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Logarithmic lower bound on the number of nodal domains

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In memory of Yuri Safarov

Abstract. We prove that the number of nodal domains of eigenfunctions grows at least logarithmically with the eigenvalue (for almost the entire sequence of eigenvalues) on certain negatively curved surfaces. The geometric model is the same as in prior joint work with J. Jung, where the number of nodal domains was shown to tend to infinity. The surfaces are assumed to be "real Riemann surfaces," i.e. Riemann surfaces with an anti-holomorphic involution σ with non-empty fixed point set. The eigenfunctions are assumed to be even or odd, which is automatically the case for generic invariant metrics. The logarithmic growth rate gives a quantitative refinement of the prior results.

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Contents

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1. Introduction

In recent articles [17, 18] (see also [16]) J. Jung and the author proved that for certain non-positively curved surfaces, the number of nodal domains of an orthonormal basis $\{u_j\}$ of Laplace eigenfunctions tends to infinity with the eigenvalue along almost the entire sequence of eigenvalues. This nodal counting result built on prior work of Ghosh, Reznikov, and Sarnak [8] in the case of the modular domain, which gave a power law lower bound on the number of nodal domains for individual Maass–Hecke eigenfunctions. Their proof uses methods of L-functions and assumes a certain Lindelöf hypothesis, while those of [17, 18] use PDE methods to obtain unconditional results for a density one subsequence of eigenfunctions. In [16], Jang and Jung used a clever Bochner positivity argument to obtain unconditional results for individual Maass–Hecke eigenfunctions of arithmetic triangle groups. However, no growth rate was specified in [17, 18] or in [16].[1] The main result of this note (Theorem 1.1) improves the qualitative result of [17] to a quantitative logarithmic lower bound for the number of nodal domains of a density one subsequence of even/odd eigenfunctions of the Laplacian on surfaces of negative curvature possessing an isometric involution. The structure of the proof is the same as in [17] but two key estimates are sharpened. The main new input is the logarithmic quantum ergodic result of Hezari and Rivière [13] and X. Han [10]. At the present time, a logarithmic growth rate is the best that can be expected due to the exponential growth rate of the geodesic flow in negative curvature and its impact on all remainder estimates and localization estimates on the spectrum. Before stating the results, we recall some background and terminology from [17].

Although the main result occurs in dimension 2, we start with some general notational conventions in any dimension. Let (M, g) be a compact C^∞ d-dimensional manifold. We denote the Laplacian of g by Δ and state the eigenvalue problem as

$$\Delta u_\lambda = -\lambda\, u_\lambda, \quad \|u_\lambda\|_{L^2(M)} = 1.$$

Eigenfunctions are always assumed to be L^2-normalized,

$$\int_M |u_\lambda|^2\, dV_g = 1,$$

where dV_g is the Riemannian volume form. We often fix an orthonormal basis $\{u_j\}_{j=1}^\infty$ with $\lambda_0 = 0 < \lambda_1 \le \lambda_2 \le \cdots$.

[1] There are two independent difficulties in the lower bounds: (i) Obtaining a quantitative lower bound for a density one subsequence, (ii) Obtaining even a qualitative unconditional lower bound for the entire sequence, i.e. "for individual eigenfunctions."

The nodal set of an eigenfunction of the Laplacian is denoted by

$$Z_{u_\lambda} = \{x: u_\lambda(x) = 0\}.$$

The key object in this note is the number $N(\varphi_\lambda)$ of nodal domains of u_λ, i.e. the number of connected components Ω_j of the complement of the nodal set,

$$M \backslash Z_{u_\lambda} = \bigcup_{j=1}^{N(u_\lambda)} \Omega_j.$$

We now restrict to setting of [17], in which M is assumed to be a Riemann surface of genus $\mathfrak{g}$ (with complex structure J) possessing an anti-holomorphic involution σ whose fixed point set $\mathrm{Fix}(\sigma)$ is non-empty.[2] We define $\mathcal{M}_{M,J,\sigma}$ to be the space of C^∞ σ-invariant *negatively curved* Riemannian metrics on a real Riemann surface (M, J, σ). As discussed in [17], $\mathcal{M}_{M,J,\sigma}$ is an open set in the space of σ-invariant metrics, and in particular is infinite dimensional. For each $g \in \mathcal{M}_{M,J,\sigma}$, the fixed point set $\mathrm{Fix}(\sigma)$ is a disjoint union

$$\mathrm{Fix}(\sigma) = \gamma_1 \cup \cdots \cup \gamma_n \tag{1.1}$$

of $0 \leq n \leq \mathfrak{g} + 1$ simple closed geodesics.

The isometry σ acts on $L^2(M, dV_g)$, and we define $L^2_{\text{even}}(M)$, resp. $L^2_{\text{odd}}(M)$, to denote the subspace of even functions

$$f(\sigma x) = f(x),$$

resp. odd elements

$$f(\sigma x) = -f(x).$$

Translation by any isometry σ commutes with the Laplacian Δ and so the even and odd parts of eigenfunctions are eigenfunctions, and all eigenfunctions are linear combinations of even or odd eigenfunctions. We denote by $\{\varphi_j\}$ an orthonormal basis of $L^2_{\text{even}}(M)$ of even eigenfunctions, resp. $\{\psi_j\}$ an orthonormal basis of $L^2_{\text{odd}}(M)$ of odd eigenfunctions, with respect to the inner product

$$\langle u, v\rangle = \int_M u\bar{v}\, dV_g,$$

ordered by the corresponding sequence of eigenvalues $\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \uparrow \infty$. We write $N(\varphi_j)$ for $N(\varphi_{\lambda_j})$. In [17] we proved that for generic metrics in $\mathcal{M}_{M,J,\sigma}$, the eigenvalues are simple (multiplicity one) and therefore all eigenfunctions are either even or odd.

[2] In [17], the fixed point set is assumed to be separating but this assumption is not necessary.

The main result of this note concerns the case $d = 2$:

Theorem 1.1. *Let (M, J, σ) be a compact real Riemann surface of genus $\mathfrak{g} \geq 2$ with anti-holomorphic involution σ satisfying $\mathrm{Fix}(\sigma) \neq \emptyset$. Let $\mathcal{M}_{M,J,\sigma}$ be the space of σ-invariant negatively curved C^∞ Riemannian metrics on M. Then for any $g \in \mathcal{M}_{(M,J,\sigma)}$ and any orthonormal Δ-eigenbasis $\{\varphi_j\}$ of $L^2_{\mathrm{even}}(M)$, resp. $\{\psi_j\}$ of $L^2_{\mathrm{odd}}(M)$, one can find a density 1 subset A of $\mathbb{N}$ and a constant $C_g > 0$ depending only on g such that, for $j \in A$*

$$N(\varphi_j) \geq C_g \ (\log \lambda_j)^K, \quad \textit{for all } K < \frac{1}{6},$$

resp.

$$N(\psi_j) \geq C_g (\log \lambda_j)^K, \quad \textit{for all } K < \frac{1}{6}.$$

Remark. The constraint on K is the one in the logarithmic scale QE (quantum ergodic) results of [13] and [10]. In [13], the authors used higher variance moments to improve the constraint to $K < \frac{1}{2d}$. It should be possible to improve Theorem 1.1 in the same way, but at the expense of additional technicalities that seem out of proportion to the improvement. The main point is that the arguments of [17] lead to quantitative estimates, and the point seems well enough established with the smaller value of K. It is not yet clear what is the threshold for small scale quantum ergodicity. An improvement from the logarithmic scale to a power scale would imply a similar improvement for the nodal count.

Remark. In [18] the authors proved a much more general qualitative result stating that the number of nodal domains tends tends to infinity for surfaces of non-positive curvature and concave boundary. As explained in a later remark, there are several obstructions to generalizing the logarithmic lower bound to such surfaces, although it should eventually be possible to over-come them.

1.1. Notations for eigenvalues and logarithmic parameters. To maintain notational consistency with [6, 13, 10] we also denote sequences of eigenfunctions by the semi-classical notation u_{h_j} or just u_h with $h = h_j = \lambda_j^{-1/2}$; equivalently, we fix E and put $\lambda_j = h_j^{-2} E$ (as in [10, 13]). Because of the homogeneity of the eigenvalue problem, there is no loss of generality in setting $E = 1$, and then we consider eigenvalues $E_j = E_j(h) = h^2 \lambda_j \in [1, 1 + h]$, or in homogeneous notation $\sqrt{\lambda_j} \in [h^{-1}, h^{-1} + 1] = [\sqrt{\lambda}, \sqrt{\lambda} + 1]$. We denote by $N(\lambda) \simeq C_d \,\mathrm{Vol}(M, g) \lambda^{(d-1)/2}$ the number of eigenvalues in the interval[3] $[\sqrt{\lambda}, \sqrt{\lambda} + 1]$.

[3] Eigenvalues are denoted by λ^2 in [6] and by λ here and in [17].

We further introduce a logarithmically small parameter for a manifold M^d of dimension d

$$\ell = |\log h|^{-K} = (\log \lambda)^{-K}, \quad \text{where } 0 < K < \frac{1}{3d}. \tag{1.2}$$

Han [10] uses the notation $\alpha = K$ and denotes the same quantity by

$$\delta(h) = |\log h|^{-\alpha} = r(\lambda_j), \quad \alpha < \frac{1}{3d}, \tag{1.3}$$

We adopt both notations. Other choices of $\delta(h)$ may arise in applications and will be specified below.

1.2. Main new steps of the proof. The main step in the proof that $N(u_j) \to \infty$ in [17, 18] was to prove that for any smooth connected arc $\beta \subset \mathrm{Fix}(\sigma)$, $u_j|_\beta$ has a sign-changing zero in β. To obtain logarithmic lower bounds, we need to prove the existence of a sign-changing zero on sequences β_j of shrinking arcs with lengths $|\beta_j| = \ell_j$, cf. (1.2). More precisely, we partition $\mathrm{Fix}(\sigma)$ into ℓ_j^{-1} open intervals of lengths ℓ_j and show that u_j has a sign changing zero in each interval.

The quantitative improvements apply to general smooth hypersurfaces $H \subset M$ of general Riemannian manifolds of any dimension d. Later we specialize the results to the surfaces in Theorem 1.1 in dimension $d = 2$ and with $H = \mathrm{Fix}(\sigma)$. In general dimensions, the partitions are defined by choosing a cover of H by $C\ell^{-1}$ balls of radius ℓ with centers $\{x_k\} \subset H$ at a net of points of H so that

$$H \subset \bigcup_{k=1}^{R(\ell)} B(x_k, C\ell) \cap H. \tag{1.4}$$

Here and hereafter, C or C_g denotes a positive constant depending only on (M, g, H) and not on λ_j. The cover may be constructed so that each point of H is contained in at most C_g of the double balls $B(x_k, 2\ell)$. The number of such balls satisfies the bounds,

$$c_1 \ell^{-d+1} \le R(\ell) \le C_2 \ell^{-d+1}.$$

There were three analytic ingredients in the proof of existence of a sign changing zero in every arc β in [17]. Two of them need to be improved to give logarithmic lower bounds.

(i) One needs to prove a QER (quantum ergodic restriction) theorem in the sense of [6] on the length scale $O(\ell_j)$, which says (roughly speaking) that there exists a subsequence of eigenfunctions u_{j_n} of density one so that matrix elements of the restricted eigenfunctions tend to their Liouville limits simultaneously for all the covering balls of (1.4). Since there are $(\log \lambda)^K$ such balls, the scale of the QER theorem is constrained by (1.2). To be more precise, we only need a weaker result giving lower bounds rather than asymptotics, as stated in Proposition 1.

(ii) One needs to prove a small scale Kuznecov asymptotic formula in the sense of [23, 11], to the effect that there exists a subsequence of density one for which $\int_\beta u_{j_k}$ is of order $|\beta|\lambda_j^{-1/4}(\log \lambda_j)^{1/3}$ when[4] $|\beta| \simeq \ell_j$. Again, one needs to show that there is a subsequence of density one for which this estimate holds simultaneously for all the balls of the cover.

(iii) The sup-norm estimate

$$\|u_j\|_\infty = O\Big(\frac{\lambda_j^{1/4}}{\sqrt{\log \lambda_j}}\Big)$$

of Bérard [3] used in [17] does not need to be modified.

