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Spectral asymptotics for first order systems

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In memory of our dear friend Yuri Safarov

Abstract. This is a review paper outlining recent progress in the spectral analysis of first order systems. We work on a closed manifold and study an elliptic self-adjoint first order system of linear partial differential equations. The aim is to examine the spectrum and derive asymptotic formulae for the two counting functions. Here the two counting functions are those for the positive and the negative eigenvalues. One has to deal with positive and negative eigenvalues separately because the spectrum is, generically, asymmetric.

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1. Introduction

This paper was inspired by Yuri Safarov's treatment of first order systems in [25]. Safarov was one of the first researchers to attempt a detailed spectral analysis of first order systems on closed manifolds. A brief historical review can be found in Section 11 of [7].

Let L be a first order linear differential operator acting on m-columns of complex-valued half-densities over a connected closed (i.e. compact and without boundary) n-dimensional manifold M. Throughout this paper we assume that $m, n \geq 2$.

Let $L_{\rm prin}(x, p)$ and $L_{\rm sub}(x)$ be the principal and subprincipal symbols of L. Here $x = (x^1, \dots, x^n)$ denotes local coordinates and $p = (p_1, \dots, p_n)$ denotes

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the dual variable (momentum). The principal and subprincipal symbols are defined in the same way as for scalar operators, see Subsection 2.1.3 in [26], only now they are $m \times m$ matrix-functions on T^*M and M respectively. As our operator L is first order and differential (as opposed to pseudodifferential), the principal and subprincipal symbols uniquely determine the operator. In other words, the principal and subprincipal symbols provide an invariant analytic description of our first order differential operator L.

We assume our operator L to be formally self-adjoint (symmetric) with respect to the standard inner product on m-columns of complex-valued half-densities, which is equivalent to the assumption that the principal and subprincipal symbols are Hermitian. We also assume that our operator L is elliptic:

$$\det L_{\mathrm{prin}}(x,p) \neq 0, \quad \text{for all } (x,p) \in T^*M \setminus \{0\}. \tag{1.1}$$

Condition (1.1) and the fact that dimension n is greater than or equal to two imply that m, the number of equations, is even. See Remark 2.1 in [13] for details.

Under the above assumptions L is a self-adjoint operator in $L^2(M;\mathbb{C}^m)$ with domain $H^1(M;\mathbb{C}^m)$ and the spectrum of L is discrete, accumulating to $+\infty$ and to $-\infty$. Let λ_k and $v_k(x)$ be the eigenvalues and the orthonormal eigenfunctions of the operator L; the particular enumeration of these eigenvalues (accounting for multiplicities) is irrelevant for our purposes. Each $v_k(x)$ is, of course, an m-column of half-densities.

The main objective of this paper is to derive asymptotic formulae for the distribution of large, in terms of modulus, eigenvalues of L. We will deal with positive and negative eigenvalues separately and we will see that the asymptotic distribution of positive and negative eigenvalues is somewhat different. Note that asymmetry of the spectrum (with respect to zero) is a major subject in geometry, see, for example, [1, 2, 3, 4].

2. The propagator

In this paper, as in [26], we use the wave equation method (Levitan's method) to derive spectral asymptotics. In our case, when we are dealing with an operator which is not semi-bounded, the wave equation method is especially natural. Furthermore, the wave equation method is the only physically meaningful way of introducing a time coordinate when dealing with a first order elliptic system.

Let $h^{(j)}(x,p)$ be the eigenvalues of the matrix-function $L_{\mathrm{prin}}(x,p)$. Throughout this paper we assume that these are simple for all $(x,p) \in T^*M \setminus \{0\}$. We enumerate the $h^{(j)}(x,p)$ in increasing order, using a negative index

$j = -m/2, \dots, -1$ for negative $h^{(j)}(x, p)$ and a positive index $j = 1, \dots, m/2$ for positive $h^{(j)}(x, p)$. By $v^{(j)}(x, p)$ we denote the corresponding normalised eigenvectors. Note that as our operator L is first order and differential (as opposed to pseudodifferential) we have the following symmetry:

$$h^{(-j)}(x, p) = -h^{(j)}(x, -p), \quad v^{(-j)}(x, p) = v^{(j)}(x, -p), \quad j = 1, \dots, m/2. \tag{2.1}$$

Now, let $x^{n+1} \in \mathbb{R}$ be the additional 'time' coordinate. Consider the Cauchy problem

$$w|_{x^{n+1}=0} = v \tag{2.2}$$

for the hyperbolic system

$$(-i\,\partial/\partial x^{n+1} + L)w = 0 \tag{2.3}$$

on $M \times \mathbb{R}$. The m-column of half-densities $v = v(x^1, \dots, x^n)$ is given and the m-column of half-densities $w = w(x^1, \dots, x^n, x^{n+1})$ is to be found. The solution of the Cauchy problem (2.2), (2.3) can be written as $w = U(x^{n+1})\, v$, where

$$\begin{aligned} U(x^{n+1}) &:= e^{-ix^{n+1}L} \\ &= \sum_k e^{-ix^{n+1}\lambda_k}\, v_k(x^1, \dots, x^n) \int_M [v_k(y^1, \dots, y^n)]^*(\cdot)\, dy^1 \dots dy^n \end{aligned} \tag{2.4}$$

is the *propagator*; here and further on the star stands for Hermitian conjugation. The propagator $U(x^{n+1})$ is a one-parameter family of unitary operators.

Remark 2.1. We chose to denote the 'time' coordinate by x^{n+1} rather than by t because some constructions presented in the current paper work in the relativistic setting, i.e. when there is no distinguished time direction and the coordinates $x^1, \dots, x^n$ and x^{n+1} are 'mixed up'. Such an approach was pursued in [12] and, to a certain extent, in [13].

