The weak Pleijel theorem with geometric control

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In memory of Y. Safarov

Abstract. Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded open set, and denote by $\lambda_j(\Omega)$, $j \geq 1$, the eigenvalues of the Dirichlet Laplacian arranged in nondecreasing order, with multiplicities. The weak form of Pleijel's theorem states that the number of eigenvalues $\lambda_j(\Omega)$, for which there exists an associated eigenfunction with precisely j nodal domains (Courant-sharp eigenvalues), is finite. The purpose of this note is to determine an upper bound for Courant-sharp eigenvalues, expressed in terms of simple geometric invariants of Ω. We will see that this is connected with one of the favorite problems considered by Y. Safarov.

1. Introduction and main result

We consider the Dirichlet Laplacian $H(\Omega)$ in a bounded open set Ω in $\mathbb{R}^d$. We denote by $\lambda_j(\Omega)$ ($j \in \mathbb{N}^*$) the sequence of eigenvalues arranged in nondecreasing order, with multiplicities. The ground state energy $\lambda_1(\Omega)$ is simply denoted by $\lambda(\Omega)$. We denote by $N(\phi_j) = \phi_j^{-1}(0)$ the *nodal set* of an eigenfunction ϕ_j associated with $\lambda_j(\Omega)$, and by $\mu(\phi_j)$ the number of connected components of $\Omega \setminus N(\phi_j)$ (*nodal domains*).

Courant's nodal domain theorem [8] (1923) says that for any $j \geq 1$, the number $\mu(\phi_j)$ is not greater than j.

An eigenvalue $\lambda_j(\Omega)$ is called *Courant-sharp* if there exists an associated eigenfunction ϕ_j with $\mu(\phi_j) = j$. In contrast with Sturm's theorem in dimension 1, the weak form of Pleijel's theorem [16] (1956) says:

Theorem 1.1. *In dimension* 2, *the number of Courant-sharp eigenvalues of* $H(\Omega)$ *is finite.*

This theorem is the consequence of a more precise theorem (strong Pleijel's theorem):

Theorem 1.2. *In dimension* 2, *for any sequence of spectral pairs* (ϕ_n, λ_n) *of* $H(\Omega)$,

$$\limsup_{n \to +\infty} \frac{\mu(\phi_n)}{n} \leq \frac{4\pi}{\lambda(\mathbb{D}_1)} = \left(\frac{2}{\mathbf{j}}\right)^2 < 1,$$

where $\mathbb{D}_1$ *is the disk of unit area, and* $\mathbf{j}$ *the least positive zero of the Bessel function* J_0.

Remark. Pleijel's theorem extends to bounded domains in $\mathbb{R}^d$, and more generally to compact d-dimensional manifolds with boundary, see Peetre [15], Bérard and Meyer [6]. More precisely for $d \geq 2$, there exists a constant $\gamma(d) < 1$ such that

$$\limsup_{n \to +\infty} \frac{\mu(\phi_n)}{n} \leq \gamma(d).$$

It is interesting to note that the constant $\gamma(d)$ only depends on the dimension and is otherwise independent of the geometry.

In view of Pleijel's theorem, it is a natural question to look for geometric upper bounds for Courant-sharp eigenvalues. The purpose of this note is to give a *geometrically controlled* version of Theorem 1.1. In dimension 2, we prove the following result.

Theorem 1.3. *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded* C^2 *domain. Then, there exists a positive constant* $\beta(\Omega)$ *depending only on the geometry of* Ω, *such that any Courant-sharp eigenvalue* $\lambda_k(\Omega)$ *of* $H(\Omega)$ *satisfies*

$$k\frac{\lambda(\mathbb{D}_1)}{|\Omega|} \leq \lambda_k(\Omega) \leq \beta(\Omega).$$

More precisely, the constant $\beta(\Omega)$ *can be computed in terms of the area* $|\Omega|$, *the perimeter* $\ell(\partial\Omega)$ *of* Ω, *as well as bounds on the curvature of* $\partial\Omega$ *and on the cut-distance*[1] $\varepsilon_0(\Omega)$ *to* $\partial\Omega$.

The result also holds with weaker regularity assumptions. For example, inspection of the proof which uses the results of van den Berg and Lianantonakis [7] gives the following non optimal but more explicit corollary.

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[1] The cut-distance is defined in Section 3.1, equation (12).

