Irregular time-dependent perturbations of quantum Hamiltonians

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Abstract. Our main goal in this paper is to prove existence (and uniqueness) of the quantum propagator for time-dependent quantum Hamiltonians $\hat{H}(t)$ when this Hamiltonian is perturbed with a quadratic white noise $\dot{\beta}\hat{K}$. β is a continuous function in time t, $\dot{\beta}$ its time-derivative and K is a quadratic Hamiltonian. $\hat{K}$ is the Weyl quantization of K.

For time-dependent quadratic Hamiltonians $H(t)$ we recover, under less restrictive assumptions, the results obtained in [3, 9, 10]. In our approach we use an exact Herman–Kluk formula [20] to deduce a Strichartz estimate for the propagator of $\hat{H}(t) + \dot{\beta}\hat{K}$.

This is applied to obtain local and global well posedness for solutions for non-linear Schrödinger equations with an irregular time-dependent linear part.

1. Introduction

The linear time-dependent Schrödinger equation was studied in [22, 23] for time dependent potential at least continuous in time. There are physical motivations to consider Schrödinger equations perturbed by quadratic potentials times a white noise in time (see for [3, 9, 10]). But the constructions elaborated in Fujiwara [11] and Yajima [22] are no more valid for time-discontinuous Hamiltonians. Nevertheless it has been shown in [3, 9, 10] that these constructions, based on Fourier integral representations of the propagator, can be revisited and extended

when the time dependence in the Hamiltonian is irregular, as for white noise. The main motivation for considering irregular time dependent Hamiltonian comes from Bose Einstein condensation or fiber optics (see [3] and its bibliography).

The main idea developed in [9, 10] to construct a propagator $U_\beta(t, s)$ for the quantum Hamiltonian $\widehat{H}(t) + \dot{\beta}\widehat{K}$, where β is in some Hölder class (so its derivative $\dot{\beta}$ is a distribution in time), is to establish a suitable representation formula when β is C^1-smooth in time and to prove that the dependence of $U_\beta(t, s)$ in β is continuous for the topology of the Hölder space $C^\mu(I_T)$, $0 \le \mu < 1$ in a time interval $I_T = [t_0 - T, t_0 + T]$, $T > 0$.

This strategy was initiated by Sussmann [21] for solving stochastic differential equations by deterministic methods.

In this paper we shall extend the main results of [3, 9, 10] to more general quadratic Hamiltonians by using a different approach. Instead of establishing a generalized Mehler formula for the time-dependent propagator we choose to use a formula inspired from the Herman–Kluk formula [20], which is more flexible. The advantage of this approach is that the link between classical and quantum mechanics is straightforward, we do not need to take care of caustics because it is not necessary to solve the classical Hamilton–Jacobi equation as in the Hörmander–Maslov approach. The quantum oscillations are represented by complex phases so that a Melher (or Van Vleck) type formula can be recovered by a stationary phase argument. This is related to complex WKB analysis and coherent states (see [7] and its bibliography).

We shall extend here several well known results in the $\mathcal{C}^1$ time regular case when β in only continuous or in an Hölderian class of order $\mu \in]0, 1[$. For example β could be a trajectory of a one dimensional Brownian motion ($\mu = \frac{1}{2} - \varepsilon$) or of a fractional Brownian motion ($\mu = H - \varepsilon$, $H \in]0, 1[$ being the Hurst index of the process).

2. Mathematical Settings and Results

Let $H(t)$ be a time-dependent real polynomial Hamiltonian, of degree at most 2 in the phase space space $\mathbb{R}_q^d \times \mathbb{R}_p^d$, with continuous coefficients in $t \in I_T = [t_0 - T, t_0 + T]$, K is a real polynomial time-independent Hamiltonian of degree at most 2.

Let β be a continuous function of time t. Denote $H_\beta(t) = H(t) + \dot\beta K$ an irregular perburbation of $H(t)$ where $\dot w = \frac{d\beta}{dt}$. H_β is here a distribution in the time t so the meaning of the classical Hamilton equation is not clear. Denote $z = (q, p)$ a generic point in the phase space $\mathbb{R}_q^d \times \mathbb{R}_p^d$. The classical Hamilton system is

$$\begin{cases} \dot q_\beta = \partial_p H_\beta(q_\beta, p_\beta), \\ \dot p_\beta = -\partial_q H_\beta(q_\beta, p_\beta), \end{cases} \tag{1}$$

where $q_\beta = q_\beta(t, s)$, $p_\beta = p_\beta(t, s)$ with initial data at time $t = s$, $q_\beta(s, s) = q(s)$, $p_\beta(s, s) = p(s)$.

If β is C^1 the classical evolution (1) is linear and so well defined. Let us denote $\Phi_\beta(t, s)z = (q_\beta(t, s), p_\beta(t, s))$ where $z = (q, p)$, $q = q(s)$, $p = p(s)$. Using the method developed in [21, 9, 10] the Hamiltonian flow $\Phi_\beta(t, s)$ can be extended in a natural way as a symplectic map for $\beta \in \mathcal{C}^0$.

Let us now consider the quantum evolution. If β is $C^1(I_T)$ denote by $\hat H_\beta(t)$ the Weyl quantization of $H_\beta(t, q, p)$, see for example [7]. Here the Planck constant is fixed, so we choose $\hbar = 1$.

It is well known that $\hat H_\beta(t)$ is a self-adjoint operator in $L^2(\mathbb{R}^d)$ and that the time-dependent Schrödinger equation generates a continuous family of unitary operators in $L^2(\mathbb{R}^d)$, which we denote by $U_\beta(t, s)$, satisfying

$$i\,\partial_t U_\beta(t, s) = \hat H_\beta(t)U_\beta(t, s), \quad U_\beta(s, s) = \mathbb{I}. \tag{2}$$

Denote

$$C_R^0(I_T) = \{\beta \in C^0(I_T), \|\beta\|_\infty \le R\} \quad \text{and} \quad C_R^1(I_T) = C_R^0(I_T) \cap C^1(I_T)$$

(equipped with the sup-norm). We have the following preliminary result.

Theorem 2.1. (i) *The map $\beta \mapsto \Phi_\beta$ is a Lipschitzian map from $C_R^1(I_T)$ into $C^0(I_T \times I_T, S(2d))$ where $S(2d)$ is the space of linear symplectic maps of $\mathbb{R}^{2d}$. In particular for any $T > 0$, the map $(\beta, t, s) \mapsto \Phi_\beta(t, s)$ can be extended in a unique continuous map from $C_R^0(I_T) \times I_T \times I_T$ into the affine symplectic group of $\mathbb{R}^{2d}$.*

(ii) *For any $\psi \in \mathcal{S}(\mathbb{R}^d)$, the map $\beta \mapsto U_\beta(t, s)\psi$ is uniformly continuous on $C_R^1(I_T)$. In particular the map $(\beta, t, s) \mapsto U_\beta(t, s)$ can be extended in a unique continuous map from $C_R^0(I_T) \times I_T \times I_T$ into the unitary group of $L^2(\mathbb{R}^d)$.*

Part (i) of Theorem 2.1 will be proved in Section 3 and part (ii) in Section 4.2.