It was shown by Safarov [25] that the propagator can be constructed explicitly, modulo C^∞, as a sum of m oscillatory integrals (Fourier integral operators)

$$U(x^{n+1}) \overset{\operatorname{mod} C^\infty}{=} \sum_j U^{(j)}(x^{n+1}), \tag{2.5}$$

where the phase function of each oscillatory integral $U^{(j)}(x^{n+1})$ is associated with the corresponding Hamiltonian $h^{(j)}(x^1, \dots, x^n, p_1, \dots, p_n)$ and summation

is carried out over all nonzero integers j from $-m/2$ to $+m/2$. The notion of a phase function associated with a Hamiltonian is defined in Section 2 of [7] and Section 2.4 of [26]. Safarov's initial exposition [25] of the construction leading up to (2.5) was quite concise. A more detailed exposition was later given in [23] and [7].

We will now state the two main results regarding the properties of the oscillatory integrals $U^{(j)}(x^{n+1})$ appearing in the right hand side of formula (2.5). From this point till the end of the section we assume the index j to be fixed.

The first result concerns the principal symbol of the oscillatory integral $U^{(j)}(x^{n+1})$. The notion of the principal symbol of an oscillatory integral is defined in accordance with Definition 2.7.12 from [26]. The principal symbol of the oscillatory integral $U^{(j)}(x^{n+1})$ is a complex-valued $m \times m$ matrix-function on $M \times \mathbb{R} \times (T^*M \setminus \{0\})$. We denote the arguments of this principal symbol by $x^1, \dots, x^n$ (local coordinates on M), x^{n+1} ('time' coordinate on $\mathbb{R}$), $y^1, \dots, y^n$ (local coordinates on M) and $q_1, \dots, q_n$ (variable dual to $y^1, \dots, y^n$).

Further on in this section and the next section we use x, y, p and q as shorthand for $x^1, \dots, x^n$, $y^1, \dots, y^n$, $p_1, \dots, p_n$ and $q_1, \dots, q_n$ respectively. The additional 'time' coordinate x^{n+1} will always be written separately.

In order to write down the principal symbol of the oscillatory integral $U^{(j)}(x^{n+1})$ we need to introduce some notation.

Curly brackets will denote the Poisson bracket on matrix-functions

$$\{P, R\} := P_{x^\alpha} R_{p_\alpha} - P_{p_\alpha} R_{x^\alpha}$$

and its further generalisation

$$\{F, G, H\} := F_{x^\alpha} G H_{p_\alpha} - F_{p_\alpha} G H_{x^\alpha}, \tag{2.6}$$

where the subscripts x^α and p_α indicate partial derivatives and the repeated index α indicates summation over $\alpha = 1, \dots, n$.

We define the scalar function $f^{(j)} \colon T^*M \setminus \{0\} \to \mathbb{R}$ in accordance with the formula

$$f^{(j)} := [v^{(j)}]^* L_{\mathrm{sub}} v^{(j)} - \frac{i}{2}\{[v^{(j)}]^*, L_{\mathrm{prin}} - h^{(j)}, v^{(j)}\} - i\,[v^{(j)}]^* \{v^{(j)}, h^{(j)}\}.$$

By $(x^{(j)}(x^{n+1}; y, q), p^{(j)}(x^{n+1}; y, q))$ we denote the Hamiltonian trajectory originating from the point (y, q), i.e. solution of the system of ordinary differential equations (the dot denotes differentiation in x^{n+1})

$$\dot{x}^{(j)} = h_p^{(j)}(x^{(j)}, p^{(j)}), \quad \dot{p}^{(j)} = -h_x^{(j)}(x^{(j)}, p^{(j)})$$

subject to the initial condition $(x^{(j)}, p^{(j)})\big|_{x^{n+1}=0} = (y, q)$.

Theorem 2.1. *The formula for the principal symbol of the oscillatory integral* $U^{(j)}(x^{n+1})$ *reads as follows*:

$$[v^{(j)}(x^{(j)}(x^{n+1}; y, q), p^{(j)}(x^{n+1}; y, q))]\,[v^{(j)}(y, q)]^* \exp\left(-i\int_0^{x^{n+1}} f^{(j)}(x^{(j)}(\tau; y, q), p^{(j)}(\tau; y, q))\,d\tau\right).$$

This principal symbol is positively homogeneous in momentum q *of degree zero.*

Theorem 2.1 is due to Safarov [25]. It was later confirmed by the more detailed analysis carried out in [23, 7].

Theorem 2.1 is insufficient for the determination of the second term in spectral asymptotics because one needs information about the lower order terms of the symbol of the oscillatory integral $U^{(j)}(x^{n+1})$. Namely, one needs information about terms positively homogeneous in momentum q of degree -1. The algorithm described in Section 2 of [7] provides a recursive procedure for the calculation of all lower order terms, of any degree of homogeneity in momentum q. However, there are two issues here. Firstly, calculations become very complicated. Secondly, describing these lower order terms in an invariant way is problematic. A few years before his untimely death Safarov had discussions with one of the authors of this paper and pointed out that, as far as he was aware, the concept of subprincipal symbol has never been defined for time-dependent oscillatory integrals (Fourier integral operators).

We overcome the problem of invariant description of lower order terms of the symbol of the oscillatory integral $U^{(j)}(x^{n+1})$ by restricting our analysis to $U^{(j)}(0)$. It turns out that knowing the properties of the lower order terms of the symbol of $U^{(j)}(0)$ is sufficient for the derivation of two-term spectral asymptotics. And $U^{(j)}(0)$ is a pseudodifferential operator, so one can use here the standard notion of subprincipal symbol of a pseudodifferential operator.

