J. Spectr. Theory 6 (2016), 837–858
DOI 10.4171/JST/144

Journal of Spectral Theory

On continuous and discrete Hardy inequalities

Lev Kapitanski[1,2] and Ari Laptev[1,3]

To the memory of our friend Yuri Safarov

Abstract. We obtain a number of Hardy type inequalities for continuous and discrete operators.

Mathematics Subject Classification (2010). Primary: 35P15; Secondary: 81Q10.

Keywords. Hardy's inequalties, discrete Schödinger operators.

1. Introduction

The original Hardy inequalities appeared first in Hardy's proof of Hilbert's theorem (related to discrete Hilbert transformation), [4]. After a few improvements (see [6] for historical details), the inequalities read as follows ([5], Theorems 326 and 327).

Discrete case. If $p > 1$ and $v_k > 0$, then

$$\sum_{n=1}^{\infty} \Big(\frac{1}{n} \sum_{k=1}^{n} v_k\Big)^p < \Big(\frac{p}{p-1}\Big)^p \sum_{n=1}^{\infty} v_n^p. \tag{1.1}$$

Continuous case. If $p > 1$ and $v(x) \geq 0$, $x \geq 0$, then

$$\int_0^\infty \Big(\frac{1}{x} \int_0^x v(t)\,dt\Big)^p\,dx < \Big(\frac{p}{p-1}\Big)^p \int_0^\infty v(x)^p\,dx, \tag{1.2}$$

unless $v \equiv 0$. The constant $(p/(p-1))^p$ is sharp.

[1] This work started during the term "Hamiltonians in Magnetic Fields" at Institut Mittag-Leffler.

[2] L. Kapitanski thanks the organisers and the Institute for kind support and hospitality.

[3] A. Laptev was supported by the grant of the Russian Federation Government to support scientific research under the supervision of leading scientist at Siberian Federal University, No. 14.Y26.31.0006.

Rewritten in a slightly different form, inequalities (1.1) and (1.2) are

$$\sum_{n=1}^{\infty} \frac{|u_n|^p}{n^p} < \Big(\frac{p}{p-1}\Big)^p \sum_{n=1}^{\infty} |u_n - u_{n-1}|^p \tag{1.3}$$

and

$$\int_0^\infty \frac{|u(x)|^p}{x^p}\, dx < \Big(\frac{p}{p-1}\Big)^p \int_0^\infty |u'(x)|^p\, dx.$$

It is assumed that $u_0 = 0$ and $u(0) = 0$.

There are many other inequalities which are also called Hardy inequalities, see for instance a survey paper by E. B. Davies [2], the books of V. G. Maz′ya [8] and Kufner and Opic [7], and a recent book by A. A. Balinsky, W. D. Evans, and R. T. Lewis [1]. The number of research papers in this area is enormous. We just mention [9], [12], [10], [11], and [3] as more closely related to our text.

In this short note we consider both discrete and continuous Hardy inequalities with $p = 2$ in dimensions $d \geq 2$. The first part deals with the standard Hardy type inequalities in $\mathbb{R}^d$. Here we present an approach allowing us to obtain inequalities with multiple singularities. It is based on the fact that the multi-dimensional Hardy inequality is intimately related to the fundamental solution of the Laplace equation (we explain the connection below). Inequalities with multiple singularities obtained in Section 2 are sometimes better and sometimes worse than the inequalities that one can derive by simply adding together standard Hardy inequalities at different points.

To our surprise Hardy inequalities for discrete operators seem to be a lot more complicated. It is still not clear what the sharp constants are. In Sections 3 and 4 we consider Hardy type inequalities for discrete operators in dimensions $d \geq 3$ and $d = 2$ respectively. We would like to mention the papers [10] and [11], where a very different method is used to obtain a discrete Hardy type inequality when $d \geq 3$.

2. Continuous case, multiple singularities

For the sake of completeness we revise here Hardy's inequality in the continuous case. One possible way of proving Hardy type inequality is as follows.

Let $A_j(x)$, $j = 1, \dots, d$, be the components of a vector-field $A(x)$ in $\mathbb{R}^d$ to be chosen later, and let λ be a real parameter. The vector field $A(x)$ will determine the type of the inequality, and λ will be chosen to optimise it. Assuming

$u(x)$ is sufficiently smooth and decaying where it should, perform the following computation:

$$\begin{aligned}
&\int \sum_{j=1}^{d}(|\partial_j u(x) - \lambda\, A_j(x)\, u(x)|^2)\, dx \\
&\quad = \int \Big(|\nabla u|^2 - 2\,\lambda\, \sum A_j\, u\, \partial_j u + \lambda^2 \sum_{j=1}^{d} |A_j|^2\, |u|^2\Big)\, dx \\
&\quad = \int \Big(|\nabla u|^2 + \Big(\lambda \operatorname{div} A + \lambda^2 \sum_{j=1}^{d} |A_j|^2\Big)\, |u|^2\Big)\, dx.
\end{aligned}$$

Thence the primeval form of the Hardy inequality is

$$\sup_{\lambda}\Big[-\int \Big(\lambda \operatorname{div} A + \lambda^2 \sum_{j=1}^{d} |A_j|^2\Big)\, |u|^2\, dx\Big] \le \int |\nabla u|^2\, dx.$$

Maximizing the left hand side with respect to λ we differentiate the integral

$$-\int \Big(\lambda \operatorname{div} A + \lambda^2 \sum_{j=1}^{d} |A_j|^2\Big)\, |u|^2\, dx$$

with respect to λ and find when the result is zero regardless of the choice of $u(x)$. This leads to the requirement

$$2\lambda\, |A|^2 + \operatorname{div} A = 0, \quad \text{for all } x \in \mathbb{R}^d.$$

This is a serious restriction on $A(x)$ implying

$$\frac{\operatorname{div} A(x)}{|A(x)|^2} = \text{const.}$$

Rescaling A, we arrive at

$$\operatorname{div} A(x) = -|A(x)|^2. \tag{2.1}$$

With this rescaling, the critical λ is

$$\lambda_* = \frac{1}{2}$$

and the corresponding Hardy inequality is

$$\frac{1}{4} \int \sum_{j=1}^{d} |A_j(x)|^2\, |u|^2\, dx \le \int |\nabla u|^2\, dx. \tag{2.2}$$

If we assume that the field A is potential, i.e., $A = \nabla\psi$, for some function ψ, then (2.1) becomes

$$\Delta\psi + |\nabla\psi|^2 = 0,$$

which in turn implies that the function

$$w = e^{\psi} \geq 0$$

is harmonic and positive valued. Then w is a constant > 0 or has a singularity.

In the 1D case, $w(x) = |x|$ (up to a translation and a scalar factor) and $\psi(x) = \ln|x|$. Then $A(x) = 1/x$ and we obtain

$$\frac{1}{4}\int \frac{|u(x)|^2}{|x|^2}\,dx \leq \int |u'(x)|^2\,dx, \tag{2.3}$$

where we assume that $u(0) = 0$.