Let $\mathcal{K}_\beta(t, s; x, y)$ be the (distribution) Schwartz kernel of $U_\beta(t, s)$. The next result is an exact formula for $\mathcal{K}_\beta(t, s; x, y)$ depending only on the classical dynamics $\Phi_\beta(t, s)$. A more precise statement will be given later in Corollary 3.3.

Theorem 2.2. *For every $\beta \in \mathcal{C}^0(I_T)$ there exist explicit complex functions $a_\beta(t, s)$ and $\Psi_\beta(t, s, z, x, y)$ where $t, s \in I_T$, $z = (q, p) \in \mathbb{R}^{2d}$, $x, y \in \mathbb{R}^d$, such that*

$$\mathcal{K}_\beta(t, s; x, y) = a_\beta(t, s) \int_{\mathbb{R}^{2d}} \exp(i \Psi_\beta(t, s, z, x, y)) dz. \tag{3}$$

Moreover Ψ_β is polynomial of degree at most 2 in z and $\Im \Psi_\beta \geq 0$.

The maps $(\beta, t, s) \mapsto a_\beta(t, s)$ and $(\beta, t, s) \mapsto \Psi_\beta(t, s, z, x, y)$ are continuous on $C_R^0(I_T) \times I_T \times I_T$.

Equation (3) is an equality between two distributions in the Schwartz space $\mathcal{S}'(\mathbb{R}_x^d \times \mathbb{R}_y^d)$.

Notice that formula (3) is valid without condition on the time interval I_T, the caustics are not obstructions here. Of course this difficulty appears again when computing the integral in $z \in \mathbb{R}^{2d}$ to get the following result. To go further we need the following hypothesis.

Hypothesis 2.3. The Hessian matrix $\partial^2_{p,p} H$ (constant here) is non singular and $\partial^2_{p,p} K = 0$.

Hypothesis 2.4. $\partial^2_{q,p} K = 0$ or $\beta \in \mathcal{C}^\mu(I_T)$ with $\mu > \frac{1}{2}$.

Remark 2.5. If $d = 3$ and if $\partial^2_{q,p} K$ is an antisymmetric matrix, it represents an angular momentum rotation term. This case was considered in [1] without noise and in [9, 10] for perturbations by noise.

The following result is a consequence of Sections 4.1 and 5.1.

Theorem 2.6. *Assume that Hypothesis 2.3 and Hypothesis 2.4 are satisfied.*

(I) *For every $R > 0$ there exists $T_R > 0$ such that for every $t, s \in I_{T_R}$ and every β such that if $\beta \in \mathcal{C}^\mu(I_T)$, $\|\beta\|_{\mathcal{C}^\mu} \leq R$ and $t \neq s$, the Schwartz kernel $\mathcal{K}_\beta(t, s)$ of $U_\beta(t, s)$ is a C^∞ function of (x, y) given by the following formula*

$$\mathcal{K}_\beta(t, s; x, y) = b_\beta(t, s)^{-d/2} \mathrm{e}^{i S_\beta(t, s; x, y)}, \tag{4}$$

where $b_\beta(t, s)$ is continuous in (β, t, s), $t \neq s$, $S_\beta(t, s; x, y)$ is the classical action along the unique classical trajectory joining y to x at time s.

Moreover there exists $\gamma > 0$ such that $|b_\beta(t, s)| \geq \gamma |t - s|$ for every $t, s \in I_{T_R}$.

(II) *There exists a constant C_R, depending only on R such that for every $t, s \in I_{TR}$ and every $x, y \in \mathbb{R}^d$, we have*

$$|\mathcal{K}_\beta(t, s; x, y)| \le C_R |t - s|^{-d/2}. \tag{5}$$

and for every $p \in [2, +\infty]$, we have for $\psi \in L^p(\mathbb{R}^d)$,

$$\|U_\beta(t, s)\psi\|_{L^p(\mathbb{R}^d)} \le C_R |t - s|^{-d(1/2-1/p)} \|\psi\|_{L^{p'}(\mathbb{R}^d)}, \quad 1/p + 1/p' = 1. \tag{6}$$

As it is well known the dispersive estimate (6) is closely related with Strichartz estimates (see [15]) and allows application to non-linear Schrödinger equations. The case with noise was considered in [3, 9].

Let us consider the non-linear Schrödinger equation (NLS):

$$i\partial_t \psi = \hat{H}_\beta(t)\psi + \lambda|\psi|^{2\sigma}\psi, \quad \psi(s) = \psi, \tag{7}$$

where $\lambda \in \mathbb{R}, \sigma > 0$.

Here $H_\beta(t)$ is irregular in time t so we have to consider the following integral mild version of (7)

$$\psi(t) = U_\beta(t, s)\psi - i\lambda \int_s^t U_\beta(t, u)|\psi(u)|^{2\sigma}\psi(s)du. \tag{8}$$

Let us introduce the Sobolev weighted spaces associated with the harmonic oscillator:

$$\mathcal{H}^k(\mathbb{R}^d) = \{\psi \in L^2(\mathbb{R}^d), \ \psi \in H^k(\mathbb{R}^d), \ |x|^k\psi \in L^2(\mathbb{R}^d)\},$$

where $H^k(\mathbb{R}^d)$, $k \in \mathbb{N}$, is the usual Hilbertian Sobolev space.

We shall see that these spaces are invariant by the quantum propagator $U_\beta(t, s)$ for any $\beta \in \mathcal{C}^0(I_T)$. In order to include the Gross–Pitaevski non-linearity ($\sigma = 1$) we have to consider initial data in the space $\mathcal{H}^1(\mathbb{R}^d)$. Here we have the following local result proved in Section 5.

Theorem 2.7. *We assume that Hypothesis 2.3 and Hypothesis 2.4 are satisfied.*

(I) *If $0 < \sigma < \frac{2}{d}$, then then for any $\psi \in L^2(\mathbb{R}^d)$ the integral equation (8) has a unique solution $\psi_\beta \in \mathcal{C}^0(I_T, L^2(\mathbb{R}^d))$. Moreover, for every $T > 0$, we have $\psi_\beta \in \mathcal{C}^0(I_T, L^2(\mathbb{R}^d)) \cap L^r(I_T, L^{2\sigma}(\mathbb{R}^d))$, with $r = \frac{4(\sigma+1)}{d\sigma}$.*

The L^2 norm is conserved: $\|\psi_\beta(t)\|_{L^2(\mathbb{R}^d)} = \|\psi\|_{L^2(\mathbb{R}^d)}$ for every $t \in I_T$. Moreover if $\psi \in \mathcal{H}^1(\mathbb{R}^d)$ then $\psi_\beta \in \mathcal{C}^0(\mathbb{R}, \mathcal{H}^1(\mathbb{R}^d))$.

