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Large bidiagonal matrices and random perturbations

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Dedicated to the memory of Yuri Safarov

Abstract. This is a first paper by the authors about the distribution of eigenvalues for random perturbations of large bidiagonal Toeplitz matrices.

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Contents

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1. Introduction and main result

It is well-known that for non-normal operators, as opposed to normal operators, the norm of the resolvent can be very large even far away from the spectrum. Equivalently, the spectrum of such operators can be highly unstable under tiny perturbations. Originating from a renewed interest in numerical analysis with the works of L. N. Trefethen and M. Embree [22, 8], spectral instability of non-self-adjoint operators has become an active subject of interest. It is the source of many interesting effects, as emphasized by the works of E. B. Davies, M. Zworski, J. Sjöstrand and many others (cf. [4, 5, 7, 3, 6]).

It is natural to study the effects of small random perturbations on the spectra of non-normal operators. A recent series of works by M. Hager, W. Bordeaux-Montrieux, J. Sjöstrand, and M. Vogel [1, 11, 12, 13, 18, 24, 23] has focused on the case of elliptic (pseudo-)differential operators subject to small random perturbations. It was shown that for a large class of (pseudo-)differential operators one obtains a probabilistic Weyl law for the eigenvalues in the bulk of the spectrum.

Another important example is the case of non-self-adjoint Toeplitz matrices. They can arise for example in models of non-Hermitian quantum mechanics, cf. [9, 14]. The spectral theory of such operators has been much discussed in the past, cf. [25, 2], and from the point of view of spectral instability in [8].

The simplest example of a truncated Toeplitz operator is the Jordan block matrix. M. Hager and E. B. Davies [6] considered the case of large Jordan block matrices subject to small Gaussian random perturbations and showed that with a sufficiently small coupling constant most eigenvalues can be found near a circle, with probability close to 1, as the dimension of the matrix N gets large. Furthermore, they give a probabilistic upper bound of order $\log N$ for the number of eigenvalues in the interior of a circle.

A recent result by A. Guionnet, P. Matched Wood and O. Zeitouni [10] implies that when the coupling constant is bounded from above and from below by (different) sufficiently negative powers of N, then the normalized counting measure of eigenvalues of the randomly perturbed Jordan block converges weakly in probability to the uniform measure on S^1 as the dimension of the matrix gets large.

In [17], J. Sjöstrand obtained a probabilistic circular Weyl law for most of the eigenvalues of large Jordan block matrices subject to small random perturbations, and in [20], we obtained a precise asymptotic formula for the average density of the residual eigenvalues in the interior of a circle, where the result of Davies and Hager yielded a logarithmic upper bound on the number of eigenvalues. The leading term is given by the hyperbolic volume form on the unit disk, independent of the dimension N.

The goal of the present work is to study the spectrum of random perturbations of the following bidiagonal $N \times N$ Toeplitz matrix:

$$\text{CASE I.} \qquad P = P_{\mathrm{I}} = \begin{pmatrix} 0 & a & 0 & \dots & \dots & 0 \\ b & 0 & a & \dots & \dots & 0 \\ 0 & b & 0 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & 0 & a \\ 0 & 0 & \dots & \dots & b & 0 \end{pmatrix}. \tag{1.1}$$

Originally we also wanted to include

$$\text{CASE II.} \qquad P = P_{\mathrm{II}} = \begin{pmatrix} 0 & a & b & 0 & \dots & \dots & 0 \\ 0 & 0 & a & b & \dots & \dots & 0 \\ 0 & 0 & 0 & a & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & a & b \\ \dots & \dots & \dots & \dots & \dots & 0 & a \\ \dots & \dots & \dots & \dots & \dots & 0 & 0 \end{pmatrix},$$

but we decided to postpone much of the study in this case.

Here $a, b \in \mathbf{C} \setminus \{0\}$ and $N \gg 1$. Identifying $\mathbf{C}^N$ with $\ell^2([1, N])$, $[1, N] = \{1, 2, \dots, N\}$ and also with $\ell^2_{[1,N]}(\mathbf{Z})$ (the space of all $u \in \ell^2(\mathbf{Z})$ with support in $[1, N]$), we have

$$P_{\mathrm{I}} = 1_{[1,N]}(a\tau_{-1} + b\tau_1) = 1_{[1,N]}(a\mathrm{e}^{iD_x} + b\mathrm{e}^{-iD_x}),$$

$$P_{\mathrm{II}} = 1_{[1,N]}(a\tau_{-1} + b\tau_{-2}) = 1_{[1,N]}(a\mathrm{e}^{iD_x} + b\mathrm{e}^{2iD_x}),$$

where $\tau_k u(j) = u(j - k)$ denotes translation by k.

The symbols of these operators are by definition,

$$P_{\mathrm{I}}(\xi) = a\mathrm{e}^{i\xi} + b\mathrm{e}^{-i\xi}, \ P_{\mathrm{II}}(\xi) = a\mathrm{e}^{i\xi} + b\mathrm{e}^{2i\xi}. \tag{1.2}$$

They are 2π-periodic in ξ and will often be viewed as functions on $\mathbf{R}/2\pi\mathbf{Z}$ with the identification $\mathbf{R}/2\pi\mathbf{Z} \ni \xi \longleftrightarrow e^{i\xi} \in S^1$. In this work, we consider the following random perturbation of $P_0 = P_{\mathrm{I}}$

$$P_\delta := P_0 + \delta Q_\omega, \quad Q_\omega = (q_{j,k}(\omega))_{1 \le j,k \le N}, \tag{1.3}$$

where $0 \le \delta \ll 1$ and $q_{j,k}(\omega)$ are independent and identically distributed complex Gaussian random variables, following complex Gaussian law $\mathcal{N}_{\mathbf{C}}(0, 1)$.

The following result shows that, with probability close to 1, most eigenvalues are in a small neighbourhood of the ellipse $E_1 = P_{\mathrm{I}}(S^1)$ with focal points $\pm 2\sqrt{ab}$ and major semi-axis of length $|a| + |b|$: let γ be a segment of E_1 and $r > 0$, put

$$\Gamma(r,\gamma) = \{z \in \mathbf{C};\ \mathrm{dist}(z, E_1) = \mathrm{dist}(z,\gamma) < r\}. \tag{1.4}$$

Theorem 1.1. *Let $P = P_{\mathrm{I}}$ be the bidiagonal matrix in (1.1) where $a, b \in \mathbf{C}$ satisfy $0 < |b| < |a|$. Let P_δ be as in (1.3). Choose $\delta \asymp N^{-\kappa}$, $\kappa > 5/2$ and consider the limit of large N. Let γ be a segment of the ellipse $E_1 = P_{\mathrm{I}}(S^1)$ and let $\Gamma = \Gamma(r,\gamma)$ be as in (1.4) with $(\ln N)/N \ll r \ll 1$. Let δ_0 be small and fixed.*

Then with probability

$$\geq 1 - \mathcal{O}(1)\Big(\frac{1}{r} + \ln N\Big) N^{2\kappa} \mathrm{e}^{-2N^{\delta_0}}, \tag{1.5}$$

we have

$$\Big|\#(\sigma(P_\delta) \cap \Gamma) - \frac{1}{2\pi}\mathrm{vol}_{]0,N]\times S^1} P_{\mathrm{I}}^{-1}(\Gamma)\Big| \leq \mathcal{O}(1) N^{\delta_0}\Big(\frac{1}{r} + \ln N\Big). \tag{1.6}$$

Here we view $]0, N]$ as an interval in $\mathbf{R}$ of length N.

If we choose $\gamma = E^1$ and view P_{I} as a function on $]0, N] \times S^1$, then, since $P_{\mathrm{I}}^{-1}(\Gamma) = P_{\mathrm{I}}^{-1}(E_1) =]0, N] \times S^1$, we have

$$\frac{1}{2\pi}\mathrm{vol}_{]0,N]\times S^1} P_{\mathrm{I}}^{-1}(\Gamma) = N$$

which is equal to the total number of eigenvalues of P_δ, so the number of eigenvalues outside of Γ is bounded be the right hand side of (1.6). With $r > 0$ fixed but arbitrarily small we get

Corollary 1.2. *Under the general assumptions in Theorem 1.1, let Γ be any fixed neighborhood of E_1. Then with probability as in (1.5), we have*

$$|\#\,(\sigma(P_\delta) \cap (\mathbf{C} \setminus \Gamma))| \leq \mathcal{O}(1) N^{\delta_0} \ln N.$$

Figure 1 illustrates the result of Theorem 1.1 by showing the eigenvalues of the $N \times N$-matrix in (1.1), with $N = 500$, $a = 1 + i$ and $b = 0.5$, perturbed with a complex Gaussian random matrix and coupling constant $\delta = 10^{-5}$. The line indicates the image of the unit circle S^1 under the symbol of the matrix (1.1). We can see that most eigenvalues are close to an ellipse with only very few in the interior. This phenomenon has been observed numerically in [8] (we refer in particular to Figures 3.2 and 3.3 in [8]). For more numerical simulations for more general Toeplitz matrices, we refer the reader to [8, Section 7].

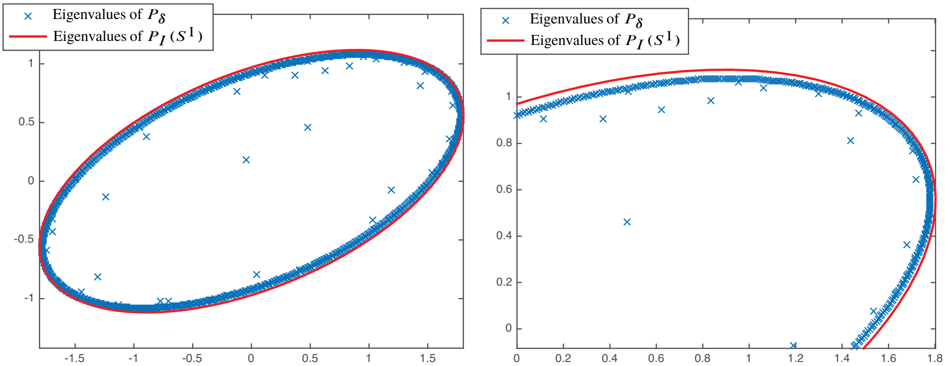

Figure 1. The spectrum of P_I with $N = 500$, $a = 1 + i$ and $b = 0.5$ perturbed with a complex Gaussian random Matrix with coupling constant $\delta = 10^{-5}$. The red line is the image of the unit circle S^1 under the symbol P_{I}.

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2. The range of the symbol

Write

$$a = |a|e^{i\alpha}, \quad b = |b|e^{i\beta}, \quad \alpha, \beta \in \mathbf{R}. \tag{2.1}$$

2.1. Case I. We have $P(\xi) = |a|e^{i(\alpha+\xi)} + |b|e^{i(\beta-\xi)}$ and the largest value of $|P(\xi)|$ (for ξ real) is attained when the two terms in the expression for $P(\xi)$ point in the same direction. This happens precisely when

$$\xi = \frac{\beta - \alpha}{2} + \pi k, \quad k \in \mathbf{Z}.$$

Write $\xi = \frac{\beta-\alpha}{2} + \eta$. Then

$$\begin{aligned} P(\xi) &= e^{i(\alpha+\beta)/2}(|a|e^{i\eta} + |b|e^{-i\eta}) \\ &= e^{i(\alpha+\beta)/2}((|a| + |b|)\cos\eta + i(|a| - |b|)\sin\eta). \end{aligned} \tag{2.2}$$

Assume, to fix the ideas, that $|b| \leq |a|$. Then $P(\mathbf{R})$ is equal to the ellipse, E_1, centered at 0 with major semi-axis of length $(|a| + |b|)$ pointing in the direction $e^{i(\alpha+\beta)/2}$ and minor semi-axis of length $|a| - |b|$. The focal points of E_1 are

$$\pm 2\sqrt{ab} = \pm e^{i(\alpha+\beta)/2} 2\sqrt{|a|\,|b|}. \tag{2.3}$$

The left hand side of Figure 2 illustrates the range of the symbol in Case I by presenting $P(S^1)$ with $b = 0.5$ and $a = 1 + i$, $a = 0.5 + 0.5i$.

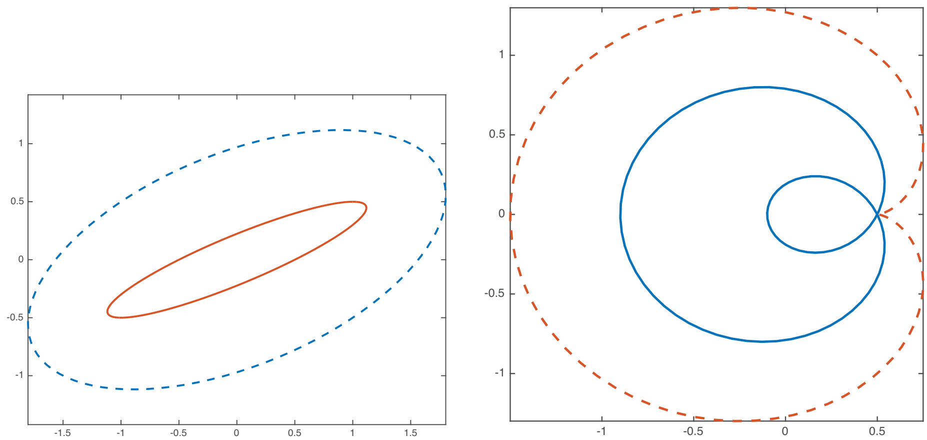

Figure 2. The left hand side shows the image of S^1 under the principal symbol of Case I (for the dashed ellipse we chose $b = 0.5$, $a = 1 + i$ and for the other ellipse $b = 0.5$, $a = 0.5 + 0.5i$). The right hand side is similar but for the principal symbol of Case II (for the dashed line we chose $b = 0.5$, $a = i$ and for the continuous line $b = 0.5$, $a = 0.4i$).