If $d = 2$ we take $w(x) = \ln|x|$ assuming that $|x| \geq 1$. Then $\psi(x) = \ln(\ln|x|)$ and

$$A(x) = \frac{x}{\ln|x|\,|x|^2}.$$

Then the corresponding Hardy inequality is

$$\frac{1}{4}\int_{\{|x|\geq 1\}} \frac{|u(x)|^2}{|x|^2\ \ln^2|x|}\,dx \leq \int_{\{|x|\geq 1\}} |\nabla u(x)|^2\,dx,$$

where we assume that $u(x) = 0$ for $|x| = 1$.

For $d \geq 3$, the choice is

$$w(x) = \frac{1}{|x|^{d-2}}.$$

Then

$$\psi(x) = -(d-2)\ln|x|$$

and $A = \nabla\psi$,

$$A_i(x) = -(d-2)\,\frac{x_i}{|x|^2}.$$

This leads to the familiar Hardy inequality

$$\Big(\frac{d-2}{2}\Big)^2 \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^2}\,dx \leq \int_{\mathbb{R}^d} |\nabla u(x)|^2\,dx.$$

This approach can be extended to Hardy inequalities with several singularities. Let $d \geq 3$ and let $a_k = (a_{k1}, a_{k2}, \ldots a_{kd}) \in \mathbb{R}^d$, $k = 1, \ldots, n$. Introduce

$$w = \sum_{k=1}^{n} \frac{1}{|x - a_k|^{d-2}}.$$

Then

$$\psi(x) = \ln(w).$$

Therefore

$$A(x) = \nabla(\ln w) = \frac{-(d-2)}{w}\Big(\sum_{k=1}^{n} \frac{x - a_k}{|x - a_k|^d}\Big)$$

and

$$|A(x)|^2 = \sum_{j=1}^{d} |A_j(x)|^2 = \Big(\frac{d-2}{w}\Big)^2 \sum_{j=1}^{d} \Big(\sum_{k=1}^{n} \frac{x_j - a_{kj}}{|x - a_k|^d}\Big)^2.$$

Inequality (2.2) implies

Theorem 2.1. *Let $d \geq 3$ and let $a_k = (a_{k1}, a_{k2}, \ldots a_{kd}) \in \mathbb{R}^d$, $k = 1, \ldots, n$. Then we obtain the following generalised Hardy inequality with multiple singularities*

$$\int_{\mathbb{R}^d} |\nabla u(x)|^2 \, dx \geq \Big(\frac{d-2}{2}\Big)^2 \int_{\mathbb{R}^d} \frac{\sum_{j=1}^{d} \big(\sum_{k=1}^{n} \frac{x_j - a_{kj}}{|x - a_k|^d}\big)^2}{w^2} |u(x)|^2 \, dx. \quad (2.4)$$

Note that if we add together the standard Hardy inequalities with singularities at different points we obtain

$$\int_{\mathbb{R}^d} |\nabla u(x)|^2 \, dx \geq \frac{1}{n}\Big(\frac{d-2}{2}\Big)^2 \int_{\mathbb{R}^d} \sum_{k=1}^{n} \Big(\frac{1}{|x - a_k|^2}\Big) |u(x)|^2 \, dx. \quad (2.5)$$

The Hardy weight in (2.4) and (2.5) asymptotically behaves as

$$\Big(\frac{d-2}{2}\Big)^2 \frac{1}{|x|^2}, \quad \text{as } x \to \infty.$$

However, in the vicinity of each singularity a_k, the Hardy weight in (2.5) behaves as

$$|A(x)|^2 \sim \frac{1}{n}\Big(\frac{d-2}{2}\Big)^2 \frac{1}{|x - a_k|^2}, \quad \text{as } x \to a_k,$$

while inequality (2.4) gives

$$|A(x)|^2 \sim \Big(\frac{d-2}{2}\Big)^2 \frac{1}{|x - a_k|^2}, \quad \text{as } x \to a_k.$$

Inequality (2.4) is not sharper than (2.5) in the sense that there is no inequality for the weight functions. Indeed, let us consider a simple example $d = 3$, $n = 2$, $a_1 = (1, 0, 0)$ and $a_2 = (-1, 0, 0)$. Then obviously at $x = 0$ we have

$$\sum_{k=1}^{2} \frac{x_j - a_{kj}}{|x - a_k|^3} = 0, \quad j = 1, 2, 3.$$

Let now $d = 2$. Then by analogy with the case $d \geq 3$ we introduce

$$w(x) = \ln\Big(\prod_{k=1}^{n} |x - a_k|\Big),$$

defined for $\prod_{k=1}^{n} |x - a_k| \geq 1$, where $a_k = (a_{k1}, a_{k2}) \in \mathbb{R}^2$.

Then $\psi(x) = \ln\ln\big(\prod_{k=1}^{n} |x - a_k|\big)$ and

$$A(x) = \nabla\psi(x) = \frac{1}{\ln\big(\prod_{k=1}^{n} |x - a_k|\big)} \Big(\sum_{k=1}^{n} \frac{x - a_k}{|x - a_k|^2}\Big).$$

Denote by $S_n = \{x \in \mathbb{R}^2 \colon \prod_{k=1}^{n} |x - a_k| \leq 1\}$. We immediately obtain the following statement:

Theorem 2.2. *Let us assume that $u(x) = 0$ for $x \in \partial S_n$. Then*

$$\int_{\mathbb{R}^2\setminus S_n} |\nabla u(x)|^2\, dx \geq \frac{1}{4} \int_{\mathbb{R}^2\setminus S_n} \frac{\sum_{j=1}^{2}\big(\sum_{k=1}^{n} \frac{x_j - a_{kj}}{|x - a_k|^2}\big)^2}{w^2} |u(x)|^2\, dx. \tag{2.6}$$

If we add the standard 2D-Hardy inequalities with different singularities we obtain

$$\int_{\mathbb{R}^2} |\nabla u(x)|^2\, dx \geq \frac{1}{4n} \int_{\mathbb{R}^2} \Big(\sum_{k=1}^{n} \frac{1}{|x - a_k|^2 \ln^2 |x - a_k|}\Big) |u(x)|^2\, dx, \tag{2.7}$$

where it is assumed that $u(x) = 0$ for $x \in \bigcup_{k=1}^{n} \{x \in \mathbb{R}^2 \colon |x - a_k| \leq 1\}$.

Similarly to the case $d \geq 3$ inequality (2.7) is not as good as (2.6) near singularities because (2.7) contains an extra term $1/n$. Moreover Theorem 2.2 requires functions to be zero only in S_n rather than in $\bigcup_{k=1}^{n} \{x \in \mathbb{R}^2 \colon |x - a_k| \leq 1\}$ which is a lot less restrictive.