Corollary 1.4. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Define the geometric quantity $D(\Omega)$ by*

$$D(\Omega) = \sup_{\varepsilon > 0} \frac{|\{x \in \Omega : d(x) < \varepsilon\}|}{\varepsilon},$$

where $d(x)$ is the distance from x to the boundary of Ω. If $D(\Omega)$ is finite, then any Courant-sharp eigenvalue $\lambda_k(\Omega)$ satisfies

$$\lambda(\mathbb{D}_1)\, k \,\leq\, |\Omega|\, \lambda_k(\Omega) \leq 2 \left(\frac{24\,\pi\,\lambda(\mathbb{D}_1)}{\lambda(\mathbb{D}_1) - 4\pi} \right)^4 \frac{(D(\Omega))^4}{|\Omega|^2}.$$

Observe that the lower and upper bounds are dilation invariant. When Ω is regular, $D(\Omega)$ can be bounded from above by

$$D(\Omega) \leq \max \left\{ \frac{|\Omega|}{\varepsilon_0(\Omega)}, 2\ell(\partial\Omega) \right\}.$$

Remarks. (i) Corollary 1.4 holds as soon as the boundary of Ω has Minkowski dimension 1, see Section 3.2. (ii) The constant $D(\Omega)$ is bigger than the upper Minkowski content. We cannot substitute $D(\Omega)$ with the upper Minkowski content because we need upper bounds on the quantities involved, not only an asymptotic behaviour.

In all the paper, we only consider the Dirichlet problem. It would also be interesting to analyze the Neumann problem in the same spirit. Looking at the proof of Polterovich in [17], the main point would be to obtain a geometric estimate of the number of nodal domains touching the boundary.

Organization of the paper. The paper is organized as follows. In Section 2, we sketch the proofs of Pleijel's theorem, and we explain the idea of how to obtain geometric upper bounds for Courant-sharp eigenvalues. In Section 3 we describe lower bounds on the counting function, using [19] or [7], and we derive upper bounds for the Courant-sharp eigenvalues. In Section 4, we compare the bounds obtained in Section 3 for three very simple examples (the disk, the annulus and the square), and the bounds one can derive for other explicit examples (rectangles, equilateral triangles, etc.).

2. Proofs of Pleijel's theorem

In this section, we sketch the proof of Theorem 1.2 for a domain Ω in $\mathbb{R}^d$. We first introduce some notation.

Let $N_\Omega(\lambda)$ denote the counting function for $H(\Omega)$,

$$N_\Omega(\lambda) = \#\{j \mid \lambda_j(\Omega) < \lambda\}.$$

The counting function can be written as

$$N_\Omega(\lambda) = C_d \, |\Omega| \, \lambda^{d/2} - R(\lambda),$$

where C_d is the Weyl constant, $|\Omega|$ denotes the d-dimensional volume of Ω, and the remainder term $R(\lambda)$ satisfies $R(\lambda) = o(\lambda^{d/2})$ according to Weyl's theorem. The Weyl constant is given by

$$C_d := (2\pi)^{-d} \omega_d,$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$,

$$\omega_d = \pi^{d/2} / \Gamma\Big(\frac{d}{2} + 1\Big). \tag{1}$$

We also denote by $\mathbb{B}_1^d$ the ball of volume 1 in $\mathbb{R}^d$.

To prove Theorem 1.2, we start with the identity

$$\frac{\mu(\phi_n)}{n} \frac{n}{\lambda_n(\Omega)^{d/2}} \frac{\lambda_n(\Omega)^{d/2}}{\mu(\phi_n)} = 1. \tag{2}$$

Applying the Faber–Krahn inequality to each nodal domain of ϕ_n and summing up, we have

$$\frac{\lambda_n(\Omega)^{d/2}}{\mu(\phi_n)} \geq \frac{\lambda(\mathbb{B}_1^d)^{d/2}}{|\Omega|}.$$

Note for later reference that

$$\mu(\phi_n) = n \implies \frac{\lambda_n(\Omega)^{d/2}}{n} \geq \frac{\lambda(\mathbb{B}_1^d)^{d/2}}{|\Omega|}. \tag{3}$$

This gives a necessary condition for $\lambda_n(\Omega)$ to be Courant-sharp, which is (up to the renormalization by the volume) independent of the geometry of Ω.

Taking a subsequence ϕ_{n_i} such that

$$\lim_{i \to +\infty} \frac{\mu(\phi_{n_i})}{n_i} = \limsup_{n \to +\infty} \frac{\mu(\phi_n)}{n},$$

and implementing in (2), we deduce

$$\frac{\lambda(\mathbb{B}_1^d)^{d/2}}{|\Omega|} \limsup_{n \to +\infty} \frac{\mu(\phi_n)}{n} \lim_{n \to +\infty} \frac{n}{N_\Omega(\lambda_n)} \lim_{\lambda \to +\infty} \frac{N_\Omega(\lambda)}{\lambda^{d/2}} \leq 1.$$

Having in mind Weyl's formula, we obtain

$$\limsup_{n \to +\infty} \frac{\mu(\phi_n)}{n} \leq \gamma(d) := \frac{1}{C_d \, \lambda(\mathbb{B}_1^d)^{d/2}}.$$

When $d = 2$, one has $C_2 = \frac{1}{4\pi}$, $\lambda(\mathbb{B}_1^2) = \pi \mathbf{j}^2$, so that $\gamma(2) = \frac{4}{\mathbf{j}^2} < 1$ since $\mathbf{j} \approx 2.40$. More generally, for $d \geq 2$, one has

$$\gamma(d) := \frac{2^{d-2} d^2 \Gamma(d/2)^2}{(j_{(d-2)/2,1})^d},$$

where $j_{\nu,1}$ denotes the first positive zero of the Bessel function J_ν (in particular $j_{0,1} = \mathbf{j}$), and it can be shown, see [6], that

$$\gamma(d) < 1. \tag{4}$$

This proves Theorem 1.2, and Theorem 1.1 follows as well.