2.2. Case II.

Write

$$P(\xi) = |a|e^{i(\alpha+\xi)} + |b|e^{i(\beta+2\xi)}.$$

By the same reasoning as in Case I, the largest value, $|a| + |b|$, of $|P(\xi)|$ is attained when

$$\xi = \xi_{\max} := \alpha - \beta + 2\pi k, \quad k \in \mathbf{Z}.$$

The smallest value $||a| - |b||$ of $|P(\xi)|$ is attained when

$$\xi = \xi_{\min} := \alpha - \beta + \pi + 2\pi k.$$

We have

$$P(\xi_{\max}) = e^{i(2\alpha-\beta)}(|a|+|b|), \qquad P(\xi_{\min}) = -e^{i(2\alpha-\beta)}(|a|-|b|).$$

Write $\xi = \alpha - \beta + \eta$, so that

$$P(\xi) = e^{i(2\alpha-\beta)} f(e^{i\eta}), \qquad f(\zeta) = |a|\zeta + |b|\zeta^2.$$

The study of $P(\mathbf{R})$ is equivalent to that of $f(S^1)$. This curve is called the "Limaçon de Pascal" after E. Pascal, father of B. Pascal.[1] Assume for notational reasons that $a, b > 0$, so that

$$f = f_{a,b}(\zeta) = a\zeta + b\zeta^2.$$

This function has the unique critical point $\zeta = \zeta_c(a,b)$, given by $a + 2b\zeta_c = 0$,

$$\zeta_c = -\frac{a}{2b}$$

and

$$f(\zeta_c) = -\frac{a^2}{4b}.$$

Since f is quadratic, we have

$$f(\zeta) = f(\zeta_c) + b(\zeta-\zeta_c)^2 = -\frac{a^2}{4b} + b\Big(\zeta + \frac{a}{2b}\Big)^2. \tag{2.4}$$

Notice that

(1) $b < a/2 \implies \zeta_c \notin \overline{D(0,1)}$,

(2) $b = a/2 \implies \zeta_c \in S^1$,

(3) $b > a/2 \implies \zeta_c \in D(0,1)$.

In the first case there is no pair of distinct points on S^1 which are symmetric to each other with respect to ζ_c so $f(S^1)$ is a simple closed curve in $\mathbf{C}$.

In the second case $f\colon S^1 \to \mathbf{C}$ is still injective but has a critical point at ζ_c. The image of S^1 is still a simple closed curve, but with a cusp at $f(\zeta_c)$.

In the third case, the critical point ζ_c is situated on the segment $]-1,0[$. There is one pair of points on S^1 that are symmetric to each other with respect to ζ_c, namely α_c and $\overline{\alpha_c}$, where $\alpha_c = \zeta_c + i\sqrt{1-\zeta_c^2}$. Thus $f(\alpha_c) = f(\overline{\alpha_c}) \in]-\infty, f(\zeta_c)[$ is the

[1] We thank A. Grigis for this information and for the link

http://www.mathcurve.com/courbes2d/limacon/limacon.shtml only point in $f(S^1)$ whose inverse image consists of more than one point. $f(S^1)$ is a closed curve with $f(\alpha_c) = f(\overline{\alpha_c})$ as its unique point of self intersection. We can write

$$f(S^1) = \{f(\alpha_c)\} \cup \gamma_{\mathrm{int}} \cup \gamma_{\mathrm{ext}},$$

where $\gamma_{\mathrm{int}} = f(S^1 \cap \{\zeta; \Re\zeta < \Re\zeta_c\})$, $\gamma_{\mathrm{ext}} = f(S^1 \cap \{\zeta; \Re\zeta > \Re\zeta_c\})$ are smooth curves, which become simple closed after adding $f(\zeta_c)$ and γ_{int} is situated in the interior of the region enclosed by the closure of γ_{ext}.

The right hand side of Figure 2 illustrates the range of the symbol in Case II by presenting $P(S^1)$ with $b = 0.5$ and $a = i$, $a = 0.4i$.

Remark 2.1. If P is a bounded operator: $\mathcal{H} \to \mathcal{H}$, where $\mathcal{H}$ is a complex Hilbert space, we define the numerical range

$$W(P) = \{(Pu|u);\ u \in \mathcal{H},\ \|u\| = 1\}.$$

It is well known that $W(P)$ is convex, see [15]. If $\Pi: \mathcal{H} \to \mathcal{H}$ is an orthogonal projection with range $\mathcal{R}(\Pi)$, then

$$W(\Pi P_{|\mathcal{R}(\Pi)}) \subset W(P),$$

for if $u \in \mathcal{R}(\Pi)$ is normalized, then $u = \Pi u$ and

$$(\Pi P u|u) = (Pu|\Pi u) = (Pu|u).$$

If

$$P = \sum_{-N}^{N} a_k \tau_k : \ell^2(\mathbb{Z}) \longrightarrow \ell^2(\mathbb{Z}), \quad a_k \in \mathbb{C},$$

is a finite difference operator with symbol $P(\xi) = \sum_{-N}^{N} a_k e^{-ik\xi}$, then P is unitarily equivalent to the multiplication operator

$$\widehat{P}: L^2(S^1) \ni \hat{u} \longmapsto P(\xi)\hat{u} \in L^2(S^1)$$

and from writing down the scalar product $(\widehat{P}\hat{u} \mid \hat{u})$ we see that $W(P) = W(\widehat{P})$ is contained in the convex hull of the range $P(S^1)$ of (the symbol of) P.

This can be applied to P_{I} and P_{II} with $\Pi u = 1_{[1,N]} u$, $u \in \ell^2(\mathbb{Z})$, and we conclude that $W(P_{\mathrm{I}})$, $W(P_{\mathrm{II}})$ are contained in the convex hulls of $P_{\mathrm{I}}(S^1)$ and $P_{\mathrm{II}}(S^1)$ respectively.[2]

[2] We thank the referee for indicating this argument.

3. Spectrum of the unperturbed operator

3.1. Case I. The spectrum of P_{I} as a set coincides with the set of eigenvalues. Consider an eigenvalue $z \in \mathbf{C}$ and a corresponding eigenvector $0 \neq u \in \ell^2([1, N])$. Extending u to $[0, N+1]$ by putting $u(0) = 0$, $u(N+1) = 0$, we have

$$au(k+1) - zu(k) + bu(k-1) = 0, \quad k = 1, \dots, N.$$

We can extend u further to all of $\mathbf{Z}$ and get a function $\tilde{u}: \mathbf{Z} \to \mathbf{C}$ such that

$$\tilde{u}(k) = u(k) \text{ on } [1, N], \quad \tilde{u}(0) = \tilde{u}(N+1) = 0 \tag{3.1}$$

and such that

$$a\tilde{u}(k+1) - z\tilde{u}(k) + b\tilde{u}(k-1) = 0, \quad k \in \mathbf{Z}. \tag{3.2}$$

The space of solutions to (3.2) is of dimension 2 and if the equation

$$a\zeta + b/\zeta - z = 0 \tag{3.3}$$

has two distinct solutions ζ_+ and ζ_-, then it is generated by the functions $\tilde{u}_\pm$, given by

$$\tilde{u}_\pm(k) = \zeta_\pm^k, \quad k \in \mathbf{Z}.$$

When the equation has a double solution, which happens precisely when

$$a\zeta = \frac{b}{\zeta} = \frac{z}{2},$$

i.e. when z is one of the focal points of E_1, the same space is generated by

$$\tilde{u}_0(k) = \zeta^k, \quad \tilde{u}_1(k) = k\zeta^k.$$

In the case when the characteristic equation has two distinct solutions, we can write

$$\tilde{u}(k) = c_+\zeta_+^k + c_-\zeta_-^k,$$

and apply the boundary conditions in (3.1), to get

$$c_+ + c_- = 0, \quad c_+\zeta_+^{N+1} + c_-\zeta_-^{N+1} = 0,$$

and

$$c_- = -c_+, \qquad \zeta_+^{N+1} = \zeta_-^{N+1}.$$

This gives the N possibilities,

$$\frac{\zeta_+}{\zeta_-} = e^{2\pi i\nu/(N+1)}, \quad \nu = 1, 2, \dots, N. \tag{3.4}$$

(The case $\nu = 0$ is excluded since we are in the case of distinct solutions of the characteristic equation.) The relation between the two solutions of (3.3) is given by

$$a\zeta_+ = b/\zeta_-,$$

and insertion of this in (3.4) gives,

$$\frac{a}{b}\zeta_+^2 = e^{2\pi i\nu/(N+1)}.$$

Fixing a branch of $\sqrt{b/a}$, we get

$$\zeta_\pm(\nu) = \sqrt{b/a}e^{\pm\pi i\nu/(N+1)}, \quad \nu = 1, \dots, N.$$

The corresponding eigenvalues are then

$$z = z(\nu) = a\zeta_+(\nu) + b/\zeta_+(\nu) = 2\sqrt{ab}\cos\left(\frac{\pi\nu}{N+1}\right).$$

These values are distinct so we conclude that the spectrum of P_{I} consists of N simple eigenvalues.

Recall the representation (2.1). We can choose the branch of the square root so that

$$\sqrt{ab} = \sqrt{|a|\,|b|}e^{i(\alpha+\beta)/2}.$$

We conclude that the eigenvalues

$$z(\nu) = 2\sqrt{|a||b|}e^{i(\alpha+\beta)/2}\cos\left(\frac{\pi\nu}{N+1}\right)$$

are situated on the major axis of the ellipse E_1, between the two focal points (2.3).

Remark 3.1. Let

$$W = \operatorname{diag}(w^k)_{0\le k\le N-1}$$

be the diagonal matrix with elements w^k, $1 \le k \le N$, where $0 \neq w \in \mathbf{C}$. Then P has the same spectrum as

$$\widetilde{P} = WPW^{-1} = \begin{pmatrix} 0 & a/w & 0 & \dots & \dots & 0 \\ bw & 0 & a/w & \dots & \dots & 0 \\ 0 & bw & 0 & a/w & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & bw & 0 & a/w \\ \dots & \dots & \dots & \dots & bw & 0 \end{pmatrix},$$

and choosing $w = (a/b)^{1/2}$ gives

$$\widetilde{P} = (ab)^{1/2} \begin{pmatrix} 0 & 1 & 0 & \dots & \dots & \dots & 0 \\ 1 & 0 & 1 & \dots & \dots & \dots & 0 \\ 0 & 1 & 0 & 1 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & 1 & 0 & 1 \\ \dots & \dots & \dots & \dots & \dots & 1 & 0 \end{pmatrix},$$

The last matrix is self-adjoint and this explains why the eigenvalues of P are situated on a segment.

3.2. Case II. $P = P_{\mathrm{II}}$ is nilpotent, so $\sigma(P) = \{0\}$.

4. Size of $|\zeta|$

For the understanding of our operators, it will be important do determine, depending on z, the number of exponential solutions $\tilde{u}(k) = \zeta^k$ that grow near $k = +\infty$ and near $k = -\infty$ respectively. Here ζ is a solution of the characteristic equation (3.3) in Case I and of the characteristic equation

$$a\zeta + b\zeta^2 = z \tag{4.1}$$

in Case II.

4.1. Case I. We recall that we have assumed for simplicity that $|a| \geq |b|$. The case $|a| = |b|$ will be obtained as a limiting case of the one when $|a| > |b|$, that we consider now. Let

$$f_{a,b}(\zeta) = a\zeta + b/\zeta$$

and observe that when $r > 0$

$$f_{a,b}(\partial D(0,r)) = f_{ar,b/r}(\partial D(0,1))$$

which gives a family of confocal ellipses E_r. The length of the major semi-axis of E_r is equal to $|a|r + |b|/r =: g(r)$. E_{r_1} is contained in the bounded domain which has E_{r_2} as its boundary, precisely when $g(r_1) \leq g(r_2)$. The function g has a unique minimum at $r = r_{\min} = (|b|/|a|)^{1/2}$. g is strictly decreasing on $]0, r_{\min}]$ and strictly increasing on $[r_{\min}, +\infty[$. It tends to $+\infty$ when $r \to 0$ and when $r \to +\infty$. We have $g_{\min} = g(r_{\min}) = 2(|a||b|)^{1/2}$ so $E_{r_{\min}}$ is just the segment between the two focal points, common to all the E_r. For $r \neq r_{\min}$, the map $\partial D(0,r) \to E_r$ is a diffeomorphism. Let r_1 be the unique value in $]0,1[$ for which $g(r_1) = |a| + |b| = g(1)$. We get the following result.

Proposition 4.1. *Let* $|b| < |a|$.

- *When* z *is strictly inside the ellipse* E_1 *described after* (2.2)*, then both solutions of* $f_{a,b}(\zeta) = z$ *belong to* $D(0,1)$.
- *When* z *is on the ellipse, one solution is on* S^1 *and the other belongs to* $D(0,1)$.
- *When* z *is in the exterior region to the ellipse, one solution fulfills* $|\zeta| > 1$ *and the other satisfies* $|\zeta| < 1$.

