbb{C}}(V)$ in all cases; every nonzero proper subspace of $\mathbb{C}^N$ is subcritical. $\square$

B. Time evolution of scattering maps

The purpose of this appendix is to give a self-contained proof that the function u defined by (1.4) solves the DS II equation for $u_0 \in \mathcal{S}(\mathbb{C})$. Previous proofs may be found, for example, in the papers of Beals-Coifman [7, 8, 9] and Sung [33], Part III. We suppose that $r \in C^1(\mathbb{R}_t; \mathcal{S}(\mathbb{C}))$ obeys a linear equation

$$\dot{r} = i\varphi r,$$

where φ is a real-valued polynomial in k and $\bar{k}$. We will obtain an effective formula for $\dot{u}$ if $u = \mathcal{I}(r)$ by differentiating

$$u = \langle e_k \bar{r}, \nu_1 \rangle$$

and exploiting solutions $(\nu_1^\#, \nu_2^\#)$ to a 'dual' problem

$$\bar{\partial}_k \nu_1^\# = \frac{1}{2} e_k \overline{r^\#}\,\overline{\nu_2^\#}, \tag{B.1a}$$

$$\bar{\partial}_k \nu_2^\# = \frac{1}{2} e_k \overline{r^\#}\,\overline{\nu_1^\#}, \tag{B.1b}$$

where $r^\# = \bar{r}$. The following lemma on symmetries of the map $\mathcal{R}$ shows that $r^\# = \mathcal{R}(u^\#)$ where $u^\#(z) = \overline{u(-z)}$.

Lemma B.1. *Let $u, u^\flat \in H^{1,1}(\mathbb{C})$ and let $r = \mathcal{R}(u)$, $r^\flat = \mathcal{R}(u^\flat)$:*

(i) *if $u^\flat(z) = -u(z)$, then $r^\flat(k) = -r(k)$,*

(ii) *if $u^\flat(z) = -u(-z)$, then $r^\flat(k) = -r(-k)$, and*

(iii) *if $u^\flat(z) = \bar{u}(z)$, then $r^\flat(k) = -\overline{r(k)}$.*

Proof. In what follows we let $(\mu_1^\flat, \mu_2^\flat)$ denote the solutions to (1.5) with u replaced by $u^\flat$.

(i) follows from (1.6) and the fact that $\mu_1^\flat = \mu_1$.

(ii) follows from (1.6) and the fact that $\mu_1^\flat(z,k) = \mu_1(-z,-k)$

(iii) From the definition (1.6) we compute (recall (2.1))

$$\begin{aligned} r^\flat(k) &= \langle e_{-k}\bar{u}, \overline{\mu_1^\flat}\rangle \\ &= \langle e_{-k}u, (I - \overline{P}_k e_{-k} u P_k e_k \bar{u})^{-1} 1\rangle \\ &= \langle (I - e_{-k} u \overline{P}_k \bar{u} e_k P_k)^{-1} e_{-k} u, 1\rangle \\ &= \langle (I - \overline{P}_k \bar{u} e_k P_k e_{-k} u)^{-1} 1, e_k \bar{u}\rangle \\ &= \overline{r(-k)} \end{aligned}$$

as claimed. □

From the formula

$$[\partial_t, T_k^2] = -\frac{i}{4} P_k e_k \bar{r} [\varphi, \overline{P}_k] e_{-k} r$$

we have

$$\begin{aligned} \dot{\nu}_1 &= [\partial_t, (I - T_k^2)^{-1}] 1 \\ &= -\frac{i}{4} (I - T_k^2)^{-1} P_k e_k \bar{r} [\varphi, \overline{P}_k] e_{-k} r \nu_1 \end{aligned}$$

so that

$$\begin{aligned}\dot{u} &= i\langle e_k\varphi\bar{r}, \nu_1\rangle - \frac{i}{4}\langle e_k\bar{r}, (I - T_k^2)^{-1}P_k e_k\bar{r}[\varphi, \overline{P}_k]e_{-k}r\nu_1\rangle\\ &= i\langle e_{-k}rf_1, \varphi g_1\rangle + i\,\langle f_2, \varphi e_{-k}r\nu_1\rangle,\end{aligned}$$

where

$$\begin{aligned}f_1 &= \overline{P}_k(I - (T_k^2)^*)^{-1}e_k\bar{r},\\ g_1 &= \overline{P}_k e_{-k}r\nu_1,\\ f_2 &= 1 + e_{-k}r\,\overline{P}_k(I - (T_k^2)^*)^{-1}e_k\bar{r}.\end{aligned}$$