Remark. In the case of general Riemannian manifolds, one needs to use an adapted isoperimetric inequality which is valid for domains with small enough volume, see [15, 6].

We now give an alternative proof of Theorem 1.1 which provides a hint on how to bound the Courant-sharp eigenvalues from above.

If λ_n is Courant-sharp, then $\lambda_{n-1} < \lambda_n$ and hence, $n = N_\Omega(\lambda_n) + 1$. Using (3), we obtain

$$\lambda_n \text{ Courant-sharp} \implies N_\Omega(\lambda_n) + 1 = n \leq |\Omega| \left(\frac{\lambda_n(\Omega)}{\lambda(\mathbb{B}_1^d)} \right)^{d/2}. \tag{5}$$

Writing the counting function as

$$N_\Omega(\lambda) = C_d \, |\Omega| \, \lambda^{d/2} - R(\lambda). \tag{6}$$

and plugging this relation into (5), we obtain that

$$\lambda_n(\Omega) \text{ Courant-sharp} \implies F_\Omega(\lambda_n(\Omega)) \leq 0, \tag{7}$$

where the function F_Ω is defined for $\lambda > 0$ by

$$F_\Omega(\lambda) = C_d(1 - \gamma(d)) |\Omega| \lambda^{d/2} - R(\lambda) + 1.$$

By Weyl's theorem, the remainder term satisfies $R(\lambda) = o(\lambda^{d/2})$. Since we have $1 - \gamma(d) > 0$, see (4), the function F_Ω tends to infinity when λ tends to infinity and hence the number of Courant-sharp eigenvalues must be finite.

As a matter of fact, the preceding proof tells us that Courant-sharp eigenvalues must be less than or equal to

$$\inf\{\mu > 0 \mid F_\Omega(\lambda) > 0 \text{ for } \lambda \geq \mu\}.$$

Although this quantity is a geometric invariant associated with Ω, it is not clear how to estimate it in terms of simple geometric invariants, even if we used Ivrii's sharp estimate $R(\lambda) = \mathcal{O}(\lambda^{(d-1)/2})$, see [12, 20]. In order to proceed, it is sufficient to have an *explicit geometric* upper bound $\bar{R}(\lambda)$ of $R(\lambda)$. Indeed, define the function

$$\bar{F}_\Omega(\lambda) = C_d \, (1 - \gamma(d)) \, |\Omega| \, \lambda^{d/2} - \bar{R}(\lambda) + 1. \tag{8}$$

Then, any Courant-sharp eigenvalue $\lambda_k(\Omega)$ must satisfy $\bar{F}_\Omega(\lambda_k(\Omega)) \leq 0$, and hence the inequality

$$\lambda_k(\Omega) \leq \inf\{\mu > 0 \mid \bar{F}_\Omega(\lambda) > 0 \text{ for } \lambda \geq \mu\}. \tag{9}$$

In the next section, we use the explicit upper bounds $\bar{R}(\lambda)$ provided by the papers of Safarov [19] and van den Berg and Lianantonakis [7] to obtain upper bounds on the Courant-sharp eigenvalues in terms of simple geometric invariants.

3. Lower bounds on the counting function and applications to Courant-sharp eigenvalues

In this section, we describe lower bounds on the counting functions derived from [19] or [7], and apply them to bounding the Courant-sharp eigenvalues.

3.1. The approach via Y. Safarov. We implement a result by Y. Safarov [19] (2001) which provides a lower bound for the spectral function on the diagonal, with an explicit control on the remainder term. This estimate is obtained by making use of finite propagation speed for the wave equation, and precise Tauberian theorems.

If $\Omega \subset \mathbb{R}^d$ is an open set, then the spectral function of the Dirichlet Laplacian

$$e(x, x, \lambda) := \frac{1}{2}\Big(\sum_{\lambda_j < \lambda} \phi_j(x)^2 + \sum_{\lambda_j \leq \lambda} \phi_j(x)^2 \Big),$$

satisfies [19, Corollary 3.1]

$$e(x, x, \lambda) \geq C_d \lambda^{d/2} - \frac{2d \, C_d \pi^{-1} v_{m_d}^2}{d(x)} \left(\lambda^{1/2} + \frac{v_{m_d}}{d(x)} \right)^{d-1},$$

for all $x \in \Omega$ and $\lambda > 0$.