Proof. When z is strictly inside E_1 it belongs to

$$E_{\rho_-} = E_{\rho_+}, \quad E_{\rho_\pm} = f(\partial D(0,\rho_\pm)),$$

for $r_1 < \rho_+ \leq \rho_- < 1$ and $\zeta_\pm \in \partial D(0,\rho_\pm)$. The other two cases are treated similarly. □

In the case $|a| = |b|$, E_1 is just the segment between the two focal points. In this case $r_{\min} = 1$ and we get the following result.

Proposition 4.2. *Assume that* $|a| = |b|$.

- *If* $z \in E_1$ *then both solutions of* $f_{a,b}(\zeta) = z$ *belong to* S^1.
- *If* z *is outside* E_1*, one solution is in* $D(0,1)$ *and the other is in the complement of* $\overline{D(0,1)}$.

Remark 4.3. Recall that E_ρ is the ellipse with focal points $\pm c = \pm 2\sqrt{ab}$ and length of major semi-axis equal to $g(\rho) := |a|\rho + |b|/\rho$. Equivalently,

$$E_\rho = \{z \in \mathbf{C};\ |z - c| + |z + c| = 2g(\rho)\}.$$

The solutions $\zeta = \zeta_\pm$ of $f_{a,b}(\zeta) = z$ belong to $\partial D(0,\rho_\pm)$, where $\rho_\pm$ are the solutions to $2g(\rho_\pm) = |z - c| + |z + c|$, situated on each side of $r_{\min} = \sqrt{|b/a|}$. Also, $g_{\min} = g(r_{\min}) = |c|$.

Assume that z is restricted to a compact subset K of $\mathbf{C}$. Then $\rho_\pm \in [1/C, C]$ for some $C = C(K) > 1$. For $\rho \in [1/C, C]$, we have

$$g(\rho) - |c| \asymp |\rho - r_{\min}|^2.$$

Consequently,

$$|\rho_\pm - r_{\min}| \asymp (g(\rho_\pm) - |c|)^{1/2} = 2^{-1/2}(|z-c| + |z+c| - 2|c|)^{1/2}.$$

Since $\rho_\pm$ are situated on opposite sides of $r_{\min}$; we get

$$|\zeta_+ - \zeta_-| \geq |\rho_+ - \rho_-| \asymp (|z-c| + |z+c| - 2|c|)^{1/2}.$$

Noticing also that $\zeta_+\zeta_- = b/a$, we have with the convention $|\zeta_+| \leq |\zeta_-|$, that $|\zeta_+|/|\zeta_-| \leq 1$ with equality precisely when z belongs to the focal segment. For every neighborhood of that segment, there exists $\theta < 1$ such that $|\zeta_+|/|\zeta_-| \leq \theta$ for z outside that neighborhood.

4.2. Case II. We write the equation (4.1) as

$$f(\zeta) = z, \text{ where } f(\zeta) := a\zeta + b\zeta^2 = b(\zeta - \zeta_c)^2 + c = z,$$

ζ_c is the critical point, given by $\zeta_c = -a/(2b)$ and

$$c = f(\zeta_c) = -a^2/(4b).$$

For any given z, the two solutions are symmetric around ζ_c, and depending on which case we are in according to the conclusion after (2.4), we just have to see if both, one or none of the symmetric solutions belong to $D(0,1)$.

When $|b| < |a|/2$, we have $\zeta_c \notin \overline{D(0,1)}$, $f(S^1)$ is a simple smooth closed curve γ and if we let Γ be the bounded open set with $\partial\Gamma = \gamma$, we conclude as follows.

Proposition 4.4. • *If $z \in \Gamma$, then one of the solutions belongs to $D(0,1)$ and the other one is in $\mathbf{C} \setminus \overline{D(0,1)}$.*

- *If $z \in \gamma$, then one of the solutions belongs to S^1 and the other is in $\mathbf{C} \setminus \overline{D(0,1)}$.*
- *If $z \in \mathbf{C} \setminus \overline{\Gamma}$, then both solutions are in $z \in \mathbf{C} \setminus \overline{D(0,1)}$.*

When $|b| = |a|/2$, we have:

Proposition 4.5. *For $z \neq f(\zeta_c)$ (i.e. for z away from the cusp of the simple closed curve $f(S^1)$) we have the same conclusion as in Proposition 4.4. When $z = f(\zeta_c)$ (i.e. at the cusp), $\zeta = \zeta_c \in S^1$ is a double solution of $z = f(\zeta)$ (and there is no other).*

When $|b| > |a|/2$, we have $\zeta_c \in D(0,1)$ and $\zeta_c \neq 0$. Draw the line through ζ_c which is perpendicular to the radius of $D(0,1)$ that passes through that point and let α_c and α_c' be the two points of intersection with S^1. We have seen in Section 2 that $f(\alpha_c) = f(\alpha_c')$ and that the short circle arcs and the long circle arcs connecting these two points are mapped by f onto two simple closed curves γ_{int} and γ_{ext}, both containing the point $f(\alpha_c) = f(\alpha_c')$ and such that if we let Γ_{int}, Γ_{ext} denote the open bounded sets bounded by γ_{int}, γ_{ext} respectively, then away from $f(\alpha_c)$, γ_{int} is contained in Γ_{ext}. Also $\Gamma_{\mathrm{int}} \subset \Gamma_{\mathrm{ext}}$.

It is geometrically clear that the set of points ζ in $\overline{D(0,1)}$ for which the symmetric point with respect to ζ_c, namely $\zeta' = \zeta_c - (\zeta - \zeta_c)$, also belongs to $\overline{D(0,1)}$, is obtained by taking the short circular segment from α_c to α_c', then the convex hull of that set and finally adding all the symmetric points with respect to ζ_c. This is a lens shaped region L inside $\overline{D(0,1)}$ whose image under f coincides with $\overline{\Gamma}_{\mathrm{int}}$. This leads to:

Proposition 4.6. *When $|b| > |a|/2$, the following holds for the solutions (counted with their multiplicity) of the equation $f(\zeta) = z$:*

- *if $z \in \Gamma_{\mathrm{int}}$, then we have two solutions in $D(0,1)$;*
- *if $z \in \gamma_{\mathrm{int}} \setminus \{f(\alpha_c)\}$, then one of the solutions is on S^1, namely on the short circular arc between α_c and α_c', while the other solution is in $D(0,1)$;*
- *if $z = f(\alpha_c)$, then there are two solutions on S^1, namely α_c and α_c';*
- *if $z \in \Gamma_{\mathrm{ext}} \setminus \overline{\Gamma}_{\mathrm{int}}$, then then there is one solution in $D(0,1)$ and one outside $\overline{D(0,1)}$;*
- *if $z \in \gamma_{\mathrm{ext}} \setminus \{f(\alpha_c)\}$, then one of the solutions is in S^1 (namely on the long circular arc from α_c to α_c') and the other one is outside $\overline{D(0,1)}$;*
- *if $z \in \mathbf{C} \setminus \overline{\Gamma}_{\mathrm{ext}}$, then both solutions are outside $\overline{D(0,1)}$.*

5. Grushin problem for the unperturbed operator

We start with a quick introduction to Grushin problems. This method is an elementary way of reducing problems to lower dimensions and basically it is just a play with 2×2-matrices with operator entries. We follow a terminology which is common in PDE, but the method appears in many areas under different names: "Lyapunov-Schmidt bifurcation," "Shur complements," "effective Hamiltonians and Feshbach reduction". The calculations are mostly simple and direct. See [21] and [16] for recent discussions.

In this paper, we limit the attention to finite matrices. Let $P: \mathcal{H} \to \mathcal{H}$ be linear, $\mathcal{H} = \mathbb{C}^N$, $1 \le N < +\infty$. Let $\mathcal{G} = \mathbb{C}^n$, $1 \le n < N$. Consider

$$\mathcal{P}(z) = \begin{pmatrix} P - z & R_- \\ R_+ & 0 \end{pmatrix} : \mathcal{H} \times \mathcal{G} \longrightarrow \mathcal{H} \times \mathcal{G},\ z \in \Omega$$

where $R_+: \mathcal{H} \to \mathcal{G}$, $R_-: \mathcal{G} \to \mathcal{H}$ are linear operators of maximal rank (n), that we assume independent of z for simplicity. $\Omega \subset \mathbf{C}$ is some open connected set.

Assume that $\mathcal{P}(z)$ is bijective for every $z \in \Omega$ and let

$$\mathcal{E}(z) = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix} : \mathcal{H} \times \mathcal{G} \longrightarrow \mathcal{H} \times \mathcal{G}$$

be the inverse, depending holomorphically on $z \in \Omega$. For each fixed $z \in \Omega$ we have

Proposition 5.1. *For every $z \in \Omega$ we have that $P - z$ is bijective iff $E_{-+}(z)$ is bijective. When bijectivity holds, we have*

$$E_{-+}(z)^{-1} = -R_+(P - z)^{-1} R_-,$$

$$(P - z)^{-1} = E(z) - E_+(z) E_{-+}(z)^{-1} E_-(z). \tag{5.1}$$

The proof follows from some direct calculations based on the 8 operator identities that we get from $\mathcal{P}\mathcal{E} = 1$ and $\mathcal{E}\mathcal{P} = 1$.

Letting $\sigma(P)$ denote the spectrum of P, it follows that

$$\sigma(P) \cap \Omega = \{z \in \Omega;\ \det E_{-+}(z) = 0\}.$$

Keeping the assumption that $\mathcal{P}(z)$ is bijective for all $z \in \Omega$, we have:

Proposition 5.2. *For every $z_0 \in \sigma(P) \cap \Omega$, the algebraic multiplicity of z_0 as an eigenvalue of P coincides with the multiplicity of z_0 as a zero of the holomorphic function $\Omega \ni z \mapsto \det E_{-+}(z)$.*

Proof. We give the proof for completeness. From the identity

$$\partial_z \mathcal{E}(z) = -\mathcal{E}(z) \partial_z \mathcal{P}(z) \mathcal{E}(z),$$

we infer that $\partial_z E_{-+}(z) = E_-(z) E_+(z)$. Let γ be the oriented boundary of the disc $D(z_0, r)$, where $r > 0$ is small enough. The multiplicity $m(z_0)$ of z_0 as an eigenvalue of P is equal to the trace of the spectral projection, and by (5.1) and the fact that $E(z)$ is holomorphic in Ω, we get

$$\begin{aligned} m(z_0) &= \operatorname{tr} \frac{1}{2\pi i} \int_\gamma (z-P)^{-1} dz \\ &= \operatorname{tr} \frac{1}{2\pi i} \int_\gamma E_+(z) E_{-+}(z)^{-1} E_-(z) dz \\ &= \frac{1}{2\pi i} \int_\gamma \operatorname{tr} (E_+(z) E_{-+}(z)^{-1} E_-(z)) dz \\ &= \frac{1}{2\pi i} \int_\gamma \operatorname{tr} (E_{-+}(z)^{-1} E_-(z) E_+(z)) dz \\ &= \frac{1}{2\pi i} \int_\gamma \operatorname{tr} (E_{-+}(z)^{-1} \partial_z E_{-+}) dz \\ &= \frac{1}{2\pi i} \int_\gamma \partial_z \ln \det E_{-+}(z) dz \end{aligned}$$

and the last expression is equal to the multiplicity of z_0 as a zero det $E_{-+}(z)$. ☐

Below it will be convenient to permute the lines in $\mathcal{P}$ and the columns in $\mathcal{E}$. Then $\mathcal{E}$ will still be the inverse of $\mathcal{P}$.

We return to the discussion of bidiagonal matrices and from now on we only consider the Case I and write $P = P_{\mathrm{I}}$. We are interested in the case when z is inside the ellipse E_1, so that the two solutions of the characteristic equation are in $D(0,1)$ and correspond to exponential solutions that decay in the direction of increasing k. Consequently, in our Grushin problem we put a condition of type "+" at the endpoint $k = 1$ of $[1, N]$ and a corresponding co-condition of type "−" at the right end point $k = N$. Define

$$R_+ \colon \mathbf{C}^N \longrightarrow \mathbf{C} \quad \text{and} \quad R_- \colon \mathbf{C} \longrightarrow \mathbf{C}^N,$$

by

$$R_+ u = a u(1) \quad \text{and} \quad R_- u_- = a u_- e_N, \tag{5.2}$$

for $u \in \mathbf{C}^N$, $u_- \in \mathbf{C}$, where e_N denotes the Nth canonical basis vector in $\mathbf{C}^N$ so that $e_N(j) = \delta_{j,N}$. We are then interested in inverting

$$\begin{pmatrix} P-z & R_- \\ R_+ & 0 \end{pmatrix} \begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} v \\ v_+ \end{pmatrix} \tag{5.3}$$

in $\mathbf{C}^N \times \mathbf{C}$.

It will be convenient (without changing the mathematics) to permute the components v, v_+, so we apply the block matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : \mathbf{C}^N \times \mathbf{C} \longrightarrow \mathbf{C} \times \mathbf{C}^N$$

to the left and get the equivalent problem

$$\mathcal{P}(z)\begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} v_+ \\ v \end{pmatrix}, \quad \text{where } \mathcal{P}(z) = \begin{pmatrix} R_+ & 0 \\ P - z & R_- \end{pmatrix} : \mathbf{C}^N \times \mathbf{C} \longrightarrow \mathbf{C} \times \mathbf{C}^N. \tag{5.4}$$

If

$$\begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix}$$

denotes the inverse of the matrix in (5.3) (when it exists), then the inverse of the matrix in (5.4) is given by

$$\mathcal{E} = \begin{pmatrix} E_+ & E \\ E_{-+} & E_- \end{pmatrix} : \mathbf{C} \times \mathbf{C}^N \longrightarrow \mathbf{C}^N \times \mathbf{C}. \tag{5.5}$$

The important quantity E_{-+} now appears in the lower left corner.