The following result was recently established in [7].

Theorem 2.2. *We have*

$$\operatorname{tr}[U^{(j)}(0)]_{\mathrm{sub}} = -i\{[v^{(j)}]^*, v^{(j)}\}, \tag{2.7}$$

where tr *stands for the matrix trace.*

It is interesting that the right hand side of formula (2.7) admits a geometric interpretation: it can be interpreted as the scalar curvature of a U(1) connection on $T^*M \setminus \{0\}$, see Section 5 of [7] for details. This connection is to do with

gauge transformations of the normalised eigenvector $v^{(j)}(x,p)$ of the principal symbol $L_{\mathrm{prin}}(x,p)$ corresponding to the eigenvalue $h^{(j)}(x,p)$. Namely, observe that if $v^{(j)}(x,p)$ is an eigenvector and $\phi^{(j)}(x,p)$ is an arbitrary smooth real-valued function, then $e^{i\phi^{(j)}(x,p)}v^{(j)}(x,p)$ is also an eigenvector. Careful analysis of the gauge transformation

$$v^{(j)} \longmapsto e^{i\phi^{(j)}} v^{(j)} \tag{2.8}$$

leads to the appearance of a curvature term.

3. Mollified spectral asymptotics

Denote by

$$u(x,x^{n+1},y) := \sum_k e^{-ix^{n+1}\lambda_k} v_k(x)[v_k(y)]^* \tag{3.1}$$

the integral kernel of the propagator (2.4). The quantity (3.1) can be understood as a matrix-valued distribution in the variable $x^{n+1} \in \mathbb{R}$ depending on the parameters $x, y \in M$. Further on in this section we will be studying the quantity

$$\hat{f}(x,x^{n+1}) := \operatorname{tr} u(x,x^{n+1},x) = \sum_k \|v_k(x)\|^2 e^{-ix^{n+1}\lambda_k}. \tag{3.2}$$

Here $\|v_k(x)\|$ is the Euclidean norm of the m-column v_k evaluated at the point $x \in M$. Of course, $\|v_k(x)\|^2$ is a real-valued density.

In order to understand the reason for our interest in (3.2), put

$$f(x,\lambda) := \sum_k \|v_k(x)\|^2 \delta(\lambda - \lambda_k). \tag{3.3}$$

Then (3.2) and (3.3) are related as $\hat{f} = \mathcal{F}_{\lambda\to x^{n+1}}[f]$ and $f = \mathcal{F}^{-1}_{x^{n+1}\to\lambda}[\hat{f}]$, where the one-dimensional Fourier transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ are defined as in Section 6 of [7]. The quantity (3.3) contains all the information on the spectrum of our operator L and it also contains some information on the eigenfunctions.

Let $\hat{\rho}\colon \mathbb{R} \to \mathbb{C}$ be a smooth function such that $\hat{\rho}(0) = 1$, $\hat{\rho}'(0) = 0$ and the support of $\hat{\rho}$ is sufficiently small. Here 'sufficiently small' means that $\operatorname{supp}\hat{\rho} \subset (-\mathbf{T},\mathbf{T})$, where $\mathbf{T}$ is the infimum of the lengths of all possible loops. See Section 6 in [7] for details. Denote also $\rho(\lambda) := \mathcal{F}^{-1}_{x^{n+1}\to\lambda}[\hat{\rho}(x^{n+1})]$.

We mollify the distributions (3.2) and (3.3) by switching to $\hat{\rho}(x^{n+1})\hat{f}(x,x^{n+1})$ and $(\rho * f)(x,\lambda)$, where the star indicates convolution in the variable λ. It was shown in [7] that Theorems 2.1 and 2.2 imply the following result.

Theorem 3.1. *We have, uniformly over $x \in M$,*

$$(\rho * f)(x,\lambda) = n\,a(x)\,\lambda^{n-1} + (n-1)\,b(x)\,\lambda^{n-2} + O(\lambda^{n-3}) \quad \text{as } \lambda \to +\infty.$$

Here the densities $a(x)$ and $b(x)$ are given by formulae

$$a(x) = (2\pi)^{-n} \sum_{j=1}^{m/2} \int_{h^{(j)}(x,p)<1} dp, \tag{3.4}$$

$$\begin{aligned} b(x) = -n(2\pi)^{-n} \sum_{j=1}^{m/2} \int_{h^{(j)}(x,p)<1} \Big(& [v^{(j)}]^* L_{\text{sub}} v^{(j)} \\ & - \frac{i}{2}\{[v^{(j)}]^*, L_{\text{prin}} - h^{(j)}, v^{(j)}\} \\ & + \frac{i}{n-1} h^{(j)} \{[v^{(j)}]^*, v^{(j)}\} \Big)(x,p)\, dp, \end{aligned} \tag{3.5}$$

where $dp = dp_1 \dots dp_n$.

Theorem 3.1 warrants the following remarks.

Remark 3.1. It is easy to see that the right hand side of formula (3.5) is invariant under gauge transformations (2.8) of the eigenvectors of the principal symbol.

Remark 3.2. Let $R\colon M \to \mathrm{U}(m)$ be an arbitrary smooth unitary matrix-function. As one would expect, the right hand side of formula (3.5) is invariant under gauge transformations $L \mapsto R^* L R$ of our operator, but establishing this is not that easy. The corresponding calculations are presented in Section 9 of [7].

Let us now leave in (3.3) only terms with positive λ_k and define the quantity

$$f_+(x,\lambda) := \sum_{\lambda_k > 0} \|v_k(x)\|^2 \delta(\lambda - \lambda_k).$$

Theorem 3.1 implies the following Corollary.