Note in the end that 1D case with multiple singularities is not interesting since the conditions $u(x) = 0$ at $x = a_k$ split the problem into one dimensional problems on intervals (a_k, a_{k+1}), $k = 1, \ldots, n-1$ with zero boundary conditions.

Remark 2.3. It would be interesting to study the shape of the subset of $\mathbb{R}^d$ where the Hardy weights in (2.4) and (2.5) (and respectively (2.6) and (2.7)) are equal. It might be also interesting to look at this problem with weighted singularities

$$\frac{b_k^2}{|x-a_k|^2}, \quad \text{where } b_k \in \mathbb{R}.$$

3. Discrete multi-dimensional Hardy's inequalities, $d \geq 3$

Denote by $\mathbf{1}_j \in \mathbb{R}^d$ the unit vector in the direction j. Then the discrete partial derivative in the direction j of a function f defined on $\mathbb{Z}^d$ equals $f(n)-f(n-\mathbf{1}_j)$, $n=(n^1,\dots,n^d)\in\mathbb{Z}^d$.

In this section we shall prove the following result.

Theorem 3.1. *Let $d \geq 3$. For a function f defined on $\mathbb{Z}^d$ we have*

$$\sum_{n\in\mathbb{Z}^d} \frac{|f(n)|^2}{|n|^2} \leq C \sum_{n\in\mathbb{Z}^d} \sum_{j=1}^{d} |f(n)-f(n-\mathbf{1}_j)|^2,$$

where

$$C \leq \frac{\pi^2}{4}\Big(\frac{2}{d-2} + \frac{\sqrt{2}\,(d-4)^2}{(d-2)\,(\sqrt{2}\,d-4\sqrt{d-2})\,\pi}\Big).$$

In order to prove this theorem we introduce a Fourier transform

$$\hat{f}(\theta) = \sum_{n\in\mathbb{Z}^d} f(n)\, e^{-\mathrm{i}\, n\,\theta},$$

where $\theta=(\theta_1,\dots,\theta_d)\in\mathbb{T}^d=[-\pi,\pi]^d$. The inverse transform is

$$f(n) = \int_{\mathbb{T}^d} \hat{f}(\theta)\, e^{\mathrm{i}\, n\,\theta}\, đ\theta,$$

where $đ\theta = (2\pi)^{-d}\, d\theta$. Given $f(n)$, define

$$f_j(n) = \begin{cases} \dfrac{n^j}{|n|^2}\, f(n), & n \neq 0, \\ 0, & n = 0. \end{cases}$$

Then,

$$\sum_j |f_j(n)|^2 = \frac{|f(n)|^2}{|n|^2}$$

and

$$\sum_j n^j\, f_j(n) = f(n), \qquad n^k\, f_j(n) = n^j\, f_k(n), \quad j \neq k.$$

Since

$$f(n) - f(n - \mathbf{1}_j) = \int_{\mathbb{T}^d} \hat{f}(\theta)(e^{\mathbf{i}\,n\,\theta} - e^{\mathbf{i}\,n\,\theta - \mathbf{i}\theta_j})\, đ\theta$$

and

$$e^{\mathbf{i}\,n\,\theta} - e^{\mathbf{i}\,n\,\theta - \mathbf{i}\theta_j} = 2\,\mathbf{i}\,e^{-\mathbf{i}\theta_j/2}\,\sin(\theta_j/2)\,e^{\mathbf{i}\,n\,\theta},$$

we obtain using the Parseval identity

$$\sum_{n\in\mathbb{Z}^d}\sum_{j=1}^{d} |f(n) - f(n-\mathbf{1}_j)|^2 = 4\int_{\mathbb{T}^d} |\hat{f}(\theta)|^2 \sum_j \sin^2(\theta_j/2)\, đ\theta.$$

Let $\hat{f}_j(\theta)$ be the Fourier transform of the sequence $\{f_j(n)\}$. Then

$$f_j(n) = \int_{\mathbb{T}^d} \hat{f}_j(\theta) e^{\mathbf{i}n\theta}\, đ\theta,$$

and

$$n^k\, f_j(n) = \int_{\mathbb{T}^d} \hat{f}_j(\theta) n^k\, e^{\mathbf{i}n\theta}\, đ\theta = \int_{\mathbb{T}^d} \mathbf{i}\,\partial_k \hat{f}_j(\theta)\, e^{\mathbf{i}n\theta}\, đ\theta\,,$$

where $\partial_k = \partial/\partial\theta_k$. Thus, we have

$$\partial_k \hat{f}_j(\theta) - \partial_j \hat{f}_k(\theta) \equiv 0, \quad \text{when } j \neq k\,, \tag{3.1}$$

and also,

$$\hat{f}(\theta) = \mathbf{i}\sum_j \partial_j \hat{f}_j(\theta).$$

Taking all this into account we arrive at

$$\sum_{n\in\mathbb{Z}^d} \frac{|f(n)|^2}{|n|^2} = \int_{\mathbb{T}^d} \sum_j |\hat{f}_j(\theta)|^2\, đ\theta,$$

and

$$\sum_{n\in\mathbb{Z}^d}\sum_{j=1}^{d} |f(n) - f(n-\mathbf{1}_j)|^2 = 4\int_{\mathbb{T}^d} \Big|\sum_k \partial_k \hat{f}_k(\theta)\Big|^2 \sum_j \sin^2(\theta_j/2)\, đ\theta.$$

Thus, the inequality we want to prove is reduced to

$$\int_{\mathbb{T}^d} \sum_j |\hat{f}_j(\theta)|^2 \, d\!\!\!{}^-\theta \ \leq 4\,C \int_{\mathbb{T}^d} \Big| \sum_k \partial_k \hat{f}_k(\theta) \Big|^2 \sum_j \sin^2(\theta_j/2) \, d\!\!\!{}^-\theta, \qquad (3.2)$$

for any $\hat{f}_j$ satisfying (3.1) and, since $f_j(0) = 0$,

$$\int_{\mathbb{T}^d} \hat{f}_j(\theta) \, d\!\!\!{}^-\theta \ = 0. \qquad (3.3)$$

Vector-valued functions $\hat{f}_j$ satisfying (3.1) and (3.3) are gradients. In other words, there exists a periodic function $\psi(\theta)$ such that (normalization)

$$\int_{\mathbb{T}^d} \psi(\theta) \, d\!\!\!{}^-\theta \ = 0$$

and

$$\hat{f}_j(\theta) = \partial_j \psi(\theta).$$

In terms of ψ, inequality (3.2) reads (we can replace $d\!\!\!{}^-\theta$ with $d\theta$ now)

$$\int_{\mathbb{T}^d} |\nabla\psi(\theta)|^2 \, d\theta \ \leq 4\,C \int_{\mathbb{T}^d} |\Delta\psi(\theta)|^2 \sum_j \sin^2(\theta_j/2) \, d\theta. \qquad (3.4)$$

This reminds us of the Poincaré–Friedrichs inequality, but here we have a stronger inequality because of $\sum_j \sin^2(\theta_j/2)$ in the right hand side.