For background on QE (quantum ergodicity) theorems we refer to [25] (and to the origins, [20]). We assume here the reader's familiarity with the basic notions and with the QER (quantum ergodic restriction) problem (see [6]).

Remark. In order to obtain the logarithmic lower bound on nodal domains, one needs to prove the QER theorem and the Kuznecov bound on shrinking balls with the same radius ℓ_j. In fact, one can shrink intervals much more in the Kuznecov bound since the estimates involve the principal term in an asymptotic expansion rather than the remainder term. This remark will be explained at the end of the proof of Proposition 3.

A logarithmic scale QE theorem asserts, roughly speaking, that matrix elements $\langle \mathrm{Op}_h(a_\ell)u_{j_k}, u_{j_k}\rangle$ of eigenfunctions with respect to logarithmically dilated symbols $a_\ell(x,\xi) = a\big(\frac{x-x_0}{\ell}, \frac{\xi-\xi_0}{\ell}\big)$ are asymptotic to their Liouville averages $⨍_{S^*M} a_\ell d\mu_L$ (see [10] for the precise formulation of the symbols). For nodal domain counting, as for nodal bounds in [13] it is crucial to have some kind of uniformity of the limits as the base point (x_0, ξ_0) varies. The simplest version would

[4] The power $\frac{1}{3}$ in $(\log \lambda_j)^{1/3}$ is chosen for later convenience. It could be any power $< \frac{1}{2}$.

be that the QE limits are uniform in the symbol and the base point, but such a uniform result is lacking at this time. Instead there exist uniform upper and lower bounds on the mass of the eigenfunctions in all of the logarithmically shrinking balls of the cover (1.4) (see Theorem 8 and Theorem 10). We will adapt these bounds to the QER problem.

The small scale QER statement is the following 'uniform comparability result,' based on Proposition 3.1 of [13] (see Proposition 8) and Corollary 1.9 of [10].

Proposition 1. *Let (M, g) be a compact negatively curved manifold of dimension d without boundary, and let $H \subset M$ be a smooth hypersurface and let dS_g denote surface measure on H. Let $\ell = \ell_j$ be defined by (1.2). Then, for any orthonormal basis of eigenfunctions $\{u_j\}$, and K as in (1.2), there exists a full density subsequence Λ_K of $\mathbb{N}$ so that for $j \in \Lambda_K$ and for every $1 \le k \le R(\ell)$, and centers x_k of (1.4),*

$$\int_{B(x_k, C\ell_j)\cap H} (|u_j|^2 + |\lambda_j^{-1/2}\partial_\nu u_j|^2)\, dS_g \ge a_1 \ell_j^{d-1} = a_1(\log\lambda_j)^{-(d-1)K},$$

where $a_1, a_2 > 0$ depend only on g and K is defined in (1.2).

Here, and hereafter, ∂_ν denotes a fixed choice of normal derivative along H. We apply the result when $\dim M = 2$ and $H = \mathrm{Fix}(\sigma)$, and the eigenfunctions are even or odd. We write $dS_g = ds$, the arc-length measure. In that case, one of the two terms above drops and we get,

Corollary 2. *Let (M, J, σ, g) be a negatively curved surface with isometric involution. Then for any orthonormal basis of even eigenfunctions $\{\varphi_j\}$, resp. odd eigenfunctions $\{\psi_j\}$, there exists a full density subsequence Λ_K so that for $j_n \in \Lambda_K$,*

$$\int_{B(x_k, C\ell_j)\cap H} |\varphi_{j_n}|^2\, ds \ge a_1(\log\lambda_{j_n})^{-K}$$

and

$$\int_{B(x_k, C\ell_j)\cap H} |\lambda_j^{-1/2}\partial_\nu\psi_{j_n}|^2\, ds \ge a_1(\log\lambda_{j_n})^{-K}$$

uniformly in k.

The statement is termed a uniform comparability result (by Han and Hezari and Rivière) because it does not assert uniform convergence of the sequences as k varies but only gives uniform upper and lower bounds. The loss of asymptotics is just due to the passage from smooth test functions to characteristic functions of balls. We only state the lower bound because it is the one which is relevant for nodal counts.

The next step is to prove a uniform logarithmic scale Kuznecov period bound in the sense of [23, 11]. It is also a general result, but for the sake of simplicity, and because it is the case relevant to this note, we assume $\dim M = 2$. As above, we denote dS_g on the curve H by ds.

Proposition 3. *In the notation of Proposition 1, let $H = \mathrm{Fix}(\sigma)$, let K be as in (1.2) and $\{x_k\}$ the centers of (1.4). Then for a subsequence of $\Lambda_K \subset \mathbb{N}$ of density one, if $j \in \Lambda_K$,*

$$\begin{aligned}\left| \int_{B(x_k, C\ell_j)\cap H} \varphi_j \, ds \right| &\leq C_0 \ell_j \lambda_j^{-1/4} (\log \lambda_j)^{1/3} \\ &= C_0 (\log \lambda_j)^{-K} \lambda_j^{-1/4} (\log \lambda_j)^{1/3},\end{aligned}$$

resp.

$$\begin{aligned}\left| \int_{B(x_k, C\ell_j)\cap H} \lambda_j^{-1/2} \partial_\nu \psi_j \, ds \right| &\leq C_0 \ell_j \lambda_j^{-1/4} (\log \lambda_j)^{1/3} \\ &= C_0 (\log \lambda_j)^{-K} \lambda_j^{-1/4} (\log \lambda_j)^{1/3},\end{aligned}$$

uniformly in k.

Remark. As mentioned in a remark above, one could improve the result by letting $\ell = \lambda^{-\varepsilon}$ for suitable ε. But the improvement is not useful for nodal counts until (or if) one can improve the small-scale quantum ergodicity result to scales of the form $\lambda^{-\varepsilon}$; such a bound would open the possibility of improving the nodal count to a power of λ.

1.3. Completion of proof. Granted the Propositions, the proof of existence of a sign-changing zero is the same as in [17, 18]. We give a sketch here for even eigenfuntions to orient the reader, with fuller details in §4.2. Combining the sup-norm estimate (iii) and the QER lower bound (i) of Proposition 1, there exists a subsequence $\mathcal{S} \in \mathbb{Z}_+$ of density one and $C > 0$ so that for $j \in \mathcal{S}$,

$$C \ell_j < \int_{B(x_k, C\ell_j)\cap H} |\varphi_j|^2 \, ds \leq \frac{\lambda_j^{1/4}}{\sqrt{\log \lambda}} \int_{B(x_k, C\ell_j)\cap H} |\varphi_j| \, ds.$$

follows that

$$\int_{B(x_k, C\ell_j)\cap H} |\varphi_j| \, ds \geq \lambda_j^{-1/4} (\log \lambda_j)^{1/2} \ell_j.$$

But by the Kuznecov upper bound (iii) of Proposition 3,

$$\left| \int_{B(x_k, C\ell_j)\cap H} \varphi_j \right| ds \leq \lambda_j^{-1/4} \ell_j (\log \lambda_j)^{1/3}.$$

Hence,

$$\left| \int_{B(x_k, C\ell_j)\cap H} \varphi_j \, ds \right| < \int_{B(x_k, C\ell_j)\cap H} |\varphi_j| \, ds,$$

and thus $\varphi_j|_{B(x_k, C\ell_j)\cap H}$ has a sign changing zero. A similar argument works for the Cauchy data of odd eigenfunctions.

The remainder of the argument is identical to that of [17, 18]. For the rest of this note we only discuss Propositions 1 and 3.

Remark. We use the same scale ℓ_j in both the quantum ergodic restriction result and the Kuznecov formula. As a result, it cancels out from the inequalities. We could use a smaller scale in the Kuznecov bound. But the size of $|\beta|$ is constrained by the more delicate QER result, so there is no gain if we shrink the interval in the Kuznecov formula.

Remark. As mentioned above, there are two aspects of the proof that are difficult to generalize to surfaces of negative curvature and concave boundary. First, the logarithmic QE result of [13, 10] is at present only proved in the boundaryless case. Second, the sup norm estimate (iii) has not been proved at this time for negatively curved surfaces with concave boundary. We conjecture that both obstacles can be overcome and that the logarithmic growth rate of nodal domains holds in the setting of [18].

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2. Energy localization and estimates of Cauchy data of eigenfunctions

In this section, we review some known results on Cauchy data of eigenfunctions that are will be used in §3.4. We assume throughout that $\partial M = \emptyset$ and let $H \subset M$ be a smooth orientable hypersurface. The relevant case is where $H \subset \mathrm{Fix}(\sigma)$ and $\dim H = 2$ so that H is a closed geodesic in the union (1.1).

As in the introduction, we denote by u_λ any L^2-normalized eigenfunction of Δ on M and by $\{u_j\}$ an orthonormal basis of eigenfunctions. Define the Cauchy data of the eigenfunctions u_j on H as

$$\begin{cases} \text{Dirichlet data:} & u_j^b = u_j|_H, \\ \text{Neumann data:} & u_j^b = \lambda_j^{-1/2} \partial_\nu u_j|_H \\ \text{Tangential Neumann data:} & u_j^{bT} := (1 - \lambda_j^{-1}\Delta_H)^{1/2} u_j|_H. \end{cases}$$

We are only concerned with Cauchy data of even eigenfunctions φ_j, resp. odd eigenfunctions ψ_j, on a component H of $\mathrm{Fix}(\sigma)$ of an involution σ. Dirichlet and tangential Neumann data thus refers to even eigenfunctions, Neumann data to odd eigenfunctions. Here, Δ_H is the Laplacian along H and ∂_ν is a choice of unit normal. Later we also denote it by D_{x^d} in Fermi normal coordinates where $x^d = 0$ on H.

2.1. Semi-classical cutoff to the energy sphere.

With the semi-classical notation $h = h_j = \lambda_j^{-1/2}$, the eigenvalue equation takes the form $(-h^2\Delta - 1)u_h = 0$. A sequence of increasingly precise results say that u_h is micro-supported on the unit co-sphere bundle $S_g^*M := \{(x,\xi): |\xi|_g = 1\}$ in T^*M. This is the characteristic variety of the semiclassical Laplacian $I + h^2\Delta$. The rigorous statements are in [25] and in Section 3.1 of [5]; we briefly summarize what we need from [5]. Let

$$A(\varepsilon) := \left\{(x,\xi): \left(1 - \frac{\varepsilon}{10}\right) < |\xi|_g < 1 + \frac{\varepsilon}{10}\right\}$$

be an "annulus" in T^*M around S_g^*M. If $\tilde{\chi} \in C_0^\infty(T^*M)$ is a cutoff equal to 1 on $A(2\varepsilon_0)$ and supported in $A(4\varepsilon_0)$, and if $\tilde{\chi}(h)$ is the quantization of $\tilde{\chi}$ as a semi-classical pseudo-differential operator, then

$$\|(I - \tilde{\chi}(h))u_h\|_{L^2(M)} = O(h^\infty). \tag{2.1}$$

Consequently, we can replace u_h by $\tilde{\chi}(h)u_h$ in matrix elements relative to u_h. More refined cutoffs are designed in [5] and give sharper localizations to S_g^*M. We refer to [5] (3.2) and [25] for further background.

2.2. Fermi normal coordinates near H.

In preparation for the Rellich identity, we introduce convenient coordinates near H and re-state the energy localization of eigenfunctions (2.1) for the Cauchy data of eigenfunctions on H. We follow [4, 6] in the following discussion.

H locally separates M into two components M_+, M_-, and separates M globally when M is a separating hypersurface, as with the separating fixed point sets in [17]. We introduce Fermi normal coordinates (x', x^d) around H, in which H is defined by $x^d = 0$. Thus, $x = \exp_x x^d \nu_{x'}$ where x' are coordinates on H and $\nu_{x'}$ is the unit normal pointing to M_+. In the tubular neighborhood

$$H(\varepsilon) := \{(x', x^d) \in U \times \mathbb{R}, |x^d| < \varepsilon\}, \tag{2.2}$$

the Riemannian integration measures are given by

$$dV_g = \sqrt{c(x)}\, dx'\, dx^d, \quad dS_g = \sqrt{c(x)} dx',$$

and the Laplacian in normal coordinates has the form,

$$\Delta = \frac{1}{\sqrt{c(x)}} \partial_{x^d} \sqrt{c(x)} \partial_{x^d} + R(x', x^d, D_{x'}), \tag{2.3}$$

where $R(0, x', hD_{x'}) = -h^2 \Delta_H$ is the induced tangential semiclassical Laplacian on H; i.e. Δ_H is the Laplacian on H for the metric induced by g. When $\dim M = 2$ and H is totally geodesic as in (1.1), then $x' = s$ (the arc-length coordinate) and $dS_g = ds$.