(II) *If* $0 < \sigma < \frac{2}{d-2}$, $d \geq 3$. *Then for any* $\psi \in \mathcal{H}^1(\mathbb{R}^d)$ *there exists* $0 < T = T(\|\psi\|_{\mathcal{H}^1(\mathbb{R}^d)}, s)$ *such that the integral equation* (8) *has a unique solution* $\psi_\beta \in \mathcal{C}^0(I_T, L^2(\mathbb{R}^d)) \cap L^r(I_T, L^{2\sigma}(\mathbb{R}^d))$, *with* $r = \frac{4(\sigma+1)}{d\sigma}$ *and such that for any* $a, b \in \mathbb{R}^d$,

$$(a \cdot x + b \cdot \nabla_x)\psi_\beta \in \mathcal{C}^0(I_T, L^2(\mathbb{R}^d)) \cap L^r((I_T), L^{2\sigma}(\mathbb{R}^d)).$$

Remark 2.8. A global well-posedness result in the $\mathcal{H}^1$-subcritical case with a rotation term for (8) is proved in [1] for time-independent Hamiltonians. For $\sigma < \frac{2}{d}$ (L^2-subcritical non-linearity) a global result in $\mathcal{H}^1(\mathbb{R}^d)$ could be obtained under the assumptions of Theorem 2.7 following [1, Theorem 2.2] but for $\frac{2}{d} \leq \sigma < \frac{2}{d-2}$, $d \geq 2$ and $\lambda \geq 0$ (defocusing case) the situation is much more involved because we cannot use the energy conservation. In [9, Theorem 3] the author uses the result of [1] to get a global $\mathcal{H}^1$-well-posedness result with a rotation term in the regular part H assuming that K is linear in (q, p).

Notice that global well-posedness results in the supercritical case are proved in [18] for the Gross–Pitaevski equation with random initial data.

Remark 2.9. The non-linear Schrödinger equation with a white noise dispersion is also considered in [2] (see also the references of [2]). These papers use the probabilistic setting of stochastic processes.

3. The smooth time-dependent case

In this section we assume that $H(t)$ is a time-dependent polynomial Hamiltonian, of degree at most 2, with continuous coefficients in $t \in I_T = [t_0 - T, t_0 + T]$, K is time-independent. Let β be a continuous function of time t. Denote $H_\beta(t) = H(t) + \dot{\beta} K$ the irregular perburbation of $H(t)$ where $\dot{\beta} = \frac{d\beta}{dt}$.

If β is C^1 the classical and quantum evolutions are well defined. We shall show in the next section that these evolutions are still well defined for $\beta \in C^0(I_T)$ following an approach inspired from [21, 9].

We shall review here some more or less well known formulas concerning quantum time-dependent quadratic Hamiltonians.

It is well known that classical and quantum evolution are well defined if $H(t)$ is continuous in time (and for $H_\beta(t)$ if β is C^1) and there exist exact formulas related the classical and quantum evolution. Denote by $\Phi_H(t, s)$ the classical flow in the phase space $\mathbb{R}^{2d}$, at time t with initial data at s and by $U_H(t, s)$ the quantum propagator generated by the Weyl quantization $\hat{H}(t)$ of $H(t)$.

Let us recall now a formulation of the exact correspondence classical–quantum. We have

$$H(t) = H_2(t) + H_1(t) + H_0(t),$$

where $H_j(t)$ is a polynomial of degree j in $(q, p) \in \mathbb{R}^{2d}$. Let us denote $S_{H_2}(t)$ the matrix of the quadratic form $H_2(t)$. Explicitly,

$$H_2(t; q, p) = \frac{1}{2}(G_H(t)q \cdot q + 2L_H(t)q \cdot p + E_H(t)p \cdot p), \quad p, q \in \mathbb{R}^d.$$

and

$$S_{H_2}(t) = \begin{pmatrix} G_H(t) & L_H(t)^\top \\ L_H(t) & E_t \end{pmatrix},$$

where $q, p \in \mathbb{R}^d$, and $G_H(t)$, $L_H(t)$, and $E_H(t)$ are real, $d \times d$ matrices, continuous in time $t \in \mathbb{R}$, $E_H(t), G_H(t)$ are symmetric, $L_H(t)^\top$ is the transposed matrix of $L_H(t)$. The classical motion defined by $H_2(t)$ in the phase space $\mathbb{R}^{2d}$, is given by the linear differential equation

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = J. \begin{pmatrix} G_H(t) & L_H(t)^T \\ L_H(t) & E_H(t) \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}, \quad J = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}, \tag{9}$$

where the matrix J defines the symplectic form

$$\sigma(z, z') := Jz \cdot z', \quad z = (q, p), z' = (q', p'),$$

where $z \cdot z'$ denotes the scalar product in $\mathbb{R}^{2d}$.

This equation defines a linear symplectic transformation, $\Phi_{H_2}(t, s)$, such that $\Phi_{H_2}(s, s) = \mathbb{I}$. It can be represented as a $2d \times 2d$ matrix which can be written as four $d \times d$ blocks:

$$\Phi_{H_2}(t, s) = \begin{pmatrix} A(t, s) & B(t, s) \\ C(t, s) & D(t, s) \end{pmatrix}. \tag{10}$$

Let us denote $\hat{H}$ the Weyl quantization of the Hamiltonian H (see [7] for the definition and properties of the Weyl quantization).

Let us denote $\mathcal{K}_{H_2}(t, t_0; x, y)$ the Schwartz kernel of the quantum propagator $U_{H_2}(t, t_0)$.

It is known that the propagator $U_H(t, t_0)$ is well defined ([7], p.67). It is unique and satisfies the properties

$$i\partial_t U_H(t, s) = \hat{H}(t)U_H(t, s), \quad U_H(s, s) = \mathbb{I}. \tag{11}$$

Let $\mathcal{K}_H(t, t_0) \in \mathcal{S}'(\mathbb{R}_x^d \times \mathbb{R}_y^d)$ be the Schwartz Kernel of $U_H(t, t_0)$. There exist many papers giving more or less explicit formula for $\mathcal{K}_H(t, t_0)$, see [13, 17, 20]. For our purpose it is convenient to use a formula closely related with coherent states and symplectic geometry of the phase space (for details see [20]).

Let us introduce the Siegel space $\Sigma_+(d)$ of $d \times d$ complex matrices Γ with imaginary part

$$\Im\Gamma := \frac{\Gamma - \Gamma^\top}{2i}$$

definite-positive. Let be Θ be a continuous map from I_T into $\Sigma_+(d)$ and

$$M_\Theta(t, t_0) = (C(t, t_0) - iD(t, t_0) - \Theta(t)(A(t, t_0) - iB(t, t_0))).$$

The exact correspondence between classical and quantum mechanics can be expressed as follows. Let us denote $\mathcal{K}(t, t_0; x, y)$ the Schwartz kernel of the quantum propagator $U_{H_2}(t, s)$.

Proposition 3.1 (Herman–Kluk formula in the quadratic case [20]). *We have the following exact formula*

$$\mathcal{K}_{H_2}(t, t_0; x, y) = 2^{d/2}(2\pi)^{-3d/2}\det^{-1/2}\Big(\frac{M_\Theta(t, t_0)}{i}\Big) \int_{\mathbb{R}^{2d}} e^{i\Psi_{\Theta,2}(t, t_0; z; x, y)} dz,$$
(12)

where

$$\Psi_{\Theta,2}(t, t_0; z; x, y) = \frac{1}{2}(q_t \cdot p_t - q \cdot p) + p_t \cdot (x - q_t) - p \cdot (y - q)$$
$$+ \frac{1}{2}(\Theta(t)(x - q_t) \cdot (x - q_t) + i(y - q) \cdot (y - q)),$$

with $z = (q, p) \in \mathbb{R}^d \times \mathbb{R}^d$.

