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On the number of Courant-sharp Dirichlet eigenvalues

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In memory of Yuri Safarov

Abstract. We consider arbitrary open sets Ω in Euclidean space with finite Lebesgue measure, and obtain upper bounds for (i) the largest Courant-sharp Dirichlet eigenvalue of Ω, (ii) the number of Courant-sharp Dirichlet eigenvalues of Ω. This extends recent results of P. Bérard and B. Helffer.

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1. Introduction

Let Ω be an open set in Euclidean space $\mathbb{R}^m$ with finite Lebesgue measure $|\Omega|$ and boundary $\partial\Omega$. We denote the spectrum of the Dirichlet Laplacian acting in $L^2(\Omega)$ by $\lambda_1(\Omega) \le \lambda_2(\Omega) \le \lambda_3(\Omega) \le \cdots$ taking the multiplicities of these eigenvalues into account. We define the counting function for Ω by

$$N_\Omega(\lambda) = \sharp\{n \in \mathbb{N}\colon \lambda_n(\Omega) < \lambda\}.$$

Weyl's law asserts that

$$N_\Omega(\lambda) = \frac{\omega_m}{(2\pi)^m}|\Omega|\lambda^{m/2} + o(\lambda^{m/2}), \quad \lambda \to \infty, \tag{1}$$

where ω_m is the measure of a ball $\mathcal{B}_m$ with radius 1 in $\mathbb{R}^m$. We refer to Theorem 2 in [16] for a proof of (1) in this generality. For a proof of Weyl's law with a

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non-trivial remainder estimate for Ω open, bounded and connected we refer to Theorem 1.8 in [12].

Let $\{\varphi_{1,\Omega}, \varphi_{2,\Omega}, \dots\}$ be an orthonormal basis in the Sobolev space $H_0^1(\Omega)$ of eigenfunctions corresponding to the Dirichlet eigenvalues. These eigenfunctions satisfy the Dirichlet boundary conditions in the usual trace sense. Let $\nu(\varphi_{n,\Omega})$ denote the number of nodal domains of $\varphi_{n,\Omega}$. Then Pleijel's theorem ([13]) states that

$$\limsup_{n\to\infty} \frac{\nu(\varphi_{n,\Omega})}{n} \le \gamma_m,$$

where

$$\gamma_m = \frac{(2\pi)^m}{\omega_m^2}(\lambda_1(\mathcal{B}_m))^{-m/2} < 1. \tag{2}$$

It is known that Pleijel's bound is not sharp. See [7], [18], and [14].

We say that $\lambda_n(\Omega)$ is Courant-sharp if $\nu(\varphi_{n,\Omega}) = n$. Courant's nodal domain theorem asserts that $\nu(\varphi_{n,\Omega}) \le n$. Courant's original proof in [8] was for the planar case. This has been subsequently stated and proved in a Riemannian manifold setting in [3]. See also [13]. Pleijel's theorem implies that for a given Ω the number of Courant-sharp Dirichlet eigenvalues is finite. Using results of [6] and [17], Bérard and Helffer, [1], obtained an upper bound for the largest Courant-sharp Dirichlet eigenvalue if Ω is bounded and has smooth boundary $\partial\Omega$.

This paper concerns arbitrary open sets in $\mathbb{R}^m$ with finite Lebesgue measure. The proofs of Courant's theorem in [8], [13], and [3] all use the fact that a restriction of an eigenfunction to a nodal domain U is the first Dirichlet eigenfunction on U. This is immediate if $(\partial\Omega) \cap (\partial U)$ is sufficiently regular. The above fact holds without that regularity requirement. See for example Theorem 1.1 in [9].

Our main result, Theorem 1 below, is for open sets Ω in $\mathbb{R}^m$ with finite Lebesgue measure. We obtain (i) an upper bound for the largest Dirichlet eigenvalue of Ω which is Courant-sharp, and (ii) an upper bound for the number of Courant-sharp eigenvalues of Ω. For $A \subset \mathbb{R}^m$, $A \ne \emptyset$ let

$$d(x, A) = \inf\{|x - y| : y \in A\}.$$

For $\epsilon \ge 0$ and $|\Omega| < \infty$ we define

$$\mu_\Omega(\epsilon) = |\{x \in \Omega : d(x, \partial\Omega) < \epsilon\}|,$$

and

$$\epsilon(\Omega) = \inf\{\epsilon : \mu_\Omega(\epsilon) \ge 2^{-1}(1 - \gamma_m)|\Omega|\}. \tag{3}$$

We denote the number of Courant-sharp eigenvalues of Ω by $\mathfrak{C}(\Omega)$.