Here $d(x)$ is the Euclidean distance to $\partial\Omega$, and v_{m_d} is a universal constant depending only on the dimension.

More precisely, let

$$m_d = \begin{cases} \dfrac{d+1}{2}, & \text{if } d \text{ is odd,} \\[2mm] \dfrac{(d+2)}{2}, & \text{if } d \text{ is even.} \end{cases}$$

Then,

$$v_m = (\tilde{v}_m)^{1/(2m)},$$

where $\tilde{v}_m$ is the ground state energy of the Dirichlet realization of $(-1)^m \frac{d^{2m}}{dt^{2m}}$ on $]-\frac{1}{2}, \frac{1}{2}[$.

Define

$$\tilde{N}_\Omega(\lambda) := \int_\Omega e(x, x, \lambda) \, dx,$$

and let $\epsilon_0(\Omega)$ be the largest number ε with the property that

$$\Omega_\epsilon^b := \{x \in \Omega : d(x) < \epsilon\}$$

is diffeomorphic to $\partial\Omega \times]0, \epsilon[$.

Then, for $0 < \epsilon < \epsilon_0(\Omega)$,

$$\begin{aligned} \tilde{N}_\Omega(\lambda) \geq{}& C_d \, |\Omega| \, \lambda^{d/2} - C_d \, |\Omega_\epsilon^b| \lambda^{d/2} \\ &- 2d \, C_d \, \pi^{-1} v_{m_d}^2 \left(\lambda^{1/2} + \frac{v_{m_d}}{\epsilon} \right)^{d-1} \left(\int_{d(x) > \epsilon} \frac{1}{d(x)} dx \right). \end{aligned}$$

This inequality is also true by semi-continuity for $N_\Omega(\lambda)$.

Writing $N_\Omega(\lambda) = C_d \, |\Omega| \, \lambda^{d/2} - R(\lambda)$ as in (6), we have

$$R(\lambda) \leq C_d \, |\Omega_\epsilon^b| \lambda^{d/2} + 2d \, C_d \, \pi^{-1} v_{m_d}^2 \left(\lambda^{1/2} + \frac{v_{m_d}}{\epsilon} \right)^{d-1} \left(\int_{d(x) > \epsilon} \frac{1}{d(x)} dx \right). \tag{10}$$

We now use our freedom for choosing ϵ. A convenient choice in order to get the right power of λ is to take

$$\epsilon := \alpha(\Omega) \lambda^{-1/2}.$$

Because we need this estimate for any λ in the spectrum of the Laplacian, and actually for $\lambda > \lambda_2(\Omega)$ (because the Courant-sharp property is already established for the two first eigenvalues), we choose

$$\alpha(\Omega) = \epsilon_0(\Omega)\lambda_2(\Omega)^{1/2}.$$

To have more explicit bounds, we could also choose

$$\alpha(\Omega) = \epsilon_0(\Omega)\underline{\lambda_2}(\Omega)^{1/2}$$

where $\underline{\lambda_2}(\Omega)$ is a geometric lower bound of $\lambda_2(\Omega)$ (using the Faber–Krahn inequality or a consequence of the Li–Yau inequality, see below (18) and (19)).

For regular domains, the right-hand side of (10) can be estimated in terms of the geometry of Ω.

For the sake of simplicity, we give the details in the case $d = 2$.

In dimension 2, the above lower bound for $\widetilde{N}_\Omega(\lambda)$ (and $N_\Omega(\lambda)$) reads

$$N_\Omega(\lambda) \geq C_2|\Omega|\lambda - C_2|\Omega^b_{\alpha\lambda^{-1/2}}|\lambda$$
$$- 4C_2\pi^{-1}\nu_2^2\left(1 + \frac{\nu_2}{\alpha}\right)\lambda^{1/2}\left(\int_{\{d(x)>\alpha\lambda^{-1/2}\}} \frac{1}{d(x)}dx\right). \tag{11}$$

When $d = 2$, we have $m_2 = 2$, and we can verify (using the quasimode $(\frac{1}{4} - x^2)^2$ of [19]) that

$$\tilde{\nu}_2 \leq 7 \times 8 \times 9 \leq 2^9,$$

which implies the rough estimate

$$\nu_2 \leq 4 \cdot 2^{1/4} \leq 5.$$

We now assume that $\partial\Omega$ is a smooth submanifold, so that $\partial\Omega$ is the union of p smooth simple closed curves. We write the proof in the case $p = 1$, the general case is similar. Let $c:[0, L] \rightarrow \mathbb{R}^2$ be a parametrization of $\partial\Omega$ by arc-length, with $L := \ell(\partial\Omega)$, the length of the boundary. The associated Frenet frame is $\{\tau(s), \nu(s)\}$. We can assume that the orientation is chosen such that $\nu(s)$ points towards the interior of Ω. The curvature $\kappa(s)$ of the curve is given by the equation $\dot{\tau}(s) = \kappa(s)\nu(s)$. Let $\kappa_-(\Omega)$ denote the infimum of κ over $[0, L]$. Define the map

$$F:[0, L] \times]-\infty, \infty[\longrightarrow \mathbb{R}^2, \quad F(s, t) = c(s) + t\nu(s).$$