Remark 3.2. Θ is a useful degree of freedom to compute $\mathcal{K}_{H_2}(t, t_0; x, y)$. The choices $\Theta = i\mathbb{I}$ and $\Theta(t) = \Gamma(t, t_0)$ can be useful, where

$$\Gamma(t, t_0) = C(t, t_0) + iD(t, t_0)(A(t, t_0) + iB(t, t_0))^{-1}.$$

In [20] Θ is supposed to be C^1 in t. The result is clearly valid for Θ only continuous. In formula (2.12) of [20] we have to read $\det^{-1/2}$ and not $\det^{1/2}$. Notice that $M_\Theta(t, t_0)$ is invertible (property of the action of symplectic matrices on the Siegel space).

Adding now lower order terms we get

Corollary 3.3. *Suppose now that* $H(t) = H_2(t) + H_1(t) + H_0(t)$ *where* $H_j(t)$ *is homogeneous of degree* j *in* $z = (q, p) \in \mathbb{R}^{2d}$.

Then the Schwartz kernel $\mathcal{K}_H(t, t_0; x, y)$ *of* $U_H(t, t_0)$ *has the following expression*:

$$
\mathcal{K}_H(t, t_0; x, y) U_{H_2}(t, t_0)
$$

$$
= 2^{d/2} (2\pi)^{-3d/2} \det{}^{-1/2} \Big(\frac{M(t, t_0)}{i} \Big) \int_{\mathbb{R}^{2d}} e^{i \Psi(t, t_0; z; x, y)} dz, \tag{13}
$$

where

$$
\Psi_\Theta(t, t_0, z, x, y)
$$

$$
= \Psi_{\Theta,2} \Big(t, t_0; z; x, y + \int_{t_0}^t \tilde{b}(s) ds \Big) - \int_{t_0}^t \tilde{a}(s) y_s ds - \int_{t_0}^t H_0(s) ds,
$$

with

$$
y_s = y + \int_t^s \tilde{b}(s) ds,
$$

and $\tilde{a}$ *and* $\tilde{b}$ *depend on* $H_1(t)$ *and are given in the proof.*

When applied to $H_\beta(t)$ *(here* $\beta \in \mathcal{C}^1(I_T)$) *we use the notations* $\mathcal{K}_\beta = \mathcal{K}_{H_\beta}$ *and* $\psi_\beta = \psi$.

Proof. It is enough to assume that $H_0(t) = 0$. Recall here a well known argument (Lagrange method). Let us compute $V(t, t_0)$ such that

$$
U_H(t, t_0) = U_{H_2}(t, t_0) \cdot V(t, t_0).
$$

We get

$$
i \partial_t V(t, t_0) = U_{H_2}(t_0, t) H_1(t) U_{H_2}(t, t_0) V(t, t_0). \tag{14}
$$

We have $H_1(t; q, p) = a(t) \cdot q + b(t) \cdot p$. Using the exact Egorov formula [7] for quadratic Hamiltonians we obtain

$$
U_{H_2}(t_0, t) \hat{H}_1(t) U_{H_2} = \hat{A}(t),
$$

where $A(t, z) = H_1(t, \Phi_{H_2}(t, t_0) z)$. Then by the characteristics method we conclude

$$
V(t, t_0) \psi(t_0, x) = \exp \Big(i \int_{t_0}^t \tilde{a}(s) x_s ds \Big) \psi \Big(t_0, x - \int_{t_0}^t \tilde{b}(s) ds \Big),
$$

where $\big(\begin{smallmatrix} \tilde{a} \\ \tilde{b} \end{smallmatrix} \big) = \Phi_{H_2}(t, t_0)^\top \big(\begin{smallmatrix} a \\ b \end{smallmatrix} \big)$, $N^\top$ is the transposed matrix of the matrix N, $x_s = x + \int_t^s \tilde{b}(s) ds$. The corollary follows. □

4. The time-irregular case

4.1. The classical evolution. What remains true of the previous computations for $H_\beta(t) = H(t) + \dot\beta K$ when β in only continuous in I_T?

For the noise part $\dot\beta K$ the classical evolution is linear:

$$\Phi_{\dot\beta K}(t, s, z) = \Phi_K(\beta_t - \beta_s)z.$$

Let $z(s) \in \mathbb{R}^{2d}$ be an initial data. Then $z_\beta(t) = \Phi_K(\beta_t, \beta_s, z_s)$ is a solution in Sussmann [21] sense of the Hamilton equation

$$\dot z_\beta(t) = \dot\beta(t) J \nabla_z K(z_\beta(t)), \quad z_\beta(s) = z(s). \tag{15}$$

We have here

$$z_\beta(t) = \exp((\beta(t) - \beta(s))J S_K)z.$$

Now let us consider the perturbed Hamiltonian $H_\beta(t) = H(t) + \dot\beta K(t)$. We want to define a classical trajectory $z_\beta(t) = \Phi_{H_\beta}(t, s, z(s))$ for the perturbed Hamilton equation

$$\dot z_\beta(t) = J \nabla_z H_\beta(z_\beta(t)), \quad z_\beta(s) = z(s). \tag{16}$$

Definition 4.1. $z_\beta(t)$ is a Sussmann solution of (16) if

(CL0) there exists a neighborhood $\mathcal{N}_\beta$ of β in $C^0(I_T)$ such that if one has $\mathcal{N}_\beta^1 = \mathcal{N}_\beta \cap C^1(I_T)$, then $\tilde\beta \mapsto z_{\tilde\beta}(t)$ is a uniformly continuous map from $\mathcal{N}_\beta^1$ into $C^0(I_T, \mathbb{R}^{2d})$;

(CL1) for every $\varepsilon > 0$, $z_{\beta^\varepsilon}(t)$ solves (16) for the C^1 function β^ε;

(CL2) $\lim_{\varepsilon \to 0} z_{\beta^\varepsilon}(t) = z_\beta(t)$ in $C^0(I_T, \mathbb{R}^d)$;

$C^0(I_T)$ is equipped with its natural norm $\|\beta\|_\infty = \sup_{t \in I_T} |\beta(t)|$.

Properties (CL0), (CL1), and (CL2) define a unique mild solution of (16). In particular $z_\beta(t)$ is independent on the C^1 approximations β^ε of β.

We have to prove that conditions (CL0), (CL1) and (CL2) are fulfilled.