Theorem 1. *Let Ω be an open set in $\mathbb{R}^m$ with finite Lebesgue measure. We have the following.*

(i) *If $\lambda_n(\Omega)$ is Courant-sharp, then*

$$\lambda_n(\Omega) \le \Big(\frac{2\pi m^2}{(1-\gamma_m)\epsilon(\Omega)}\Big)^2. \tag{4}$$

(ii)

$$\mathfrak{C}(\Omega) \le \frac{\omega_m}{(1-\gamma_m)^m}(m^3(m+2))^{m/2}\frac{|\Omega|}{\epsilon(\Omega)^m}. \tag{5}$$

(iii) *If $n \in \mathbb{N}, n > \frac{\omega_m}{(1-\gamma_m)^m}(m^3(m+2))^{m/2}\frac{|\Omega|}{\epsilon(\Omega)^m}$, then $\lambda_n(\Omega)$ is not Courant-sharp.*

In Section 2 below we prove Theorem 1. In Section 3 we analyse some examples including the von Koch snowflake.

2. Proof of Theorem 1

Suppose $\lambda_n(\Omega)$ is Courant-sharp with eigenfunction $\varphi_{n,\Omega}$. Let $U_1, \dots, U_n$ be the nodal domains of $\varphi_{n,\Omega}$ so that $\lambda_n(\Omega) = \lambda_1(U_1) = \cdots = \lambda_1(U_n)$. Without loss of generality we may assume that $|U_1| \le |U_2| \le \cdots \le |U_n|$. Hence $|U_1| \le |\Omega|/n$. By Faber–Krahn we have that

$$\lambda_n(\Omega) = \lambda_1(U_1) \ge \lambda_1(\mathcal{B}_m)\Big(\frac{n\omega_m}{|\Omega|}\Big)^{2/m}.$$

It follows that, since $\lambda_{n-1}(\Omega) < \lambda_n(\Omega)$,

$$\begin{aligned}(\lambda_n(\Omega))^{m/2} &\ge (\lambda_1(\mathcal{B}_m))^{m/2}\frac{n\omega_m}{|\Omega|}\\ &\ge (\lambda_1(\mathcal{B}_m))^{m/2}\frac{\omega_m}{|\Omega|}(n-1)\\ &= (\lambda_1(\mathcal{B}_m))^{m/2}\frac{\omega_m}{|\Omega|}N_\Omega(\lambda_n(\Omega)).\end{aligned}$$

This gives that

$$\frac{\omega_m}{(2\pi)^m}(1-\gamma_m)|\Omega|(\lambda_n(\Omega))^{m/2} \le R_\Omega(\lambda_n(\Omega)), \tag{6}$$

where $R_\Omega\colon \mathbb{R}^+ \to \mathbb{R}$ is defined by

$$R_\Omega(\lambda) = \frac{\omega_m}{(2\pi)^m}|\Omega|\lambda^{m/2} - N_\Omega(\lambda). \tag{7}$$

See (15) and (16) in [1]. Below we obtain an upper bound for $R_\Omega(\lambda)$. Let $\epsilon > 0$ be arbitrary. Consider the collection $\mathfrak{M}_\epsilon$ of open cubes of measure ϵ^m with vertices in the set of m-tuples $\{\mathbb{Z}\epsilon, \dots, \mathbb{Z}\epsilon\}$. Let $M_\Omega(\epsilon)$ be the number of open cubes of side-length ϵ in $\mathfrak{M}_\epsilon$ which are contained in Ω,

$$M_\Omega(\epsilon) = \sharp\{N \in \mathfrak{M}_\epsilon : N \subset \Omega\}.$$

We have that

$$|\Omega| - M_\Omega(\epsilon)\epsilon^m \ge 0. \tag{8}$$

In order to obtain an upper bound for the left hand-side of (8), we let $x \in \Omega$. If $d(x, \partial\Omega) > m^{1/2}\epsilon$, then x belongs to an open ϵ-cube in $\mathfrak{M}_\epsilon$ contained in Ω. Hence the measure of the set which is not covered by the ϵ-cubes in $\mathfrak{M}_\epsilon$ that are entirely contained in Ω is bounded from above by $\mu_\Omega(m^{1/2}\epsilon)$. So

$$|\Omega| - M_\Omega(\epsilon)\epsilon^m \le \mu_\Omega(m^{1/2}\epsilon). \tag{9}$$