2.3. Energy localization of Cauchy data. Define the elliptic ($\mathcal{E}$), hyperbolic ($\mathcal{H}$), resp. glancing ($\mathcal{G}$) sets in T^*H by

$$\mathcal{E} = \{(x', \xi') \colon R(x', \xi', 0) > 1\} = \{(x', \xi') \colon \|\xi'\|_g > 1\},$$

$$\mathcal{G} = \{(x', \xi') \colon R(x', \xi', 0) = 1\} = \{(x', \xi') \colon \|\xi'\|_g = 1\},$$

$$\mathcal{H} = \{(x', \xi') \colon R(x', \xi', 0) < 1\} = \{(x', \xi') \colon \|\xi'\|_g < 1\}.$$

Here, $\mathcal{H} \cup \mathcal{G}$ is the projection of $S^*_g M$ and $\mathcal{E}$ is the exterior of the projection.

Cauchy data of eigenfunctions are microsupported in the image of the projection to

$$S^*M \longrightarrow T^*H,$$

i.e. to the energy ball

$$B^*H = \{(x', \xi') \colon |\xi'|_g \leq 1\}.$$

If A is a semi-classical pseudo-differential operator on H supported in the elliptic region $\mathcal{E}$ then $\|Au_j^b\|_{H^s(\partial M)} \to 0$. This is parallel to (2.1) on M.

Although it is not used in our results, we mention that there is a more refined localization result recently proved in [5]. For for any $\varepsilon > 0$ and for $\delta \leq \frac{2}{3} - \varepsilon$, the microlocal L^2 mass of u_h outside $B^*_{(1+h^\delta)}H$ is $O(h^\infty)$. To prove this, the authors introduce a partition of unity of the form

$$\chi_{\text{in}} + \chi_{\tan} + \chi_{\text{out}},$$

with χ_{in} supported in $\mathcal{H}$ and equal to 1 up to a small shell around $\mathcal{G}$, where $\chi_{\tan}$ is a cutoff to a small shell around $\mathcal{G}$ and where χ_{out} is supported in $\mathcal{E}$. Each cutoff function $\chi(x', \xi')$ is rescaled to $\chi_{\delta,h}(x', \xi') := (\chi(h^{-\delta}(R(x', \xi') - 1))$. We refer to [5] (p. 1641) for further details of the cutoff functions (see also figure 1). In (3.3) of Section 3.1, in Proposition 3.1, and in Corollary 3.2 of [5] it is proved that

$$\|(\chi_{\text{out}})^w_{h,\delta} u^b_h\|_{C^k(H)} = O(h^\infty).$$

For our purposes it is only necessary to cutoff the infinite exterior region $\mathcal{E}$.

2.4. Rellich identity and energy localization. In this section we review the Rellich identity for various choices of pseudo-differential operators on M and on H. We refer to [12, 4, 6, 5] for further background.

Let $A(x, hD_x) \in \Psi^0_{sc}(M)$ be an order zero semiclassical pseudodifferential operator on M (see [6]). By Green's formula we get the Rellich identity,

$$\begin{aligned} &\frac{i}{h}\int_{M_+} ([-h^2\Delta, A(x, hD_x)]u_h(x))\overline{u_h(x)}\, dV_g \\ &\quad = \int_H (hD_\nu A(x', x^d, hD_x)u_h|_H)\overline{u_h}|_H\, dS_g \\ &\qquad + \int_H (A(x', x^d, hD_x)u_h|_H)\overline{hD_{x^d}u_h}|_H\, dS_g. \end{aligned} \tag{2.4}$$

Here,

$$D_{x_j} = \frac{1}{i}\frac{\partial}{\partial x^j}, \quad D_{x'} = (D_{x^1}, \ldots, D_{x^{d-1}}), \quad D_{x^d}|_H = \frac{1}{i}\partial_\nu,$$

where ∂_ν is the interior unit normal to M_+. In the integral

$$\frac{i}{h}\int_{M_+} ([-h^2\Delta, A(x, hD_x)]u_h(x))\overline{u_h(x)}\, dV_g \tag{2.5}$$

on the left side of (2.4), the commutator is h times a second degree polynomial in hD_{x_j}; thus, the extra factor cancels the factor of $\frac{1}{h}$ outside the integral. Using (2.3) and (3.8), the principal symbol of the commutator is given by

$$\text{p. s.}([-h^2\Delta, A(x, hD_x)] = \{(\xi^d)^2 + R(x^d, x', \xi'), a(x, x^d, \xi', \xi^d)\}.$$

There are two complications in applying the Rellich formula to semi-classical limits of matrix elements. One is that the integral over H on the right side involves terms with two normal derivatives of u_h, which are not tangential pseudo-differential operators and do not give Neumann data of u_h. But the second order term in the normal derivative hD_{x^d} can be eliminated using the eigenvalue equation. A second complication is caused by the fact that the symbol of Δ on the left side is a second order polynomial in ξ, giving rise to terms of positive order in ξ in the commutator. Results on interior matrix elements on M of zeroth order semi-classical pseudo-differential operators do not *a priori* apply to such terms. But we can exploit energy localization on $S^*_g M$ on the left side and in B^*H on the right side to show that both symbols can be cutoff to compactly supported symbols with only an h^∞ remainder on both sides.

Using (2.1), we can replace the semi-classical pseudo-differential operator $A = \mathrm{Op}_h(a)$ on M in the left side by

$$A^{\tilde{\chi}} := \tilde{\chi}^*(h)\,\mathrm{Op}_h(a)\tilde{\chi}(h), \tag{2.6}$$

modulo errors of order $O(h^\infty)$. The same cutoff changes A on the right side to $A^{\tilde{\chi}}$, whose complete symbol is supported in the ball bundle $B^*_3 H$ of radius 3 in H.

We record the known bounds on the Cauchy data on H which exploit the energy localization.

2.5. Bounds on Neumann data of odd (Dirichlet) eigenfunction. The first result is that if (M, g) is a smooth Riemannian manifold and $H \subset M$ is a smooth embedded orientable separating hypersurface, then there exists $M > 0$ so that for any L^2-normalized eigenfunction φ_λ,

$$\|\lambda^{-1/2}\partial_\nu u_\lambda\|_{L^2(H)} \leq C_g.$$

We only need the estimate for odd eigenfunctions. When H is a separating hypersurfaces it follows from the well-known estimate for Dirichlet eigenfunctions on manifolds with boundary using the Rellich identity. See e.g. [1, 12] and its references to the earlier literature. In the boundaryless case, a new proof is given in Theorem 1.1 of [5] for general eigenfunctions.

2.6. Bounds on Dirichlet data of even eigenfunctions. In general,

$$\|u_\lambda|_H\|_{L^2(H)} \leq C_g \lambda^{1/2}$$

and the estimate is achieved for certain eigenfunctions on a surface of revolution.

When u_λ is a Neumann eigenfunction on a Euclidean domain Ω with boundary, then $\|u_\lambda\|_{L^2(\partial\Omega)} \leq \lambda^{1/6}$. As discussed in [6, 2] and elsewhere, the most relevant analogue for even/Neumann eigenfunctions of Neumann data of odd/Dirichlet eigenfunctions is the "tangential Neumann" data

$$u_j^{bT} := (1 - \lambda_j^{-1}\Delta_{\partial\Omega})_+^{1/2} u_j.$$

Here, the $+$ represents the Riesz mean, i.e. one takes the square root on the part of subspace of $L^2(H)$ where $(I + h^2\Delta_H) \geq 0$ and defines the operator to be zero on the orthogonal complement.

In Remark 4.1 of [5] it is pointed out that

$$\langle (I + h^2\Delta_H)u_h^b, u_h^b\rangle_{L^2(H)} = O(1),$$

and in particular this applies to even eigenfunctions in our setting.

For QER eigenfunctions, the Neumann data is bounded for the quantum ergodic sequence.

3. Proof of Proposition 1

The purpose of this section is to prove the uniform lower bound of Proposition 1. We use the Rellich identity argument of [6] to extract a logarithmic scale QER (quantum ergodic restriction) result from the global ones of [13] and [10]. However, as mentioned above, a key new issue is to obtain uniformity in the centers $\{x_k\}$ of the covering balls (1.4). In principle, one would like to prove existence of a subsequence of eigenfunctions of density one for which one has Liouville weak* limits with uniform remainders in all the balls $B(x_k, C\ell) \cap H$, but this has not yet been established globally in [13, 10]. For the proof of Theorem 1.1, it is only necessary to obtain the uniform lower bounds in Proposition 1.

The proof of Proposition 1 is based on estimates of variances of restricted matrix elements with respect to logarithmic scale pseudo-differential operators on H, and then on extraction of density one subsequences by feeding variance bounds into Chebyshev inequalities. The variance bounds are obtained by applying Rellich identities as in [6] between variance sums on M and on H.We then use the small scale global variance bounds of [13, 10]. We begin by defining the variance sums on M and on H, and then review the results of [13, 10] before going on to the proof of Proposition 1.

3.1. Variance sums on M and on H. Given a semi-classical symbol a on T^*M and its semi-classical Weyl quantization $\mathrm{Op}_h(a) = a^w$ on $L^2(M)$, we define the variance sum on M to be

$$\begin{aligned} V_2(h, \mathrm{Op}_h(a)) &:= V_2(h,a) \\ &:= h^{d-1} \sum_{E_j \in [1,1+h]} |\langle \mathrm{Op}_h(a) u_j, u_j\rangle - \omega(a_0)|^2 \\ &:= \frac{1}{N(\lambda)} \sum_{j: \sqrt{\lambda_j} \in [\sqrt{\lambda}, \sqrt{\lambda}+1]} |\langle Op(a) u_j, u_j\rangle - \omega(a_0)|^2 \end{aligned} \tag{3.1}$$

where $\omega(a_0) = \fint_{S^*M} a_0 d\mu_L$ is average of a_0 relative to normalized Liouville measure and a_0 is the principal symbol of a. For background on semi-classical symbols and pseudo-differential operators we refer to [6, 25].

The restricted variance sums on H have a somewhat different from (3.1). First, in place of the Liouville integral of a one has the restricted state,

$$\omega_H(a) := \int_{B^*H} a(y', \xi')(1 - |\xi'|^2)^{1/2}\, dy'\, d\xi'.$$

The notation ω_H is adopted from [15, 6] and we refer there for further discussion. Secondly, the restriction QER analogue of the matrix element $\langle \mathrm{Op}_h(a) u_j, u_j\rangle$ is the matrix element of the Cauchy data

$$CD(u_h|_H) = (u_h|_H, h\partial_\nu u_h|_H)$$

of u on H. with respect to a semi-classical pseudo-differential operator $\mathrm{Op}_h(a)$ on $L^2(H)$:

$$\begin{aligned} &\langle \mathrm{Op}_h(a) CD(u_h|_H), CD(u_h|_H)\rangle_{L^2(H)} \\ &\quad := \langle \mathrm{Op}_h(a) h\partial_\nu u_h|_H, h\partial_\nu u_h|_H\rangle_{L^2(H)} \\ &\qquad + \langle \mathrm{Op}_h(a)(1 + h^2\Delta_H) u_h|_H, u_h|_H\rangle_{L^2(H)}. \end{aligned}$$

We therefore define restricted variance sums by

$$\begin{aligned} V_{2,H}(h,a) &:= h^{d-1} \sum_{E_j \in [1,1+h]} |\langle \mathrm{Op}_h(a) CD(u_h|_H), CD(u_h|_H)\rangle_{L^2(H)} - \omega_H(a)|^2 \\ &:= \frac{1}{N(\lambda)} \sum_{j: \sqrt{\lambda_j} \in [\sqrt{\lambda}, \sqrt{\lambda}+1]} |\langle Op(a) CD(u_j), CD(u_j)\rangle - \omega_H(a)|^2 \end{aligned} \tag{3.2}$$

Here we use the two different notations that are employed in [17, 6].