Let us prove (3.4). Denote

$$w(\theta) = \Big(\sum_j \sin^2(\theta_j/2) \Big)^{1/2}.$$

Then

$$\int_{\mathbb{T}^d} |\nabla\psi(\theta)|^2 \, d\theta = - \int_{\mathbb{T}^d} \psi \cdot \overline{\Delta\psi} \, d\theta \leq \|\Delta\psi \ w\|_{L^2(\mathbb{T}^d)} \left\| \frac{\psi}{w} \right\|_{L^2(\mathbb{T}^d)}.$$

Since

$$w^2(\theta) = \sum_j \sin^2(\theta_j/2) = \Big[4 \frac{\sum_j \sin^2(\theta_j/2)}{|\theta|^2} \Big] \frac{|\theta|^2}{4} \geq \frac{|\theta|^2}{\pi^2},$$

we obtain

$$\int_{\mathbb{T}^d} \frac{|\psi(\theta)|^2}{w^2(\theta)} \, d\theta \leq \pi^2 \int_{\mathbb{T}^d} \frac{|\psi(\theta)|^2}{|\theta|^2} \, d\theta.$$

It remains to prove a Hardy inequality on the torus.

Lemma 3.2. *Let* $d \geq 3$. *Then there exists a constant* $C(d) > 0$ *such that*

$$\int_{\mathbb{T}^d} \frac{|\psi(\theta)|^2}{|\theta|^2}\, d\theta \leq C(d) \int_{\mathbb{T}^d} |\nabla\psi(\theta)|^2\, d\theta, \tag{3.5}$$

for all smooth ψ *with zero average*:

$$\int_{\mathbb{T}^d} \psi(\theta)\, d\theta = 0. \tag{3.6}$$

In fact, we can take

$$C(d) = \frac{2}{d-2} + \frac{\sqrt{2}\,(d-4)^2}{(d-2)\,(\sqrt{2}\,d - 4\sqrt{d-2})\,\pi}.$$

Proof. We will use our assumption (3.6) to have the Poincaré–Friedrichs inequality:

$$\|\psi\|^2 = (2\pi)^{-d} \sum_{n\in\mathbb{Z}^d\setminus\{0\}} |\hat{\psi}(n)|^2 \leq (2\pi)^{-d} \sum_{n\in\mathbb{Z}^d\setminus\{0\}} |n|^2 |\hat{\psi}(n)|^2 = \|\nabla\psi\|^2. \tag{3.7}$$

Here $\|\cdot\|$ is the L^2 norm over $\mathbb{T}^d$ and $\{\hat{\psi}(n)\}$ are the Fourier coefficients of ψ,

$$\psi(\theta) = (2\pi)^{-d} \sum_{n\in\mathbb{Z}^d} \hat{\psi}(n)\, e^{\mathbf{i}n\theta}.$$

Sometimes it will be useful to think of ψ as a function on $\mathbb{R}^d$, which is 2π-periodic in each variable. Let $\zeta(\theta)$ be a product of one-dimensional cut-off functions, $\zeta(\theta) = \prod_{j=1}^d \tilde{\zeta}(\theta_j)$, where each $\tilde{\zeta}(t)$ is piece-wise linear, $0 \leq \tilde{\zeta}(t) \leq 1$ for all t, $\tilde{\zeta}(t) = 1$, when $|t| \leq r$, and $\tilde{\zeta}(t) = 0$, when $|t| \geq r + \epsilon$ (where r and ϵ are positive numbers, and $r + \epsilon < \pi$). We have

$$\left\| \frac{\psi}{|\theta|} \right\| \leq \left\| \frac{\zeta\,\psi}{|\theta|} \right\| + \left\| \frac{(1-\zeta)\,\psi}{|\theta|} \right\| \tag{3.8}$$

Viewing $\zeta\,\psi$ as a compactly supported function on $\mathbb{R}^d$, apply the standard Hardy inequality in $\mathbb{R}^d$:

$$\left\| \frac{\zeta\,\psi}{|\theta|} \right\| = \left\| \frac{\zeta\,\psi}{|\theta|} \right\|_{L^2(\mathbb{R}^d)} \leq \frac{2}{d-2} \|\nabla(\zeta\,\psi)\|_{L^2(\mathbb{R}^d)} = \frac{2}{d-2}\, \|\nabla(\zeta\,\psi)\|. \tag{3.9}$$

Since

$$\|\nabla(\zeta\,\psi)\| \le \|\zeta\,\nabla\psi\| + \|\psi\,\nabla\zeta\| \le \|\nabla\psi\| + \frac{1}{\epsilon}\,\|\psi\|,$$

we invoke (3.7) to obtain

$$\|\nabla(\zeta\,\psi)\| \le \Big(1+\frac{1}{\epsilon}\Big)\,\|\nabla\psi\|. \tag{3.10}$$

The second term in (3.8) is even simpler, since $|\theta| \ge r$ on the support of $1-\zeta$. Thus, applying again (3.7),

$$\left\|\frac{(1-\zeta)\,\psi}{|\theta|}\right\| \le \frac{1}{r}\,\|\psi\| \le \frac{1}{r}\,\|\nabla\psi\|.$$

Combining this with (3.9) and (3.10), we estimate the right hand side in (3.8) as follows

$$\le \left[\frac{2}{d-2}\Big(1+\frac{1}{\epsilon}\Big)+\frac{1}{r}\right]\|\nabla\psi\|.$$

In order to minimize the quantity

$$\frac{2}{d-2}\,\frac{1}{\epsilon}+\frac{1}{r},$$

take $r=\pi-\epsilon$. The minimum of

$$\frac{2}{d-2}\,\frac{1}{\epsilon}+\frac{1}{\pi-\epsilon} \tag{3.11}$$

is attained at

$$\epsilon = \frac{\sqrt{2d-4}-2}{d-4}\,\pi$$

(with $\epsilon=\pi/2$ for $d=4$), and the corresponding value of (3.11) is

$$\frac{\sqrt{2}\,(d-4)^2}{(d-2)\,(\sqrt{2}\,d-4\sqrt{d-2})\,\pi}.$$

This proves (3.5) $\square$

4. Hardy's inequality on $\mathbb{Z}^2$

The main result of this section is the following theorem.

Theorem 4.1. *There exists a constant* $C>0$ *such that*

$$\sum_{x\in\mathbb{Z}^2,\,|x|\ge 2}\frac{|f(x)|^2}{|x|^2(\ln|x|)^2} \le C\sum_{x,y\in\mathbb{Z}^2,\,x\sim y}|f(x)-f(y)|^2, \tag{4.1}$$

for all compactly supported $f(x)$ *with* $f(x)=0$ *for* $|x|\le 1$.