By Dirichlet bracketing (see [15]), we have that

$$N_\Omega(\lambda) \ge M_\Omega(\epsilon) N_{C_\epsilon}(\lambda), \tag{10}$$

where C_ϵ is an open cube in $\mathbb{R}^m$ with side-length ϵ. The following standard estimate is attributed to Gauss:

$$\begin{aligned} N_{C_\epsilon}(\lambda) &= \sharp\Big\{(k_1, \dots, k_m) \in \mathbb{N}^m : \sum_{i=1}^m k_i^2 < \pi^{-2}\epsilon^2\lambda\Big\} \\ &\ge \frac{\omega_m}{2^m}(\pi^{-1}\epsilon\lambda^{1/2} - m^{1/2})_+^m \\ &\ge \frac{\omega_m}{(2\pi)^m}\epsilon^m\lambda^{m/2}\Big(1 - \frac{\pi m^{3/2}}{\epsilon\lambda^{1/2}}\Big), \end{aligned} \tag{11}$$

where $+$ denotes the positive part. By (10) and (11),

$$\begin{aligned} N_\Omega(\lambda) &\ge M_\Omega(\epsilon) N_{C_\epsilon}(\lambda) \\ &\ge M_\Omega(\epsilon)\frac{\omega_m}{(2\pi)^m}\epsilon^m\lambda^{m/2} - M_\Omega(\epsilon)\frac{\omega_m}{(2\pi)^m}\pi m^{3/2}\epsilon^{m-1}\lambda^{(m-1)/2} \\ &= \frac{\omega_m}{(2\pi)^m}|\Omega|\lambda^{m/2} - (|\Omega| - M_\Omega(\epsilon)\epsilon^m)\frac{\omega_m}{(2\pi)^m}\lambda^{m/2} \\ &\quad - M_\Omega(\epsilon)\frac{\omega_m}{(2\pi)^m}\pi m^{3/2}\epsilon^{m-1}\lambda^{(m-1)/2}. \end{aligned} \tag{12}$$

We bound the second and third terms in the right hand-side of (12), using (9) and (8) respectively. This then gives, by (7), that

$$R_\Omega(\lambda) \le \frac{\omega_m}{(2\pi)^m}\mu_\Omega(m^{1/2}\epsilon)\lambda^{m/2} + \frac{\pi m^{3/2}\omega_m}{(2\pi)^m}\frac{|\Omega|\lambda^{(m-1)/2}}{\epsilon}. \tag{13}$$

By (6) and (13) we have that if $\lambda_n(\Omega)$ is Courant-sharp, then

$$\frac{\omega_m}{(2\pi)^m}(1-\gamma_m)|\Omega|(\lambda_n(\Omega))^{m/2} \le \frac{\omega_m}{(2\pi)^m}\mu_\Omega(m^{1/2}\epsilon)(\lambda_n(\Omega))^{m/2} + \frac{\pi m^{3/2}\omega_m}{(2\pi)^m}\frac{|\Omega|(\lambda_n(\Omega))^{(m-1)/2}}{\epsilon}. \tag{14}$$

We now choose ϵ such that the second term in the right hand-side of (14) equals half of the left hand-side of (14). That is

$$\epsilon = 2\pi m^{3/2}(1-\gamma_m)^{-1}(\lambda_n(\Omega))^{-1/2}. \tag{15}$$

By (14) and the choice of ϵ in (15) we have that if $\lambda_n(\Omega)$ is Courant-sharp, then

$$2^{-1}(1-\gamma_m)|\Omega| \le \mu_\Omega(2\pi m^2(1-\gamma_m)^{-1}(\lambda_n(\Omega))^{-1/2}). \tag{16}$$

Since $\epsilon \mapsto \mu_\Omega(\epsilon)$ is continuous and onto $[0, |\Omega|]$, the infimum in (3) is over a non-empty set which is bounded from below and therefore exists. So if $\lambda_n(\Omega)$ is Courant-sharp, then, by (3) and (16), $\frac{2\pi m^2}{(1-\gamma_m)(\lambda_n(\Omega))^{1/2}} \ge \epsilon(\Omega)$. This proves Theorem 1(i).