Identifying $\mathbf{C}^N \times \mathbf{C} \simeq \mathbf{C} \times \mathbf{C}^N \simeq \mathbf{C}^{N+1}$ in the natural way we see that $\mathcal{P}(z)$ has a lower triangular matrix with

- all entries on the main diagonal equal to a,
- all entries on the "subdiagonal" (i.e. with indices j, k satisfying $j - k = 1$) equal to $-z$,
- all entries on the "subsubdiagonal" (with indices satisfying $j - k = 2$) equal to b,
- all other entries equal to zero.

Equivalently, we can write

$$\mathcal{P}(z) = 1_{[1,N+1]}(a - z\tau + b\tau^2),$$

where τ denotes translation to the right by one unit, when identifying

$$\mathbf{C}^{N+1} \simeq \ell^2_{[1,N+1]}(\mathbf{Z}) = \{u \in \ell^2(\mathbf{Z}); u(k) = 0 \text{ for } k \notin [1, N+1]\}.$$

We see that $\mathcal{P}(z)$ is invertible with inverse $\mathcal{E}(z)$ given by a lower triangular matrix with constant entries c_ν on the νth subdiagonal (i.e. the entries with index (j,k) for which $j-k=\nu$). Further $c_0 = 1/a$. Equivalently,

$$\mathcal{E}(z) = 1_{[1,N+1]}(c_0 + c_1\tau + \cdots + c_N\tau^N). \tag{5.6}$$

Also notice that c_ν is independent of N. The first column u in the matrix of $\mathcal{E}(z)$ is equal to $(c_0, c_1, \ldots, c_N)^{\mathrm{t}}$ and it solves the problem

$$\mathcal{P}(z)u = e_1, \quad \text{where } e_1(j) = \delta_{1,j}.$$

This gives the equations,

$$\begin{aligned} au(1) &= 1, \\ -zu(1) + au(2) &= 0, \\ bu(1) - zu(2) + au(3) &= 0, \\ bu(2) - zu(3) + au(4) &= 0, \\ &\vdots \\ bu(N-1) - zu(N) + au(N+1) &= 0. \end{aligned}$$

Extend u to $u \in \ell^2([0, N+1])$, by putting $u(0) = 0$. Then we get,

$$\begin{cases} bu(k-1) - zu(k) + au(k+1) = 0, & k = 1, \ldots, N, \\ u(0) = 0, \\ u(1) = 1/a. \end{cases} \tag{5.7}$$

Here u can be extended uniquely to all of $\mathbf{Z}$ by solving successively the first equation in (5.7) for $k = 0, -1, \ldots$ and for $k = N+1, N+2, \ldots$ and the extended function u has to be of the form

$$u(k) = c_+\zeta_+^k + c_-\zeta_-^k, \tag{5.8}$$

where $\zeta_\pm$ are the solutions of (3.3), and we assume that z is not a focal point of E_1, so that $\zeta_+ \neq \zeta_-$. The last two equations in (5.7) give

$$c_+ + c_- = 0, \quad \zeta_+c_+ + \zeta_-c_- = 1/a,$$

and we conclude that

$$c_+ = \frac{1}{a(\zeta_+ - \zeta_-)}, \quad c_- = -\frac{1}{a(\zeta_+ - \zeta_-)} \tag{5.9}$$

By (5.5), we know that E_{-+} is the last component of u, and hence

$$E_{-+}(z) = \frac{\zeta_+^{N+1} - \zeta_-^{N+1}}{a(\zeta_+ - \zeta_-)}. \tag{5.10}$$

Recall that by a general identity for Grushin problems,

$$(-1)^N E_{-+}(z) \det \mathcal{P}(z) = \det(P - z). \tag{5.11}$$

See (2.12) in [13] and also (4.3) in [16] for an indication of a quick proof. Here the factor $(-1)^N$ comes from the displacement of the 1st line when going from the "standard" matrix in (5.3) to $\mathcal{P}(z)$. Now,

$$\det \mathcal{P}(z) = a^{N+1}. \tag{5.12}$$

The last three equations give,

$$\det(P - z) = (-a)^N \frac{\zeta_+^{N+1} - \zeta_-^{N+1}}{\zeta_+ - \zeta_-}. \tag{5.13}$$

We observe a symmetry property of equation (5.13): reversing the orientation and permuting a and b, we could replace R_+ by $\widetilde{R}_+ u = bu(N)$ and R_- by $\widetilde{R}_- u = bu_- e_1$. We should then replace $\zeta_\pm$ by $1/\zeta_\pm$ and (5.13) becomes

$$\det(P - z) = (-b)^N \frac{\zeta_+^{-(N+1)} - \zeta_-^{-(N+1)}}{\zeta_+^{-1} - \zeta_-^{-1}}. \tag{5.14}$$

Using that $\zeta_+\zeta_- = b/a$, we check directly that the right hand sides in (5.13) and (5.14) are equal.

6. Estimates on the Grushin problem and the resolvent

The aim of this section is to obtain estimates on the Grushin problem and the resolvent for the unperturbed operator. In the following we will work with the convention that the two solutions $\zeta_\pm$ to the characteristic equations are such that

$$|\zeta_+| \le |\zeta_-|.$$

Since $\zeta_+\zeta_- = b/a$, this means that $|\zeta_+| \le |b/a|^{1/2} \le |\zeta_-|$.

6.1. When ζ_+ and ζ_- both belong to $D(0,1)$. Here we give estimates on (5.5) which is the inverse of (5.4), the Grushin problem for the unperturbed operator P_{I} in the case when z is inside the ellipse E_1.

By (5.5) and (5.6)

$$E = 1_{[1,N]}(c_0 + c_1\tau + \cdots + c_{N-2}\tau^{N-2})1_{[2,N+1]}, \tag{6.1}$$

$$E_+ = \begin{pmatrix} c_0 \\ \vdots \\ c_{N-1} \end{pmatrix}, \quad E_- = (c_{N-1}, \dots, c_0), \tag{6.2}$$

where, using (5.8) and (5.9),

$$c_k = \frac{\zeta_+^{k+1} - \zeta_-^{k+1}}{a(\zeta_+ - \zeta_-)}, \quad k = 0, \dots, N. \tag{6.3}$$

For $k \in \mathbb{N}$, $t \in \overline{D(0,1)}$, let

$$F_{k+1}(t) = 1 + t + \cdots + t^k = \begin{cases} k+1 & \text{when } t = 1, \\ \dfrac{1-t^{k+1}}{1-t} & \text{when } t \neq 1. \end{cases}$$

By the triangle inequality,

$$|F_{k+1}(t)| \le F_{k+1}(|t|).$$

We have

$$|F_{k+1}(t)| \le \min\Big(k+1, \frac{2}{|1-t|}\Big).$$

From (6.3) we get

$$c_k = \frac{\zeta_-^k}{a} F_{k+1}\Big(\frac{\zeta_+}{\zeta_-}\Big), \tag{6.4}$$

$$|c_k| \le \frac{|\zeta_-|^k}{|a|} \min\Big(k+1, \frac{2}{|1-\zeta_+/\zeta_-|}\Big). \tag{6.5}$$

By (5.10), (6.4), and (6.5) we have that

$$|E_{-+}| = \frac{|\zeta_-|^N}{|a|}|F_{N+1}(\zeta_+/\zeta_-)| \le \frac{|\zeta_-|^N}{|a|} \min\Big(N+1, \frac{2}{|1-\zeta_+/\zeta_-|}\Big).$$

Proposition 6.1. *If* $\zeta_\pm \in D(0,1)$, *then*

$$\|E\| \le |a|^{-1} \min\Big(N, \frac{2}{1-|\zeta_-|}\Big) \min\Big(N, \frac{2}{1-|\zeta_-|}, \frac{2}{|1-\zeta_+/\zeta_-|}\Big) \tag{6.6}$$

and

$$\|E_+\| = \|E_-\| \le |a|^{-1} \min\Big(N, \frac{2}{1-|\zeta_-|}\Big)^{1/2} \min\Big(N, \frac{2}{1-|\zeta_-|}, \frac{2}{|1-\zeta_+/\zeta_-|}\Big). \tag{6.7}$$

Proof. From 6.1, we infer that

$$\|E\| \le |c_0| + \cdots + |c_{N-2}|.$$

Then, by (6.5),

$$\|E\| \le \frac{1}{|a|} \min\Big(\sum_0^{N-2} |\zeta_-|^k (k+1), \frac{2}{|1-\zeta_+/\zeta_-|} F_{N-1}(|\zeta_-|)\Big).$$

Here

$$\sum_0^{N-2} |\zeta_-|^k (k+1) \le \min\Big((N-1)^2, \sum_0^{\infty} |\zeta_-|^k (k+1)\Big)$$

and

$$\sum_0^{\infty} |\zeta_-|^k (k+1) = \partial_t \Big(\sum_0^{\infty} t^{k+1}\Big)_{t=|\zeta_-|} = \partial_t \Big(\frac{t}{1-t}\Big)_{t=|\zeta_-|} = \frac{1}{(1-|\zeta_-|)^2},$$

leading to

$$\|E\| \le \frac{1}{|a|} \min\Big((N-1)^2, \frac{1}{(1-|\zeta_-|)^2}, \frac{2(N-1)}{|1-\zeta_+/\zeta_-|}, \frac{4}{|1-\zeta_+/\zeta_-|(1-|\zeta_-|)}\Big)$$

which implies (6.6). Continuing, we see by (6.2) and (6.5), that

$$\begin{aligned} \|E_-\|^2 = \|E_+\|^2 &= |c_0|^2 + \cdots + |c_{N-1}|^2 \\ &\le \frac{1}{|a|^2} \min\Big(\sum_0^{N-1} |\zeta_-|^{2k} (k+1)^2, \frac{4}{|1-\zeta_+/\zeta_-|^2} \sum_0^{N-1} |\zeta_-|^{2k}\Big). \end{aligned}$$

Here,

$$\sum_{0}^{N-1} |\zeta_-|^{2k} = F_N(|\zeta_-|^2),$$

and for $0 \le t \le 1$,

$$F_N(t^2) = \frac{1-t^{2N}}{1-t^2} = F_N(t)\frac{1+t^N}{1+t} \le F_N(t)$$

so

$$\sum_{0}^{N-1} |\zeta_-|^{2k} \le F_N(|\zeta_-|).$$

Furthermore,

$$\sum_{0}^{N-1} |\zeta_-|^{2k}(k+1)^2 \le \min\Big(N^3, \sum_{0}^{\infty} |\zeta_-|^{2k}(k+1)^2\Big).$$

Here,

$$\begin{aligned}\sum_{0}^{\infty} t^k(k+1)^2 &= \partial_t t \partial_t \sum_{0}^{\infty} t^{k+1}\\ &= \partial_t t \partial_t \frac{t}{1-t}\\ &= \partial_t t \partial_t \frac{1}{1-t}\\ &= \partial_t \frac{t}{(1-t)^2}\\ &= \frac{1+t}{(1-t)^3}.\end{aligned}$$

Hence

$$\begin{aligned}\sum_{0}^{N-1} |\zeta_-|^{2k}(k+1)^2 &\le \min\Big(N^3, \frac{1+|\zeta_-|^2}{(1-|\zeta_-|^2)^3}\Big)\\ &\le \min\Big(N^3, \frac{1}{(1+|\zeta_-|)(1-|\zeta_-|)^3}\Big)\\ &\le \min\Big(N^3, \frac{1}{(1-|\zeta_-|)^3}\Big)\\ &\le \min\Big(N, \frac{1}{(1-|\zeta_-|)}\Big)^3.\end{aligned}$$

Thus,

$$
\begin{aligned}
\|E_+\|^2 &= \|E_-\|^2 \\
&\le \frac{1}{|a|^2} \min\Big(\min\Big(N, \frac{1}{(1-|\zeta_-|)}\Big)^3, \frac{4}{|1-\zeta_+/\zeta_-|^2} F_N(|\zeta_-|)\Big) \\
&\le \frac{1}{|a|^2} \min\Big(\min\Big(N, \frac{1}{(1-|\zeta_-|)}\Big)^3, \frac{4}{|1-\zeta_+/\zeta_-|^2} \min\Big(N, \frac{2}{1-|\zeta_-|}\Big)\Big) \\
&\le \frac{1}{|a|^2} \min\Big(N, \frac{2}{1-|\zeta_-|}\Big) \min\Big(N, \frac{1}{(1-|\zeta_-|)}, \frac{2}{|1-\zeta_+/\zeta_-|}\Big)^2,
\end{aligned}
$$

and we conclude (6.7). □

In the following we concentrate on the case when E_1 is a true non-degenerate ellipse, i.e. when

$$0 < |b| < |a|. \tag{6.8}$$

The degeneration $\zeta_+/\zeta_- \approx 1$ takes place near the focal points $z = \pm 2\sqrt{ab}$ where $\zeta_+ \approx \zeta_- \approx \pm\sqrt{b/a}$ and hence $|\zeta_-| < 1$ so we are away from the degeneration $|\zeta_-| \approx 1$, which takes place near E_1. Until further notice we assume that z is not in a neighbourhood of the focal segment.