We have
$$\partial_s F(s,t) \wedge \partial_t F(s,t) = (1 - t\kappa(s))\tau(s) \wedge \nu(s).$$

The map F is a local diffeomorphism for $|t| < t_+$ with
$$t_+ := \left(\sup_{[0,L]} |\kappa(s)|\right)^{-1}.$$

The injectivity of F is determined by the infimum $\underline{\delta}_+$ of the *cut-distance* $\delta_+(s)$ to the submanifold $\partial\Omega$, where
$$\delta_+(s) := \sup\{t > 0 : t = \text{dist}(F(s,t), \partial\Omega)\}. \tag{12}$$

In this case, we have
$$\epsilon_0(\Omega) = \inf\{t_+, \underline{\delta}_+\}$$

so that F is a diffeomorphism from $[0,L] \times]0, \epsilon_0(\Omega)[$ onto its image (i.e. so that F is both a local diffeomorphism and injective). For $\epsilon < \epsilon_0(\Omega)$, we have
$$|\Omega_\epsilon^b| = \int_0^L \int_0^\epsilon (1 - t\kappa(s)) \, ds \, dt.$$

It follows that
$$\begin{cases} C_2 \, |\Omega_{\alpha\lambda^{1/2}}^b| \, \lambda \leq \beta_1(\Omega)\lambda^{1/2}, \text{ where} \\ \beta_1(\Omega) = \dfrac{1}{4\pi}(1 + \epsilon_0(\Omega)|\kappa_-(\Omega)|)\epsilon_0(\Omega)\lambda_2(\Omega)^{1/2}\ell(\partial\Omega). \end{cases}$$

The third term in the right-hand side of (11) can be written as
$$\begin{cases} \beta_2(\Omega)\lambda^{1/2} \displaystyle\int_{\{d(x)>\alpha\lambda^{-1/2}\}} \dfrac{1}{d(x)}dx, \text{ where} \\ \beta_2(\Omega) := \pi^{-2}v_2^2(1 + v_2\epsilon_0(\Omega)^{-1}\lambda_2(\Omega)^{-\frac{1}{2}}). \end{cases}$$

Write
$$\int_{\{d(x)>\alpha\lambda^{-1/2}\}} \frac{1}{d(x)}dx = \int_{\{d(x)>\epsilon_0(\Omega)\}} \frac{1}{d(x)}dx + \int_0^L \int_{\alpha\lambda^{-1/2}}^{\epsilon_0(\Omega)} \frac{(1 - t\kappa(s))}{t}ds\,dt$$

We can estimate the second integral in the right-hand side as we did above. The first integral can be estimated from above by $|\Omega|/\varepsilon_0(\Omega)$. It follows that there exist positive constants $\beta_3(\Omega)$ and $\beta_4(\Omega)$ such that, for all $\lambda > \lambda_2(\Omega)$, $N_\Omega(\lambda) = \frac{|\Omega|}{4\pi}\lambda - R(\lambda)$, with

$$R(\lambda) \leq \bar{R}(\lambda) = \beta_3(\Omega)\lambda^{1/2}\ln\left(\frac{\lambda}{\lambda_2(\Omega)}\right) + \beta_4(\Omega)\lambda^{1/2}. \tag{13}$$

Note that the constants only depend on the geometry of the domain Ω. More precisely, the constants can be computed in terms of $|\Omega|, \ell(\partial\Omega), \kappa_-(\Omega), \epsilon_0(\Omega)$, and $\lambda_2(\Omega)$.

Remarks. (i) The preceding proof shows that one can alternatively estimate the constants in terms of $|\Omega|, \ell(\partial\Omega), \epsilon_0(\Omega), \lambda_2(\Omega)$, and the number of holes of the domain (through the integral $\int_{\partial\Omega} \kappa$).

(ii) In higher dimensions, one can state a similar result in which the curvature κ of the curve is replaced by the mean curvature h of the hypersurface $\partial\Omega$. For this purpose, one uses the Heintze-Karcher comparison theorem [9].