By [11], we also have that

$$\lambda_n(\Omega) \ge \frac{m}{m+2}\frac{(2\pi)^2}{\omega_m^{2/m}}\left(\frac{n}{|\Omega|}\right)^{2/m}. \tag{17}$$

This, together with (4), implies (5) and proves Theorem 1(ii).

To prove Theorem 1(iii) we just note that by (17),

$$\max\left\{n \in \mathbb{N}: \lambda_n(\Omega) \le \left(\frac{2\pi m^2}{(1-\gamma_m)\epsilon(\Omega)}\right)^2\right\} \le \frac{\omega_m}{(1-\gamma_m)^m}(m^3(m+2))^{m/2}\frac{|\Omega|}{\epsilon(\Omega)^m}.$$

□

We note that if we were to use the lower bounds for the counting function from Section 2 in [6], then we would have to assume a weak integrability condition on μ_Ω of the form $\int \epsilon^{-1}\, d\mu_\Omega(\epsilon) < \infty$. Such an integrability condition may fail if the interior Minkowski dimension of $\partial\Omega$ is equal to m. The procedure above avoids this integrability condition.

3. Examples

In this section we analyse three examples where explicit computations seem out of reach.

Example 1. Let Ω be an open, bounded, convex set in $\mathbb{R}^m$. Let $\mathcal{H}^{m-1}(\partial\Omega)$ denote the $(m-1)$-dimensional Hausdorff measure of $\partial\Omega$. Then

$$\mathfrak{C}(\Omega) \le \frac{\omega_m}{(1-\gamma_m)^{2m}}(4m^3(m+2))^{m/2}\frac{(\mathcal{H}^{m-1}(\partial\Omega))^m}{|\Omega|^{m-1}}. \tag{18}$$

Proof. By convexity of Ω, we have that

$$\mu_\Omega(\epsilon) \le \mathcal{H}^{m-1}(\partial\Omega)\epsilon.$$

By (3),

$$\epsilon(\Omega) \ge 2^{-1}(1-\gamma_m)\frac{|\Omega|}{\mathcal{H}^{m-1}(\partial\Omega)}, \tag{19}$$

and (18) follows from Theorem 1 and (19). □

It was shown in [10] that only the first, second and fourth Dirichlet eigenvalues for $\mathcal{B}_2$ are Courant-sharp. Hence $\mathfrak{C}(\mathcal{B}_2) = 3$, and the largest Courant-sharp eigenvalue for $\mathcal{B}_2$ is equal to $j_{0,2}^2$. Here $j_{0,2} \asymp 5.520..$ is the second positive zero of the Bessel function J_0. A straightforward computation using (4) and (19) shows that the largest Courant-sharp eigenvalue of $\mathcal{B}_2$ is strictly less than $1.2 \cdot 10^6$. This compares well with the bound $7.1 \cdot 10^6$ obtained in [1]. For the unit square $\mathcal{C}_2$ it is known ([13] and [2]) that only the first, second and fourth Dirichlet eigenvalues are Courant-sharp. Hence $\mathfrak{C}(\mathcal{C}_2) = 3$, and the largest Courant-sharp eigenvalue for $\mathcal{C}_2$ is equal to $8\pi^2$. Using (4) and (19), we have that the largest Courant-sharp eigenvalue of the unit square is strictly less than $4.5 \cdot 10^6$, whereas [1] gives a bound $5.9 \cdot 10^6$. These examples illustrate that the bounds obtained in Theorem 1 are very crude.

The second example is a von Koch snowflake K with similarity ratio $\frac{1}{3}$. We recall its construction. Let the basic square (generation 0) in K have side-length 1. The first generation consists of 4 squares with side-length $\frac{1}{3}$ each attached symmetrically to the basic square. Proceeding inductively we have that the j'th generation in K, $j \in \mathbb{N}$, consists of $4 \cdot 5^{j-1}$ squares with side-length 3^{-j}. We let K be the interior of its closure. Then K is connected, has Lebesgue measure $|K| = 2$, and both the Hausdorff dimension of ∂K and the interior Minkowski dimension of ∂K are equal to $\log 5/\log 3$. See Figure 1, and [4] for further details.