Recall that $H(t)$ and K are quadratic forms on $\mathbb{R}^{2d}$. If $\beta \in C^1(I_T)$ then it is well known that $\Phi_{H_\beta}(t, s)$ is a symplectic linear transformation of the phase space $\mathbb{R}^{2d}$. It is convenient here to consider that the Hamiltonian H_β is a perturbation of the noise term $\dot\beta K$. Then it solves the following integral equation

$$\Phi_{H_\beta}(t, s) = \Phi_K(\beta_t - \beta_s) + \int_s^t \Phi_K(\beta_t - \beta_\tau)J S_H(\tau)\Phi_{H_\beta}(\tau, s)d\tau, \tag{17}$$

where $S_H(t)$ is the symmetric matrix of the quadratic form $H(t)$.

Now the trick is that we can solve equation (17) using the Picard fixed theorem. Denote

$$C_R^0(I_T) = \{\beta \in C^0(I_T), \|\beta\|_\infty \leq R\} \quad \text{and} \quad C_R^1(I_T) = C_R^0(I_T) \cap C^1(I_T)$$

(equipped with the sup-norm).

Proposition 4.2 (see also [10], Proposition 2.29). (1) *There exists $T_R > 0$ small enough such that for $T \leq T_R$ and $\beta \in C_R^0(I_T)$, equation 17 has a unique solution defined for $(t, s) \in I_T \times I_T$.*

(2) $\beta \mapsto \Phi_{H_\beta}$ *is a Lipschitzian map from $C_R^1(I_T)$ into $C^0(I_T \times I_T, S(2d))$ where $S(2d)$ is the space of linear symplectic maps of $\mathbb{R}^{2d}$.*

(3) $\Phi_{H_\beta}(t, s)$ *satisfies*

$$\Phi_{H_\beta}(t, t_1) = \Phi_{H_\beta}(t, s)\Phi_{H_\beta}(s, t_1), \quad \text{for all } t, t_1, s \in I_T. \tag{18}$$

In particular for any $T > 0$ $\Phi_{H_\beta}(t, s)$ can be extended to $I_T \times I_T$ in a unique way such that for every $z \in \mathbb{R}^{2d}$, $z_\beta(t) = \Phi_{H_\beta}(t, s)z$ satisfies (CL0), (CL1), *and* (CL2).

Proof. (1) is a direct application of the Picard fixed point theorem. First from a well known estimate for linear ODE we have, for some $\Gamma > 0$,

$$\|\Phi_K(\beta_t - \beta_s)\| \leq e^{\Gamma|\beta_t - \beta_s|}. \tag{19}$$

For $X \in C^0(I_T \times I_T, S(2d))$ denote

$$F_\beta(X) = \Phi_K(\beta_t - \beta_s) + \int_s^t \Phi_K(\beta_t - \beta_\tau)JS_H(\tau)X(\tau, s)d\tau. \tag{20}$$

So if $\beta \in C_R^0(I_T)$, F_β has unique fixed point X_β in $C^0(I_T \times I_T, S(2d))$ for $T \leq T_R$. Moreover there exists $C > 0$ such that

$$\|X_\beta(t, s)\| \leq Ce^{2\Gamma R}$$

and if $\beta \in C_R^1(I_T)$ then $X_\beta = \Phi_{H_\beta}$.

(2) $\beta \mapsto \Phi_K(\beta_t, \beta_s)$ is C^1 from $C_R^0(I_T)$ into $S(2d)$. Choosing $T_R > 0$ small enough, the derivative $D_X F_W(X_\beta)$ satisfies $\|D_X F_W(X_\beta)\| \leq \frac{1}{2}$. Applying the implicit function theorem we get that $\beta \mapsto \Phi_{H_\beta}$ is also C^1.

(3) is now easy to prove using that it is true for $\beta \in C^1$. $\qquad\square$

We can now add the contribution of order one. We have

$$H_\beta(t) = H_2(t) + H_1(t) + \dot{\beta}(K_2 + K_1).$$

Denote

$$H_{\beta,2}(t) = H_2(t) + \dot{\beta}K_2(t),$$

$$H_{\beta,1}(t) = H_1(t) + \dot{\beta}K_1.$$

$$H_{\beta,1}(t, z) = (V_H(t) + \dot{\beta}V_K) \cdot z.$$

We have, using the Duhamel formula, for every $z \in \mathbb{R}^{2d}$,

$$\Phi_{H_\beta}(t, s)z = \Phi_{H_{\beta,2}}(t, s)z + \int_s^t \Phi_{H_{\beta,2}}(t, u)J(V_H(u) + \dot{\beta}V_K)du, \qquad (21)$$

$\Phi_{H_{\beta,2}}(t, s)$ solves the integral equation (17). So plugging (17) for $H = H_2$ in (21) and integrating by parts we get

Corollary 4.3. *The map $\beta \mapsto \Phi_{H_\beta}(t, s)z$ given by* (21) *is C^1 from $C^0(I_T)$ into* $\mathbb{R}^{2d}$ *and $z_\beta(t)$ satisfies properties* (CL0), (CL1) *and* (CL2). *In particular there exists $C_R > 0$ such that for all $z \in \mathbb{R}^{2d}$, $\beta_1, \beta_2 \in C_R(I_T)$, we have*

$$|z_{\beta_1}(t) - z_{\beta_2}(t)| \leq C_R\|\beta_1 - \beta_2\|_\infty |z|. \qquad (22)$$

4.2. The quantum evolution. Following [10] we define the quantum evolution for the Hamiltonian $\hat{H}_\beta(t)$ when $\beta \in C^0(I_T)$ as follows.

Definition 4.4. $t \mapsto \psi_\beta(t) \in L^2(\mathbb{R}^d)$, $t \in I_T$, is a mild solution of the Schrö-dinger equation

$$i\partial_t\psi(t) = \hat{H}_\beta(t)\psi(t), \quad \psi(t_0) = \psi_0. \qquad (23)$$

if the following conditions are satisfied:

(QM0) there exists a neighborhood $\mathcal{N}_\beta$ in $C^0(I_T)$ such that if one has $\mathcal{N}_\beta^1 = \mathcal{N}_\beta \cap C^1(I_T)$, then $\tilde{\beta} \mapsto \psi_{\tilde{\beta}}(t)$ is a uniformly continuous map from $\mathcal{N}_\beta^1$ into $C^0(I_T, L^2(\mathbb{R}^d))$;

(QM1) for every $\varepsilon > 0$, $\psi_{\beta^\varepsilon}(t)$ solves (23) for $\beta = \beta^\varepsilon$;

(QM2) $\lim_{\varepsilon \to 0} \psi_{\beta^\varepsilon}(t) = \psi_\beta(t)$ in $C^0(I_T)$.

Recall that β^ϵ are C^1 approximations of β in $C^0(I_T)$. As in the classical case $\psi_\beta(t)$ is independent on the C^1 approximations β^ε of β.

For simplicity we shall assume that that H_β is homogenous of degree 2. Adding terms of degree 1 and 0 is easy to check using the Duhamel formula as in Corollary 4.3. As in the time-regular case we shall show now that the quantum evolution is completely determined by the quantum evolution studied above. In order to go from the regular to the irregular case we use the following proposition.

We use now the notation $U_\beta = U_{H_\beta}$.