Corollary 3.1. *We have, uniformly over $x \in M$, the following two results*:

$$(\rho * f_+)(x,\lambda) = n\,a(x)\,\lambda^{n-1} + (n-1)\,b(x)\,\lambda^{n-2} + O(\lambda^{n-3}) \quad \text{as } \lambda \to +\infty$$

and $(\rho * f_+)(x,\lambda)$ vanishes faster than any negative power of $|\lambda|$ as $\lambda \to -\infty$.

Let us define the two local counting functions

$$N_{\pm}(x,\lambda) := \begin{cases} 0 & \text{if } \lambda \le 0, \\ \displaystyle\sum_{0<\pm\lambda_k<\lambda} \|v_k(x)\|^2 & \text{if } \lambda > 0. \end{cases} \tag{3.6}$$

The function $N_+(x,\lambda)$ counts the eigenvalues λ_k between zero and λ, whereas the function $N_-(x,\lambda)$ counts the eigenvalues λ_k between $-\lambda$ and zero. In both cases counting eigenvalues involves assigning them weights $\|v_k(x)\|^2$.

We have $(\rho * N_+)(x,\lambda) = \int_{-\infty}^{\lambda} (\rho * f_+)(x,\mu)\, d\mu$, so Corollary 3.1 implies

$$(\rho * N_+)(x,\lambda) = a(x)\lambda^n + b(x)\lambda^{n-1} + \begin{cases} O(\lambda^{n-2}) & \text{if } n \ge 3, \\ O(\ln\lambda) & \text{if } n = 2, \end{cases} \quad \text{as } \lambda \to +\infty. \tag{3.7}$$

The asymptotics for $(\rho * N_-)(x,\lambda)$ is obtained by applying the above result to the operator $-L$ and using the symmetries (2.1). This gives

$$(\rho * N_-)(x,\lambda) = a(x)\lambda^n - b(x)\lambda^{n-1} + \begin{cases} O(\lambda^{n-2}) & \text{if } n \ge 3, \\ O(\ln\lambda) & \text{if } n = 2, \end{cases} \quad \text{as } \lambda \to +\infty. \tag{3.8}$$

Note that the second terms in the asymptotic formulae (3.7) and (3.8) have opposite signs and that the remainders are uniform in $x \in M$.

Finally, let us define the two global counting functions

$$N_{\pm}(\lambda) := \begin{cases} 0 & \text{if } \lambda \le 0, \\ \displaystyle\sum_{0<\pm\lambda_k<\lambda} 1 & \text{if } \lambda > 0. \end{cases} \tag{3.9}$$

We have $N_{\pm}(\lambda) = \int_M N_{\pm}(x,\lambda)\, dx$, where $dx = dx^1 \dots dx^n$. Therefore, formulae (3.7) and (3.8) imply

$$(\rho * N_{\pm})(\lambda) = a\lambda^n \pm b\lambda^{n-1} + \begin{cases} O(\lambda^{n-2}) & \text{if } n \ge 3, \\ O(\ln\lambda) & \text{if } n = 2, \end{cases} \quad \text{as } \lambda \to +\infty, \tag{3.10}$$

where

$$a = \int_M a(x)\, dx, \quad b = \int_M b(x)\, dx. \tag{3.11}$$

4. Unmollified spectral asymptotics

In this section we write down asymptotic formulae for the local and global counting functions without mollification. These can be obtained from the mollified asymptotic formulae (3.7), (3.8), and (3.10) by applying appropriate Fourier Tauberian theorems, see Section 8 of [7].

Theorem 4.1. *We have, uniformly over $x \in M$,*

$$N_{\pm}(x,\lambda) = a(x)\,\lambda^n + O(\lambda^{n-1}) \quad \text{as } \lambda \to +\infty. \tag{4.1}$$

Corollary 4.1. *We have*

$$N_{\pm}(\lambda) = a\lambda^n + O(\lambda^{n-1}) \quad \text{as } \lambda \to +\infty. \tag{4.2}$$

Theorem 4.2. *If the point $x \in M$ is nonfocal then*

$$N_{\pm}(x,\lambda) = a(x)\,\lambda^n \pm b(x)\,\lambda^{n-1} + o(\lambda^{n-1}) \quad \text{as } \lambda \to +\infty. \tag{4.3}$$

Theorem 4.3. *If the nonperiodicity condition is fulfilled then*

$$N_{\pm}(\lambda) = a\lambda^n \pm b\lambda^{n-1} + o(\lambda^{n-1}) \quad \text{as } \lambda \to +\infty. \tag{4.4}$$

A point $x \in M$ is said to be nonfocal if there are not too many Hamiltonian loops originating from this point. 'Nonperiodicity' means that there are not too many periodic Hamiltonian trajectories. See Subsection 8.2 in [7] for details.

The results presented in this section were first obtained by Victor Ivrii [18, 19] but without an explicit formula for the second asymptotic coefficient.

Asymptotic formulae of the type (4.1)–(4.4) are called *Weyl-type formulae* and the coefficients in such formulae are often referred to as *Weyl coefficients*.

5. The eta function

The (global) eta function of our operator L is defined as

$$\eta(s) := \sum_{\lambda_k \neq 0} \frac{\operatorname{sgn} \lambda_k}{|\lambda_k|^s} = \int_0^{+\infty} \lambda^{-s}\,(N'_+(\lambda) - N'_-(\lambda))\,d\lambda, \tag{5.1}$$

where summation is carried out over all nonzero eigenvalues λ_k of L and $s \in \mathbb{C}$ is the independent variable. The series (5.1) converges absolutely for $\operatorname{Re} s > n$ and defines a holomorphic function in this half-plane. Moreover, it is known [4] that it

extends meromorphically to the whole s-plane with simple poles which can only occur at real integer values of s. The eta function is the accepted way of describing the asymmetry of the spectrum.