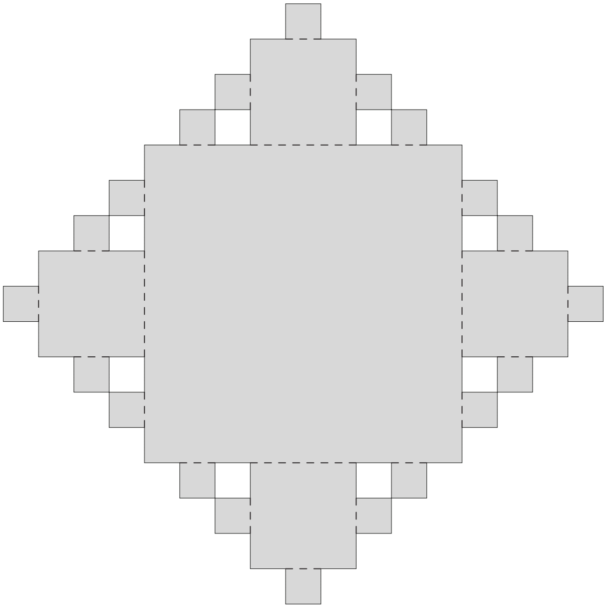

Figure 1. The first two generations of K

Example 2. Let K be the von Koch snowflake generated by the unit square and similarity ratio $\frac{1}{3}$. Then

$$\mathfrak{C}(K) \leq 15 \cdot 10^7. \tag{20}$$

Proof. By Theorem 1, (2), and $|K| = 2$, we find that

$$\mathfrak{C}(K) \leq \frac{64\pi j_0^4}{(j_0^2 - 4)^2} \epsilon(K)^{-2}, \tag{21}$$

where we have used that

$$\lambda_1(\mathcal{B}_2) = j_0^2,$$

where $j_0 = 2.405...$ is the first positive zero of the Bessel function J_0. It remains to find a lower bound for $\epsilon(K)$. We obtain an upper bound for $\mu_\Omega(\epsilon)$ by adding all edges between squares of different generations. This gives a disjoint union of 1 unit square and $4 \cdot 5^{j-1}$ squares with side-lengths 3^{-j}, $j \in \mathbb{N}$. Let $\epsilon < \frac{1}{18}$, and let $J \in \mathbb{N}$ be such that

$$J < \frac{\log\left(\frac{1}{2\epsilon}\right)}{\log 3} \leq J + 1.$$

Then $J \geq 2$. The contribution to the upper bound for $\mu_\Omega(\epsilon)$ from the squares in generations $1, \dots, J-1$ is bounded from above by

$$\left(4 + 16\sum_{j=1}^{J-1} 5^{j-1}3^{-j}\right)\epsilon \leq \frac{24\epsilon}{5}\left(\frac{5}{3}\right)^J \leq \frac{48}{5}2^{-\frac{\log 5}{\log 3}}\epsilon^{2-\frac{\log 5}{\log 3}}. \tag{22}$$

The first term in the left-hand side above is the contribution from the unit square. The contribution to the upper bound for $\mu_\Omega(\epsilon)$ from the squares in generations $J, J+1, \dots$ is bounded from above by

$$\sum_{j\geq J} 4\cdot 5^{j-1}9^{-j} = \left(\frac{5}{9}\right)^{J-1} \leq \frac{36}{5}2^{-\frac{\log 5}{\log 3}}\epsilon^{2-\frac{\log 5}{\log 3}}. \tag{23}$$

We recognise the interior Minkowski dimension $\frac{\log 5}{\log 3}$ of ∂K. By (22) and (23), we have that

$$\mu_\Omega(\epsilon) \leq \frac{84}{5}2^{-\frac{\log 5}{\log 3}}\epsilon^{2-\frac{\log 5}{\log 3}}, \quad 0 < \epsilon < \frac{1}{18}.$$