In fact instead of (4.1) we shall prove an equivalent inequality

$$\sum_{x\in\mathbb{Z}^2,\,\|x\|\geq 2} \frac{|f(x)|^2}{\|x\|^2(\ln\|x\|)^2} \leq C \sum_{x\in\mathbb{Z}^2} \sum_{j=1,2} |f(x)-f(x-\mathbf{1}_j)|^2, \tag{4.2}$$

where as before $\mathbf{1}_j$ is the unit vector in the direction j. In this section we denote by $\|x\|$ the ℓ^∞-norm of x

$$\|x\| = \max\{|x_1|, |x_2|\}.$$

Let us introduce the *discrete polar coordinates* on $\mathbb{Z}^2$ based on this norm. Denote by D_R the closed disk $\{x \in \mathbb{Z}^2\colon \|x\| \leq R\}$ and by ∂D_R denote its boundary. Thus, if $n > 0$ is an integer,

$$\partial D_n = \{x \in \mathbb{Z}^2\colon |x_1| = n, -n \leq x_2 \leq n \;\text{ or }\; |x_2| = n, -n \leq x_1 \leq n\}.$$

Define a non-linear operation, U, of turning one step counterclockwise along the circle:

$$U\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{cases} \begin{bmatrix} x_1 \\ x_2+1 \end{bmatrix} = x + \mathbf{1}_2, & \text{if } x_1 > 0 \text{ and } -|x_1| \leq x_2 < |x_1|, \\ \begin{bmatrix} x_1-1 \\ x_2 \end{bmatrix} = x - \mathbf{1}_1, & \text{if } x_2 > 0 \text{ and } -|x_2| < x_1 \leq |x_2|, \\ \begin{bmatrix} x_1 \\ x_2-1 \end{bmatrix} = x - \mathbf{1}_2, & \text{if } x_1 < 0 \text{ and } -|x_1| < x_2 \leq |x_1|, \\ \begin{bmatrix} x_1+1 \\ x_2 \end{bmatrix} = x + \mathbf{1}_1, & \text{if } x_2 < 0 \text{ and} -|x_2| \leq x_1 < |x_2|. \end{cases}$$

For $x = 0$ set $U0 = 0$. Note that U is invertible on ∂D_n for $n > 0$. Also, $U^{8\cdot n} = \mathrm{Id}$. If

$$x = U^m\langle n\rangle, \quad \text{where } \langle n\rangle = n\,\mathbf{1}_1 + n\,\mathbf{1}_2,$$

we call $\langle n, m\rangle$ the (*discrete*) *polar coordinates* of x. Each circle ∂D_n has four corners:

$$U^0\langle n\rangle, \quad U^{2n}\langle n\rangle, \quad U^{4n}\langle n\rangle, \quad U^{6n}\langle n\rangle.$$

To describe other points $U^m\langle n\rangle = \langle n, m\rangle$ on ∂D_n, we sometimes decompose m as follows

$$m = 2sn + t, \quad s = 0, 1, 2, 3, \quad 0 < t < 2n$$

(but sometimes, for convenience, t will be equal to 0 or $2n$). For a function f on $\mathbb{Z}^2$, we write its value at x as $f(x)$ or, when using polar coordinates, $f\langle n, m\rangle$. We say that $x \sim y$ if x and y are neighbours in the sense that $y = x \pm \mathbf{1}_j$ for $j = 1$ or $j = 2$.

Let us begin with an useful observation that allows one to take care of *finite* sums of $|f(x)|^2$. This is a discrete analog of the Poincaré–Friedrichs lemma.

Lemma 4.2. *For any integer $N \geq 1$,*

$$\sum_{\|x\|\leq N} |f(x)|^2 \leq \frac{4}{3} N(N+1)(2N+1) \sum_{x,y\in D_N, x\sim y} |f(x) - f(y)|^2 \tag{4.3}$$

for any function f such that $f(0) = 0$.

Proof. We have

$$\sum_{\|x\|\leq N} |f(x)|^2 = \sum_{n=1}^{N} \sum_{x\in\partial D_n} |f(x)|^2.$$

For any point $x_0 \in \partial D_n$, there is a path $x_0 \sim x_1 \sim \cdots \sim x_k \sim 0$ from x_0 to the origin. The smallest k is at most $2n - 1$. We have

$$|f(x_0)| \leq |f(x_0) - f(x_1)| + \cdots + |f(x_k) - f(0)|$$

and therefore,

$$|f(x_0)|^2 \leq 4n \left(|f(x_0) - f(x_1)|^2 + \cdots + |f(x_k) - f(0)|^2\right).$$

When x_0 runs over ∂D_n, the corresponding paths can be chosen so that the maximum number of times an edge is repeated in different paths is $2n$. A very rough estimate is thus

$$\sum_{x_0\in\partial D_n} |f(x_0)|^2 \leq 8n^2 \sum_{x,y\in D_n, x\sim y} |f(x) - f(y)|^2.$$

Summing it from $n = 1$ to $n = N$ results in (4.3). □

Plan of the proof of (4.2)**.** In polar coordinates, the sum we are going to estimate is

$$A_0 := \sum_{n=2}^{\infty} \frac{1}{n^2(\ln n)^2} \sum_{m=0}^{8n-1} |f\langle n,m\rangle|^2.$$

Subtract from $f\langle n,m\rangle$ and add back the average of $f\langle n,m\rangle$ over the circle ∂D_n,

$$\underline{f}(n) = \frac{1}{8n} \sum_{m=0}^{8n-1} f\langle n,m\rangle.$$

Then

$$A_0 \le 2 \sum_{n=2}^{\infty} \frac{1}{n^2(\ln n)^2} \sum_{m=0}^{8n-1} |f\langle n,m\rangle - \underline{f}(n)|^2 + 2 \sum_{n=2}^{\infty} \frac{1}{n^2(\ln n)^2} \sum_{m=0}^{8n-1} |\underline{f}(n)|^2$$
$$\le 2 \sum_{n=2}^{\infty} \frac{1}{n^2(\ln n)^2} \sum_{m=0}^{8n-1} |f\langle n,m\rangle - \underline{f}(n)|^2 + 16 \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} |\underline{f}(n)|^2.$$

Since we are not going to find sharp constants, it is convenient to use the notation $\lesssim$ and write simply $a \lesssim b$ when there is an absolute constant $C > 0$ such that $a \le Cb$. Thus, the above inequality can be written as

$$A_0 \lesssim A_1 + A_2,$$

where

$$A_1 := \sum_{n=2}^{\infty} \frac{1}{n^2(\ln n)^2} \sum_{m=0}^{8n-1} |f\langle n,m\rangle - \underline{f}(n)|^2,$$

and

$$A_2 := \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} |\underline{f}(n)|^2. \tag{4.4}$$

The sum A_1 is estimated with the help of Lemma 4.3, which implies that even without ln in the denominator, we have

$$\sum_{n=2}^{\infty} \frac{1}{n^2} \sum_{m=0}^{8n-1} |f\langle n,m\rangle - \underline{f}(n)|^2 \lesssim \sum_{n=2}^{\infty} \sum_{m=0}^{8n-1} |f\langle n,m+1\rangle - f\langle n,m\rangle|^2. \tag{4.5}$$

(This is because the average of $f\langle n,m\rangle - \underline{f}(n)$ over ∂D_n is zero.) The sum on the right is (a part of) the angular part of the sum

$$B_0 := \sum_{x,y\in\mathbb{Z}^2,\, x\sim y} |f(x) - f(y)|^2. \tag{4.6}$$

The sum A_2 is harder to estimate. We will show that, for any sufficiently small $\epsilon_0 > 0$,

$$A_2 \leq C(\epsilon_0)\, B_0 + \epsilon_0\, A_0, \tag{4.7}$$

where the constant $C(\epsilon_0)$ goes to ∞ as $\epsilon_0 \searrow 0$. A choice of ϵ_0 then will conclude the proof of $A_0 \lesssim B_0$, which is (4.2).