Noting that $(T_k^2)^* = \frac{1}{4}e_k\bar{r}\,\overline{P}_k e_{-k}rP$, it is not difficult to see that

$$\begin{aligned}f_1(z,k) &= \overline{\nu_2^\#(-z,k)},\\ g_1(z,k) &= \overline{\nu_2(z,k)},\\ f_2(z,k) &= \nu_1^\#(-z,k),\end{aligned}$$

so that

$$\dot{u}(z,t) = i\langle e_{-k}r\overline{\nu_2^\#(-z,\cdot)}, \varphi\overline{\nu_2(z,\cdot)}\rangle + i\langle\nu_1^\#(-z,\cdot), \varphi e_{-k}r\nu_1(z,\cdot)\rangle,$$

where we have suppressed the t-dependence of ν and $\nu^\#$. Setting

$$\eta(z,k) = \frac{1}{2}e_k(z)\bar{r}(k)\nu_2^\#(-z,k)\overline{\nu_2(z,k)} + \frac{1}{2}\overline{\nu_1^\#(-z,k)}e_{-k}(z)r(k)\nu_1(z,k)$$

we have

$$\dot{u}(z) = 2i\int\varphi(k)\eta(z,k)\,dA(k).$$

Using (1.8) and (B.1), we can write

$$\eta(z,k) = \bar{\partial}_k[\nu_2^\#(-z,k)\nu_1(z,k)]$$

and

$$\overline{\eta(z,k)} = \bar{\partial}_k[\nu_1^\#(-z,k)\nu_2(z,k)]$$

so that, if $\varphi(k) = 4\,\mathrm{Re}(k^2)$, we conclude that

$$\dot{u}(z) = 4i(I_1 + \overline{I_2})$$

(the complex conjugate on I_2 is intentional), where

$$I_1 = \int k^2\bar{\partial}_k[\nu_2^\#(-z,k)\nu_1(z,k)]\,dA(k),$$

$$I_2 = \int k^2 \bar{\partial}_k [\nu_1^{\#}(-z,k)\nu_2(z,k)]\, dA(k).$$

The integrands in I_1 and I_2 are exact differentials and, for $r \in \mathcal{S}(\mathbb{C})$, vanish rapidly at infinity. We can evaluate I_1 and I_2 using the fact that, if h is a smooth function with $\bar{\partial} h$ of rapid decay and

$$h \sim \sum_{j\geq 0} \frac{h_j}{k^{j+1}} \tag{B.2}$$

then

$$\int k^n \bar{\partial}_k h \, dA(k) = 2\pi i h_n.$$

We compute the large-k asymptotic expansions of ν_1 and ν_2 in Appendix C. Write $[h]_j$ for h_j in the expansion (B.2). In terms of the expansion (C.2) we have

$$[\nu_2^{\#}(-z,k)\nu_1(z,k)]_2 = \nu_{2,0}^{\#}\nu_{1,2} + \nu_{2,1}^{\#}\nu_{1,1} + \nu_{2,2}^{\#}\nu_{1,0},$$

$$[\nu_1^{\#}(-z,k)\nu_2(z,k)]_2 = \nu_{2,0}^{\#}\nu_{1,2} + \nu_{2,1}^{\#}\nu_{1,1} + \nu_{2,2}^{\#}\nu_{1,0},$$