Formula (4.2) implies that the eta function does not have a pole at $s = n$ and formula (3.10) implies that at $s = n - 1$ the residue is

$$\operatorname{Res}(\eta, n-1) = 2(n-1)b, \tag{5.2}$$

where b is the coefficient from (3.10). Thus, for a generic operator L the first pole of the eta function is at $s = n - 1$ and formulae (3.5), (3.11), and (5.2) give us an explicit expression for the residue.

It is known [4, 14] that the eta function does not have a pole at $s = 0$. The real number $\eta(0)$ is called the *eta invariant*. It can be interpreted as the number of positive eigenvalues minus the number of negative eigenvalues. This interpretation is based on the observation that if we were dealing with an Hermitian matrix L, then $\eta(0)$ would indeed be the number of positive eigenvalues minus the number of negative eigenvalues. Our differential operator L has infinitely many positive eigenvalues and infinitely many negative eigenvalues, and the concept of the eta function allows us to regularise the expression 'number of positive eigenvalues minus the number of negative eigenvalues'.

The eta function may have poles at

$$s = n - 1, \dots, 1, -1, -2, \dots. \tag{5.3}$$

However, a more careful analysis [22] shows that poles may occur only at values of s of the form

$$s = n - 1 - 2l, \quad l = 0, 1, \dots. \tag{5.4}$$

The authors of [22] call values of s from the intersection of the sets (5.3) and (5.4) *admissible*. It was shown in [22] that residues of the eta function at positive admissible integers are generically nonzero. This agrees with our explicit calculation of the residue at $s = n - 1$.

6. Systems of two equations

From now on we assume that

$$m = 2 \tag{6.1}$$

and that

$$\operatorname{tr} L_{\text{prin}}(x, p) = 0. \tag{6.2}$$

In other words, we now restrict our analysis to 2×2 operators with trace-free principal symbols. The logic behind the assumptions (6.1) and (6.2) is that they single out the simplest class of first order systems and we expect to extract more geometry out of our asymptotic analysis and simplify the results.

It is easy to see that formulae (1.1), (6.1), and (6.2) imply that the dimension n of our manifold M is less than or equal to three. Further on we assume that

$$n = 3. \tag{6.3}$$

Remark 6.1. It was shown in [8] that a 3-manifold is parallelizable if and only if there exists a self-adjoint elliptic first order linear differential operator with trace-free principal symbol acting on 2-columns of complex-valued half-densities over this manifold. This means that once we restricted our analysis to the special case (6.1)–(6.3) we are working on a parallelizable manifold.

Remark 6.2. It is well known that a 3-manifold is orientable if and only if it is parallelizable, see Theorem 1 in Chapter VII of [21].

Observe that under the assumption (6.1) the determinant of the principal symbol is a quadratic form in the dual variable (momentum) p:

$$\det L_{\mathrm{prin}}(x, p) = -g^{\alpha\beta}(x)\, p_\alpha p_\beta. \tag{6.4}$$

Furthermore, formulae (1.1) and (6.2) imply that the quadratic form $g^{\alpha\beta}(x)\, p_\alpha p_\beta$ is positive definite. We interpret the real coefficients $g^{\alpha\beta}(x) = g^{\beta\alpha}(x)$, $\alpha, \beta = 1, 2, 3$, as components of a (contravariant) metric tensor. Thus, 2×2 operators with trace-free principal symbols are special in that the concept of a Riemannian metric is encoded within such operators. This opens the way to the geometric interpretation of our analytic results.

Note also that under the assumptions (6.1) and (6.2) the principal symbol of the operator L^2 is automatically proportional to the 2×2 identity matrix I:

$$(L^2)_{\mathrm{prin}}(x, p) = (L_{\mathrm{prin}})^2(x, p) = (g^{\alpha\beta}(x)\, p_\alpha p_\beta)\, I. \tag{6.5}$$

Operators possessing the property (6.5) are called *Dirac-type operators*.

Now take an arbitrary smooth matrix-function

$$R\colon M \longrightarrow \mathrm{SU}(2) \tag{6.6}$$

and consider the transformation of our 2×2 differential operator

$$L \longmapsto R^* L R. \tag{6.7}$$

We interpret (6.7) as a gauge transformation because it does not affect our counting functions (3.6) and (3.9) and the eta function (5.1). Note also that the transformation (6.7) preserves the condition (6.2).

The transformation (6.7) of the differential operator L induces the following transformations of its principal and subprincipal symbols:

$$L_{\mathrm{prin}} \longmapsto R^* L_{\mathrm{prin}} R, \tag{6.8}$$

$$L_{\mathrm{sub}} \longmapsto R^* L_{\mathrm{sub}} R + \frac{i}{2}(R^*_{x^\alpha}(L_{\mathrm{prin}})_{p_\alpha} R - R^*(L_{\mathrm{prin}})_{p_\alpha} R_{x^\alpha}). \tag{6.9}$$

Comparing formulae (6.8) and (6.9) we see that, unlike the principal symbol, the subprincipal symbol does not transform in a covariant fashion due to the appearance of terms with the gradient of the matrix-function $R(x)$.

It turns out that one can overcome the non-covariance in (6.9) by introducing the *covariant subprincipal symbol* $L_{\mathrm{csub}}(x)$ in accordance with formula

$$L_{\mathrm{csub}} := L_{\mathrm{sub}} - \frac{i}{16}\, g_{\alpha\beta}\{L_{\mathrm{prin}}, L_{\mathrm{prin}}, L_{\mathrm{prin}}\}_{p_\alpha p_\beta}, \tag{6.10}$$

where subscripts p_α and p_β indicate partial derivatives and curly brackets denote the generalised Poisson bracket on matrix-functions (2.6).