As above, we introduce Fermi normal coordinates (x', x^d) around H, in which H is defined by $x^d = 0$. For a given k, we center coordinates x'_k at the centers x_k of the cover (1.4). We then consider restricted symbols of the form $f_k(\ell^{-1}x'_k)$ where f_k is obtained by transplanting to $B(x_k, C\ell)$ a fixed C^∞ cutoff function on $\mathbb{R}^d$ which equals 1 on the ball $B(0, 1)$ of radius 1 and zero on the complement of twice the ball. That is, with ℓ defined in (1.2),

$$f_k(\ell^{-1}x'_k) = \text{ the pullback of the } \ell^{-1} \text{ dilate of } f_k \text{ under the chart } x'_k. \tag{3.3}$$

We often drop the subscript k on the chart. As in [6] (see the discussion around e.q. (3.8)) we convert the multiplication operator on H defined by $f_k(\ell^{-1}x'_k)$ to an associated pseudo-differential operator on M given by

$$A_k(x', x^d, hD_x) = \chi\Big(\frac{x^d}{\varepsilon}\Big)hD_{x^d}\, f_k(\ell^{-1}x'), \tag{3.4}$$

where χ is a C_0^∞ cutoff equal to 1 near 0, and ε is a parameter to be chosen later. We further introduce the cutoff (2.6).

The following Proposition asserts that the restricted Cauchy data matrix elements (3.2) with respect to $f_k(\ell^{-1}x')$ on H are asymptotic to the globalized matrix elements on M, with a certain dependence on the parameters ℓ, h:

Proposition 4. *Let* $\dim M = d$ *and let* H *be a hypersurface. Let* $\{x_k\}$ *denote the centers of the cover* (1.4) *and let* $f = f_k \in C_0^\infty(B(x_k, 2C\ell)$ *be defined as in* (3.3)*. Let* $V_2(h, f_k)$ *be the restricted variance sum* (3.2)*, let* A *be as in* (3.4) *and let* $V_{2,h}(A)$ *be as in* (3.1)*. Then,*

$$V_{2,H}(h, f_k) = V_{2,h}(A(x', x^d, hD_x)) + \mathcal{O}(\ell^{-2}h) + \mathcal{O}(\varepsilon^{-2}h) + \mathcal{O}(\varepsilon^{-1}\ell^{-1}h), \quad h \to 0,$$

uniformly in k.

If one picks $\varepsilon = \ell$ (1.2) then the remainder is $\mathcal{O}(|\log h|^{2K}h)$.

We then apply the the global small scale variance estimates on M in [13, 10], which are recalled below in Proposition 7 and Proposition 9. Their results imply that, for any [5] $\beta > 0$,

$$V_{2,h}(A(x', x_d, hD_x)) \leq \frac{C}{|\log\lambda|^{(1-\beta)}}. \tag{3.5}$$

[5] The exponent is written $1 - 2K\beta$ in Proposition 2.1 of [13]. We drop the $2K$ since β is an arbitrarily small positive quantity.

Since the remainders in Proposition 4 are smaller than the right side of (3.5), we obtain:

Corollary 5. *With the same notations and assumptions as in Proposition 4, for any $\beta > 0$,*

$$V_{2,H}(h, f_k) \leq \frac{C}{|\log \lambda|^{(1-\beta)}}.$$

The Corollary implies Proposition 1. The main application of the result is to even resp. odd eigenfunctions of the negatively curved surface (M, g) with orientation reversing involution σ. For even eigenfunctions, the $\partial_\nu \varphi_h$ term is zero, while for odd eigenfunctions the $(1 + h^2 \Delta)\varphi_h$ term is zero. Hence we have,

Corollary 6. *Let (M, J, σ, g) be a negatively curved surface as in Theorem 1.1. Then for any $\beta > 0$, the variances for the even eigenfunctions satisfy*

$$h^{d-1} \sum_{E_j \in [1,1+h]} |\langle f_k(\ell^{-1} x'_k)(1 + h^2 \Delta_H)\varphi_j|_H, \varphi_j|_H\rangle_{L^2(H)} - \omega_H(a)|^2$$
$$\leq \frac{C}{|\log \lambda_j|^{(1-\beta)}},$$

resp. the variances of the odd eigenfunctions satisfy

$$h^{d-1} \sum_{E_j \in [1,1+h]} |\langle f(\ell^{-1} x'_k) h D_\nu \psi_j|_H, h D_\nu \psi_j|_H\rangle_{L^2(H)} - \omega_H(a)|^2$$
$$\leq \frac{C}{|\log \lambda_j|^{(1-\beta)}}.$$

Both remainders are uniform in k.

In §3.6 we use Corollary 6 to extract density one subsequences for which one has uniform QER lower bounds as stated in Corollary 2, following the analogous results of [13, 10].

3.2. Review of QE on the logarithmic scale. In this section we review the results of [13, 10]. The first result is Proposition 2.1 of [13]. Given $x_0 \in M$ and[6] $0 < \varepsilon < \frac{\text{inj}(M,g)}{10}$), define

$$\chi_{x_0,\varepsilon} = \chi\Big(\frac{\| \exp_{x_0}^{-1}(x)\|_{x_0}}{\varepsilon}\Big).$$

[6] inj(M, g) denotes the injectivity radius.

Let $\{u_j\}$ denote an orthonormal basis of eigenfunctions and let

$$V_\lambda(x_0, \varepsilon) := \frac{1}{N(\lambda)} \sum_{j:\lambda_j \le \lambda} \left| \int_M \chi_{x_0,\varepsilon} |u_j|^2 \, dV_g - \int_M \chi_{x_0,\varepsilon} \, dV_g \right|^2.$$

Proposition 7. *If (M, g) has negative sectional curvature, $\beta > 0$ and*[7] $0 < K < \frac{1}{2d}$, *then*

$$V_\lambda(x_0, (\log \lambda)^{-K}) \le \frac{C}{|\log \lambda|^{(1-2K\beta)}}.$$

Note that the symbols $\chi_{x_0,\varepsilon}$ are very special in this result, particularly because they are independent of the ξ variable and thus do not involve dilations in ξ.

In Section 3.1 of [13], the authors cover M,

$$M \subset \bigcup_{k=1}^{R(\varepsilon)} B(x_k, \varepsilon),$$

with balls $\{B(x_k, \varepsilon)\}_{k=1}^{R(\varepsilon)}$ of radius

$$\varepsilon = |\log \lambda|^{-K}, \tag{3.6}$$

The cover has the property that each point of M is contained in C_g many of the double balls $B(x_k, 2\varepsilon)$. The number of such balls satisfies the bounds,

$$c_1 \varepsilon^{-d} \le R(\varepsilon) \le C_2 \varepsilon^{-d}.$$

The main QE result of [13] gives uniform upper and lower bounds:

Theorem 8. *Let (M, g) have negative sectional curvature. Let ε be defined by (3.6), with K is constrained by (1.2). Then, given any orthonormal basis of eigenfunctions $\{u_j\}$, there exists a full density subsequence Λ_K of $\mathbb{N}$ so that for $j \in \Lambda_K$, and for every $1 \le k \le R(\varepsilon)$,*

$$a_1 \varepsilon^d \le \int_{B(x_k,\varepsilon)\cap H} |u_j|^2 \, dS_g \le \int_{B(x_k,50\varepsilon)\cap H} |u_j|^2 \, dS_g \le a_2 \varepsilon^d,$$

where $a_1, a_2 > 0$ depend only on g.

A key point is the uniformity of the estimates in the centers x_k. We will go over the argument in §3.6 when we adapt it to the QER setting.

[7] The $\frac{1}{2d}$ constraint on K in the variance estimate is weaker than the constraint $\frac{1}{3d}$ in the uniform QE (and QER) theorems stated in (1.2). See §3.6.

3.3. Review of the result of X. Han. X. Han [10] proves a more general small scale QE theorem for semi-classical pseudo-differential operators with $\delta(h)$-microlocalized symbols, where $\delta(h)$ depends on the way symbols are dilated. Theorem 1.5 of [10] is the main result on small-scale quantum ergodicity. It applies to small scale pseudo-differential operators $Op(a)$ where a satisfies symbol estimates of the form

$$\sup |D_x^\alpha D_\xi^\beta a| \leq C_{\alpha,\beta}\delta(h)^{-|\alpha|-|\beta|}\langle\xi\rangle^{-|\beta|},$$

were $\delta(h) = |\log h|^{-\alpha}$ for $\alpha > 0$ satisfying the constraints in (1.3).

Define the associated variance sums by

$$V_2(h,a)) := h^{d-1}\sum_{E_j\in[1,1+h]} |\langle \mathrm{Op}_h(a)u_j, u_j\rangle - \omega(a^b_{(x_0,\xi_0)})|^2.$$

In Theorem 1.5, Han proves the following result:

Proposition 9. *Let (M,g) be negatively curved and let a be a small scale pseudo-differential symbol with $\delta(h) = |\log h|^{-\alpha}$. Then*

$$V_2(h,a) = O(|\log h|^{-1+\varepsilon}), \quad \textit{for all } \varepsilon > 0.$$

Han states the estimate as $O(|\log h|^{-\beta})$ for $\beta < 1$ when $\alpha > 0$. When $\alpha = 0$, one can let $\beta = 1$ and then the result agrees with [24] for non-dilated symbols.

The symbols we will be using have the form

$$a_{x_0}^{b_1,b_2}(x,\xi;h) = \delta(h)^{-d}b_1\Big(\frac{x-x_0}{\delta(h)}\Big)b_2(x,\xi)\chi(|\xi|_g-1), \tag{3.7}$$

where x_0 will be chosen to be a center x_k of one of the balls of (1.4). Here, b_1 is called the 'base symbol'. The rescaled symbols are called $\delta(h)$-localized symbols in [10]. Special cases have the form

$$a_{x_0}^b(x,\xi;h) = \delta(h)^{-d}b\Big(\frac{x-x_0}{\delta(h)}\Big)\chi(|\xi_g|-1).$$

Theorem 1.7 of [10] shows that for $\alpha < \frac{1}{2d}$, $\beta < 1-2\alpha d$, then

$$V_2(h,a_{x_0}^b) = O_b(\delta(h)^{2d}|\log h|^{-\beta}), \quad h\to 0,$$

uniformly in x_0.

Han's uniform comparability result (Corollary 1.9 of [10]) is analogous to Theorem 8.

Theorem 10. *Let (M, g) be negatively curved and of dimension d. Then for any orthonormal basis $\{u_j\}$ of eigenfunctions, there exists a density one subsequence u_{j_n} and a uniform constant $C > 0$ so that for all x_k as in (1.4), and with $r(\lambda_{j_n})$ defined in (1.2),*

$$\int_{B(x_k, r(\lambda_{j_n}))} |u_{j_n}|^2 \, dV_g \geq C \operatorname{Vol}(B(x_k, r_{\lambda_{j_n}})).$$

There is a slight difference between the integral in Theorem 10 and the matrix elements above, namely that we are replacing a smooth symbol by the characteristic function of a ball. This is possible by the Portmanteau theorem for weak* convergence, i.e. the statement that if a sequence $\nu_j \to \nu$ as continuous linear functionals on $C^0(X)$ then $\nu_j(E) \to \nu(E)$ for all sets E with $\nu(\partial E) = 0$. However in the use of this theorem, rates of approach to the limit get destroyed, and as in Theorem 10 one can conclude an inequality rather than an asymptotic with a remainder.

3.4. Rellich identity and proof of Proposition 4. We now start the proof of Proposition 4 and of Proposition 1. We begin by recalling the Rellich identity as in [6] (based ideas of [7, 4]) to convert global QE statements into restricted QER statements. With no loss of generality, we assume H is a separating hypersurface, so that H is the boundary $H = \partial M_+$ of a smooth open submanifold $M_+ \subset M$.