From (6.6), (6.7), and the fact that $|\zeta_+/\zeta_-|$ is bounded by a constant < 1 (cf. the end of Remark 4.3), we get

$$\|E\| \le \mathcal{O}(1) F_N(|\zeta_-|), \tag{6.9}$$

$$\|E_+\|,\ \|E_-\| \le \mathcal{O}(1) F_N(|\zeta_-|)^{1/2}. \tag{6.10}$$

Here, we used that $F_{k+1}(t) \asymp \min\big(k+1, \frac{1}{1-t}\big)$ for $0 \le t \le 1$.

6.2. When one of $|\zeta_\pm|$ is larger than 1 and the other one smaller than 1. In this case

$$|\zeta_+| \le |b/a| < 1 \le |\zeta_-|$$

and we estimate the resolvent of $P = P_{\mathrm{I}}$ directly. Recall that we work with

$$P = P_{\mathrm{I}} = 1_{[1,N]}(a\tau^{-1} + b\tau), \quad \tau = \tau_1,$$

with the identification $\ell^2([1,N]) \simeq \ell^2_{[1,N]}(\mathbf{Z})$. We start by deriving a fairly explicit expression for the resolvent, valid under the sole assumption that z is not in the spectrum.

a) We first invert $a\tau^{-1} + b\tau - z$ on $\ell^2(\mathbf{Z})$. This is a convolution operator and we look for a fundamental solution $F:\mathbf{Z} \to \mathbf{C}$ solving

$$(a\tau^{-1} + b\tau - z)F = \delta_0, \tag{6.11}$$

where $\delta_0(j) = \delta_{0,j}$. As before, we assume that $|\zeta_+| \le |\zeta_-|$. When $|\zeta_+| < 1 < |\zeta_-|$ our function F will belong to ℓ^1. Try

$$F(k) = c \begin{cases} \zeta_+^k, & k \ge 0, \\ \zeta_-^k, & k \le 0, \end{cases} \tag{6.12}$$

where c will be determined. (6.11) means that

$$au(k+1) + bu(k-1) - zu(k) = \delta_{0,k}, \quad k \in \mathbf{Z}.$$

With the choice (6.12), this holds for $k \neq 0$ and for $k = 0$ we get

$$c(a\zeta_+ + b\zeta_-^{-1} - z) = 1,$$

i.e.

$$c = \frac{1}{a\zeta_+ + b/\zeta_- - z}. \tag{6.13}$$

Using that $a\zeta_\pm + b/\zeta_\pm - z = 0$, we have $b/\zeta_- - z = -a\zeta_-$ and (6.13) becomes

$$c = \frac{1}{a(\zeta_+ - \zeta_-)}. \tag{6.14}$$

Thus, with $P_\infty = a\tau^{-1} + b\tau$ acting on functions on $\mathbf{Z}$, we get

$$(P_\infty - z) \circ (F*)v = v, v \in \ell^2_{\mathrm{comp}}(\mathbf{Z}), \tag{6.15a}$$

$$(F*) \circ (P_\infty - z)u = u, u \in \ell^2_{\mathrm{comp}}(\mathbf{Z}), \tag{6.15b}$$

where ℓ^2_{comp} is the space of functions on $\mathbf{Z}$ that vanish outside a bounded interval, and $F*$ denotes the convolution operator, defined by

$$F * v(j) = \sum_k F(j-k)v(k).$$

When

$$|\zeta_+| < 1 < |\zeta_-|,$$

F belongs to ℓ^1, $F*$ is bounded on ℓ^2, eq. (6.15) extends to the case when $u, v \in \ell^2(\mathbf{Z})$ and then expresses that $F*$ is a bounded 2-sided inverse of $P_\infty - z: \ell^2 \to \ell^2$.

For future reference we combine (6.12) and (6.14) to

$$F(k) = \frac{1}{a(\zeta_+ - \zeta_-)} \begin{cases} \zeta_+^k, & k \geq 0, \\ \zeta_-^k, & k \leq 0. \end{cases}$$

b) We next solve

$$(a\tau^{-1} + b\tau - z)u = 0 \quad \text{on } \mathbf{Z},$$

with one of the two sets of “Dirichlet” conditions,

$$u(0) = 1, \quad u(N+1) = 0$$

or

$$u(0) = 0, \quad u(N+1) = 1$$

Denote the solutions by $u = u_L$, $u = u_R$ respectively, when they exist and are unique.

In both cases we know that u has to be of the form

$$u(j) = c_+\zeta_+^j + c_-\zeta_-^j,$$

and it suffices to see when $c_\pm$ exist and are unique. After some straightforward calculations, we get existence and uniqueness under the condition

$$\zeta_+^{N+1} - \zeta_-^{N+1} \neq 0, \tag{6.16}$$

and then

$$u_L(j) = \frac{1}{1 - (\zeta_+/\zeta_-)^{N+1}} (\zeta_+^j - \zeta_+^{N+1}(1/\zeta_-)^{N+1-j}),$$

$$u_R(j) = \frac{1}{1 - (\zeta_+/\zeta_-)^{N+1}} ((1/\zeta_-)^{N+1-j} - (1/\zeta_-)^{N+1}\zeta_+^j).$$

c) Solution of $(P - z)u = v$ in $\ell^2([1, N])$. We adopt the assumption (6.16) from now on and recall, that this is equivalent to the assumption that z avoids the spectrum of $P = P_{\mathrm{I}}$ and the two focal points. With the usual identification $\ell^2([1, N]) \simeq \ell^2_{[1,N]}(\mathbf{Z})$ it is now clear that the unique solution is

$$u = \tilde{u}_{|[1,N]}, \quad \text{where } \tilde{u} = F * v - (F * v)(0)u_L - (F * v)(N+1)u_R.$$

Let $E = (P - z)^{-1}$ (in line with our general notation for Grushin problems, now in the case when $R_\pm$, $E_\pm$, E_{-+} are absent) and let $E(j,k)$, $1 \le j,k \le N$ be the matrix elements of E. Then $E(j,k) = \tilde{u}(j)$ where $\tilde{u}$ is the function above associated to $v = \delta_k$. Writing $\zeta^j_{\mathrm{sgn(j)}} = \zeta^j_+$ for $j \ge 0$ and $= \zeta^j_-$ for $j < 0$, $|\zeta_{\mathrm{sgn}(j)}|^j = |\zeta^j_{\mathrm{sgn}(j)}|$, we get first

$$E(j,k) = F(j-k) - F(-k)u_L(j) - F(N+1-k)u_R(j),$$

and after substitution of the above expressions for F, u_L and u_R,

$$\begin{aligned} E(j,k) = \frac{1}{a(\zeta_+ - \zeta_-)}\Big(\zeta^{j-k}_{\mathrm{sgn}(j-k)} - \frac{1}{1-\big(\frac{\zeta_+}{\zeta_-}\big)^{N+1}}\Big(\Big(1-\Big(\frac{\zeta_+}{\zeta_-}\Big)^{N+1-j}\Big)\zeta^j_+\Big(\frac{1}{\zeta_-}\Big)^k \\ + \Big(1-\Big(\frac{\zeta_+}{\zeta_-}\Big)^j\Big)\Big(\frac{1}{\zeta_-}\Big)^{N+1-j}\zeta_+^{N+1-k}\Big)\Big). \end{aligned}$$

d) In addition to (6.16), we now assume

$$|\zeta_+| \le 1 \le |\zeta_-|.$$

Then we get

$$\begin{aligned} |E(j,k)| \le \frac{1}{|a||\zeta_+ - \zeta_-|}\Big(|\zeta_{\mathrm{sgn}(j-k)}|^{j-k} \\ + \frac{2}{\big|1-\big(\frac{\zeta_+}{\zeta_-}\big)^{N+1}\big|}\Big(|\zeta_+|^j\Big(\frac{1}{|\zeta_-|}\Big)^k \\ + \Big(\frac{1}{|\zeta_-|}\Big)^{N+1-j}|\zeta_+|^{N+1-k}\Big)\Big). \end{aligned}$$

In the big parenthesis the first term corresponds to a convolution and the second term corresponds to the sum of two rank 1 operators. Letting $\|\cdot\|$ denote the norm in ℓ^2 or in $\mathcal{L}(\ell^2,\ell^2)$, depending on the context, we get

$$\|E\| \le \frac{1}{|a|\,|\zeta_+ - \zeta_-|}\Big(\sum_{1-N}^{N-1}|\zeta_{\mathrm{sgn}(j)}|^j + \frac{4}{\big|1-\big(\frac{\zeta_+}{\zeta_-}\big)^{N+1}\big|}\|1_{[1,N]}\zeta_+^{\cdot}\|\|1_{[1,N]}\zeta_-^{-\cdot}\|\Big). \tag{6.17}$$

Recall that $F_N(t) = 1 + t + \cdots + t^{N-1}$ and that

$$F_N(t) \asymp \min(1/(1-t), N), \quad 0 < t \le 1.$$

We have

$$\sum_{1-N}^{N-1}|\zeta_{\mathrm{sgn}(j)}|^j = 1 + |\zeta_+|F_{N-1}(|\zeta_+|) + \frac{1}{|\zeta_-|}F_{N-1}(1/|\zeta_-|).$$

Also,

$$\|1_{[1,N]}\zeta_+^{\cdot}\|^2 = |\zeta_+|^2 F_N(|\zeta_+|^2).$$

Here,

$$F_N(t^2) = \frac{1-t^{2N}}{1-t^2} = \frac{1+t^N}{1+t} F_N(t),$$

so

$$F_N(t)/2 \leq F_N(t^2) \leq F_N(t).$$

Similarly,

$$\|1_{[1,N]}\zeta_-^{-\cdot}\|^2 = \frac{1}{|\zeta_-|^2} F_N(1/|\zeta_-|^2)$$

and using these facts in (6.17), we get

$$\begin{aligned}\|E\| \leq \frac{\mathcal{O}(1)}{|a|\,|\zeta_+ - \zeta_-|}\Big(1 + |\zeta_+| F_N(|\zeta_+|) + \frac{1}{|\zeta_-|} F_N\Big(\frac{1}{|\zeta_-|}\Big) \\ + \frac{|\zeta_+/\zeta_-|}{|1-(\zeta_+/\zeta_-)^{N+1}|} F_N(|\zeta_+|)^{1/2} F_N\Big(\frac{1}{|\zeta_-|}\Big)^{1/2}\Big).\end{aligned} \tag{6.18}$$

Recall here, that $|\zeta_+| \leq |b/a|^{1/2} < 1$ and we have assumed that z avoids a fixed neighborhood of the focal segment, so $|\zeta_+/\zeta_-| \leq \text{const.} < 1$.

7. Grushin problem for the perturbed operator

We interested in the following random perturbation of $P_0 = P_{\mathrm{I}}$:

$$P_\delta := P_0 + \delta Q_\omega, \quad Q_\omega = (q_{j,k}(\omega))_{1\leq j,k\leq N}, \tag{7.1}$$

where $0 \leq \delta \ll 1$ and $q_{j,k}(\omega)$ are independent and identically distributed complex Gaussian random variables, following the complex Gaussian law $\mathcal{N}_{\mathbf{C}}(0,1)$.

The Markov inequality implies that if $C_1 > 0$ is large enough, then for the Hilbert–Schmidt norm,

$$\mathbb{P}\,[\|Q_\omega\|_{\mathrm{HS}} \leq C_1 N] \geq 1 - e^{-N^2}, \tag{7.2}$$

where $\mathbb{P}(A)$ denotes the probability of the event A. This has already been observed by W. Bordeaux-Montrieux in [1].