Lemma 6.1. *The transformation* (6.7) *of the differential operator induces the transformation* $L_{\mathrm{csub}} \mapsto R^* L_{\mathrm{csub}} R$ *of its covariant subprincipal symbol.*

The proof of Lemma 6.1 was given in [12]. Note that the analysis in [12] was performed in a more general, 4-dimensional Lorentzian setting.

In our 3-dimensional Riemannian setting the correction term in the right hand side of (6.10) turns out to be proportional to the 2×2 identity matrix I:

$$-\frac{i}{16}\, g_{\alpha\beta}\{L_{\mathrm{prin}}, L_{\mathrm{prin}}, L_{\mathrm{prin}}\}_{p_\alpha p_\beta} = If, \tag{6.11}$$

where $f\colon M \to \mathbb{R}$ is some scalar function. The function $f(x)$ in (6.11) admits a geometric interpretation [8, 10] but we do not discuss this in the current paper.

Theorem 6.1. *In the special case* (6.1)–(6.3) *formulae* (3.4) *and* (3.5) *read*

$$a(x) = \frac{1}{6\pi^2}\sqrt{\det g_{\alpha\beta}(x)}, \tag{6.12}$$

$$b(x) = -\frac{1}{4\pi^2}\left((\operatorname{tr} L_{\mathrm{csub}})\sqrt{\det g_{\alpha\beta}}\,\right)(x). \tag{6.13}$$

Theorem 6.1 was established in [8], though the density $b(x)$ was written in [8] in a slightly different way. The use of the concept of covariant subprincipal symbol introduced in [12] allows us to replace formula (1.19) from [8] by the more compact expression (6.13).

Formula (6.12) tells us that the first local Weyl coefficient is proportional to the standard Riemannian density. The first global Weyl coefficient is obtained from the local one by integration, see formula (3.11), and is proportional to the Riemannian volume V of our manifold M: $a = \frac{1}{6\pi^2} V$.

In order to understand the geometric meaning of formula (6.13) we observe that the covariant subprincipal symbol can be uniquely represented in the form

$$L_{\mathrm{csub}}(x) = L_{\mathrm{prin}}(x, A(x)) + I A_4(x), \tag{6.14}$$

where $A = (A_1, A_2, A_3)$ is some real-valued covector field, A_4 is some real-valued scalar field, $x = (x^1, x^2, x^3)$ are local coordinates on M (we are working in the nonrelativistic setting) and I is the 2×2 identity matrix. Applying the results of [12] to the relativistic operator appearing in the left hand side of (2.3) we conclude that $A = (A_1, A_2, A_3)$ is the magnetic covector potential and A_4 is the electric potential. Note that Lemma 6.1 and formulae (6.8) and (6.14) tell us that the magnetic covector potential and electric potential are invariant under gauge transformations (6.7).

Substituting (6.14) into (6.13) and making use of (6.2) we get

$$b(x) = -\frac{1}{2\pi^2} \left(A_4 \sqrt{\det g_{\alpha\beta}}\right)(x). \tag{6.15}$$

Thus, the second Weyl coefficient is proportional to the electric potential and does not depend on the magnetic covector potential.

A number of researchers have studied the effect of the electromagnetic field on the spectrum of the first order differential operator L under the assumptions (6.1)–(6.3) and our formula (6.15) is a further contribution to this line of research. However, we believe that such results do not have a physical meaning because our 2×2 first order differential operator L describes a massless particle and no known massless particle has an electric charge. In the absence of an electric charge the particle cannot interact with the electromagnetic field.

The electron is an example of a charged massive particle but it is described by a 4×4 first order differential operator. Also, in the case of the electron it is more natural to do asymptotic analysis in a different setting, with Planck's constant tending to zero. Spectral problems for the electron in 3-dimensional Euclidean

space in the presence of magnetic and electric potentials were extensively studied by Ivrii [18, 19]. An analytic (i.e. based on the concepts of principal symbol and covariant subprincipal symbol) representation of the massive Dirac equation in curved 4-dimensional Lorentzian spacetime was given in [12].

7. Spin structure

Let M be a connected closed oriented Riemannian 3-manifold. Let us consider all possible self-adjoint elliptic first order 2×2 linear differential operators L with trace-free principal symbols corresponding, in the sense of formula (6.4), to the prescribed metric. See also Remarks 6.1 and 6.2. In this section our aim is to classify all such operators L.

We define the topological charge as

$$\mathbf{c} := -\frac{i}{2}\sqrt{\det g_{\alpha\beta}}\ \operatorname{tr}((L_{\mathrm{prin}})_{p_1}(L_{\mathrm{prin}})_{p_2}(L_{\mathrm{prin}})_{p_3}), \tag{7.1}$$

with the subscripts p_1, p_2, p_3 indicating partial derivatives with respect to the components of momentum $p = (p_1, p_2, p_3)$. It was shown in Section 3 of [8] that the number $\mathbf{c}$ defined by formula (7.1) can take only two values, $+1$ or -1, and describes the orientation of the principal symbol relative to the chosen orientation of local coordinates $x = (x^1, x^2, x^3)$. Of course, the transformation $L \mapsto -L$ inverts the topological charge.

Further on we work with operators whose topological charge is $+1$.

We say that the operators L and $\tilde{L}$ are equivalent if there exists a smooth matrix-function (6.6) such that $\tilde{L}_{\mathrm{prin}} = R^* L_{\mathrm{prin}} R$. The equivalence classes of operators obtained this way are called *spin structures*.