As in §2.2, we let $x = (x^1, \ldots, x^{d-1}, x_d) = (x', x^d)$ be Fermi normal coordinates in a small tubular neighbourhood $H(\varepsilon)$ of H (see (2.2)) defined near a center x_k of a ball in the cover (1.4). To lighten the notation we drop the subscript in x_k. Thus, $x = \exp_x x^d \nu_{x'}$ where x' are coordinates on H and $\nu_{x'}$ is the unit normal pointing to M_+. We let $\chi \in C_0^\infty(\mathbb{R})$ be a cutoff with $\chi(t) = 0$ for $|t| \geq 1$ and $\chi(t) = 1$ for $|t| \leq 1/2$. The main result of this section is,

Lemma 11. *Let $f \in C^\infty(H)$. Let $\varepsilon > 0$ and define ℓ as in (1.2). With the above Fermi normal coordinates around each center x_k of the balls of (1.4),*

$$\begin{aligned} &\langle f(\ell^{-1}x') h\partial_\nu \varphi_h|_H, h\partial_\nu u_h|_H \rangle_{L^2(H)} + \langle f(\ell^{-1}x')(1 + h^2\Delta_H) u_h|_H, u_h|_H \rangle_{L^2(H)} \\ &\quad = \left\langle \operatorname{Op}_h \left(\left\{ (\xi^d)^2 + R(x^d, x', \xi'), \chi\Big(\frac{x^d}{\varepsilon}\Big) \xi^d f(\ell^{-1}x') \right\} \right)^{\tilde{\chi}} u_h, u_h \right\rangle_{L^2(M_+)} \\ &\qquad + \mathcal{O}(\ell^{-2}h) + \mathcal{O}(\varepsilon^{-2}h) + \mathcal{O}(\varepsilon^{-1}\ell^{-1}h), \end{aligned}$$

where the remainders are uniform in k.

On the right side,

$$\mathrm{Op}_h\left(\left\{(\xi^d)^2+R(x^d,x',\xi'),\chi\left(\frac{x^d}{\varepsilon}\right)\xi^d f(\ell^{-1}x')\right\}\right)^{\tilde{\chi}}$$

is defined in (2.6) and as discussed in §2.4, the complete symbol is compactly supported in T^*M and in particular is bounded.

Remark. Throughout the proof, as above, we implicitly cutoff the symbol using (2.6) to $|\xi|\leq 3$. To simplify the notation we do not include the cutoff $\tilde{\chi}$ on the left side in all of the computations.

Proof. We introduce a small parameter ε and choose the pseudo-differential operator (3.4) in (2.4), namely

$$A(x',x_d,hD_x)=\chi\left(\frac{x^d}{\varepsilon}\right)hD_{x^d}f(\ell^{-1}x'). \tag{3.8}$$

We then cut it off to $A^{\tilde{\chi}}$ (2.6). We temporarily suppress the cutoff $\tilde{\chi}$ to simplify notation. Since $\chi(0)=1$ it follows that the second term on the right side of (2.4) is just

$$\left\langle f(\ell^{-1}x')hD_{x^d}u_h|_H, hD_{x^d}u_h|_H\right\rangle. \tag{3.9}$$

The first term on right hand side of (2.4) equals

$$\begin{aligned}
&\int_H hD_{x^d}\left(\chi\left(\frac{x^d}{\varepsilon}\right)hD_{x^d}f(\ell^{-1}x')u_h\right)\Big|_{x^d=0}\overline{u_h}\Big|_{x_d=0}\,dS\\
&\quad=\int_H\left(\chi\left(\frac{x^d}{\varepsilon}\right)f(\ell^{-1}x')(hD_{x^d})^2u_h\right.\\
&\qquad\qquad\left.+\frac{h}{i\varepsilon}\chi'\left(\frac{x_d}{\varepsilon}\right)hD_{x^d}f(\ell^{-1}x')u_h\right)\Big|_{x^d=0}\overline{u_h}\Big|_{x^d=0}\,dS\\
&\quad=\int_H\left(\chi\left(\frac{x^d}{\varepsilon}\right)f(\ell^{-1}x')(1-R(x^d,x',hD'))u_h\right)\Big|_{x^d=0}\overline{u_h}\Big|_{x^d=0}\,dS,
\end{aligned}\tag{3.10}$$

since $\chi'(0)=0$ and $((hD_{x^d})^2+R+O(h))u_h=u_h$ in these coordinates. Thus, the left side of the stated formula follows from (2.4)-(3.10).

To compute the integral (2.5) on the left side, we recall that its principal symbol is given by (3.11) and with our choice of A it equals

$$\text{p. s.}([-h^2\Delta,A(x,hD_x)])=\left\{(\xi^d)^2+R(x^d,x',\xi'),\chi\left(\frac{x^d}{\varepsilon}\right)\xi^d f(\ell^{-1}x')\right\} \tag{3.11}$$

The cutoff $\tilde{\chi}$ as in (2.6) essentially multiplies this expression and makes the complete symbol compactly supported in $|\xi|\leq 3$.

The quantization of the principal symbol symbol is simply the naive one taking $\xi_j \to hD_{x_j}$ in the ordering specified by (2.3). The additional non-principal terms are of one higher order in h but may involve two derivatives of $\chi(\frac{x^d}{\varepsilon})$, two derivatives of $f(\ell^{-1}x')$ or a product of one mixed derivatives of these functions. This accounts for the remainder and completes the proof of the Lemma. □

The fact that the remainders are uniform in k is evident from the proof of the variance estimates, and is stated explicitly in Proposition 2.1 of [13] and Theorem 1.7 of [10].

This completes the proof of Proposition 4. The conclusion is the following result.

Corollary 12. *We have*

$$V_{2,H}(h, f_k) = V_{2,h}\Big(h, \Big\{(\xi^d)^2 + R(x^d, x', \xi'), \chi\Big(\frac{x^d}{\varepsilon}\Big)\xi^d f(\ell^{-1}x')\Big\}\Big) + \mathcal{O}(\ell^{-2}h) + \mathcal{O}(\varepsilon^{-2}h) + \mathcal{O}(\varepsilon^{-1}\ell^{-1}h), \quad h \to 0.$$

The remainders are uniform in k.

3.5. Decomposition of the global variance sums. The variance sums on the right side of Corollary 12 can be simplified and made more explicit, because only one term in the Poisson bracket of (3.11) dominates. To see this, we observe that

$$\begin{aligned}
&\Big\{(\xi^d)^2 + R(x^d, x', \xi'), \chi\Big(\frac{x^d}{\varepsilon}\Big)\xi^d f(\ell^{-1}x')\Big\} \\
&= 2\Big\{\xi^d, \chi\Big(\frac{x^d}{\varepsilon}\Big)\Big\}(\xi^d)^2 f(\ell^{-1}x') \\
&\quad + \{R(x^d, x', \xi'), \xi^d\}\chi\Big(\frac{x^d}{\varepsilon}\Big) f(\ell^{-1}x') \\
&\quad + \{R(x^d, x', \xi'), f(\ell^{-1}x')\}\xi^d \chi\Big(\frac{x^d}{\varepsilon}\Big) \\
&= \chi\Big(\frac{x^d}{\varepsilon}\Big) f(\ell^{-1}x') R_3(x', x^d, \xi'), \\
&\quad + \frac{2}{\varepsilon}\chi'\Big(\frac{x_d}{\varepsilon}\Big)(\xi^d)^2 f(\ell^{-1}x') \\
&\quad + \ell^{-1}\chi\Big(\frac{x^d}{\varepsilon}\Big) f'(\ell^{-1}x')\xi^d R_2(x', x^d, \xi') \\
&= \mathrm{I} + \mathrm{II} + \mathrm{III},
\end{aligned} \tag{3.12}$$

where R_2, R_3 are zero order symbols which are polynomial of degree ≤ 2 in ξ. The new type of term not encountered in [6] is the term III in which one takes the Poisson bracket $\{R(x^d, x', \xi'), f(\ell^{-1}x')\}$, where ξ' is paired with x'.

Remark. As remarked above, when we introduce the cutoff $\tilde{\chi}$ as in (2.6), it multiplies these expressions and cuts each one off to $|\xi| \leq 3$.

We introduce a further small parameter δ and $\chi_2(t) \in \mathcal{C}^\infty(\mathbb{R})$ satisfying $\chi_2(t) = 0$ for $t \leq -1/2$, $\chi_2(t) = 1$ for $t \geq 0$, and $\chi_2'(t) > 0$ for $-1/2 < t < 0$, and let ρ be a boundary defining function for M_+, i.e. $M_+ = \{\rho \geq 0\}$, $\rho = 0$ on $\partial M_+ = H$ and $d\rho \neq 0$ on H. For instance one may take $\rho = x^d$. By construction, $\chi_2(\rho/\delta) = 1$ on M_+ and $= 0$ outside a $\delta/2$ neighbourhood of H in $M_- = M \backslash M_+$. We choose χ_2, δ so that $\chi_2(\frac{\rho}{\delta})\chi(\frac{x^d}{\varepsilon}) = 1$.

Further, $\chi'(\frac{x_n}{\varepsilon})|_{M_+} = \tilde{\chi}'(\frac{x_n}{\varepsilon})$ for a smooth function $\tilde{\chi} \in \mathcal{C}^\infty(M)$ satisfying $\tilde{\chi} = 1$ in a neighbourhood of $M \setminus M_+$ and zero inside a neighbourhood of H. The purpose of the cutoff $\tilde{\chi}$ is to express matrix elements on M_+ as matrix elements on M, a manifold without boundary.

In summary, we now have four small parameters: $h, \ell, \varepsilon, \delta$ with h, ℓ related by (1.2) and cutoffs

- $\chi(\frac{x_d}{\varepsilon})$, where $\chi(t) = 0$ for $|t| \geq 1$ and $\chi(t) = 1$ for $|t| \leq 1/2$;
- $\chi_2(\frac{\rho}{\delta})$ $(\rho = x^d)$, $\chi_2(t) = 0$ for $t \leq -1/2$, $\chi_2(t) = 1$ for $t \geq 0$, and $\chi_2'(t) > 0$ for $-1/2 < t < 0$;
- $\tilde{\chi}(\frac{x^d}{\varepsilon})$, $\tilde{\chi} \in \mathcal{C}^\infty(M)$ satisfying $\tilde{\chi} = 1$ in a neighbourhood of $M \setminus M_+$ and zero inside a neighbourhood of H.

Further, we will be applying Proposition 9 to the following types of small-scale symbols (3.7).

- The base symbol of (3.7) has the form $b_1 = f(\ell^{-1}x')$, or $b_1 = f'(\ell^{-1}x')$, or $b_1 = \chi(\frac{x^d}{\varepsilon})$ (or its derivative).
- We choose b_2 to be one of the symbols $b_2 = R_3, R_2$, resp. $(\xi^d)^2$ defined in (3.12).
- The base symbols only involve rescaling in the x variables, while the factors $R_3, R_2, (\xi^d)^2$ are classical un-scaled symbols.

The variance sums on the right side of Corollary 12 are bounded above by the sum of the three sub-sums involving I, II, and III. Proposition 9 applies to all of them. We are interested in the variance sums where f_k is centered at the center x_k of a ball in the cover (1.4). In addition, there are the parameters ε, δ. We introduce some new notation to emphasize this dependence.

We also explicitly put in the cutoff $\tilde{\chi}$ (2.6) to emphasize that all the symbols are compactly supported.

3.5.1. Term I. We fix k and center the coordinates x' at x_k as discussed above. Then we let

$$b_1 = \chi\Big(\frac{x_d}{\varepsilon}\Big) f(\ell^{-1}x'),$$

where x' is a local coordinate around x_k giving x_k the value 0 and where $b_2 = R_3(x, \xi')$.

We then define (writing $x = (x', x^d)$ as above as Fermi coordinates centered at x_k)

$$\begin{aligned}
&\mathrm{Var}_{\mathrm{I}}(h, x_k, \varepsilon, \delta)\\
&\quad := V_2(h, \chi(x^d/\varepsilon) f(\ell^{-1}x') R_3(x, \xi'))\\
&\quad := h^{d-1} \sum_{E_j \in [1,1+h]} \bigg| \Big\langle \mathrm{Op}\Big(\chi\Big(\frac{x^d}{\varepsilon}\Big) f(\ell^{-1}x') R_3(x, \xi')\Big)^{\tilde{\chi}} u_h, u_h \Big\rangle_{L^2(M_+)}\\
&\qquad\qquad\qquad - \fint_{S^*M} \chi\Big(\frac{x^d}{\varepsilon}\Big) f(\ell^{-1}x') R_3(x, \xi')\, d\mu_L \bigg|^2
\end{aligned}$$

By the estimate of Proposition 9,

$$\mathrm{Var}_{\mathrm{I}}(h, x_k, \varepsilon, \delta) = O_{b_1, f}(|\log h|^{-1+\varepsilon}).$$

Moreover,

$$\fint_{S^*M} \chi(x^d/\varepsilon) f(\ell^{-1}x') R_3(x, \xi')\, d\mu_L = \mathcal{O}(\varepsilon \ell^{d-1}).$$

Due to the factor of ε, these matrix elements will make a negligible contribution to the limit $h \to 0$.