Solving the equation

$$\frac{84}{5}2^{-\frac{\log 5}{\log 3}}\epsilon^{2-\frac{\log 5}{\log 3}} = 1 - \frac{4}{j_0^2}$$

gives that

$$\epsilon(K) \geq 0.00379. \tag{24}$$

The bound of (20) follows by (21) and (24). □

Below we construct an open set $D_s \subset \mathbb{R}^3$. Let $Q_0 \subset \mathbb{R}^3$ be an open cube of side-length 1. Let $0 < s \leq \sqrt{2} - 1$. Attach a regular open cube $Q_{1,i}$ of side-length s to the centre $c_{1,i}, i = 1, \dots, 6$, of each face of ∂Q_0, and such that all the faces are pairwise-parallel. Now proceed by induction. For $j = 2, 3, \dots$, attach $N(j) = 6\cdot 5^{j-1}$ open cubes $Q_{j,1}, \dots, Q_{j,N(j)}$, of side-length s^j to the centres of the boundary faces of the cubes $Q_{j-1,1}, \dots, Q_{j-1,N(j-1)}$, again with pairwise-parallel faces. We define the polyhedron D_s as

$$D_s = \text{interior}\left\{\overline{Q_0 \cup \left[\bigcup_{j\geq 1}\ \bigcup_{1\leq i\leq N(j)} Q_{j,i}\right]}\right\}.$$

See Figure 2. We note that for $0 < s \leq \sqrt{2} - 1$ no cubes in the construction of D_s overlap.

The asymptotic behaviour of the heat content of D_s in $\mathbb{R}^3$ for small time was analysed in [5]. Here we have the following.

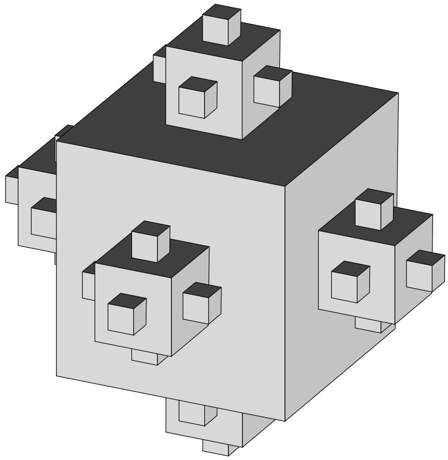

Figure 2. The first two generations of D_s with $s = \frac{1}{3}$.

Example 3. Let $s \in (0, \sqrt{2} - 1]$, and let D_s be the polyhedron in $\mathbb{R}^3$ defined above. Then

$$\mathfrak{C}(D_s) \leq 25 \cdot 10^{10}. \tag{25}$$

Proof. We have that

$$|D_s| = \frac{1 + s^3}{1 - 5s^3},$$

and that the two-dimensional Hausdorff measure of the boundary is given by

$$\mathcal{H}^2(\partial D_s) = 6\Big(\frac{1 - s^2}{1 - 5s^2}\Big).$$

By Theorem 1, we have that

$$\mathfrak{C}(D_s) \leq 36(15)^{3/2}\pi\Big(1 - \frac{9}{2\pi^2}\Big)^{-3} \frac{|D_s|}{\epsilon(D_s)^3}, \tag{26}$$

where we have used that

$$\lambda_1(\mathcal{B}_3) = j_{1/2}^2 = \pi^2,$$

where $j_{1/2} = \pi$ is the first positive zero of the Bessel function $J_{1/2}$. We obtain an upper bound for $\mu_\Omega(\epsilon)$ by adding all faces between cubes of different generations. This gives a disjoint union of 1 unit cube and $6 \cdot 5^{j-1}$ cubes of side-length s^j, $j \in \mathbb{N}$. Hence

$$\mu_\Omega(\epsilon) \leq \Big(6 + 36 \sum_{j=1}^{\infty} 5^{j-1} s^{2j}\Big)\epsilon = \frac{6(1 + s^2)}{1 - 5s^2}\epsilon. \tag{27}$$

By (3) and (27), we have that

$$\epsilon(D_s) \geq \frac{1}{12}\Big(1 - \frac{9}{2\pi^2}\Big)\frac{1-5s^2}{1+s^2}|D_s|. \tag{28}$$

Finally by (26), (28), the fact that $0 < s \leq \sqrt{2} - 1$, and $|D_s| \geq 1$, we obtain that

$$\mathfrak{C}(D_s) \leq 6(12)^4(15)^{3/2}(140 + 99\sqrt{2})\pi\Big(1 - \frac{9}{2\pi^2}\Big)^{-6}.$$

This implies (25). □

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References

[1] P. Bérard and B. Helffer, The weak Pleijel theorem with geometric control. Preprint 2015. arXiv:1512.07089 [math.SP]