where $\nu^{\#}$ corresponds to the potential $u^{\#}$, and, since $\nu^{\#}$ is evaluated at $-z$, we replace u by $-\bar{u}$, P by $-P$, and ∂ by $-\partial$ in (C.3) and (C.4)–(C.6) to find the expansion coefficients for $\nu^{\#}$. Straightforward computation using (C.3) and (C.4)–(C.6) gives

$$[\nu_2^{\#}(-z,k)\nu_1(z,k)]_2 = \frac{1}{4}u(\mathcal{S}(|u|^2)) - \frac{1}{2}\partial^2 u,$$

where we used the identity $(\bar{\partial}^{-1} f)^2 = 2\bar{\partial}^{-1}(f \bar{\partial}^{-1} f)$ with $f = |u|^2$ to eliminate terms of fifth order in u. Similarly,

$$[\nu_1^{\#}(-z,k)\nu_2(z,k)]_2 = -\frac{1}{4}u(\mathcal{S}(|u|^2)) + \frac{1}{2}\partial^2 \bar{u}.$$

Finally, we obtain

$$i\dot{u}(z) = -2(\partial^2 u + \bar{\partial}^2 u) - u(g + \bar{g}),$$

where

$$g = -\mathcal{S}(|u|^2).$$

This is exactly the DS II equation.

C. Asymptotic expansions

In this section we compute large-parameter asymptotic expansions of the solutions $\nu = (\nu_1, \nu_2)$ of (1.8). Exploiting the fact that $\nu = (\mu_1, e_k \overline{\mu_2})$, we conclude from (1.5) that

$$\begin{aligned} \bar{\partial}_z \nu_1 &= \frac{1}{2} u \nu_2, \\ (\partial_z + k)\nu_2 &= \frac{1}{2} \bar{u} \nu_1. \end{aligned} \tag{C.1}$$

For $r \in \mathcal{S}(\mathbb{C})$, the functions (ν_1, ν_2) admit a large-k asymptotic expansion of the form

$$\nu \sim (1, 0) + \sum_{\ell \geq 0} k^{-(\ell+1)} \nu^{(\ell)}, \tag{C.2}$$

where $\nu^{(\ell)} = (\nu_{1,\ell}, \nu_{2,\ell})^T$. From the system (C.1) we easily deduce that

$$\nu_{1,0} = \frac{1}{4} \bar{\partial}^{-1}(|u|^2), \quad \nu_{2,0} = \frac{1}{2} \bar{u}, \tag{C.3}$$

while for $\ell \geq 1$,

$$\begin{aligned} \nu_{2,\ell} &= \frac{1}{2} \bar{u} \nu_{1,\ell-1} - \partial \nu_{2,\ell-1}, \\ \nu_{1,\ell} &= \frac{1}{2} P(u \nu_{2,\ell}). \end{aligned}$$

It easily follows that

$$\nu_{1,1} = \frac{1}{16} P(|u|^2 P(|u|^2)) - \frac{1}{4} P(u \partial \bar{u}), \tag{C.4}$$

$$\nu_{2,1} = \frac{1}{8} \bar{u} P(|u|^2) - \frac{1}{2} \partial \bar{u}, \tag{C.5}$$

$$\begin{aligned} \nu_{2,2} &= \frac{1}{32} \bar{u}\ P(|u|^2 P(|u|^2)) \\ &\quad + \frac{1}{8} \partial(\bar{u}\ P(|u|^2)) - \frac{1}{8} \bar{u} P(u \partial \bar{u}) + \frac{1}{2} \partial^2 \bar{u}. \end{aligned} \tag{C.6}$$

Remark C.1. In a similar way one can show that for $r \in \mathcal{S}(\mathbb{C})$, μ has a large-z asymptotic expansion whose coefficients are computed in terms of r and its derivatives. Thus for example

$$\begin{aligned} \mu_1(z, k) &= 1 + \frac{1}{z} \left(\frac{1}{4} \bar{\partial}_k^{-1}(|r|^2) \right) + \mathcal{O}(|z|^{-2}), \\ \mu_2(z, k) &= \frac{1}{z} \left(\frac{1}{2} r \right) + \mathcal{O}(|z|^{-2}). \end{aligned}$$

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