3.5.2. Term II. We define

$$\begin{aligned}
&\mathrm{Var}_{\mathrm{II}}(h, x_k, \varepsilon, \delta)\\
&\quad := h^{d-1} \sum_{E_j \in [1,1+h]} \bigg| \Big\langle \Big(\frac{2}{\varepsilon}\chi'\Big(\frac{x^d}{\varepsilon}\Big) f(\ell^{-1}x')(\xi^d)^2\Big)^{\tilde{\chi}} u_h, u_h \Big\rangle_{L^2(M_+)}\\
&\qquad\qquad\qquad - \fint_{S^*M} \frac{2}{\varepsilon}\chi'\Big(\frac{x^d}{\varepsilon}\Big) f(\ell^{-1}x')((\xi^d)^2)\, d\mu_L \bigg|^2.
\end{aligned}$$

Note that

$$\Big\langle \Big(\frac{1}{\varepsilon}\chi'\Big(\frac{x^d}{\varepsilon}\Big)(\xi^d)^2 f(\ell^{-1}x')\Big)^{\tilde{\chi}} u_h, u_h\Big\rangle_{L^2(M_+)}$$
$$= \Big\langle \Big(\frac{1}{\varepsilon}\tilde{\chi}'\Big(\frac{x^d}{\varepsilon}\Big)(\xi^d)^2 f(\ell^{-1}x')\Big)^{\tilde{\chi}} u_h, u_h\Big\rangle_{L^2(M)}.$$

Again by Proposition 9,

$$\mathrm{Var}_{\mathrm{II}}(h, x_k, \varepsilon, \delta) = O_{b_1,f}(|\log h|^{-1+\varepsilon}).$$

But the presence of $\frac{1}{\varepsilon}$ in $\frac{1}{\varepsilon}\chi'\big(\frac{x^d}{\varepsilon}\big)$ ensures that the matrix elements in this variance sum have non-trivial limits. Indeed,

$$\int_{S^*M} \frac{2}{\varepsilon}\chi'\Big(\frac{x^d}{\varepsilon}\Big) f(\ell^{-1}x')(\xi^d)^2\, d\mu_L$$
$$\simeq \ell^{d-1}\int_{B^*H} f(y')(1-|\xi'|^2)^{1/2}\, dy'\, d\xi'.$$

3.5.3. Term III. We define

$$\mathrm{Var}_{\mathrm{III}}(h, x_k, \varepsilon, \delta)$$
$$:= h^{d-1}\sum_{E_j\in[1,1+h]} \Big| \ell^{-1}\Big\langle \mathrm{Op}_h\Big(\chi\Big(\frac{x^d}{\varepsilon}\Big) f'(\ell^{-1}x')R_2(x', x^d, \xi')\Big)^{\tilde{\chi}} u_h, u_h\Big\rangle_{M_+}$$
$$- \ell^{-1}\fint_{S^*M} \chi\Big(\frac{x^d}{\varepsilon}\Big) f'(\ell^{-1}x')R_2(x', x^d, \xi')\, d\mu_L\Big|^2.$$

Again by Proposition 9,

$$\mathrm{Var}_{\mathrm{III}}(h, x_k, \varepsilon, \delta) = O_{b_1,f}(|\log h|^{-1+\varepsilon}).$$

The matrix elements in the variance sum III *a priori* have non-zero limits due to the factor ℓ^{-1} in

$$\ell^{-1}\Big\langle \mathrm{Op}_h\Big(\chi\Big(\frac{x^d}{\varepsilon}\Big) f'(\ell^{-1}x')R_2(x', x^d, \xi')\Big)^{\tilde{\chi}} u_h, u_h\Big\rangle_{M_+}.$$

The power of ℓ is one higher than in the other terms. However, the factor of ε can be chosen here (and consistently elsewhere) to be ℓ and then the term balances the term II. Moreover, $\int_{\mathbb{R}^{d-1}} f'(y')dy' = 0$, so the limit vanishes and the matrix elements in this term are of order $o(\ell^{d-1})$.

In summary, the means have the following asymptotics:

$$\begin{cases} \text{I:} \quad \int_{S^*M} \chi\Big(\frac{x^d}{\varepsilon}\Big) f(\ell^{-1}x') R_3(x, \xi')\, d\mu_L \\ \qquad \simeq \varepsilon \ell^{d-1} \int_{S^*M} \chi(y^d) f(y') R_3(\varepsilon y^d, \ell y')\, d\mu_L, \\ \text{II:} \quad \int_{S^*M} \frac{2}{\varepsilon} \chi'\Big(\frac{x^d}{\varepsilon}\Big) f(\ell^{-1}x')(\xi^d)^2)\, d\mu_L \\ \qquad \simeq 2\ell^{d-1} \int_{S^*M} \chi'(y^d) f(y')(\xi^d)^2\, d\mu_L, \\ \text{III:} \ \ell^{-1} \int_{S^*M} \chi\Big(\frac{x^d}{\varepsilon}\Big) f'(\ell^{-1}x') R_2(x', x^d, \xi')\, d\mu_L \\ \qquad \simeq \varepsilon \ell^{d-1} \int_{S^*M} \chi(y^d) f'(y') R_2(0, 0, \xi')\, d\mu_L. \end{cases}$$

3.6. Uniform lower bounds in the centers x_k of the cover: proof of Proposition 1. To complete the proof of Proposition 1, we employ the diagonal argument of [13] (Section 3.1) or [10] (Proof of Corollary 1.9) to extract a subsequence of density one for which the lower bound of Proposition 1 is valid. The main point is that one is intersecting $|\log h|$ subsequences, and one needs to use the rate of variance decay to construct a subsequence of density one satisfying all $|\log h|$ conditions. Since the argument is given in Section 3.1 of [13] or in Section 5.2 of [10], we only sketch it and explain the modifications necessary for the proof of Proposition 1.

To clarify the logic of the final argument, we are applying the Chebychev inequality to the three variance sums I, II, and III above. We use it to extract a subsequence of indices j of density one so that the jth summand tends to zero at a certain rate uniformly in k. Since there are $(\log |h|)^K$ values of k, the radii $r(\lambda_j)$ of the shrinking balls must be slightly larger than would be the case for one fixed k. More precisely as in (1.2), $r(\lambda_j) = (\log \lambda_j)^{-K}$, where $0 < K < \frac{1}{3d}$, as opposed to $K < \frac{1}{2d}$ for one fixed k. (Compare Corollary 1.8 and Corollary 1.9 of [10] or the discussion on page 3266). Once one has extracted the subsequence, one goes back to the analysis of the means in I, II, and III to see that II contributes the dominant asymptotics of the matrix elements. This determines the asymptotics of the restricted matrix elements by Lemma 11.

We work separately with the three variance sums $\mathrm{Var}_{\mathrm{I}}, \mathrm{Var}_{\mathrm{II}}, \mathrm{Var}_{\mathrm{III}}$ of §3.5. Proposition 9 applies to all of them, with remainders uniform in x_k.

In the notation of [10] (Step 2, p. 3283), the logarithmic dilation scale is set as in (1.3) at $\delta(h) = (|\log h|)^{-\alpha} = r(\lambda_j)$ where $\alpha < \frac{1}{3d}$. Also fix $\beta > 0$. We consider a symbol a of the form (3.7), or more precisely one of the symbols that arises in I, II, and III above, and define the 'exceptional sets'[8]

$$\begin{aligned}\Lambda^b_{x_k}(h) := \{j : E_j \in [1, 1+h], |\langle \mathrm{Op}^b_h(a^b_{x_k})u_j, u_j\rangle - \mu_L(a^b_{x_k})|^2 \\ \geq \delta(h)^{2d} |\log h|^{-2\beta}\}.\end{aligned}$$

Remark. In [13] the exceptional set is defined by the condition,

$$\left| \int_M \chi_{x_k} |u_j|^2 \, dV_g - \int_M \chi_{x_k} \, dV_g \right| \geq |\log h|^{-K\beta} \int_M \chi_{x_k} \, dV_g,$$

where $\beta > 0$. The definitions are equivalent because

$$\int_M \chi_{x_k} \, dV_g \simeq \delta(h)^d.$$

We then apply Chebyshev's inequality

$$\mathrm{Prob}\{X \geq C\} \leq \frac{1}{C} \mathbf{E} X$$

with

$$X = |\langle \mathrm{Op}^b_h(a^b_{x_k})u_j, u_j\rangle - \mu_L(a^b_{x_k})|^2$$

respect to normalized counting measure of $E_j \in [1, 1+h]$ and with

$$C = \delta(h)^{2d} |\log h|^{-2\beta}.$$

The variance estimate of Proposition 9 says that the expected valued of X is $O(|\log h|^{-1+\varepsilon})$. It follows that

$$\frac{\#\Lambda^b_{x_k}(h)}{N(h)} \leq (|\log h|)^{2d\alpha} |\log h|^{2\beta} |\log h|^{-1+\varepsilon}.$$

Remark. In (7) of [13], with $p = 1$ the authors get $C|\log h|^{K(4\beta+2d)-1}$, which is equivalent since α of [10] is K of [13] and because β is an arbitrarily small number whose exact definition changes in each occurrence.

[8] The exceptional sets are denoted by $J_{k,K}(h)$ and the generic sets are denoted $\Lambda_{k,K}$ in [13].

The key point is to obtain uniform upper bounds as x_k varies. For $0 < \alpha < \frac{1}{3d}$, define

$$\Lambda^b(h) = \bigcup_{k=1}^{N(h)} \Lambda^b_{x_k}(h), \quad N(h) \le C|\log h|^{\alpha d}.$$

We further define the 'generic sets'

$$\Gamma^b(h) := \{j : E_j \in [1, 1+h]\} \backslash \Lambda^b(h).$$

Adding the Chebychev upper bounds for the $|\log h|^{\alpha d}$ exceptional sets gives

$$\frac{\#\Lambda^b(h)}{N(h)} \le (|\log h|)^{\alpha d} (|\log h|)^{2d\alpha} |\log h|^{2\beta} |\log h|^{-1+\varepsilon},$$

hence

$$\frac{\#\Gamma^b(h)}{\#\{j : E_j \in [1, 1+h]\}} \ge 1 - \frac{C}{|\log h|^{-\alpha((4\beta+2d)+d)} |\log h|},$$

as stated in ([13], above Lemma 3.1; [10], p. 3263), and with $|\log h| = \log \lambda$. The remainder tends to zero if

$$-\alpha((4\beta + 2d) + d) + 1 > 0.$$

Since β is arbitrarily small, this requires

$$-\alpha 3d + 1 > 0 \quad \text{or} \quad \alpha = K < \frac{1}{3d}.$$

In this case,

$$\frac{\#\Gamma^b(h)}{\#\{j : E_j \in [1, 1+h]\}} \longrightarrow 1, \quad h \to 0,$$

thus giving a subsequence of density one.

If $j \in \Gamma^b(h)$ then for any $\beta > 0$,

$$|\langle \mathrm{Op}_h(a^b_{x_k}) u_j, u_j \rangle - \mu_L(a^b_{x_k})| \le C\delta(h)^d |\log h|^{-\beta}$$

uniformly for all x_k.