[2] P. Bérard and B. Helffer, Dirichlet eigenfunctions of the square membrane: Courant's property, and A. Stern's and Å. Pleijel's analyses. In A. Baklouti, A. El Kacimi, S. Kallel, and N. Mir (eds.), *Analysis and geometry.* Selected papers presented at the MIMS (Mediterranean Institute for the Mathematical Sciences)—GGTM (Geometry and Topology Grouping for the Maghreb) Conference held at the Cité des Sciences, Tunis, March 24–27, 2014. In honour of M. S. Baouendi. Springer Proceedings in Mathematics & Statistics, 127. Springer, Cham, 2015, 69–114. MR 3445517 Zbl 06499037

[3] P. Bérard and D. Meyer, Inégalités isopérimétriques et applications. *Ann. Sci. École Norm. Sup.* (4) **15** (1982), no. 3, 513–541. MR 0690651 Zbl 0527.35020

[4] M. van den Berg, Heat equation on the arithmetic von Koch snowflake. *Probab. Theory Related Fields* **118** (2000), no. 1, 17–36. MR 1785451 Zbl 0963.35072

[5] M. van den Berg and K. Gittins, On the heat content of a polygon. *J. Geom. Anal.* **26** (2016), no. 3, 2231–2264. MR 3511476 Zbl 1342.35127

[6] M. van den Berg and M. Lianantonakis, Asymptotics for the spectrum of the Dirichlet Laplacian on horn-shaped regions. *Indiana Univ. Math. J.* **50** (2001), no. 1, 299–333. MR 1857038 Zbl 1040.35052

[7] J. Bourgain, On Pleijel's nodal domain theorem. *Int. Math. Res. Not. IMRN* **2015**, no. 6, 1601–1612. MR 3340367 Zbl 1317.35145

[8] R. Courant and D. Hilbert, *Methods of mathematical physics.* Vol. I. Translated and revised from the German original. Interscience Publishers, New York, N.Y., 1953. MR 0065391 Zbl 0051.28802 Zbl 0053.02805

[9] L. I. Hedberg, Spectral synthesis in Sobolev spaces, and uniqueness of solutions of the Dirichlet problem. *Acta Math.* **147** (1981), no. 3-4, 237–264. MR 0639040 Zbl 0504.35018

[10] B. Helffer, T. Hoffmann-Ostenhof, and S. Terracini, Nodal domains and spectral minimal partitions. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **26** (2009), no. 1, 101–138. MR 2483815 Zbl 1171.35083

[11] P. Li and S. T. Yau, On the Schrödinger equation and the eigenvalue problem. *Comm. Math. Phys.* **88** (1983), no. 3, 309–318. MR 0701919 Zbl 0554.35029

[12] Yu. Netrusov and Yu. Safarov, Weyl asymptotic formula for the Laplacian on domains with rough boundaries. *Comm. Math. Phys.* **253** (2005), no. 2, 481–509. MR 2140257 Zbl 1076.35085

[13] Å. Pleijel, Remarks on Courant's nodal line theorem. *Comm. Pure Appl. Math.* **9** (1956), 543–550. MR 0080861 Zbl 0070.32604

[14] I. Polterovich, Pleijel's nodal domain theorem for free membranes. *Proc. Amer. Math. Soc.* **137** (2009), no. 3, 1021–1024. MR 2457442 Zbl 1162.35005

[15] M. Reed and B. Simon, Methods of modern mathematical physics. IV. Analysis of operators. Academic Press [Harcourt Brace Jovanovich, Publishers], New York and London, 1978. MR 0493421 Zbl 0401.47001

[16] G. V. Rozenbljum, On the eigenvalues of the first boundary value problem in unbounded domains. *Mat. Sb.* (*N.S.*) **89** (131) (1972), 234–247, 350. In Russian. English transl., *Math. USSR-Sb.* **18** (1972), 235–248. MR 0348295 Zbl 0267.35063

[17] Yu. Safarov, Fourier Tauberian theorems and applications. *J. Funct. Anal.* **185** (2001), no. 1, 111–128. MR 1853753 Zbl 1200.35204

[18] S. Steinerberger, A geometric uncertainty principle with an application to Pleijel's estimate. *Ann. Henri Poincaré* **15** (2014), no. 12, 2299–2319. MR 3272823 Zbl 1319.35132

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