Proposition 4.5. *For any $R > 0$ there exists $C_R > 0$ and $T_R > 0$ such that for all $\beta, \beta^{(1)}, \beta^{(0)} \in C_R^0(I_{TR}) \cap C^1(I_{TR})$, all $\psi \in \mathcal{S}(\mathbb{R}^d)$ and $t, s \in I_{TR}$, we have*

$$\|U_\beta(t,s)\psi\|_{L^2(\mathbb{R}^d)} \le C_R\|\psi\|_{L^2(\mathbb{R}^d)}, \tag{24}$$

$$\|U_{\beta^{(1)}}(t,t_0)\psi - U_{\beta^{(0)}}(t,t_0)\psi\|_{L^2(\mathbb{R}^d)} \le C_R\|\beta^{(1)} - \beta^{(0)}\|_\infty\|\psi\|_{\mathcal{H}^2(\mathbb{R}^d)}. \tag{25}$$

Corollary 4.6. *$\beta \mapsto U_\beta(t,s)$ can be extended in a unique way to $C_R^0(I_T)$ such that properties* (QM0), (QM1), *and* (QM2) *are satisfied. Moreover the Schwartz kernel of $U_\beta(t,s)$ is given by the Herman–Kluk formula* (12) *with the generalized classical flow $\Phi_{H_\beta}(t,s)$ determined by* (21). *Notice that the linear part $\Phi_{H_{\beta,2}}(t,s)$ of the affine map $\Phi_{H_\beta}(t,s)$ is a symplectic matrix denoted by*

$$\Phi_{H_{\beta,2}}(t,s) = \begin{pmatrix} A_\beta(t,s) & B_\beta(t,s) \\ C_\beta(t,s) & D_\beta(t,s) \end{pmatrix}.$$

Corollary 4.7. *For every $k \ge 0$ we have $U_\beta(t,s)\mathcal{H}^k(\mathbb{R}^d) \subseteq \mathcal{H}^k(\mathbb{R}^d)$. Moreover there exists $C_{R,t_0,T} > 0$ such that for $\psi \in \mathcal{H}^k(\mathbb{R}^d)$, $t, s \in I_T$, $\beta \in \mathcal{C}_R^0(I_T)$, we have*

$$\|U_\beta(t,s)\psi\|_{\mathcal{H}^k(\mathbb{R}^d)} \le C_{R,t_0,T}\|\psi\|_{\mathcal{H}^k(\mathbb{R}^d)}. \tag{26}$$

Proof. This a consequence of (24) and of the Egorov property:

$$U_\beta(s,t)\hat{A}\,U_\beta(t,s) = \widehat{A \circ \Phi_\beta(t,s)}. \tag{27}$$

We start by proving the corollary for $k = 1$ hence by induction we get the result for any $k \in \mathbb{N}$. Consider the linear symbol $A(q,p) = a \cdot q + b \cdot p$ and let $\psi \in \mathcal{H}^1(\mathbb{R}^d)$. Then using (27) and that $\Phi_\beta(t,t_0)$ is an affine map we get that

$$\hat{A}\,U_\beta(t,s)\psi = U_\beta(t,s)U_\beta(s,t)\hat{A}\,U_\beta(t,s)\psi \in L^2(\mathbb{R}^d).$$

So $U_\beta(t,s)\psi \in \mathcal{H}^1(\mathbb{R}^d)$ and (26) for $k = 1$. $\qquad\square$

We shall use coherent states with the notations of [7, Chapter 1] (here we choose the Planck constant $\hbar = 1$). The next two lemmas will be used to prove Proposition 5.1

Lemma 4.8. *There exists $T_R > 0$ small enough, $C_R > 0$ such that for $t, s \in I_{TR}$ and $\beta \in \mathcal{C}_R^1(I_{TR})$ we have*

$$\sup_{Y \in \mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} |\langle \varphi_Y, U_\beta(t,s)\varphi_X \rangle| dX \leq M_R, \tag{28}$$

$$\sup_{X \in \mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} |\langle \varphi_Y, U_\beta(t,s)\varphi_X \rangle| dY \leq M_R. \tag{29}$$

Proof. From [8], Proposition (5.7), we have

$$\begin{aligned} &\langle \varphi_{z+X}, U_\beta(t,s)\varphi_z \rangle \\ &= a_\beta \exp\left(-\left| z + \frac{X}{2} \right|^2 + \Lambda_\beta\left(z + \frac{X - iJX}{2} \right) \cdot \left(z + \frac{X - iJX}{2} \right) \right), \end{aligned} \tag{30}$$

where

$$\Lambda_\beta = (\mathbb{I} + F_\beta)(\mathbb{I} + F_\beta + iJ(\mathbb{I} - F_\beta))^{-1},$$

$$F_\beta = \Phi_{H_\beta}(t,s),$$

$$a_\beta = 2^d \det(\mathbb{I} + F_\beta + iJ(\mathbb{I} - F_\beta))^{1/2}.$$

Then using [8, Lemma 5.ll] and (30), we have

$$|\langle \varphi_Y, U_\beta(t,s)\varphi_0 \rangle| \leq \exp\left(-\frac{|Y|^2}{2(1 + \lambda_\beta(t,s))} \right),$$

where $\lambda_\beta(t,s)$ is the largest eigenvalue of $\Phi_{H_\beta}(t,s)\Phi_{H_\beta}(t,s)^\top$.

But we have $\|\Phi_{H_\beta}(t,s)\| \leq C e^{2\Gamma R}$. So for some $C_R > 0$ we have

$$|\langle \varphi_Y, U_\beta(t,s)\varphi_0 \rangle| \leq e^{-\frac{|Y|^2}{C_R}}. \tag{31}$$

But $U_\beta(t,s)$ is the metaplectic transformation associated with F_β. More precisely, recall that we have (see [8]) $U_\beta(t,s) = \widehat{R}(F_\beta)$ and $\varphi_X = \widehat{T}(X)\varphi_0$, where $\widehat{R}$ denotes the metaplectic representation and $\widehat{T}$ the Weyl translation representation. In particular we have the useful property

$$\widehat{R}(F_\beta)\widehat{T}(z)\widehat{R}(F_\beta)^* = \widehat{T}(F_\beta z). \tag{32}$$

So we get

$$\langle \varphi_Y, U_\beta(t,s)\varphi_X \rangle = e^{g(\beta)} \langle \varphi_{Y-F_\beta X}, U_\beta(t,s)\varphi_0 \rangle, \tag{33}$$

where $g(\beta) = \frac{\sigma}{2}(F_\beta X, Y)$, $\sigma(Z, Y) = JZ \cdot Y$ is the symplectic form. From (32), (31), and (33) we get, $t, s \in I_{T R}$,

$$|\langle \varphi_Y, U_\beta(t,s)\varphi_X \rangle| \le e^{-\frac{|Y - \Phi_{H_\beta}(t,s)X|^2}{C_R}}. \tag{34}$$

Now choosing T_R small enough we have $\|\Phi_{H_\beta}(t,s)^{-1}\| \le 2$. Hence (28) follows from (34). □

We come now to a continuity property of U_β in β (25). For proving this property we shall use again coherent states.