Applying (8) and (9), we obtain that any Courant-sharp eigenvalue $\lambda_k(\Omega)$ satisfies

$$f_\Omega(\lambda_k) \leq 0, \tag{14}$$

where the function f_Ω is defined for $\mu > \lambda_2(\Omega)$ by

$$f_\Omega(\mu) := \frac{\lambda(\mathbb{D}_1) - 4\pi}{4\pi\lambda(\mathbb{D}_1)} |\Omega|\mu - \beta_3(\Omega)\mu^{1/2}\ln\left(\frac{\mu}{\lambda_2}\right) - \beta_4(\Omega)\mu^{1/2} + 1.$$

Since $\lambda(\mathbb{D}_1) > 4\pi$, see (4), the coefficient of the term μ in the expression of f_Ω is positive, so that the function tends to infinity when μ tends to infinity. Hence $I_\Omega := f_\Omega^{-1}(]-\infty, 0])$ is either empty or bounded from above.
Define

$$\beta_S(\Omega) = \max\{\lambda_2(\Omega), \beta_0(\Omega)\},$$

where $\beta_0(\Omega)$ is the supremum of I_Ω if I_Ω is non empty and 0 otherwise. From equation (14) we conclude that

$$\lambda_k(\Omega) \text{ Courant-sharp} \implies \lambda_k(\Omega) \leq \beta_S(\Omega).$$

We have proved Theorem 1.3.

Starting from the inequality $f_\Omega(\lambda_k) \leq 0$ in the above proof, we conclude that any Courant-sharp eigenvalue λ_k satisfies

$$A_2 |\Omega|\lambda_k^{1/2} \leq \beta_3(\Omega)\ln\frac{\lambda_k}{\lambda_2} + \beta_4(\Omega),$$

where $A_2 = \frac{1}{4\pi} - \frac{1}{\lambda(\mathbb{D}_1)}$. Using the inequality $\ln\frac{\mu}{\lambda_2} \leq 4\left(\frac{\mu}{\lambda_2}\right)^{1/4}$ which holds for any $\mu \geq \lambda_2$, we obtain the following more explicit bound.

Corollary 3.1. *In dimension* 2, *any Courant-sharp eigenvalue* $\lambda_k(\Omega)$ *of* $H(\Omega)$ *satisfies*

$$\lambda_k(\Omega) \leq \max \left\{ \lambda_2(\Omega), \left(\frac{16\pi \lambda(\mathbb{D}_1)}{\lambda(\mathbb{D}_1) - 4\pi} \right)^4 \frac{(\beta_3(\Omega) + \beta_4(\Omega))^4}{|\Omega|^4 \lambda_2(\Omega)} \right\}. \tag{15}$$

Remarks. (i) For the unit disk, the bound (15) is sharper than Corollary 1.4, see Section 4. (ii) Pólya's conjecture for Dirichlet eigenvalues (see [18]) does not go in the right direction. Indeed lower bounds on the Dirichlet eigenvalues correspond to upper bounds on $N(\lambda)$. This would be good for Neumann eigenvalues, but in this case, there are other problems, see [16] and more recently [17].

3.2. Approach via van den Berg–Lianantonakis. Prior to Y. Safarov, van den Berg and Lianantonakis have given lower bounds for the counting function $N_\Omega(\lambda)$ depending on the Minkowski dimension of $\partial\Omega$. When this dimension is $(d-1)$, they prove [7, Theorem 2.1] that if

$$\lambda \geq 4|\Omega|^{-2/d}, \tag{16}$$

then

$$N(\lambda) \geq C_d|\Omega|\,\lambda^{d/2} - 3D(\Omega)\lambda^{(d-1)/2}\log((2|\Omega|)^{2/d}\,\lambda), \tag{17}$$

where the geometric constant $D(\Omega)$ is defined by

$$D(\Omega) := \sup_\epsilon \frac{|\Omega_\epsilon^b|}{\epsilon}.$$

To apply (17) to Pleijel's theorem, one needs to compare condition (16) with the condition $\lambda > \lambda_2(\Omega)$. One can for example observe that the Faber–Krahn inequality applied to the second eigenvalue gives (see [1] or (3) for $d = 2$)

$$\lambda_2(\Omega) \geq (2\omega_d)^{2/d}|\Omega|^{-2/d}\,j_{d/2-1,1}^2. \tag{18}$$

For $d = 2$, since $2\pi j_{0,1}^2 > 4$, the condition $\lambda > \lambda_2(\Omega)$ implies (16). For $d \geq 2$, we can use the following lower bound for $\lambda_2(\Omega)$ which is a consequence of the Li–Yau inequality, see [1, eq. (11.5)],

$$\lambda_2(\Omega) > \frac{d}{d+2} \frac{4\pi^2 2^{2/d}}{(\omega_d|\Omega|)^{2/d}}. \tag{19}$$

Hence it is enough to verify that

$$\frac{d}{d+2} \frac{4\pi^2 2^{2/d}}{(\omega_d)^{2/d}} \geq 4,$$

which is easy to establish. Indeed, using (1), we obtain

$$\frac{d}{d+2} \pi \, 2^{2/d} \, \Gamma\left(\frac{d}{2} + 1\right)^{2/d} \geq 1,$$

which follows from the inequality $\frac{d}{d+2}\pi \geq 1$ for $d \geq 1$.