7.1. A general discussion. We begin with a formal discussion of the natural Grushin problem for P_δ. Recall from Section 5 that the Grushin problem is of the form

$$\mathcal{P}_0 = \begin{pmatrix} R_+ & 0 \\ P_0 - z & R_- \end{pmatrix} : \mathbf{C}^N \times \mathbf{C} \longrightarrow \mathbf{C} \times \mathbf{C}^N,$$

where we added a subscript 0 to indicate that we deal with the unperturbed operator. Recall that $\mathcal{P}_0$ is bijective with inverse

$$\mathcal{E}_0 = \begin{pmatrix} E_+^0 & E^0 \\ E_{-+}^0 & E_-^0 \end{pmatrix} : \mathbf{C} \times \mathbf{C}^N \longrightarrow \mathbf{C}^N \times \mathbf{C},$$

where we added a superscript 0 for the same reason. If $\delta \| Q_\omega \| \| E^0 \| < 1$, we see using a Neumann series that

$$\mathcal{P}_\delta = \begin{pmatrix} R_+ & 0 \\ P_\delta - z & R_- \end{pmatrix} : \mathbf{C}^N \times \mathbf{C} \longrightarrow \mathbf{C} \times \mathbf{C}^N,$$

is bijective and admits the inverse

$$\mathcal{E}_\delta = \begin{pmatrix} E_+^\delta & E^\delta \\ E_{-+}^\delta & E_-^\delta \end{pmatrix} : \mathbf{C} \times \mathbf{C}^N \longrightarrow \mathbf{C}^N \times \mathbf{C}.$$

where

$$\begin{aligned} E_+^\delta &= E_+^0 - \delta E^0 Q_\omega E_+^0 + \delta^2 (E^0 Q_\omega)^2 E_+^0 + \cdots = (1 + \delta E^0 Q_\omega)^{-1} E_+^0, \\ E_-^\delta &= E_-^0 - \delta E_-^0 (Q_\omega) E^0 + \delta^2 E_-^0 (Q_\omega E^0)^2 + \cdots = E_-^0 (1 + \delta Q_\omega E^0)^{-1}, \\ E^\delta &= E^0 - \delta E^0 (Q_\omega E^0) + \delta^2 E^0 (Q_\omega E^0)^2 + \cdots = E^0 (1 + \delta Q_\omega E^0)^{-1}, \\ E_{-+}^\delta &= E_{-+}^0 - \delta E_-^0 Q_\omega E_+^0 + \delta^2 E_-^0 Q_\omega E^0 Q_\omega E_+^0 + \cdots \\ &= E_{-+}^0 - \delta E_-^0 Q_\omega (1 + \delta E^0 Q_\omega)^{-1} E_+^0. \end{aligned}$$

One obtains the following estimates:

$$\| E^\delta \| \leq \frac{\| E^0 \|}{1 - \delta \| Q_\omega \| \| E^0 \|}, \tag{7.3a}$$

$$\| E_\pm^\delta \| \leq \frac{\| E_\pm^0 \|}{1 - \delta \| Q_\omega \| \| E^0 \|}, \tag{7.3b}$$

$$| E_{-+}^\delta - E_{-+}^0 | \leq \frac{\delta \| E_+^0 \| \| E_-^0 \| \| Q_\omega \|}{1 - \delta \| Q_\omega \| \| E^0 \|}. \tag{7.3c}$$

Differentiating the equation $\mathcal{E}^\delta \mathcal{P}^\delta = 1$ with respect to δ yields

$$\partial_\delta \mathcal{E}^\delta = -\mathcal{E}^\delta (\partial_\delta \mathcal{P}^\delta) \mathcal{E}^\delta = -\begin{pmatrix} E^\delta Q_\omega E_+^\delta & E^\delta Q_\omega E^\delta \\ E_-^\delta Q_\omega E_+^\delta & E_-^\delta Q_\omega E^\delta \end{pmatrix}. \tag{7.4}$$

Integrating this relation from 0 to δ yields

$$\|E^\delta - E^0\| \le \frac{\delta \|Q_\omega\| \|E^0\|^2}{(1-\delta\|Q_\omega\|\|E^0\|)^2}, \tag{7.5a}$$

$$\|E_\pm^\delta - E_\pm^0\| \le \frac{\delta \|Q_\omega\| \|E_\pm^0\| \|E^0\|}{(1-\delta\|Q_\omega\|\|E^0\|)^2}. \tag{7.5b}$$

Since $\mathcal{P}^\delta$ is invertible and of finite rank, we know that

$$|\partial_\delta \ln \det \mathcal{P}^\delta| = |\mathrm{tr}(\mathcal{E}^\delta \partial_\delta \mathcal{P}^\delta)|.$$

Letting $\|\cdot\|_{\mathrm{tr}}$ denote the trace class norm, we get

$$|\partial_\delta \ln \det \mathcal{P}^\delta| = |\mathrm{tr}(Q_\omega E^\delta)| \le \|Q_\omega\|_{\mathrm{tr}} \|E^\delta\| \le \frac{\|E^0\|\|Q_\omega\|_{\mathrm{tr}}}{1-\delta\|Q_\omega\|\|E^0\|},$$

where $\|Q_\omega\|_{\mathrm{tr}} \le N^{1/2}\|Q_\omega\|_{\mathrm{HS}}$. Integration from 0 to δ yields

$$|\ln|\det \mathcal{E}^\delta| - \ln|\det \mathcal{E}^0|| = |\ln|\det \mathcal{P}^\delta| - \ln|\det \mathcal{P}^0|| \le \frac{\delta\|E^0\|\|Q_\omega\|_{\mathrm{tr}}}{1-\delta\|Q_\omega\|\|E^0\|}. \tag{7.6}$$

Sharpening the assumption $\delta\|Q_\omega\|\|E^0\| < 1$ to

$$\delta\|Q_\omega\|\|E^0\| < \frac{1}{2}, \tag{7.7}$$

we get

$$\|E^\delta\| \le 2\|E^0\|, \tag{7.8a}$$

$$\|E_\pm^\delta\| \le 2\|E_\pm^0\|, \tag{7.8b}$$

$$|E_{-+}^\delta - E_{-+}^0| \le 2\delta \|E_+^0\|\,\|E_-^0\|\,\|Q_\omega\|. \tag{7.8c}$$

By (7.4) we know that

$$\partial_\delta E_{-+}^\delta = -E_-^\delta Q_\omega E_+^\delta.$$

Therefore, using (7.3), (7.5), and (7.8) we get

$$\begin{aligned}|\partial_\delta E^\delta_{-+} + E^0_- Q_\omega E^0_+| &\le \|E^0_- Q_\omega\| \|E^\delta_+ - E^0_+\| + \|Q_\omega E^\delta_+\| \|E^\delta_- - E^0_-\| \\ &\le 12\delta \|Q_\omega\|^2 \|E^0_-\| \|E^0_+\| \|E^0\|.\end{aligned} \tag{7.9}$$

By integration from 0 to δ, we conclude

$$E^\delta_{-+} = E^0_{-+} - \delta E^0_- Q_\omega E^0_+ + \mathcal{O}(1)\delta^2 \|Q_\omega\|^2 \|E^0_-\| \|E^0_+\| \|E^0\|. \tag{7.10}$$

7.2. More specific estimates

a) The case where z is inside the ellipse E_1. We adopt the non-degeneracy condition (6.8): $0 < |b| < |a|$ and keep the assumption that z avoids a neighbourhood of the focal points. In view of (6.9) and the fact that $\|Q_\omega\|_{\mathrm{HS}} \le C_1 N$ (cf. (7.2)) we replace assumption (7.7) by the stronger and more explicit condition

$$\delta N F_N(|\zeta_-|) \ll 1. \tag{7.11}$$

Notice that this is fulfilled for all z inside E_1, if we make the even stronger assumption

$$\delta N^2 \ll 1. \tag{7.12}$$

(Recall that $N \gg 1$).

We conclude from the discussion above and (6.9), (6.10):

Proposition 7.1. *Let $0 \le \delta \ll 1$ satisfy (7.11) and let P_δ be as in (7.1), $R_\pm$ be as in (5.2) and assume that $\|Q_\omega\|_{\mathrm{HS}} \le C_1 N$ (cf. (7.2)). Then,*

$$\mathcal{P}_\delta = \begin{pmatrix} R_+ & 0 \\ P_\delta - z & R_- \end{pmatrix} : \mathbf{C}^N \times \mathbf{C} \longrightarrow \mathbf{C} \times \mathbf{C}^N,$$

is bijective with bounded inverse

$$\mathcal{E}_\delta = \begin{pmatrix} E^\delta_+ & E^\delta \\ E^\delta_{-+} & E^\delta_- \end{pmatrix} : \mathbf{C} \times \mathbf{C}^N \longrightarrow \mathbf{C}^N \times \mathbf{C},$$

where

$$\|E^\delta - E^0\| \le \mathcal{O}(1)\delta N F_N(|\zeta_-|)^2,$$

$$\|E^\delta_\pm - E^0_\pm\| \le \mathcal{O}(1)\delta N F_N(|\zeta_-|)^{3/2},$$

$$E^\delta_{-+} = E^0_{-+} - \delta E^0_- Q_\omega E^0_+ + \mathcal{O}(1)(\delta N F_N(|\zeta_-|))^2.$$

Here $E^0_\pm$, E^0, E^0_{-+} are as in (6.1), (6.2), and (5.10).

Using (5.10), (6.2), and (6.3) we get with $Q_\omega = (q_{j,k}(\omega))$,

$$E^\delta_{-+} = \frac{\zeta_+^{N+1} - \zeta_-^{N+1}}{a(\zeta_+ - \zeta_-)} - \delta \sum_{j,k=1}^{N} q_{j,k}(\omega) \frac{\zeta_+^{N+1-j} - \zeta_-^{N+1-j}}{a(\zeta_+ - \zeta_-)} \frac{\zeta_+^k - \zeta_-^k}{a(\zeta_+ - \zeta_-)} \tag{7.13}$$
$$+ \mathcal{O}(1)(\delta N F_N(|\zeta_-|))^2.$$

From (7.6), we get

Proposition 7.2. *Under the same assumptions as in Proposition 7.1, we have*

$$|\ln|\det \mathcal{P}_\delta| - \ln|\det \mathcal{P}_0|| \le \mathcal{O}(1)\delta N^{3/2} F_N(|\zeta_-|). \tag{7.14}$$

Here we also used that $\|Q_\omega\|_{\mathrm{tr}} \le N^{1/2}\|Q_\omega\|_{\mathrm{HS}} \le \mathcal{O}(N^{3/2})$. Also recall that $\det \mathcal{P}_0 = a^{N+1}$.

b) The case when z belongs to a compact set outside E_1. We see from (6.18) and the subsequent sentence that

$$\|(P-z)^{-1}\| \le \mathcal{O}(1) F_N\Big(\frac{1}{|\zeta_-|}\Big).$$

Assume

$$\delta N F_N\Big(\frac{1}{|\zeta_-|}\Big) \ll 1, \tag{7.15}$$

which like (7.11) is a weaker condition than (7.12). Then,

$$\|\delta Q_\omega (P-z)^{-1}\| \le \mathcal{O}(1)\delta N F_N \ll 1$$

and $P_\delta - z$ is bijective satisfying

$$\|(P_\delta - z)^{-1}\| \le \mathcal{O}(1) F_N\Big(\frac{1}{|\zeta_-|}\Big),$$

$$\|(P_\delta - z)^{-1} - (P-z)^{-1}\| \le \mathcal{O}(1)\delta N (F_N(|\zeta_-|^{-1}))^2,$$

In analogy with (7.6), we have

$$|\ln|\det(P_\delta - z)| - \ln|\det(P-z)|| \le \mathcal{O}(1)\delta \|Q_\omega\|_{\mathrm{tr}} F_N\Big(\frac{1}{|\zeta_-|}\Big),$$

leading to

$$|\ln|\det(P_\delta - z)| - \ln|\det(P-z)|| \le \mathcal{O}(\delta) N^{3/2} F_N\Big(\frac{1}{|\zeta_-|}\Big), \tag{7.16}$$

under the assumption $\|Q_\omega\|_{\mathrm{HS}} \le \mathcal{O}(N)$. Recall that $\det(P_0 - z)$ is given by (5.13).

c) Estimation of the probability that E_{-+}^{δ} is small. We now return to the situation in a), i.e. when z is inside the ellipse E_1, so that $|\zeta_-| \le 1$. We assume (7.11) (to be strengthened later on). We shall follow Section 13.5 in [17] with only small changes. Write (7.13) as

$$E_{-+}^{\delta} = \frac{\zeta_+^{N+1} - \zeta_-^{N+1}}{a(\zeta_+ - \zeta_-)} - \delta(Q_\omega|\overline{Z}) + \mathcal{O}(1)(\delta N F_N)^2, \tag{7.17}$$

$F_N = F_N(|\zeta_-|)$, where

$$Z = \frac{1}{a^2}\Big(\frac{\zeta_+^{N+1-j} - \zeta_-^{N+1-j}}{a(\zeta_+ - \zeta_-)} \frac{\zeta_+^k - \zeta_-^k}{a(\zeta_+ - \zeta_-)}\Big)_{1\le j,k\le N}.$$

In the following we often write $|\cdot|$ for the Hilbert–Schmidt norm (i.e. the l^2-norm of the matrix). Write

$$Z = (F_{N+1-j}(\zeta_+/\zeta_-)F_k(\zeta_+/\zeta_-)\zeta_-^{N-j+k-1})_{1\le j,k\le N}$$

and assume

$$z \notin \mathrm{neigh}([-2\sqrt{ab}, 2\sqrt{ab}], \mathbf{C})$$

so that

$$|\zeta_+/\zeta_-| \le 1 - \frac{1}{\mathcal{O}(1)}.$$

(In fact, using that $\zeta_- = b/(a\zeta_+)$, the assumption $|\zeta_-| = |\zeta_+|$ leads to $\zeta_+ = \sqrt{b/a}e^{i\theta}$, for some $\theta \in \mathbf{R}$, so $z = a\zeta_+ + b/\zeta_+ = 2\sqrt{ab}\cos\theta$ belongs to the focal segment.) Then $|F_{N+1-j}(\zeta_+/\zeta_-)|, |F_k(\zeta_+/\zeta_-)| \asymp 1$ and a straightforward calculation shows that

$$\frac{1}{\mathcal{O}(1)}F_N(|\zeta_-|) \le |Z| \le \mathcal{O}(1)F_N(|\zeta_-|). \tag{7.18}$$

Working still under the assumption that $|Q_\omega| \le \mathcal{O}(N)$, we get (cf. (6.10) and (7.8))

$$|E_{-+}^{\delta} - E_{-+}^{0}| \le \mathcal{O}(1)\delta N F_N(|\zeta_-|). \tag{7.19}$$