Let us denote $\delta U = U_{\beta^{(1)}}(t,s) - U_{\beta^{(0)}}(t,s)$. We have to establish an estimate for the Bargman kernel $\widetilde{K}_{\delta U}(X, Y) := \langle \varphi_Y, \delta U \varphi_X \rangle$.

Lemma 4.9. *For any $R > 0$ there exists $C_R > 0$ and T_R such that for all $\beta^{(1)}, \beta^{(2)} \in C_R^0(I_{T R}) \cap C^1(I_{T R})$, $X, Y \in \mathbb{R}^{2d}$,*

$$|\widetilde{K}_{\delta U}(X, Y)| \le C_R \|\beta^{(1)} - \beta^{(2)}\|_\infty (1 + |F_{\beta^{(\theta)}} X||Y| + |Y - F_{\beta^{(\theta)}} X|^2) e^{-\frac{|F_{\beta^{(\theta)}} X - Y|^2}{C_R}}. \tag{35}$$

Proof. We use the same method as in the proof of Lemma 4.8. For $\theta \in [0, 1]$ denote $\beta^{(\theta)} = \theta\beta^{(1)} + (1 - \theta)\beta^{(0)}$. So we have

$$\widetilde{K}_{\delta U}(X, Y) = \int_0^1 \frac{\partial}{\partial \theta} \langle \varphi_Y, U_{\beta^{(\theta)}} \varphi_X \rangle d\theta. \tag{36}$$

Using (30) and known estimates on F_{β_θ} we shall easily get (35). Let us begin with the particular case $X = 0$. We have to compute $\frac{\partial}{\partial \theta} \langle \varphi_Y, U_{\beta^{(\theta)}} \varphi_0 \rangle$ using (30). Then applying Corollary 4.3 and (31) we get for every $\theta \in [0, 1]$, $C_R > 0$ large enough,

$$\left| \frac{\partial}{\partial \theta} \langle \varphi_Y, U_{\beta^{(\theta)}} \varphi_0 \rangle \right| \le C_R \|\beta^{(1)} - \beta^{(2)}\|_\infty (1 + |Y|^2) e^{-\frac{|Y|^2}{C_R}}. \tag{37}$$

Now from estimate on the derivative of $g(\beta^{(\theta)})$, using (33) and (34), we get the estimate (35). □

Proof of Proposition 4.5. The estimate (24) is a direct consequence of Lemma 4.8. We can get (25) from estimate (35) as follows. Let us introduce the space

$$L^{2,s}(\mathbb{R}^{2d}) = \{u \in L^2(\mathbb{R}^{2d}), \langle X \rangle^s u(X) \in L^2(\mathbb{R}^{2d})\},$$

where $\langle X \rangle^s = (1 + |X|^2)^{s/2}$. Recall the useful estimate

$$\langle X + Y \rangle^{-2} \leq 2 \langle Y \rangle^{-2} \langle X \rangle^2.$$

From (35) we can deduce that the linear operator $\widetilde{\delta U}$ with kernel $\widetilde{K}_{\delta U}$ is continuous from $L^{2,2}(\mathbb{R}^{2d})$ into $L^2(\mathbb{R}^{2d})$. Let us consider the integral kernel $\mathcal{K}_2(X, Y) = \widetilde{K}_{\delta U}(X, Y) \langle Y \rangle^{-2}$. We have to prove that $\mathcal{K}_2(X, Y)$ is the kernel of a bound operator $\mathcal{T}_{\mathcal{K}_2}$ in $L^2(R^{2d})$. Denote

$$M_{\mathcal{K}_2} = \max \left\{ \sup_X \int |\mathcal{K}_2(X, Y| dY, \sup_Y \int |\mathcal{K}_2(X, Y| dX \right\}, \qquad (38)$$

We have the well known L^2-norm estimate

$$\|\mathcal{T}_{\mathcal{K}_2}\| \leq M_{\mathcal{K}_2}. \qquad (39)$$

Then using (35) and (39) we get that $\widetilde{\delta U}$ is continuous from $L^{2,2}(\mathbb{R}^{2d})$ in $L^2(\mathbb{R}^{2d})$, with a norm estimate as follows.

Introduce the Fourier–Bargmann transform

$$\widetilde{\psi}(X) = (2\pi)^{-d/2} \langle \varphi_X, \psi \rangle,$$

which is well defined for every $\psi \in \mathcal{S}'(\mathbb{R}^d)$. Recall that $\psi \mapsto \widetilde{\psi}$ is an isometry from $L^2(\mathbb{R}^d)$ into $L^2(\mathbb{R}^{2d})$ and that φ_X is an eigenvector for the creation operators $\mathbf{a}_j = \frac{1}{\sqrt{2}}\left(x_j + \frac{\partial}{\partial x_j}\right)$ with eigenvalue $\alpha_j = \frac{q_j + i p_j}{\sqrt{2}}$ if $X = (q, p) \in \mathbb{R}^d \times \mathbb{R}^d$.

Then for every $k \geq 0$ there exists C_k such that

$$\|\widetilde{\psi}\|_{L^{2,k}(\mathbb{R}^{2d})} \leq C_k \|\psi\|_{\mathcal{H}^k(\mathbb{R}^d)}.$$

So we get, under the conditions of (4.9),

$$\|\delta U \psi\|_{L^2(\mathbb{R}^d)} \leq C_R \|\beta^{(1)} - \beta^{(2)}\|_\infty \|\psi\|_{\mathcal{H}^2(\mathbb{R}^d)}.$$

This proves (25).   □

Finally we have proved Corollary 4.6 which is the main result of this section.

5. Application to Strichartz estimate and NLS

In this section we give a proof for Theorem 2.6 and Theorem 2.7.

A motivation for studying linear quantum dynamics with noise is the get results for non-linear Schrödinger equations. For that it is now well known that Strichartz inequality is very useful. For quadratic Hamiltonian with noise β this inequality is derived from the a Mehler–Van Vleck formula for the Schwartz kernel of the propagator $U_{H_\beta}(t, s)$ for $0 < |t - s| < T$, with T small enough.

5.1. A local dispersive estimate. We start with an almost explicit expression for the kernel of the propagator valid with noise.

Proposition 5.1. *If the Hypothesis 2.3 and Hypothesis 2.4 are satisfied then for every $R > 0$ there exists $T_R > 0$ such that for every $t, s \in]t_0, t_0 + T_R]$ and every β such that $\|\beta\|_{\mathcal{C}^\mu} \leq R$ the Schwartz kernel $\mathcal{K}_\beta(t, s)$ of $U_{H_\beta}(t, s)$ is a C^∞ function of (x, y) given by the following formula*

$$\mathcal{K}_\beta(t, s; x, y) = (2i\pi)^{-d/2}\det^{-1/2}(B_\beta(t, s))e^{iS_\beta(t,s;x,y)}, \tag{40}$$

where $S_\beta(t, s; x, y)$ is the classical action along the unique classical trajectory joining y to x at time s.

In particular there exists $\gamma > 0$ such that $\det B_\beta(t, s) \geq \gamma |t - s|^d$ for every $t \in I_{T R}$.