We now let b be one of the symbols occurring in I, II, and III and denote by $\Gamma_I(h)$, $\Gamma_{\mathrm{II}}(h)$, resp. $\Gamma_{\mathrm{III}}(h)$ the associated index sets. Recalling that

$$\delta(h) = (|\log h|)^{-\alpha} = r(\lambda_j)$$

where $\alpha < \frac{1}{3d}$, and that $\beta = 1 - \varepsilon$ is a positive number < 1 and arbitrarily close to 1 (cf. Theorem 1.5 of [10]), we have

$$\begin{cases} \text{I:} \quad j \in \Gamma_{\mathrm{I}}(h) \iff \left| \left\langle \left(\chi\left(\frac{x^d}{\varepsilon}\right) f(\ell^{-1}x') R_3(x,\xi') \right)^{\tilde{\chi}} u_h, u_h \right\rangle_{L^2(M_+)} \right. \\ \qquad\qquad\qquad\qquad \left. - \int_{S^*M} \chi\left(\frac{x^d}{\varepsilon}\right) f(\ell^{-1}x') R_3(x,\xi')\, d\mu_L \right| \\ \qquad\qquad\qquad \leq C(|\log h|)^{-d\alpha} |\log h|^{-\beta}, \\ \text{II:} \quad j \in \Gamma_{\mathrm{II}}(h) \iff \left| \left\langle \frac{2}{\varepsilon} \chi'\left(\frac{x^d}{\varepsilon}\right) f(\ell^{-1}x') (\xi_d^2)^{\tilde{\chi}} u_h, u_h \right\rangle_{L^2(M_+)} \right. \\ \qquad\qquad\qquad\qquad \left. - \int_{S^*M} \frac{2}{\varepsilon} \chi'\left(\frac{x^d}{\varepsilon}\right) f(\ell^{-1}x') (\xi^d)^2)\, d\mu_L \right| \\ \qquad\qquad\qquad \leq C(|\log h|)^{-d\alpha} |\log h|^{-\beta}, \\ \text{III:} \quad j \in \Gamma_{\mathrm{III}}(h) \iff \left| \ell^{-1} \left\langle \mathrm{Op}_h \left(\chi\left(\frac{x^d}{\varepsilon}\right) f'(\ell^{-1}x') R_2(x',x^d,\xi') \right)^{\tilde{\chi}} u_h, u_h \right\rangle_{M_+} \right. \\ \qquad\qquad\qquad\qquad \left. - \ell^{-1} \int_{S^*M} \chi\left(\frac{x^d}{\varepsilon}\right) f'(\ell^{-1}x') R_2(x',x^d,\xi')\, d\mu_L \right| \\ \qquad\qquad\qquad \leq C(|\log h|)^{-d\alpha} |\log h|^{-\beta}. \end{cases} \tag{3.13}$$

All three conditions hold for indices in $\Gamma_{\mathrm{I}}(h) \cap \Gamma_{\mathrm{II}}(h) \cap \Gamma_{\mathrm{III}}(h)$ and the estimates are uniform in k. Thus, there exists a subsequence of density one so that the above asymptotics and remainders are valid.

3.7. Implications for restricted matrix elements

Lemma 13. *If $j \in \Gamma_{\mathrm{I}}(h) \cap \Gamma_{\mathrm{II}}(h) \cap \Gamma_{\mathrm{III}}(h)$, then*

$$\left| \langle f(\ell^{-1}x') CD\varphi_h|_H, CD\varphi_h|_H \rangle_{L^2(H)} - \int_{B^*H} f_k(\ell^{-1}x')(1 - |\xi'|^2)^{1/2}\, dx'\, d\xi' \right| \\ = O(\ell^d).$$

Proof. We use Lemma 11 to convert the restricted matrix elements into global ones. We then set $\varepsilon = \ell$. By Lemma 13, the remainders are of smaller order than the means, as we now verify by evaluating the means asymptotically. Throughout we use that $\ell = |\log h|^{-K}$ with $K < \frac{1}{3d}$, hence the integrals are of order $|\log h|^{-(d-1)K} > |\log h|^{-(d-1)/(3d)}$, thus are larger than the remainder.

We claim that the means have the following asymptotics:

$$\begin{cases} \text{I:} \quad \displaystyle\int_{S^*M} \chi\Big(\frac{x^d}{\varepsilon}\Big) f(\ell^{-1}x') R_3(x,\xi')\, d\mu_L \\ \qquad \simeq \varepsilon \ell^{d-1} \displaystyle\int_{S^*M} \chi(y^d) f(y') R_3(\varepsilon y^d, \ell y')\, d\mu_L, \\ \text{II:} \quad \displaystyle\int_{S^*M} \frac{2}{\varepsilon}\chi'\Big(\frac{x^d}{\varepsilon}\Big) f(\ell^{-1}x')(\xi^d)^2\, d\mu_L \\ \qquad \simeq 2\ell^{d-1} \displaystyle\int_{S^*M} \chi'(y^d) f(y')(\xi^d)^2\, d\mu_L, \\ \text{III:} \ \ell^{-1} \displaystyle\int_{S^*M} \chi\Big(\frac{x^d}{\varepsilon}\Big) f'(\ell^{-1}x') R_2(x', x^d, \xi')\, d\mu_L \\ \qquad \simeq \varepsilon \ell^{d-1} \displaystyle\int_{S^*M} \chi(y^d) f'(y') R_2(0,0,\xi')\, d\mu_L. \end{cases}$$

When we set $\varepsilon = \ell$, we find that the mean in II has the leading order, thus the restricted matrix element is asymptotic to the mean of II.

First we consider the sequence of matrix elements for term I. The mean value of the matrix elements is

$$\int_{S^*M} \chi\Big(\frac{x^d}{\varepsilon}\Big) f(\ell^{-1}x') R_3(x,\xi')\, d\mu_L = \mathcal{O}(\varepsilon \ell^{d-1}).$$

We may (and will) set $\varepsilon = \ell$ and then the mean is $O(\ell^d)$, while the square root of the variance estimate in (3.13) is $(|\log h|)^{-d\alpha}|\log h|^{-\beta} = O(\ell^d |\log h|^{-\beta})$, which is smaller. It follows that with $\varepsilon = \ell$,

$$\Big\langle \Big(\chi\Big(\frac{x^d}{\varepsilon}\Big) f(\ell^{-1}x') R_3(x,\xi')\Big)^{\tilde{\chi}} u_h, u_h \Big\rangle_{L^2(M_+)} = O(\ell^d).$$

3.7.1. Term II. Note that

$$\Big\langle \Big(\frac{1}{\varepsilon}\chi'\Big(\frac{x^d}{\varepsilon}\Big)(\xi^d)^2 f(\ell^{-1}x')\Big)^{\tilde{\chi}} u_h, u_h \Big\rangle_{L^2(M_+)}$$
$$= \Big\langle \Big(\frac{1}{\varepsilon}\tilde{\chi}'\Big(\frac{x^d}{\varepsilon}\Big)(\xi^d)^2 f(\ell^{-1}x')\Big)^{\tilde{\chi}} u_h, u_h \Big\rangle_{L^2(M)}$$

Again with $\varepsilon = \ell$, we get

$$\int_{S^*M} \frac{2}{\varepsilon}\chi'\Big(\frac{x^d}{\varepsilon}\Big) f(\ell^{-1}x')(\xi^d)^2\, d\mu_L \simeq \ell^{d-1} \int_{B^*H} f(y')(1-|\xi'|^2)^{1/2}\, dy'\, d\xi'.$$

Since this is larger than the variance bound, we obtain

$$\Big\langle\Big(\frac{1}{\varepsilon}\chi'\Big(\frac{x^d}{\varepsilon}\Big)(\xi^d)^2 f(\ell^{-1}x')\Big)^{\tilde{\chi}} u_h, u_h\Big\rangle \simeq \ell^{d-1} \int_{B^*H} f(y')(1-|\xi'|^2)^{1/2}\, dy'\, d\xi'.$$

3.7.2. Term III. Setting $\varepsilon = \ell$ and changing variables shows that

$$\int_{\mathbb{R}^{d-1}} f'(y')\, dy' = 0,$$

so the limit vanishes and the matrix elements of term II are of order $O(\ell^d)$.

Thus, the full matrix element on H is asymptotic to the mean of term II plus a remainder of order ℓ^d. □

Specializing to the special surfaces and applying the Rellich identities, we conclude:

Corollary 14. *Let (M, J, σ, g) be a negatively curved surface with isometric involution. Then If $j \in \Gamma_{\mathrm{I}}(h) \cap \Gamma_{\mathrm{II}}(h) \cap \Gamma_{\mathrm{III}}(h)$, and if φ_j is an orthonormal basis of even eigenfunctions, resp. $\{\psi_j\}$ is an orthonormal basis of odd eigenfunctions, then for the center x_k of every ball in (1.4),*

$$\begin{aligned}&\langle f_k(\ell^{-1}x')(1+h^2\Delta_H)\varphi_h|_H, \varphi_h|_H\rangle_{L^2(H)}\\ &\quad = \int_{B^*H} f_k(\ell^{-1}x')(1-|\xi'|^2)^{1/2}\, dx'\, d\xi'| + O(\ell^2),\end{aligned}$$

$$\begin{aligned}&\langle f_k(\ell^{-1}x')hD_{x^d}|_H, hD_{x^d}\psi_h|_H\rangle_{L^2(H)}\\ &\quad = \int_{B^*H} f_k(\ell^{-1}x')(1-|\xi'|^2)^{1/2}\, dx'\, d\xi' + O(\ell^2).\end{aligned}$$

uniformly in k.

3.8. Completion of the proof of Proposition 1. Next we show that we can eliminate the Δ_H in the operator $(1+h^2\Delta_H)$ in at the expense of getting a lower bound.

Let μ_j be a positive microlocal lift of $u_j|_H$ to B^*H in the sense that $d\mu_j$ are positive measures and[9]

$$\langle \mathrm{Op}_h(a)u_j|_H, u_j|_H\rangle \simeq \int_{B^*H} a\, d\mu_j.$$

[9] Sometimes referred to as a *Wigner measure*.

In particular,

$$\langle f_k(\ell^{-1}x')(1+h^2\Delta_H)u_j|_H, u_j|_H\rangle_{L^2(H)} \simeq \int_{B^*H} (1-|\xi'|^2)\, d\mu_j.$$

If $CD(u_j)|_H$ is quantum ergodic on the logarithmic scale at all centers x_k of (1.4), then

$$\begin{aligned}\langle f_k(\ell^{-1}x')(1+h^2\Delta_H)u_j|_H, u_j|_H\rangle_{L^2(H)} + \int_H f_k(\ell^{-1}x')|\lambda_j^{-1/2}\partial_\nu u_j|^2\, dS_g\\ \simeq \int_{B^*H} f_k(\ell^{-1}x')(1-|\xi'|^2)\, d\mu_j.\end{aligned} \tag{3.14}$$

Since

$$\begin{aligned}\int_H f_k(\ell^{-1}x')u_j^2\, dS_g &= \int_{B^*H} f_k(\ell^{-1}x')\, d\mu_j\\ &\geq \int_{B^*H} f_k(\ell^{-1}x')(1-|\xi'|^2)\, d\mu_j\\ &\simeq \langle f_k(\ell^{-1}x')(1+h^2\Delta_H)u_h|_H, u_h|_H\rangle_{L^2(H)},\end{aligned}$$

it follows from (3.14) that for all x_k and large enough λ_j,

$$\begin{aligned}\int_H f_k(\ell^{-1}x')u_j^2\, dS_g + \int_H f_k(\ell^{-1}x')|\lambda_j^{-1/2}\partial_\nu u_j|^2\, dS_g\\ \geq \int_{B^*H} f_k(\ell^{-1}x')(1-|\xi'|^2)^{1/2}\, dx'\, d\xi'.\end{aligned} \tag{3.15}$$

Proposition 1 is an immediate consequence of this inequality. Also Corollary 2 is an immediate consequence of Corollary 14 and this inequality.

Remark. As J. Toth also observed,

$$\int_H f_k(\ell^{-1}x')|(u_j|_H)|^2 dS_g \geq \langle f_k(I+h^2\Delta_H)u_j, u_j\rangle_H$$

since $h^2\Delta_H \leq 0$. Thus we could also use that

$$\begin{aligned}\int_H f_k(\ell^{-1}x')u_j^2 &= \int_{B^*H} f_k(\ell^{-1}x')\, d\mu_j\\ &\geq \langle f_k(\ell^{-1}x')(1+h^2\Delta_H)u_j|_H, u_j|_H\rangle_{L^2(H)}\\ &\simeq \int_{B^*H} f_k(\ell^{-1}x')(1-|\xi'|^2)\, d\mu_j.\end{aligned}$$

4. Proof of Proposition 3

The proof is similar to that in [18] but we must keep track of the dependence of the remainder estimate on ℓ, i.e. on the number of derivatives of $f_k(x/\ell)$.