Assuming $d = 2$ for the sake of simplicity, and using (7) together with (17), we obtain that any Courant-sharp eigenvalue $\lambda_k(\Omega)$, with $\lambda_k > \lambda_2$, satisfies $g_\Omega(\lambda_k) \leq 0$, where g_Ω is defined by

$$g_\Omega(\mu) = \left(\frac{1}{4\pi} - \frac{1}{\lambda(\mathbb{D}_1)}\right)|\Omega|\,\lambda - 3D(\Omega)\,\mu^{1/2}\ln(2|\Omega|\mu) + 1,$$

for $\mu \geq \lambda_2(\Omega)$. Define $\beta_1(\Omega)$ to be 0 if $g_\Omega(\mu) \geq 0$, and $\sup\{\mu > \lambda_2 : g_\Omega(\mu) \leq 0\}$ otherwise, and define

$$\beta_B(\Omega) := \max\{\lambda_2(\Omega), \beta_1(\Omega)\}.$$

Then,

$$\lambda_k(\Omega) \text{ Courant-sharp} \implies \lambda_k(\Omega) \leq \beta_B(\Omega).$$

This proves Theorem 1.3 using the lower bound for the counting function provided by [7].

From the inequality $g_\Omega(\lambda_k) \leq 0$ in the preceding proof, we have that any Courant-sharp eigenvalue $\lambda_k(\Omega)$ satisfies the inequality

$$\left(\frac{1}{4\pi} - \frac{1}{\lambda(\mathbb{D}_1)}\right)|\Omega|\,\lambda_k^{1/2} - 3D(\Omega)\ln(2|\Omega|\lambda_k) \leq 0,$$

for $\lambda_k \geq \lambda_2$. Using the inequality

$$\ln \mu \leq 2\,\mu^{1/4} \quad \text{for } \mu \geq 16,$$

and the fact that $2|\Omega|\lambda_k > 16$ (Faber–Krahn), we obtain the more explicit bound given in Corollary 1.4.

As kindly communicated by M. van den Berg, in dimension 2, when Ω is sufficiently regular, the geometric invariant $D(\Omega)$ can be bounded from above by

$$D(\Omega) \leq \max\left(\frac{|\Omega|}{\epsilon_0(\Omega)}, \ell(\partial\Omega) + \pi\,\varepsilon_0(\Omega)\,h(\Omega)\right),$$

where $h(\Omega)$ is the number of holes of Ω, or by

$$D(\Omega) \leq \max\left(\frac{|\Omega|}{\epsilon_0(\Omega)}, 2\ell(\partial\Omega)\right).$$

4. Examples and particular cases

4.1. Examples. In some 2-dimensional cases, it is possible to compute the upper bounds for Courant-sharp eigenvalues arising from the preceding sections explicitly. Consider the following domains:

$$\Omega_1 = B(0,1), \qquad \text{the unit disc in } \mathbb{R}^2,$$

$$\Omega_2 = B(0,1) \setminus B(0,a), \quad 0 < a < 1, \quad \text{the annulus } A(0,a,1) \subset \mathbb{R}^2,$$

$$\Omega_3 = {]}0,\pi[\times {]}0,\pi[, \qquad \text{the square in } \mathbb{R}^2 \text{ with side } \pi.$$

For the unit disc, one finds that $\beta_S(\Omega_1) \approx 7.1 \cdot 10^6$ and $\beta_B(\Omega_1) \approx 2.1 \cdot 10^7$.

For the annulus, one finds that $\beta_B(\Omega_2) \approx 4.2 \cdot 10^8$ when $a = 0.75$, and $\beta_B(\Omega_2) \approx 4 \cdot 10^7$ when $a = 0.25$. This indicates that the cut-distance to the boundary does matter in the upper bound on Courant-sharp eigenvalues.

For the square with side π, one finds that $\beta_B(\Omega_3) \approx 5.9 \cdot 10^6$. It turns out that this bound is much bigger than the bound which is deduced in the next sub-section, namely 51.

This is not surprising. The general lower bounds for the counting functions used in the preceding sections, equations (13) and (17), are worse than the sharp 2-dimensional estimate $R(\lambda) = \mathcal{O}(\lambda^{1/2})$, see [12], by a $\ln(\lambda)$ factor. On the other-hand, the estimate (22) has the right powers, and almost the right second constant.

Generally speaking one should therefore expect that the bounds $\beta_S(\Omega)$ and $\beta_B(\Omega)$ are not sharp.