Let $p_\beta(t, s; x, y)$ be the momentum of the trajectory $(q_s, p_s) = \Phi_{H_\beta}(t, s)(q, p)$. Then we have

$$S_\beta(t, s; x, y) = \int_s^t (\dot{q}_u \cdot p_u - H_\beta(u, q_u, p_u))du, \quad where \ p = p(t, s; x, y). \tag{41}$$

Proof. The computation is well known, the new fact here is that we need to control the validity of this computation with the noise term in β.

First of all let us remark that the action S_β is continuous in β for the $\mathcal{C}^0$ topology.

To obtain this property it is enough to assume that $H_\beta(t)$ is quadratic. From Euler identity we have $H_\beta(t) = \frac{1}{2}(q \cdot \partial_q H_\beta + p \cdot \partial_p H_\beta)$. So, using the Hamilton equations, $\dot{q} = \partial_p H_\beta$, $\dot{p} = -\partial_q H_\beta$ we have

$$S_\beta(t, s; x, y) = \frac{1}{2}(p_t \cdot q_t - p_s \cdot q_s).$$

Recall that $(q_t, p_t) = \Phi_{H_\beta}(t, s)(q_s, p_s)$, so continuity properties in β for S_β is a consequence of continuity for the flow Φ_{H_β}.

Now, we shall use here computations taken from [4]. Notice that in formula (13) and (12) the phases Ψ_Θ and $\Psi_{\Theta,2}$ are quadratic in z. So the integral is the integral of a Gaussian and we have to compute a Gaussian integral (a particular case of the stationary theorem). For this computation we use the simpler notations $A = A_\beta(t, s)$ and the same for B, C, D.

We choose here in (12) the complex matrices $\Theta(t) = \Gamma(t)$ where

$$\Gamma(t) := (C + iD)(A + iB)^{-1}$$

(see Section 3). The matrix of the quadratic part of Ψ_Θ was computed in [4]:

$$\partial^2_{z,z}\Psi = \begin{pmatrix} 2i\mathbb{I} + (A + iB)^{-1}B & i(A + iB)^{-1}B \\ i(A + iB)^{-1}B & -(A + iB)^{-1}B \end{pmatrix}. \tag{42}$$

In particular we get

$$\det(\partial^2_{z,z}\Psi) = \det(-2i(A + iB)^{-1}B) \tag{43}$$

and $\partial^2_{z,z}\Psi$ is invertible if and only if $B := B_\beta(t, s)$ is invertible.

This is checked using the following lemma (in [9, Proposition 2.30] a similar result is proved).

Lemma 5.2. *We have the following estimate of the flow* $\Phi_{H_\beta}(t, s)$, *for* $|t - s|$ *small enough,*

$$\Phi_{H_\beta}(t, s) = \mathbb{I} + ((\beta(t) - \beta(s))JS_K + (t - s)JS_H(s) \\ + O(|t - s|^2 + \sup_{|t-u| \leq |t-s|} |\beta(t) - \beta(u)|^2). \tag{44}$$

In particular if $\beta \in \mathcal{C}^\mu(I_T)$ *with* $\mu > \frac{1}{2}$ *then we have*

$$B_\beta(t, s) = (t - s)\partial^2_{pp}H(s) + O(|t - s|^2). \tag{45}$$

Moreover if $\partial^2_{q,p}K = 0$ *then the estimate* (45) *remains true for any* $\beta \in \mathcal{C}^0_R(I_{TR})$.
In estimates (44) *and* (45), $O(\cdot)$ *is uniform for* $\|\beta\|_{\mathcal{C}^\mu} \leq R$.

Using lemma 5.2 and choosing $T_R > 0$ small enough we get that $B_\beta(t, t_0)$ is invertible for $t \in I_{TR}$. So under the same conditions we have that

$$\det(\partial^2_{z,z}\Psi(t, s)) \neq 0.$$

So we get (40) by computing a Gaussian integral.

Proof of lemma 5.2. Using (17) we get

$$\Phi_{H_\beta}(t,s) = e^{(\beta_t - \beta_s)JS_K} + \int_s^t e^{(\beta_t - \beta_u)JS_K} JS_H(u) e^{(\beta_u - \beta_s)JS_K} du$$

$$+ \int_s^t e^{(\beta_t - \beta_u)JS_K} JS_H(u) \left(\int_s^s e^{(\beta_u - \beta_\sigma)JS_K} \Phi_{H_\beta}(\sigma, s) d\sigma \right) du.$$

$$(46)$$

The last term is clearly $O(|t-s|^2)$. To estimate the first we use the Taylor formula

$$e^{uJS_K} = 1 + uJS_K + u^2(JS_K)^2 \int_0^1 (1-\theta)e^{\theta uJS_K}. \qquad (47)$$

Notice that we have

$$JS_K = \begin{pmatrix} L_K & 0 \\ G_K & L_K^\top \end{pmatrix} \quad \text{and} \quad JS_H(t) = \begin{pmatrix} L(t) & E_H(t) \\ G_H(t) & L_H(t)^\top \end{pmatrix}.$$

Notice that $E_H(t)$ is invertible for t close to t_0. Moreover $JS_K^2 = 0$ if $L_K = 0$. So the lemma can be easily obtained from (46) and (47). $\triangle$

This concludes the proof of Proposition 5.1. $\square$

The next corollary is very useful in applications to get Strichartz estimates, as explained in [15].

Corollary 5.3 (dispersive estimate). *There exists a constant C_R, depending only on R such that for every $t \in I_{TR}$ and every $x, y \in \mathbb{R}^d$, we have*

$$|\mathcal{K}_\beta(t, s; x, y)| \leq C_R |t-s|^{-d/2} \qquad (48)$$

and, for every $p \in [2, +\infty]$, we have, for $\psi \in L^p(\mathbb{R}^d)$,

$$\|U_\beta(t,s)\psi\|_{L^p(\mathbb{R}^d)} \leq C_R |t-s|^{-d(1/2-1/p)} \|\psi\|_{L^{p'}(\mathbb{R}^d)}, \quad 1/p + 1/p' = 1. \quad (49)$$

Let us notice that the principal symbol of $\hat{H}(t)$ is not necessary elliptic, the important property to get the local dispersive estimate (49) is that the quadratic form $\partial_{p,p}^2 H$ is non degenerate (for $d = 2$ we may have $H(q, p) = p_1^2 - p_2^2 + q_1^2 + q_2^2$).

5.2. About the proof of Theorem 2.7. As already remarked in [3, 10], using Strichartz estimate (49) it is possible to extend the results proved in [5] concerning non-linear Schrödinger equations for quadratic linear parts with noise. The proofs follows closely [5] so we do not repeat the details here (see also [6, 16]) for the regular case.

In a first step the result is proved locally in time by a fixed point argument such that $\beta \mapsto \psi_\beta(t)$ is continuous from $\mathcal{C}^\mu(I_T)$ into $L^2(\mathbb{R}^d)$. Then we get the conservation of the L^2 norm (this is true for $\beta \in C^1(I_T)$ and also for $\beta \in \mathcal{C}^\mu(I_T)$ by continuity). Using the conservation law we can extend the local solution in a global solution for initial data in $L^2(\mathbb{R}^d)$ for subcritical non-linear terms.

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