Let us start with the proof of (4.5).

Lemma 4.3. *For functions $g\langle n, m\rangle$ whose averages over the circles ∂D_n vanish,*

$$\sum_{n=2}^{\infty} \frac{1}{n^2} \sum_{m=0}^{8n-1} |g\langle n, m\rangle|^2 \lesssim \sum_{x,y \in \mathbb{Z}^2 \setminus D_1,\, x \sim y} |g(x) - g(y)|^2.$$

This result is a consequence of the following discrete Poincaré–Friedrichs type inequality on the circle.

Lemma 4.4. *For any function ϕ on the finite set $\{0, 1, \ldots, N-1\}$ such that $\sum_{m=0}^{N-1} \phi(m) = 0$,*

$$4\, \sin^2(\pi/N) \sum_{m=0}^{N-1} |\phi(m)|^2 \leq \sum_{m=0}^{N-1} |\phi(m+1) - \phi(m)|^2, \tag{4.8}$$

where we set $\phi(N) = \phi(0)$.

Proof. Use the discrete Fourier transform:

$$\hat{\phi}(k) = \sum_{m=0}^{N-1} \phi(m)\, e^{-2\pi \mathbf{i} m k/N}.$$

Then

$$\phi(m) = \frac{1}{N} \sum_{k=0}^{N-1} \hat{\phi}(k)\, e^{2\pi \mathbf{i} m k/N},$$

and

$$\begin{aligned} \phi(m+1) - \phi(m) &= \frac{1}{N} \sum_{k=0}^{N-1} \hat{\phi}(k)\, [e^{2\pi \mathbf{i} k/N} - 1]\, e^{2\pi \mathbf{i} m k/N} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} 2\,\mathbf{i}\, e^{\pi \mathbf{i} k/N}\, \sin(\pi k/N)\, \hat{\phi}(k)\, e^{2\pi \mathbf{i} m k/N}. \end{aligned}$$

By Plancherel,

$$\sum_{m=0}^{N-1} |\phi(m+1)-\phi(m)|^2 = \frac{4}{N}\sum_{k=0}^{N-1} \sin^2(\pi k/N)\,|\hat{\phi}(k)|^2$$

Since $\hat{\phi}(0)=0$ due to our assumption on the average of ϕ,

$$\sum_{k=0}^{N-1} \sin^2(\pi k/N)\,|\hat{\phi}(k)|^2 \geq \sin^2(\pi/N)\sum_{k=0}^{N-1} |\hat{\phi}(k)|^2,$$

and inequality (4.8) follows. □

Next we prove (4.2) in the case of radial functions on $\mathbb{Z}^2$. A function f is radial if $f\langle n,m\rangle$ does not depend on the angular variable m. If f is radial, and if we write $f\langle n,m\rangle = g(n)$, then

$$\sum_{x\in\mathbb{Z}^2,\,\|x\|\geq 2} \frac{|f(x)|^2}{\|x\|^2(\ln\|x\|)^2} = 8\sum_{n=2}^{\infty}\frac{|g(n)|^2}{n\,(\ln\,n)^2}.$$

Lemma 4.5. *We have*

$$\sum_{n=2}^{\infty}\frac{|g(n)|^2}{n\,(\ln\,n)^2} \leq C\sum_{n=2}^{\infty} n\,|g(n+1)-g(n)|^2. \tag{4.9}$$

Proof. Using an elementary inequality

$$\frac{1}{n\,(\ln n)^2} < \frac{1}{\ln(n-1)} - \frac{1}{\ln n}, \tag{4.10}$$

we obtain

$$\begin{aligned}\sum_{n=3}^{\infty}\frac{|g(n)|^2}{n\,(\ln\,n)^2} &< \sum_{n=3}^{\infty}\frac{|g(n)|^2}{\ln(n-1)} - \sum_{n=3}^{\infty}\frac{|g(n)|^2}{\ln n}\\ &= \sum_{k=2}^{\infty}\frac{|g(k+1)|^2}{\ln k} - \sum_{k=3}^{\infty}\frac{|g(k)|^2}{\ln k}\\ &= \frac{|g(3)|^2}{\ln 2} + \sum_{k=3}^{\infty}\frac{|g(k+1)|^2-|g(k)|^2}{\ln k}\end{aligned}$$

and continue, using Cauchy–Schwarz followed by Young's inequality,

$$\le \frac{|g(3)|^2}{\ln 2} + \Big(\sum_{n=3}^{\infty} n\,|g(n+1)-g(n)|^2\Big)^{1/2}\Big(\sum_{k=3}^{\infty}\frac{(|g(k+1)|+|g(k)|)^2}{k\,(\ln k)^2}\Big)^{1/2}$$

$$\le \frac{|g(3)|^2}{\ln 2} + \frac{1}{4\epsilon}\sum_{n=3}^{\infty} n\,|g(n+1)-g(n)|^2 + \epsilon\sum_{k=3}^{\infty}\frac{(|g(k+1)|+|g(k)|)^2}{k\,(\ln k)^2}$$

Choose the right ϵ and apply Lemma 4.2 to obtain (4.9). □

Let us now estimate A_2. Since $\underline{f}(n)$ is the average of f we have

$$64A_2 = A_3 := \sum_{n=2}^{\infty}\frac{1}{n^3\,(\ln n)^2}\Big|\sum_{m=0}^{8n-1} f\langle n,m\rangle\Big|^2,$$