4.2. Particular cases. As already mentioned, improved Weyl's formulas with control of the remainder which are only asymptotic are not sufficient for an explicit version of Pleijel's theorem. We nevertheless mention for comparison a formula due to V. Ivrii in 1980 (cf [11, Chapter XXIX, Theorem 29.3.3 and Corollary 29.3.4]) which reads

$$N(\lambda) = \frac{\omega_d}{(2\pi)^d}|\Omega|\,\lambda^{d/2} - \frac{1}{4}\frac{\omega_{d-1}}{(2\pi)^{d-1}}\,|\partial\Omega|\,\lambda^{(d-1)/2} + r(\lambda), \qquad (20)$$

where $r(\lambda) = \mathcal{O}(\lambda^{(d-1)/2})$ in general, but can also be shown to be $o(\lambda^{(d-1)/2})$ if the boundary is C^∞, and under some generic conditions on the geodesic billiards (the measure of periodic trajectories should be zero). For piecewise smooth boundaries, see [21]. The second term is meaningful in this case only.

Formula (20) is also established for irrational rectangles as a very special case in [12], but more explicitly in [13] without any assumption of irrationality.

See also [3] for some 2-dimensional domains with negative curvature. We do not discuss here the case of "rough" boundaries which was in particular analyzed by Netrusov et Safarov in [14] (and references therein).

Note that when $d = 2$, the second term in (20) is

$$W_2(\lambda) := -\frac{1}{4\pi}|\partial\Omega|\,\lambda^{1/2}. \tag{21}$$

The Dirichlet (and Neumann) eigenvalues are explicitly given for few domains. In dimension 2 these domains include the rectangles, the right-angled isosceles triangle, the equilateral triangle and the hemiequilateral triangle. In these cases, estimating the counting function amounts to estimating the number of points with integer coordinates inside some ellipse (these domains are obtained as quotient of a torus). The estimates which are obtained in this manner are compatible with Weyl's two terms asymptotic formula (20)), involving the area of the domain and the length of it boundary. Similarly, in higher dimensions, one can explicitly describe the Dirichlet (and Neumann) eigenvalues of the fundamental domains of crystallographic affine Weyl groups, [2]. As far as the asymptotic estimate is concerned, this is possible because the remainder term in Weyl's estimate has order $\lambda^{(d-2)/2+1/(d+1)}$ for a d-dimensional torus.

Rectangle. Following (and improving) a remark in a course of R. Laugesen [4], one has a lower bound of $N(\lambda)$ in the case of the rectangle

$$\mathcal{R} = \mathcal{R}(a, b) := (0, a\pi) \times (0, b\pi),$$

which can be expressed in terms of area and perimeter. One can indeed observe that the area of the intersection of the ellipse $\{\frac{(x+1)^2}{a^2} + \frac{(y+1)^2}{b^2} < \lambda\}$ with $\mathbb{R}^+ \times \mathbb{R}^+$ is a lower bound for $N(\lambda)$.

The formula reads

$$N_\mathcal{R}(\lambda) > \frac{1}{4\pi}|\mathcal{R}|\,\lambda - \frac{1}{2\pi}|\partial\mathcal{R}|\,\sqrt{\lambda} + 1, \quad \text{for } \lambda \geq \frac{1}{a^2} + \frac{1}{b^2}. \tag{22}$$

Here we can observe that the second term is $2\,W_2(\lambda)$, see (21).

Equilateral triangle ([5]). We consider the equilateral triangle with side 1:

$$N_\mathcal{T}(\lambda) \geq \frac{\sqrt{3}}{4}\frac{\lambda}{4\pi} - \frac{3}{2\pi}\sqrt{\lambda} + 1.$$

Again we observe that the second term is $2\,W_2(\lambda)$, see (21).

Right-angled isosceles triangle ([5]). Let $\mathcal{B}_\pi$ denote the right-angled isosceles triangle,

$$\mathcal{B}_\pi = \{(x, y) \in \,]0, \pi[^2 |\ y < x\},$$

$$N_\mathcal{B}(\lambda) \geq \frac{\pi\lambda}{8} - \frac{(4 + \sqrt{2})\sqrt{\lambda}}{4} - \frac{1}{2}.$$

The cube ([10]). For the cube $]0, \pi[\,^3$, we have, for $\lambda \geq 3$,

$$N(\lambda) > \frac{\pi}{6}\lambda^{3/2} - \frac{3\pi}{4}\lambda + 3\sqrt{\lambda - 2} - 1.$$

Added in proof. We point out the following recent paper. M. van den Berg and K. Gittins, On the number of Courant-sharp Dirichlet eigenvalues. *J. Spectr. Theory* **6** (2016), no. 4, 735–745.

Concerning the Neumann problem, we point out the following recent paper. C. Léna, Pleijel's nodal domain theorem for Neumann eigenfunctions. Preprint 2016. arXiv:1609.02331 [math.AP]

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