An example of non-equivalent operators L and $\tilde{L}$ on the 3-torus was given in Appendix A of [8]. Furthermore, using the above definition of spin structure one can show that there are eight distinct spin structures on the 3-torus whereas the spin structure on the 3-sphere is unique.

We see that an operator L is uniquely determined, modulo a gauge transformation (6.7), by the metric, topological charge, spin structure, magnetic covector potential and electric potential.

We claim that in dimension three our analytic definition of spin structure is equivalent to the traditional topological definition. We will provide a rigorous proof of this claim in a separate paper.

8. The massless Dirac operator

In this section we continue dealing with the special case (6.1)–(6.3) but make the additional assumption that the magnetic covector potential and electric potential vanish. The resulting operator L is called the *massless Dirac operator on half-densities*. It is uniquely determined, modulo a gauge transformation (6.7), by the metric, topological charge and spin structure.

In practice most researchers work with the massless Dirac operator which acts on 2-columns of complex-valued scalar fields (components of a Weyl spinor) rather than on 2-columns of complex-valued half-densities. As we have a Riemannian metric encoded in the principal symbol of our operator, scalar fields can be identified with half-densities: it is just a matter of multiplying or dividing by $(\det g_{\alpha\beta})^{1/4}$. Hence, the 'traditional' massless Dirac operator and the massless Dirac operator on half-densities are related by a simple formula, see formula (A.19) in [8], and their spectra are the same. For spectral theoretic purposes it is more convenient to work with half-densities because in this case the inner product does not depend on the metric.

The massless Dirac operator describes the massless neutrino. We are looking at a single neutrino living in a closed 3-dimensional Riemannian universe. The eigenvalues are the energy levels of the particle. The tradition is to associate positive eigenvalues with the energy levels of the neutrino and negative eigenvalues with the energy levels of the antineutrino.

Formula (6.15) tells us that the second Weyl coefficient for the massless Dirac operator is zero, both locally and globally. Formula (5.2), in turn, tells us that the eta function of the massless Dirac operator does not have a pole at $s = 2$.

The natural question is where is the first pole of the eta function? It was shown in [5] that the eta function of the massless Dirac operator is holomorphic in the half-plane $\operatorname{Re} s > -2$. This agrees with formulae (5.3) and (5.4).

Furthermore, Branson and Gilkey [6] have shown that generically the eta function of the massless Dirac operator has a pole at $s = -2$ and calculated the residue. Consider the covariant rank three tensor $(\nabla_\alpha \operatorname{Ric}_{\beta\nu}) \operatorname{Ric}_\gamma{}^\nu$, where ∇ stands for the covariant derivative and Ric for Ricci curvature (both are understood in terms of the Levi-Civita connection), and antisymmetrize it. This gives a totally antisymmetric covariant rank three tensor which is equivalent to a 3-form. According to [6], the integral of this 3-form over the 3-manifold M gives, up to a particular nonzero constant factor, the residue of the eta function of the massless Dirac operator at $s = -2$.

The fact that the first pole of the eta function of the massless Dirac operator is at $s = -2$ indicates that with a very high accuracy the large (in terms of modulus) positive and negative eigenvalues are distributed in the same way. This, in turn, means that the massless Dirac operator is special and has hidden symmetries encoded in it.

We end this section by highlighting one particular symmetry of the massless Dirac operator. Consider the following antilinear operator acting on 2-columns of complex-valued half-densities:

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mapsto \begin{pmatrix} -\overline{v_2} \\ \overline{v_1} \end{pmatrix} =: \mathrm{C}(v). \tag{8.1}$$

The operator C defined by formula (8.1) is called the *charge conjugation operator*. It is known, see Appendix A in [8], that the linear massless Dirac operator on half-densities L and the antilinear charge conjugation operator C commute:

$$\mathrm{C}(Lv) = L\,\mathrm{C}(v). \tag{8.2}$$

Formula (8.2) implies, in particular, that all eigenvalues of the massless Dirac operator have even multiplicity.

The addition of an electric potential preserves the symmetry (8.2), but the addition of a magnetic covector potential destroys it.

9. Small eigenvalues

Up till now we dealt with large, in terms of modulus, eigenvalues. In this section we will deal with small eigenvalues of the massless Dirac operator.

Suppose that we are working on a connected closed oriented Riemannian 3-manifold and let $\lambda^{(0)}$ be a double eigenvalue of the massless Dirac operator. As explained in the end of the previous section, multiplicity two is the lowest possible. We now perturb the metric, i.e. consider an arbitrary metric $g_{\alpha\beta}(x;\epsilon)$ the components of which are smooth functions of local coordinates x^α, $\alpha = 1,2,3$, and small real parameter ϵ; here we assume that for $\epsilon = 0$ we get the original metric. In this case one can expand the eigenvalue into an asymptotic series in powers of the small parameter ϵ: $\lambda(\epsilon) = \lambda^{(0)} + \lambda^{(1)}\epsilon + \lambda^{(2)}\epsilon^2 + \cdots$ with some constants $\lambda^{(1)}, \lambda^{(2)}, \ldots$. This asymptotic construction was described in Sections 3–5 of [9]. The construction is somewhat nontrivial because we are dealing with a double eigenvalue that cannot split.

We now consider two special cases. In both cases the unperturbed spectrum is symmetric but symmetry is broken under generic perturbations of the metric.

9.1. The 3-torus with standard spin structure. Here the unperturbed metric is assumed to be Euclidean and standard spin structure means that our unperturbed massless Dirac operator can be written as an operator with constant coefficients in the natural 2π-periodic cyclic coordinates parameterizing the 3-torus, see formula (1.1) in [9].