(see (4.4)). Use (4.10) to proceed as follows:

$$\sum_{n=2}^{\infty}\frac{1}{n^3\,(\ln n)^2}\Big|\sum_{m=0}^{8n-1} f\langle n,m\rangle\Big|^2$$

$$< \frac{1}{2^3\,(\ln 2)^2}\Big|\sum_{m=0}^{8\cdot 2-1} f\langle 2,m\rangle\Big|^2 + \sum_{n=3}^{\infty}\frac{1}{n^2}\Big(\frac{1}{\ln(n-1)}-\frac{1}{\ln n}\Big)\Big|\sum_{m=0}^{8n-1} f\langle n,m\rangle\Big|^2$$

$$= \frac{1}{2^3\,(\ln 2)^2}\Big|\sum_{m=0}^{8\cdot 2-1} f\langle 2,m\rangle\Big|^2 + \sum_{n=2}^{\infty}\frac{1}{(n+1)^2\ln n}\Big|\sum_{m=0}^{8(n+1)-1} f\langle n+1,m\rangle\Big|^2$$

$$-\sum_{n=3}^{\infty}\frac{1}{n^2\ln n}\Big|\sum_{m=0}^{8n-1} f\langle n,m\rangle\Big|^2.$$

Splitting the term with $n=2$ from the second sum we continue

$$= \frac{1}{2^3\,(\ln 2)^2}\Big|\sum_{m=0}^{8\cdot 2-1} f\langle 2,m\rangle\Big|^2 + \frac{1}{3^2\ln 2}\Big|\sum_{m=0}^{8\cdot 3-1} f\langle 3,m\rangle\Big|^2$$

$$+\sum_{n=3}^{\infty}\frac{1}{(n+1)^2\ln n}\Big|\sum_{m=0}^{8(n+1)-1} f\langle n+1,m\rangle\Big|^2 - \sum_{n=3}^{\infty}\frac{1}{n^2\ln n}\Big|\sum_{m=0}^{8n-1} f\langle n,m\rangle\Big|^2$$

$$\le \frac{1}{2^3\,(\ln 2)^2}\Big|\sum_{m=0}^{8\cdot 2-1} f\langle 2,m\rangle\Big|^2 + \frac{1}{3^2\ln 2}\Big|\sum_{m=0}^{8\cdot 3-1} f\langle 3,m\rangle\Big|^2$$

$$+\sum_{n=3}^{\infty}\frac{1}{n^2\ln n}\Big|\sum_{m=0}^{8(n+1)-1} f\langle n+1,m\rangle\Big|^2 - \sum_{n=3}^{\infty}\frac{1}{n^2\ln n}\Big|\sum_{k=0}^{8n-1} f\langle n,k\rangle\Big|^2.$$

By Lemma 4.2, we can take care of the first two terms and write

$$A_3 \lesssim B_0 + \sum_{n=3}^{\infty} \frac{1}{n^2 \ln n} \Big| \sum_{m=0}^{8(n+1)-1} f\langle n+1, m\rangle \Big|^2 - \sum_{n=3}^{\infty} \frac{1}{n^2 \ln n} \Big| \sum_{k=0}^{8n-1} f\langle n, k\rangle \Big|^2 .$$

In the sum

$$\Big| \sum_{m=0}^{8(n+1)-1} f\langle n+1, m\rangle \Big|^2 ,$$

separate eight terms corresponding to the following values of m:

$$m = 2s(n+1) + t, \;\; s = 0, 1, 2, 3, \;\; t = 2n+1, 2n+2.$$

The remaining $8n$ terms will be paired with the corresponding terms in the sum

$$\Big| \sum_{k=0}^{8n-1} f\langle n, k\rangle \Big|^2 .$$

Figure 1 shows (thick lines) the sample edges pairing $f\langle 3,1\rangle$ with $f\langle 2,0\rangle$, $f\langle 3,2\rangle$ with $f\langle 2,1\rangle$, $f\langle 3,3\rangle$ with $f\langle 2,2\rangle$, $f\langle 3,4\rangle$ with $f\langle 2,3\rangle$, and $f\langle 3,8\rangle$ with $f\langle 2,5\rangle$. The excluded nodes on ∂D_3 are circled.

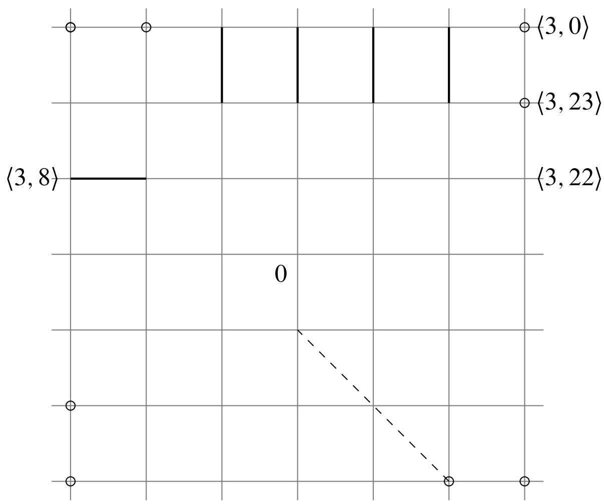

Figure 1. Paths.

Since

$$\Big|\sum_{m=0}^{8(n+1)-1} f\langle n+1,m\rangle\Big|^2 \lesssim \sum_{s=0}^{3} |f\langle n+1, 2s(n+1)+2n+1|^2 + \sum_{s=0}^{3} |f\langle n+1, 2s(n+1)+2n+2\rangle|^2 + \Big|\dot{\sum_{m=0,\dots,8(n+1)-1}} f\langle n+1,m\rangle\Big|^2,$$

where $\dot{\sum}$ stands for the sum without eight terms, we have

$$A_3 \lesssim B_0 + A_4 + A_5,$$

where

$$A_4 := \sum_{n=3}^{\infty} \frac{1}{n^2 \ln n} \sum_{s=0}^{3} |f\langle n+1, 2s(n+1)+2n+1\rangle|^2 + \sum_{n=3}^{\infty} \frac{1}{n^2 \ln n} \sum_{s=0}^{3} |f\langle n+1, 2s(n+1)+2n+2\rangle|^2$$

and

$$A_5 := \sum_{n=3}^{\infty} \frac{1}{n^2 \ln n}\bigg(\Big|\dot{\sum_{m=0,\dots,8(n+1)-1}} f\langle n+1,m\rangle\Big|^2 - \Big|\sum_{k=0}^{8n-1} f\langle n,k\rangle\Big|^2\bigg).$$

The sum A_5 can be estimated as follows. Since

$$\Big(\sum_{1}^{N} a_j\Big)^2 - \Big(\sum_{1}^{N} b_j\Big)^2 = \Big(\sum_{j=1}^{N}(a_j - b_j)\Big)\Big(\sum_{k=1}^{N}(a_k + b_k)\Big) \le N\,\frac{1}{4\epsilon_N} \sum_{j=1}^{N}(a_j - b_j)^2 + N\,\epsilon_N \sum_{j=1}^{N}(a_j + b_j)^2,$$

we have

$$\Big|\dot{\sum_{m=0,\dots,8(n+1)-1}} f\langle n+1,m\rangle\Big|^2 - \Big|\sum_{k=0}^{8n-1} f\langle n,k\rangle\Big|^2 \le 8n\,\frac{1}{4\epsilon_n} \sum_{\sim}(f\langle n+1,m\rangle - f\langle n,k\rangle)^2 + 8n\,\epsilon_n \sum_{\sim}(f\langle n+1,m\rangle + f\langle n,k\rangle)^2$$

$$\leq 8n\,\frac{1}{4\epsilon_n}\sum_{\sim}(f\langle n+1,m\rangle - f\langle n,k\rangle)^2$$
$$+\,16n\,\epsilon_n\sum_{\sim}|f\langle n+1,m\rangle|^2+|f\langle n,k\rangle|^2.$$