Let $f \in C^\infty(H)$ and consider

$$N(\sqrt{\lambda}, f_k(\ell^{-1}x')) := \sum_{j:\sqrt{\lambda_j} \leq \sqrt{\lambda}} \left| \int_H f(\ell^{-1}s) u_j^b|_H(s)\, dS(s) \right|^2.$$

Proposition 15. *we have*

$$N(\sqrt{\lambda}, f_k(\ell^{-1}x')) = \lambda^{1/2} \int_H f_\ell^2\, dS + O(\lambda^{-1/2+\gamma}) \quad \textit{for all } \gamma > 0.$$

This is proved by the standard Tauberian method. Let $E(t, x, y)$ denote the kernel of $\cos t\sqrt{-\Delta}$. We further denote by $E^b(t, q, q')$ the Dirichlet, resp. Neumann data of the wave kernel on H. First we consider the cosine transform of

$$dN(\sqrt{\lambda}, f_k(\ell^{-1}x')) := \sum_j \left| \int_H f_k(\ell^{-1}s) u_j^b|_H(s)\, dS(s) \right|^2 \delta_{\sqrt{\lambda_j}},$$

given by

$$\begin{aligned} S_{f_k(\ell^{-1}x')}(t) &:= \int_H \int_H E^b(t, q, q') f_k(\ell^{-1}q) f_k(\ell^{-1}q')\, dS(q)\, dS(q') \\ &= \sum_j \cos t\sqrt{\lambda_j} \left| \int_H f_k(\ell^{-1}q) u_j^b(q)\, dS(q) \right|^2. \end{aligned}$$

We then introduce a smooth cutoff $\rho \in \mathcal{S}(\mathbb{R})$ with supp$\hat{\rho} \subset (-\varepsilon, \varepsilon)$, where $\hat{\rho}$ is the Fourier transform of ρ, and consider

$$S_{f_k(\ell^{-1}x')}(\sqrt{\lambda}, \rho) = \int_{\mathbb{R}} \hat{\rho}(t)\; S_{f_k(\ell^{-1}x')}(t) e^{it\sqrt{\lambda}}\, dt.$$

Lemma 16. *If supp $\hat{\rho}$ is contained in a sufficiently small interval around* 0*, with $\hat{\rho} \equiv 1$ in a smaller interval, then for both Dirichlet and Neumann data u_j^b,*

$$\begin{aligned} S_{f_k}(\sqrt{\lambda}, \rho) &= \frac{\pi}{2} \sum_j \rho\big(\sqrt{\lambda} - \sqrt{\lambda_j}\big) |\langle u_j^b, f_k(\ell^{-1}x')\rangle|^2 \\ &= \| f_k(\ell^{-1}x')\|_{L^2(H)}^2 + O(\lambda^{-1/2+\gamma}), \quad \textit{for all } \gamma > 0. \end{aligned}$$

Proof. Until the end, the proof is the same as in [17, 18]. In [18] we assumed that the boundary of M was (weakly) concave and here we assume that the curve H is a geodesic.

By wave front set considerations (see [23, 11]), there exists $\delta_0 > 0$ so that the

$$\text{sing supp}\, S_{f_k(\ell^{-1}x')}(t) \cap (-\delta, \delta) = \{0\}.$$

The singular times $t \neq 0$ are the lengths of H-orthogonal geodesics, i.e. geodesics which intersect H orthogonally at both endpoints. The singularities at $t \neq 0$ are of lower order than the singularity at $t = 0$.

We now determine the singularity at $t = 0$ when we introduce the small-scale f_ℓ, using a Hörmander-style small time parametrix for the even wave kernel $E(t, x, q)$.There exists an amplitude A so that modulo smoothing operators,

$$E(t, x, q) \equiv \int_{T_q^* X} A(t, x, q, \xi) \exp(i(\langle \exp_q^{-1}(x), \xi\rangle - t|\xi|))\, d\xi.$$

The amplitude has order zero. We then take the Cauchy data on H.

Changing variables $\xi \to \sqrt{\lambda}\xi$, the trace may be expressed in the form,

$$\begin{aligned}\lambda \int_{\mathbb{R}} \int_H \int_H \int_{T_q^* X} & \hat{\rho}(t) e^{it\sqrt{\lambda}} \chi_q(\xi) \\ & A(t, q', q, \lambda\xi) \exp\left(i\sqrt{\lambda}(\langle \exp_q^{-1}(q'), \xi\rangle - t|\xi|)\right) \\ & f_k(\ell^{-1}q) f_k(\ell^{-1}q)\, d\xi\, dt\, dS(q)\, dS(q').\end{aligned} \tag{4.1}$$

We now compute the asymptotics by the stationary phase method. As in [17], we calculate the expansion by putting the integral over $T_q^* X$ in polar coordinates,

$$\begin{aligned}\int_{\mathbb{R}} \int_0^\infty \int_H \int_H \int_{S_q^* X} & \hat{\rho}(t) e^{it\sqrt{\lambda}} \chi_q(\xi) A(t, q', q) \\ & \exp\left(i\sqrt{\lambda} r(\langle \exp_q^{-1}(q'), \omega\rangle - t)\right) \\ & f_k(\ell^{-1}q) f_k(\ell^{-1}q') r^{n-1}\, dr\, d\omega\, dt\, dS(q)\, dS(q'),\end{aligned}$$

and in these coordinates the phase becomes,

$$\Psi(q, \rho, t, \omega, q') := t + r\langle \exp_q^{-1}(q'), \omega\rangle - tr.$$

We get a non-degenerate critical point in the variables (t, r) when $r = 1$, $t = \langle \exp_q^{-1}(q'), \omega\rangle$.

We are mainly interested in the singularity at $t = 0$ and consider that first. If $t = 0$ then $\langle \exp_q^{-1}(q'), \omega\rangle = 0$. Since ω is an arbitrary unit co-vector at q, this implies $q = q'$. Due to this constraint we need to consider the stationary phase asymptotics of the full integral, and find that there is a non-degenerate critical manifold given by the diagonal $\mathrm{diag}(H \times H) \times S^*_q M$. We write $\omega = (\xi', \xi_d)$ where $\xi_d = \sqrt{1 - |\xi'|^2}\nu$ where ν is the unit normal to H.

When H is totally geodesic (our main application being a closed geodesic of a negatively curved surface), then $\exp_q^{-1} q'$ integral over the unit coball bundle B^*H of H and replace $\omega = (\xi', \xi_d)$ by ξ'. Rescaling, our integral for fixed q is

$$\frac{1}{\sqrt{\lambda}} \int_0^\infty \int_{\mathbb{R}} \int_{B^*H} e^{i\sqrt{\lambda}(t + r\langle \exp_q^{-1}(q'), \xi'\rangle - tr)} \chi_q(\omega) \widetilde{A}(t, q', q, r\omega) f_k(\ell^{-1} q) f_k(\ell^{-1} q') \hat{\rho}(t)\, d\xi'\, dS(q')\, dt\, dr,$$

for another amplitude $\widetilde{A}$. We fix q and consider the oscillatory integral in (q', ξ', t, r).

In addition to the non-degenerate (r, t) block there is the (q', ξ') block, which is non-degenerate and has Hessian determinant one. Thus, the singularity at $t = 0$ produces the term,

$$(4.1) = \int_H A(0, q, q, \nu_q) |f_k(\ell^{-1} q)|^2\, dS(q) + O(\lambda^{-1/2} (\log \lambda)^M),$$

for some $M > 0$.

The remainder estimate in the method of stationary phase [14] has two contributions. To localize near the critical point, one needs to integrate by parts, and each time one gets $O(\ell^{-1}\lambda^{-1/2})$. Thus, when ℓ is given by (1.2), this part of the remainder is $O(\lambda^{-1/2+\gamma})$ for any $\gamma > 0$. Secondly the remainder in the stationary phase

expansion to leading order is

$$O(\lambda^{-1/2} \| f_k(\ell^{-1} q)\|_{C^6}) = O(\lambda^{-1/2} \ell^{-6}) = O(\lambda^{-1/2+\gamma}), \quad \text{for all } \gamma > 0.$$

Thus, the remainders are as stated in the Lemma. □

Proposition 15 then follows by a standard Tauberian theorem [19] (Appendix B).

4.1. Uniformity for small balls. As in the logarithmic QER proof, we need to extract a density one subsequence for which the Kuznecov bounds hold uniformly for all x_k. We use the same Chebyshev argument as in §3.6, but based on Proposition 16 rather than on variance sums. The terms are positive and therefore the same Chebyshev argument gives a subsequence of density one for which the limits.

We only consider Dirichlet data of Neumann eigenfunctions, since the same argument is valid for Neumann data of Dirichlet eigenfunctions. In the notation of [10] (Step 2, p. 3283) we define the 'exceptional sets'

$$\Lambda_{x_k}(h) := \{j : E_j \in [1, 1+h],$$
$$|\langle u_j^b, f_k(\ell^{-1} x_k') \rangle|^2 \geq (\log \lambda_j)^{1/3} \lambda_j^{-1/2} \| f_k(\ell^{-1}(q) \|^2_{L^2(H)}\}.$$

We again apply Chebyshev's inequality

$$\text{Prob}\{X \geq C\} \leq \frac{1}{C} \mathbf{E} X$$

with

$$X = \frac{|\langle u_j^b, f_k(\ell^{-1} x_k') \rangle|^2}{\| f_\ell \|^2_{L^2(H)}}$$

with respect to normalized counting measure of $E_j \in [1, 1+h]$ and with[10]

$$C = \lambda_j^{-1/2} |\log \lambda_j|^{1/3}.$$

The Kuznecov sum estimate of Proposition 15 says that the expected valued of X is $O(\lambda^{-1})$. It follows that

$$\frac{\#\Lambda^b_{x_k}(h)}{N(h)} \leq (|\log \lambda|)^{-1/3}.$$

For $0 < \alpha < \frac{1}{3d}$ define

$$\Lambda(h) = \bigcup_{k=1}^{N(h)} \Lambda_{x_k}(h), \quad N(h) \leq C |\log h|^{\alpha d},$$

and

$$\Gamma(h) := \{j : E_j \in [1, 1+h]\} \backslash \Lambda(h).$$

Adding the Chebychev upper bounds for the $|\log h|^{\alpha d}$ exceptional sets gives

$$\frac{\#\Lambda(h)}{N(h)} \leq (|\log h|)^{\alpha d} |\log h|^{-1/3},$$

hence

$$\frac{\#\Gamma(h)}{\#\{j : E_j \in [1, 1_h]} \geq 1 - (|\log h|)^{\alpha d - 1/3}.$$

[10] Here, the choice of $1/3$ is rather arbitrary: it could be any number $< \frac{1}{2}$. We pick it to be $\frac{1}{3}$ to obtain the same constraints on α as for variance sums.

Again if $\alpha < \frac{1}{3d}$ we obtain a subsequence of density one.

If $j \in \Gamma(h)$ then

$$|\langle u_j, f_k(\ell^{-1}x_k')\rangle|^2 \leq (\log\lambda_j)^{1/3}\lambda_j^{-1/2}\|f_\ell\|^2_{L^2(H)}$$

uniformly for all x_k.

This completes the proof of Proposition 3.

4.2. Conclusion of the proof. We now restrict to the surfaces of Theorem 1.1 in dimension $d = 2$, and conclude the proof along the lines sketched in §1.3 of the Introduction. We only consider Dirichlet data of even eigenfuntions φ_j; the proof for Neumann data of odd eigenfunctions is the same. The lower bound of Proposition 3 and 8 proves that for a density one subsequence,

$$\int_{B(x_k,\ell_j)\cap H} |\varphi_j|^2 dS_g \geq C_g \ell_j.$$

Combining with the sup norm estimate

$$|\varphi_j|_{L^\infty} \leq C_g \frac{\lambda^{1/4}}{\sqrt{\log\lambda}},$$

we find that along the density one subsequence,

$$\int_{B(x_k,\ell_j)\cap H} |\varphi_j| \geq \lambda_j^{-1/4}\sqrt{\log\lambda_j}\,\ell_j. \tag{4.2}$$

But by Proposition 3, along the density one subsequence

$$\left|\int_{B(x_k,\ell_j)\cap H} \varphi_j\, ds\right| \leq C_0 \ell_j \lambda_j^{-1/4}(\log\lambda_j)^{1/3}. \tag{4.3}$$

Comparing (4.2) and (4.3) shows that

$$\int_{B(x_k,\ell_j)\cap H} |\varphi_j| dS_g > \int_{B(x_k,\ell_j)\cap H} \varphi_j\, dS_g$$

for all balls in the cover (1.4) for sufficiently large j in the density one subsequence. It follows that $\varphi_j|_{B(x_k,\ell)\cap H}$ must have a zero for every k.

The rest of the proof of Theorem 1.1 is the same as in [17].

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