The spectrum of the unperturbed operator is known, see, for example, Appendix B in [8] or Section 1 in [9]. The smallest eigenvalue is the double eigenvalue $\lambda^{(0)} = 0$. It was shown in [9] that

$$\lambda(\epsilon) = \lambda^{(2)}\epsilon^2 + O(\epsilon^3) \quad \text{as } \epsilon \to 0 \tag{9.1}$$

with an explicit expression for the constant $\lambda^{(2)}$. Examination of this explicit expression shows that under a generic perturbation of the metric we get $\lambda^{(2)} \neq 0$ which is an indication of spectral asymmetry.

Furthermore, two special families of metrics were identified in [9] for which the eigenvalue closest to zero, $\lambda(\epsilon)$, can be evaluated explicitly. Formula (9.1) was tested against explicit results for these two families of metrics.

9.2. The 3-sphere. Here the unperturbed metric is obtained by restricting the Euclidean metric from $\mathbb{R}^4$ to $\mathbb{S}^3$. There is no issue with spin structure because for the 3-sphere the spin structure is unique.

The spectrum of the unperturbed operator is known, see, for example, Appendix B in [8]. The smallest, in terms of modulus, eigenvalues are the double eigenvalues $\lambda_+^{(0)} = +\frac{3}{2}$ and $\lambda_-^{(0)} = -\frac{3}{2}$. We get

$$\lambda_\pm(\epsilon) = \pm\frac{3}{2} + \lambda_\pm^{(1)}\epsilon + \lambda_\pm^{(2)}\epsilon^2 + O(\epsilon^3) \quad \text{as } \epsilon \to 0. \tag{9.2}$$

In order to write down the coefficients $\lambda_\pm^{(1)}$ we consider the Riemannian volume $V(\epsilon)$ of our manifold M and expand it in powers of ϵ:

$$V(\epsilon) = V^{(0)} + V^{(1)}\epsilon + O(\epsilon^2) \quad \text{as } \epsilon \to 0, \tag{9.3}$$

where $V^{(0)} = 2\pi^2$ is the volume of the unperturbed 3-sphere. It turns out that

$$\lambda_\pm^{(1)} = \mp\frac{1}{4\pi^2}V^{(1)}. \tag{9.4}$$

Formulae (9.2)–(9.4) tell us that in the first approximation in ϵ spectral symmetry is preserved and the increments of the two eigenvalues closest to zero are determined by the increment of volume. If the volume increases then the moduli of the two eigenvalues closest to zero decrease and in the first approximation in ϵ they decrease in the same way.

Arguing along the lines of [9] one can write down explicit expressions for the constants $\lambda_{\pm}^{(2)}$. Examination of these explicit expressions shows that under a generic perturbation of the metric we get spectral asymmetry in the ϵ^2 terms: $\lambda_{+}^{(2)} + \lambda_{-}^{(2)} \neq 0$. A detailed exposition is provided in [11].

Note that there is a family of metrics for which the two eigenvalues closest to zero, $\lambda_{+}(\epsilon)$ and $\lambda_{-}(\epsilon)$, can be evaluated explicitly. These are the so-called generalized Berger metrics: see Proposition 3.1 in [16] or Definition 4 in [15] or Section 6 in [11].

10. Issue with eigenvalues of the principal symbol

Throughout this paper we assumed that the eigenvalues of the matrix-function $L_{\mathrm{prin}}(x, p)$, the principal symbol of our operator L, are simple for all $(x, p) \in T^*M \setminus \{0\}$. In this section we briefly examine the issues that arise if one drops this assumption.

Ivrii showed that Theorem 4.1 holds without any assumptions on the eigenvalues of the principal symbol, see Theorem 0.1 in [17] or Theorem 0.1 in [18]. However, establishing analogues of Theorems 4.2 and 4.3 without the assumption that the eigenvalues of the principal symbol are simple is not, by any means, straightforward and there are two issues that have to be addressed. These are highlighted in the following two subsections.

10.1. Geometric conditions for the existence of two-term spectral asymptotics. If the multiplicity of eigenvalues of the principal symbol varies as a function of $(x, p) \in T^*M \setminus \{0\}$ then the expectation is that one needs to consider 'generalised' Hamiltonian trajectories, with branching occurring at points in the cotangent bundle where multiplicities of eigenvalues of the principal symbol change.

Ivrii [17, 18] dealt with the issue of variable multiplicities of eigenvalues of the principal symbol by assuming that the set of Hamiltonian trajectories encountering points where multiplicities of eigenvalues of the principal symbol change is, in some sense, small.

G. V. Rozenblyum [24] and later I. Kamotski and M. Ruzhansky [20] considered 'generalised' Hamiltonian trajectories with branching assuming that the principal symbol of the operator is well behaved at points where multiplicities of eigenvalues of the principal symbol change. Here good behaviour is understood as smooth microlocal diagonalisability of the principal symbol plus some conditions on the Poisson brackets of eigenvalues.

10.2. Explicit formulae for the second Weyl coefficient. In the case when the eigenvalues of the principal symbol are not simple explicit formulae for the second Weyl coefficient are not known.

A good starting point for the derivation of such formulae would be the analysis of the case when eigenvalues of the principal symbol have constant multiplicities for all $(x, p) \in T^*M \setminus \{0\}$. Let $L_{\text{prin}}(x, p)$, be our $m \times m$ principal symbol and let l_j, $j = 1, \dots, k$, be the multiplicities of its positive eigenvalues, so that $l_1 + \cdots + l_k = m/2$. Then one can, by analogy with Section 5 of [7], introduce a $\mathrm{U}(l_j)$ connection associated with the jth positive eigenspace of the principal symbol. It is natural to conjecture that the curvature of this $\mathrm{U}(l_j)$ connection will appear in the explicit formula for the second Weyl coefficient.

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