Choose ϵ_n so that $16n\,\epsilon_n = \epsilon_0/\ln n$, i.e.,

$$\epsilon_n = \frac{\epsilon_0}{16n\ \ln n},$$

with some small ϵ_0 to be chosen later. Then,

$$\left|\sum^{\bullet}_{m=0,\ldots,8(n+1)-1} f\langle n+1,m\rangle\right|^2 - \left|\sum_{k=0}^{8n-1} f\langle n,k\rangle\right|^2$$
$$\leq 32n^2\ \ln n\ \frac{1}{\epsilon_0}\ \sum_{\sim}(f\langle n+1,m\rangle - f\langle n,k\rangle)^2\ \cdot$$
$$+\,\frac{\epsilon_0}{\ln n}\sum_{\sim}|f\langle n+1,m\rangle|^2+|f\langle n,k\rangle|^2.$$

This leads to the following estimate

$$A_5 \leq \frac{32}{\epsilon_0}\sum_{n=3}^{\infty}\sum_{\sim}(f\langle n+1,m\rangle - f\langle n,k\rangle)^2$$
$$+\,\epsilon_0\sum_{n=3}^{\infty}\frac{1}{n^2(\ln n)^2}\ \sum_{\sim}|f\langle n+1,m\rangle|^2+|f\langle n,k\rangle|^2,$$

which implies

$$A_5 \lesssim \frac{32}{\epsilon_0}\,B_0+\epsilon_0\,A_0.$$

Now look at the excluded eight terms comprising A_4. To each of them we can apply the one-dimensional Hardy inequality (1.3). Consider, for example,

$$\sum_{n=3}^{\infty}\frac{1}{n^2\ln n}|f\langle n+1,2s(n+1)+2n+1\rangle|^2.$$

Define temporarily $u(n) = f\langle n+1, 2s(n+1)+2n+1\rangle$ for $n \geq 3$ and $u(n) = 0$ for $n = 0, 1, 2$. Then,

$$\sum_{n=3}^{\infty}\frac{1}{n^2\ln n}|f\langle n+1,2s(n+1)+2n+1\rangle|^2 \leq 4\sum_{n=1}^{\infty}|u(n)-u(n-1)|^2.$$

In the case $s = 3$, the discrete function $u(n)$ picks up the values of $f(x)$ from the lattice points on the dashed line on Fig. 1. Although each difference $u(n)-u(n-1)$

is not between the values at the neighboring lattice points, it can be modified by adding and subtracting a value at one of the two lattice points on either side of the dashed line segment. In addition, we must remember that the sum is from $n = 3$. Hence,

$$\sum_{n=1}^{\infty} |u(n) - u(n-1)|^2 \leq 2\, B_0 + |f\langle 3+1, 2s(3+1) + 2\cdot 3 + 1\rangle|^2.$$

The last term is bounded by B_0 in view of Lemma 4.2. Thus,

$$A_4 \lesssim B_0.$$

Putting all the estimates together, we complete the proof of Theorem 4.1.

References

[1] A. A. Balinsky, W. D. Evans, and R. T. Lewis, *The analysis and geometry of Hardy's inequality.* Universitext. Springer, Cham, 2015. MR 3408787 Zbl 1332.26005

[2] E. B. Davies, A review of Hardy inequalities. In J. Rossmann, P. Takáč, and G. Wildenhain (eds.), *The Maz′ya anniversary collection.* Vol. 2. Rostock Conference on Functional Analysis, Partial Differential Equations and Applications. Papers from the conference held at the University of Rostock, Rostock, August 31–September 4, 1998. Operator Theory: Advances and Applications, 110. Birkhäuser Verlag, Basel, 1999, 55–67. MR 1747888 Zbl 0936.35121

[3] V. Felli, E. M. Marchini, and S. Terracini, On Schrödinger operators with multisingular inverse-square anisotropic potentials. *Indiana Univ. Math. J.* **58** (2009), no. 2, 617–676. MR 2514383 Zbl 1169.35013

[4] G. H. Hardy, Note on a theorem of Hilbert. *Math. Z.* **6** (1920), no. 3-4, 314–317. MR 1544414 JFM 47.0207.01

[5] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities.* Second edition. Cambridge University Press, Cambridge, 1952. MR 0046395 Zbl 0047.05302

[6] A. Kufner, L . Maligranda, and L.-E. Persson, The prehistory of the Hardy inequality. *Amer. Math. Monthly* **113** (2006), no. 8, 715–732. MR 2256532 Zbl 1153.01015

[7] A. Kufner and B. Opic, *Hardy-type inequalities.* Pitman Research Notes in Mathematics Series, 219. Longman Scientific & Technical, Harlow, 1990. MR 1069756 Zbl 0698.26007

[8] V. G. Maz′ya, *Sobolev spaces.* Translated from the Russian by T. O. Shaposhnikova. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1985. MR 0817985 Zbl 0692.46023

[9] E. Mitidieri, A simple approach to Hardy inequalities. *Mat. Zametki* **67** (2000), no. 4, 563–572. In Russian. English translation, *Math. Notes* **67** (2000), no. 3-4, 479–486. MR 1769903 Zbl 0964.26010

[10] G. Rozenblum and M. Solomyak, On the spectral estimates for the Schrödinger operator on $\mathbb{Z}^d$, $d \geq 3$. *Probl. Mat. Anal.* **41** (2009), 107–126. In Russian. English translation, *J. Math. Sci.* (*N.Y.*) **159** (2009), no. 2, 241–263. MR 2544038 Zbl 1207.35237

[11] G. Rozenblum and M. Solomyak, On the spectral estimates for the Schrödinger operators in global dimension 2. *Algebra i Analiz* **25** (2013), no. 3, 185–199. Reprint, *St. Petersburg Math. J.* **25** (2014), no. 3, 495–505. MR 3184603 Zbl 1304.35454

[12] F. Takahashi, A simple proof of Hardy's inequality in a limiting case. *Arch. Math.* (*Basel*) **104** (2015), no. 1, 77–82. MR 3299153 Zbl 1308.35010

Received February 7, 2016

Lev Kapitanski, Department of Mathematics, University of Miami, Coral Gables, FL 33124, USA

e-mail: levkapit@math.miami.edu

Ari Laptev, Imperial College London, 180 Queen's Gate, London SW7 2AZ, UK

e-mail: a.laptev@imperial.ac.uk