From (7.17) and the Cauchy inequalities, we get

$$d_Q E_{-+}^{\delta} = \delta|Z|(dQ|e_1) + \mathcal{O}(N^{-1})(\delta N F_N)^2 \tag{7.20}$$

in $\mathbf{C}^{N^2}$, where

$$e_1 = \frac{1}{|Z|}\overline{Z}.$$

Here the remainder is measured in the $\mathbf{C}^{N^2}$-norm. It is obtained from letting d_Q act on the remainder in (7.17) (holomorphic in Q) and as usual we need a slight shrinking of the ball in $\mathbf{C}^{N^2}$. More precisely, we may assume that (7.17) holds for $|Q| < \widetilde{C}_1 N$, where $\widetilde{C}_1$ is slightly larger than C_1 and we get (7.20) for $|Q| < C_1 N$. Complete e_1 into an orthonormal basis $e_1, e_2, \dots, e_{N^2}$ in $\mathbf{C}^{N^2}$ and write

$$Q = Q' + Q_1 e_1, \quad Q' = \sum_2^{N^2} Q_k e_k \in (e_1)^\perp.$$

Then (7.17) and (7.20) read

$$E_{-+}^\delta = E_{-+}(0) + \delta|Z|Q_1 + \mathcal{O}(1)(\delta N F_N)^2, \tag{7.21}$$

$$d_Q E_{-+}^\delta = \delta|Z|dQ_1 + \mathcal{O}(N^{-1})(\delta N F_N)^2. \tag{7.22}$$

As in [17, Chapter 13] we can extend $Q \mapsto E_{-+}^\delta(Q)$ to a smooth function $F\colon \mathbf{C}^{N^2} \to \mathbf{C}$ such that

$$F(Q) = E_{-+}(0) + \delta|Z|Q_1 + \mathcal{O}(1)(\delta N F_N)^2 =: F(0) + \delta|Z|f(Q) \tag{7.23}$$

$$d_Q F(Q) = \delta|Z|dQ_1 + \mathcal{O}(N^{-1})(\delta N F_N)^2$$

and such that the remainders vanish outside $B_{\mathbf{C}^{N^2}}(0, 2C_1 N)$, where $B_{\mathbf{C}^{N^2}}(0, C_1 N)$ is the ball of validity for (7.21) and (7.22). The function f satisfies

$$f(Q) = Q_1 + \mathcal{O}(\delta N^2 F_N)$$

$$d_Q f = dQ_1 + \mathcal{O}(\delta N F_N).$$

From the assumption (7.11) it follows that the map $\mathbf{C} \ni Q_1 \mapsto f(Q_1, Q') \in \mathbf{C}$ is bijective for every Q' and has a smooth inverse $g = g(\zeta, Q')$, satisfying

$$g(\zeta, Q') = \zeta + \mathcal{O}(\delta N^2 F_N),$$

$$d_{\zeta,Q'} g(\zeta, Q') = d\zeta + \mathcal{O}(\delta N F_N). \tag{7.24}$$

Let $\mu(d\zeta)$ be the direct image under f of the measure $\pi^{-N^2} \mathrm{e}^{-|Q|^2} L(dQ)$. We study μ in $D(0, C)$ for any fixed $C > 0$. For $\varphi \in \mathcal{C}_0(D(0, C))$, we get

$$\begin{aligned}
\int \varphi(\zeta)\mu(d\zeta) &= \int \varphi(f(Q))\pi^{-N^2}\mathrm{e}^{-|Q|^2} L(dQ) \\
&= \int_{\mathbf{C}^{N^2-1}} \pi^{1-N^2}\mathrm{e}^{-|Q'|^2} \left(\int_{\mathbf{C}} \pi^{-1}\mathrm{e}^{-|Q_1|^2}\varphi(f(Q))L(dQ_1)\right) L(dQ') \\
&= \int_{\mathbf{C}^{N^2-1}} \pi^{1-N^2}\mathrm{e}^{-|Q'|^2} \left[\int_{\mathbf{C}} \pi^{-1}\mathrm{e}^{-|g(\zeta,Q')|^2}\varphi(\zeta)L(d_\zeta g)\right] L(dQ'),
\end{aligned}$$

where

$$L(d_\zeta g) = L(dQ_1) = \det\Big(\frac{\partial(Q_1,\overline{Q}_1)}{\partial(\zeta,\bar{\zeta})}\Big)L(d\zeta).$$

We get for $\varphi \in \mathcal{C}_0(D(0,C))$,

$$\int \varphi(\zeta)\mu(d\zeta) = \int_{\mathbf{C}} \varphi(\zeta)\Big(\int_{\mathbf{C}^{N^2-1}} \pi^{-1}\mathrm{e}^{-|g(\zeta,Q')|^2}\pi^{1-N^2}\mathrm{e}^{-|Q'|^2} \det\Big(\frac{\partial(Q_1,\overline{Q}_1)}{\partial(\zeta,\bar{\zeta})}\Big)L(dQ')\Big)L(d\zeta),$$

so that in $D(0,C)$

$$\mu(d\zeta) = \Big(\int_{\mathbf{C}^{N^2-1}} \pi^{-1}\mathrm{e}^{-|g(\zeta,Q')|^2}\pi^{1-N^2}\mathrm{e}^{-|Q'|^2} \det\Big(\frac{\partial(Q_1,\overline{Q}_1)}{\partial(\zeta,\bar{\zeta})}\Big)L(dQ')\Big)L(d\zeta).$$

We conclude that for $|\zeta_0|, r \le \mathcal{O}(1)$, the probability that $|Q| \le C_1 N$ and $f(Q) \in D(\zeta_0, r)$ is bounded from above by

$$\int_{\mathbf{C}^{N^2-1}} \int_{\zeta\in D(\zeta_0,r)} \pi^{-1}e^{-|g(\zeta,Q')|^2}L(d_\zeta g)\pi^{1-N^2}e^{-|Q'|^2}L(dQ'). \tag{7.25}$$

From (7.24) we infer that

$$\{g(\zeta,Q');\ \zeta\in D(\zeta_0,r)\} \subset D(g(\zeta_0,Q'),\tilde{r}), \quad \tilde{r} = (1+\mathcal{O}(\delta N F_N))r$$

and the last integral is

$$\le \int_{\mathbf{C}^{N^2-1}} \int_{D(g(\zeta_0,Q'),\tilde{r})} \pi^{-1}e^{-|\omega|^2}L(d\omega)\pi^{1-N^2}e^{-|Q'|^2}L(dQ').$$

Here the inner integral is

$$\le \int_{D(0,\tilde{r})} \frac{1}{\pi}\mathrm{e}^{-|\omega|^2}L(d\omega) = 1-\mathrm{e}^{-\tilde{r}^2}.$$

Indeed, by rotation symmetry, we may assume that $g(\zeta_0, Q') = t \ge 0$ and by Fubini's theorem, we are reduced to show that $F(t) \le F(0)$, where

$$F(t) = \int_{t-\tilde{r}}^{t+\tilde{r}} \mathrm{e}^{-s^2}ds.$$

It then suffices to observe that $F'(t) \le 0$.

Thus the integral in (7.25) is bounded by

$$1-\mathrm{e}^{-\tilde{r}^2} \le (1+\mathcal{O}(F_N\delta N))(1-\mathrm{e}^{-r^2}).$$

In terms of E_{-+}^{δ}, we get under the assumption (7.11):

Lemma 7.3. *We recall* (7.23)*. For* $0 \le t$, $|E^0_{-+}| \le C\delta F_N(|\zeta_-|)$*, the probability that* $|Q| \le C_1 N$ *and* $|E^\delta_{-+}| \le t$ *is*

$$\le (1 + \mathcal{O}(F_N N\delta))\Big(1 - \exp\Big[-\Big(\frac{t}{\delta|Z|}\Big)^2\Big]\Big).$$

In view of (7.2) *it follows that*

$$\begin{aligned}
&\mathbb{P}(|Q| \le C_1 N \textit{ and } |E^\delta_{-+}| > t)\\
&\quad = \mathbb{P}(|Q| \le C_1 N) - \mathbb{P}(|Q| \le C_1 N \textit{ and } |E^\delta_{-+}| \le t)\\
&\quad \ge 1 - e^{-N^2} - (1 + \mathcal{O}(F_N N\delta))\Big(1 - \exp\Big[-\Big(\frac{t}{\delta|Z|}\Big)^2\Big]\Big).
\end{aligned}$$

From the bound

$$|Q| \le C_1 N,$$

that we adopt from now on, and the Cauchy-Schwartz inequality for the singular values of Q, we know that

$$\|Q\|_{\mathrm{tr}} \le C_1 N^{3/2}.$$

8. Counting eigenvalues

8.1. Estimates on $\det(P_\delta - z)$ inside E_1. In this section we assume (7.12), implying (7.11) when $|\zeta_-| \le 1$. We identify the eigenvalues of P_δ with the zeros of the function

$$D_\delta(z) = \det(P_\delta - z).$$

We sum up the various estimates and identities for this function:

When $\delta = 0$, we have (5.13)

$$\det(P_0 - z) = (-a)^N \frac{\zeta_+^{N+1} - \zeta_-^{N+1}}{\zeta_+ - \zeta_-} = (-a)^N \zeta_-^N F_{N+1}(\zeta_+/\zeta_-),$$

and for the Grushin problem (5.3) we have (5.12)

$$\det \mathcal{P}_0(z) = a^{N+1} \tag{8.1}$$

and (5.10)

$$E^0_{-+}(z) = \frac{\zeta_+^{N+1} - \zeta_-^{N+1}}{a(\zeta_+ - \zeta_-)} = \frac{\zeta_-^N}{a} F_{N+1}(\zeta_+/\zeta_-). \tag{8.2}$$

For z in the interior of E_1, by (7.11), $\delta N F_N(|\zeta_-|) \ll 1$ and the assumption that $\|Q\|_{\mathrm{HS}} \le C_1 N$ we have (7.14)

$$\ln|\det \mathcal{P}_\delta| = \ln|\det \mathcal{P}_0| + \mathcal{O}(1)\delta N^{3/2} F_N(|\zeta_-|). \tag{8.3}$$

We also have the general identity (cf. (5.11))

$$\det(P_\delta - z) = (-1)^N E_{-+}^\delta(z) \det \mathcal{P}_\delta(z). \tag{8.4}$$

From (7.19) and (8.2), we infer that

$$|E_{-+}^\delta| \le \frac{|\zeta_-|^N}{a} |F_{N+1}(\zeta_+/\zeta_-)| + \mathcal{O}(1)\delta N F_N(|\zeta_-|). \tag{8.5}$$

We will also assume that $z \notin \mathrm{neigh}([-2\sqrt{ab}, 2\sqrt{ab}], \mathbf{C})$ so that

$$|\zeta_+| \le (1 - 1/\mathcal{O}(1))|\zeta_-|.$$

Then (8.5) implies that

$$|E_{-+}^\delta| \le \mathcal{O}(1),$$

and (8.1),(8.3), and (8.4) give

$$\ln|\det(P_\delta - z)| \le (N+1)\ln|a| + \mathcal{O}(1)(1 + \delta N^{3/2} F_N(|\zeta_-|)). \tag{8.6}$$

Still under the assumption that z is in the interior of E_1, we give a probabilistic lower bound on $\ln|\det(P_\delta - z)|$, starting from

$$\begin{aligned} \ln|\det(P_\delta - z)| &= \ln|\det \mathcal{P}_\delta| + \ln|E_{-+}^\delta| \\ &\ge \ln|\det \mathcal{P}_0| + \ln|E_{-+}^\delta| - \mathcal{O}(1)\delta N^{3/2} F_N(|\zeta_-|) \\ &= (N+1)\ln|a| + \ln|E_{-+}^\delta| - \mathcal{O}(1)\delta N^{3/2} F_N(|\zeta_-|). \end{aligned} \tag{8.7}$$

In order to apply Lemma 7.3, we analyze the condition

$$|E_{-+}^0(z)| \le C\delta F_N(|\zeta_-|),$$

which by (8.2) amounts to

$$|\zeta_-|^N |F_{N+1}(\zeta_+/\zeta_-)| \le C\delta F_N(|\zeta_-|).$$

Since $F_{N+1}(\zeta_+/\zeta_-) = \mathcal{O}(1)$ (by the assumption that z avoids a neighborhood of the focal segment), this would follow from

$$|\zeta_-|^N \le C\delta F_N(|\zeta_-|). \tag{8.8}$$

Recall that

$$F_N(|\zeta_-|) \asymp \min\Big(N, \frac{1}{1-|\zeta_-|}\Big).$$

We know by (7.12) that $\delta F_N(|\zeta_-|) \ll N^{-1}$ so if (8.8) holds, we necessarily have

$$|\zeta_-|^N \ll \frac{1}{N},$$

i.e.

$$\ln|\zeta_-| \le \frac{\ln(N^{-1}) - (\gg 1)}{N},$$

where $(\gg 1)$ indicates a sufficiently large constant and hence

$$1 - |\zeta_-| \ge \frac{\ln N + (\gg 1)}{N}.$$

In this region $F_N(|\zeta_-|) = 1 = (1 - |\zeta_-|)$, and to understand (8.8) amounts to understanding for which $s\ (= |\zeta_-|)$ in $\big]\frac{1}{\mathcal{O}(1)}, 1 - \frac{\ln N + (\gg 1)}{N}\big]$ we have

$$m(s) \le C\delta,$$

where

$$m(s) = s^N(1-s), \quad 0 \le s \le 1.$$

This function increases from $s = 0$ to $s = s_{\max} = N/(N+1) = 1 - 1/(N+1)$, and then decreases. We have

$$m(s_{\max}) = \Big(1 - \frac{1}{N+1}\Big)^N \frac{1}{N+1} = \frac{1 + \mathcal{O}(\frac{1}{N})}{eN}$$

while $\delta \ll \frac{1}{N}$. The solution $s = s_\delta < s_{\max}$ of

$$m(s) = C\delta \tag{8.9}$$

satisfies

$$s^N \frac{1}{N+1} \le s^N(1-s) = C\delta \le s^N,$$

so

$$(C\delta)^{1/N} \le s_\delta \le (C(N+1)\delta)^{1/N}. \tag{8.10}$$

We now apply Lemma 7.3 to (8.7) and get with the help of (7.18).

Proposition 8.1. *Restrict z to a region inside E_1, where $\delta N F_N(|\zeta_-|) \ll 1$, $|\zeta_-| \leq s_\delta$ as in (8.9) and (8.10). Then for each such z and for $t \leq C\delta F_N(|\zeta_-|)$, we have $|Q| \leq C_1 N$ and*

$$\ln|\det(P_\delta - z)| \geq (N+1)\ln|a| + \ln t - \mathcal{O}(1)\delta N^{3/2} F_N(|\zeta_-|) \tag{8.11}$$

with probability

$$\begin{aligned} &\geq 1 - (1 + \mathcal{O}(\delta N F_N))\Big(1 - \exp\Big(-\Big[\frac{t}{\delta|Z|}\Big]^2\Big)\Big) - \mathrm{e}^{-N^2} \\ &\geq 1 - \frac{t^2}{\mathcal{O}(1)\delta^2 F_N(|\zeta_-|)^2} - \mathrm{e}^{-N^2}. \end{aligned}$$

8.2. Estimates for $\det(P_\delta - z)$ in the exterior of E_1. We just recall (7.16): with probability $\geq 1 - \mathrm{e}^{-N^2}$, we have

$$\ln|\det(P_\delta - z)| = N\ln|a| + \ln\frac{|\zeta_+^{N+1} - \zeta_-^{N+1}|}{|\zeta_+ - \zeta_-|} + \mathcal{O}(\delta)N^{3/2}F_N(|\zeta_-|^{-1}) \tag{8.12}$$

for all z satisfying (7.15):

$$\delta N F_N\Big(\frac{1}{|\zeta_-|}\Big) \ll 1, \tag{8.13}$$

which is guaranteed for all z in the exterior region by (7.12). Here we also used the formula (5.13):

$$\det(P - z) = (-a)^N \frac{\zeta_+^{N+1} - \zeta_-^{N+1}}{\zeta_+ - \zeta_-}.$$

If $\delta N^2 \ll 1$, then (8.13) is satisfied in the whole exterior region. For larger values of δ, (8.13) says that

$$\frac{\delta N}{1 - 1/|\zeta_-|} \ll 1,$$

which means that $|\zeta_-| - 1 \gg \delta N$.

8.3. Choice of parameters. We chose

$$\delta \asymp N^{-\kappa}, \quad \kappa > \frac{5}{2}, \tag{8.14}$$

as in Theorem 1.1. Then (7.12) holds and by (8.6) we have with probability

$\geq 1 - \mathrm{e}^{-N^2}$ the upper bound

$$\ln|\det(P_\delta - z)| \leq N(\ln|a| + \mathcal{O}(N^{-1})), \tag{8.15}$$

for all z in the interior of E_1, away from any fixed neighborhood of the focal segment.

Proposition 8.1 is applicable for $|\zeta_-| \leq (C\delta)^{1/N}$ and hence for each ζ_- with

$$|\zeta_-| \leq \left(\frac{1}{\mathcal{O}(1)}\right)^{1/N} N^{-\frac{\kappa}{N}} = \mathrm{e}^{-\frac{\kappa}{N}(\ln N + \mathcal{O}(1))}$$

or equivalently

$$|\zeta_-| \leq 1 - \frac{\kappa}{N}(\ln N + \mathcal{O}(1)), \tag{8.16}$$

we have (8.11)

$$\ln|\det(P_\delta - z)| \geq N\ln|a| - \mathcal{O}(1) + \ln t$$

with probability

$$\geq 1 - \mathcal{O}(1)N^{2\kappa}t^2 - \mathrm{e}^{-N^2}$$

for $t \leq C^{-1}N^{-\kappa}$. Choose $t = \mathrm{e}^{-N^{\delta_0}}$ for a small fixed $\delta_0 > 0$. Then, for each ζ_- satisfying (8.16), we have

$$\ln|\det(P_\delta - z)| \geq N(\ln|a| - 2N^{\delta_0 - 1}), \tag{8.17}$$

with probability

$$\geq 1 - \mathcal{O}(1)N^{2\kappa}\mathrm{e}^{-2N^{\delta_0}}. \tag{8.18}$$

As for the exterior region we write

$$\begin{aligned}\ln\frac{|\zeta_+^{N+1} - \zeta_-^{N+1}|}{|\zeta_+ - \zeta_-|} &= N\ln|\zeta_-| + \ln\frac{|1 - (\zeta_+/\zeta_-)^{N+1}|}{|1 - \zeta_+/\zeta_-|}\\ &= N\ln|\zeta_-| + \mathcal{O}(1).\end{aligned}$$

and from (8.12) and (8.13) we get with probability $\geq 1 - \mathrm{e}^{-N^2}$,

$$\ln|\det(P_\delta - z)| = N(\ln|a| + \ln|\zeta_-| + \mathcal{O}(N^{-1})) \tag{8.19}$$

for all z in any fixed bounded region with $|\zeta_-| \geq 1$. (Recall that (7.12) and hence (8.13) hold here by the choice of δ in (8.14).) Put

$$\varphi(z) = \ln|a| + \max(\ln|\zeta_-|, 0) + \frac{C}{N}, \tag{8.20}$$

for $C > 0$ large enough. Then by (8.15) and (8.19) we have with probability $\geq 1 - \mathrm{e}^{-N^2}$ that

$$\ln|\det(P_\delta - z)| \leq N\varphi(z),$$

for all z in any fixed compact subset of $\mathbf{C}$ which does not intersect the focal segment.

Moreover,

$$\ln|\det(P_\delta - z)| \geq N(\varphi(z) - \mathcal{O}(N^{-1}))$$

in the exterior region. For each z with $|\zeta_-| \leq 1 - \frac{\kappa}{N}(\ln N + \mathcal{O}(1))$, we have (8.17) with probability as in (8.18):

$$\ln|\det(P_\delta - z)| \geq N(\varphi(z) - \varepsilon), \quad \varepsilon = \frac{2N^{\delta_0}}{N}. \tag{8.21}$$

Let γ be a segment of E_1 and $\frac{C\ln N}{N} \leq r \ll 1$, put

$$\Gamma(r,\gamma) = \{z \in \mathbf{C};\ \mathrm{dist}(z, E_1) < r,\ \Pi(z) \in \gamma\}$$

where $\Pi(z) \in E_1$ is the point in E_1 with $|\Pi(z) - z| = \mathrm{dist}(z, E_1)$. We want to estimate the number of eigenvalues of P_δ in Γ. (With probability $\geq 1 - \mathrm{e}^{-N^2}$ we know that P_δ has no eigenvalues in the exterior region to E_1 and we are free to modify Γ there. However, there seems to be no point to do so in the present situation.)

Choose $z_j^0 \in \partial\Gamma$, $r_j = \max\left(\frac{1}{2}\mathrm{dist}(z_j^0,\gamma), 4C(\ln N)/N\right)$, $j = 1,\dots,M$ such that

- $\partial\Gamma \subset \cup_{j=1}^{M} D(z_j^0, r_j/2)$,
- $\mathrm{dist}\,(z_j^0,\gamma) \geq C(\ln N)/N$ for all j,
- $\#\{j;\ r_j = 4C(\ln N)/N\} = \mathcal{O}(1)$,
- $M \leq \mathcal{O}\left(\frac{1}{r}\right) + \mathcal{O}(1)\ln\left(\frac{r}{C(\ln N)/N}\right)$

For any choice of $z_j \in D(z_j^0, r_j/8)$, $\zeta_-(z_j)$ satisfies (8.16) (for N large enough) and (8.21) holds for $z = z_1,\dots,z_M$ with probability

$$\geq 1 - \mathcal{O}(1)MN^{2\kappa}\mathrm{e}^{-2N^{\delta_0}} \geq 1 - \mathcal{O}(1)\left(\frac{1}{r} + \ln N\right)N^{2\kappa}\mathrm{e}^{-2N^{\delta_0}}. \tag{8.22}$$

Applying Theorem 1.2 in [19], we get

$$\begin{aligned}&\left|\#(\sigma(P_\delta)\cap\Gamma) - \frac{N}{2\pi}\int_\Gamma \Delta\varphi L(dz)\right| \\ &\qquad \leq \mathcal{O}(N)\left(\sum_j \varepsilon + \sum_{D(z_j^0,r_j)\cap E_1\neq\emptyset} \int_{D(z_j^0,r_j)} \Delta\varphi L(dz)\right),\end{aligned}$$

where $\varepsilon(z_j) = \frac{2N^{\delta_0}}{N}$ is given in (8.21). Here we also used that φ is harmonic away from E_1. In view of (8.24) and (8.25) below, we have

$$\sum_{D(z_j^0, r_j)\cap E_1 \neq \emptyset} \int_{D(z_j^0, r_j)} \Delta\varphi L(dz) = \mathcal{O}(1)\frac{\ln N}{N},$$

and the number of points z_j^0 for which $D(z_j^0, r_j) \cap E_1 \neq \emptyset$ is $\mathcal{O}(1)$, so finally, with probability as in (8.22),

$$\begin{aligned} &\left|\#(\sigma(P_\delta)\cap\Gamma) - \frac{N}{2\pi}\int_\Gamma \Delta\varphi L(dz)\right| \\ &\quad\le \mathcal{O}(N)\Big(\Big(\frac{1}{r} + \ln N\Big)N^{\delta_0 - 1} + \frac{\ln N}{N}\Big) \\ &\quad\le \mathcal{O}(1)N^{\delta_0}\Big(\frac{1}{r} + \ln N\Big). \end{aligned} \tag{8.23}$$

The measure $\Delta_z\varphi(z)L(dz)$ is invariant under holomorphic changes of coordinates and in particular, we can replace z by ζ_-:

$$\Delta_z\varphi(z)L(dz) = \Delta_{\zeta_-}\varphi L(d\zeta_-). \tag{8.24}$$

Here we recall that φ is given by (8.20) and compute the right hand side of (8.24). Let $\psi \in C_0^\infty(\mathbf{C}\setminus\{0\})$ be a test function. Then in the sense of distributions,

$$\begin{aligned} &\int \psi(\zeta_-)\Delta\varphi(\zeta_-)L(d\zeta_-) \\ &\quad= \int \Delta\psi(\zeta_-)\varphi(\zeta_-)L(d\zeta_-) \\ &\quad= \int_{D(0,1)} \Delta\psi(\zeta_-)\varphi(\zeta_-)L(d\zeta_-) + \int_{\mathbf{C}\setminus D(0,1)} \Delta\psi(\zeta_-)\varphi(\zeta_-)L(d\zeta_-), \end{aligned}$$

and by Green's formula, this is equal to

$$\int_{S^1} \psi(\zeta_-)\left(\partial_n\varphi_{\mathrm{ext}}(\zeta_-) - \partial_n\varphi_{\mathrm{int}}(\zeta_-)\right)|d\zeta_-|,$$

where n denotes the exterior unit normal to the unit disc,

$$\begin{cases} \varphi_{\mathrm{int}} = \ln|a| + C/N, \\ \varphi_{\mathrm{ext}} = \ln|a| + \ln|\zeta_-| + C/N. \end{cases}$$

Since $\partial_n \varphi_{\mathrm{int}}(\zeta_-) = 0$, $\partial_n \varphi_{\mathrm{ext}}(\zeta_-) = 1$ on S^1, we get

$$\int \psi(\zeta_-)\Delta\varphi(\zeta_-)L(d\zeta_-) = \int_{S^1} \psi(\zeta_-)|d\zeta_-|,$$

i.e.

$$\Delta\varphi(\zeta_-)L(d\zeta_-) = L_{S^1}(ds), \text{the length measure on } S^1. \tag{8.25}$$

Recall that $\Gamma \cap E_1 = \gamma$. Letting γ also denote the corresponding arc in $S^1_{\zeta_-}$, depending on the context, we see that

$$\int_\Gamma \Delta\varphi L(dz) = \int_\gamma L_{S^1}(d\zeta_-)$$

i.e. the length of γ with respect to the ζ_--coordinates. To make the connection with Weyl's formula, we write $\zeta_- = e^{i\xi}$, so that $S^1_{\zeta_-}$ corresponds to $\xi \in \mathbf{R}/2\pi\mathbf{Z}$. Then $\gamma \subset S^1$ can be identified with $\{\xi \in \mathbf{R}/2\pi\mathbf{Z};\ e^{i\xi} \in \gamma\}$. Viewing again γ as a segment in E_1, we get

$$\int_\Gamma \Delta\varphi L(dz) = \text{length}\left(\{\xi \in \mathbf{R}/2\pi\mathbf{Z};\ P_{\mathrm{I}}(\xi) \in \gamma\}\right),$$

where the right hand side does not change if we replace γ by Γ and where P_{I} is the symbol given in (1.2). Equivalently,

$$\int_\Gamma \Delta\varphi L(dz) = \mathrm{vol}_{S^1_\xi} P_{\mathrm{I}}^{-1}(\Gamma).$$

If we view P_{I} as a function on $]0,N]_x \times S^1_\xi$, where $]0,N] = \{x \in \mathbf{R};\ 0 < x \le N\}$, then

$$N\int_\Gamma \Delta\varphi L(dz) = \mathrm{vol}_{]0,N]\times S^1} P_{\mathrm{I}}^{-1}(\Gamma)$$

Using this in (8.23), we get Theorem 1.1.

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