

**Dispersive estimates
for higher dimensional Schrödinger operators
with threshold eigenvalues
II: The even dimensional case**

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Abstract. We investigate $L^1(\mathbb{R}^n) \rightarrow L^\infty(\mathbb{R}^n)$ dispersive estimates for the Schrödinger operator $H = -\Delta + V$ when there is an eigenvalue at zero energy in even dimensions $n \geq 6$. In particular, we show that if there is an eigenvalue at zero energy then there is a time dependent, rank one operator F_t satisfying $\|F_t\|_{L^1 \rightarrow L^\infty} \lesssim |t|^{2-\frac{n}{2}}$ for $|t| > 1$ such that

$$\|e^{itH} P_{ac} - F_t\|_{L^1 \rightarrow L^\infty} \lesssim |t|^{1-\frac{n}{2}}, \quad \text{for } |t| > 1.$$

With stronger decay conditions on the potential it is possible to generate an operator-valued expansion for the evolution, taking the form

$$e^{itH} P_{ac}(H) = |t|^{2-\frac{n}{2}} A_{-2} + |t|^{1-\frac{n}{2}} A_{-1} + |t|^{-\frac{n}{2}} A_0,$$

with A_{-2} and A_{-1} mapping $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$ while A_0 maps weighted L^1 spaces to weighted L^∞ spaces. The leading-order terms A_{-2} and A_{-1} are both finite rank, and vanish when certain orthogonality conditions between the potential V and the zero energy eigenfunctions are satisfied. We show that under the same orthogonality conditions, the remaining $|t|^{-\frac{n}{2}} A_0$ term also exists as a map from $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$, hence $e^{itH} P_{ac}(H)$ satisfies the same dispersive bounds as the free evolution despite the eigenvalue at zero.

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1. Introduction

In this paper we examine dispersive properties of the operator e^{itH} , where $H := -\Delta + V$ with V a real-valued potential on \mathbb{R}^n . The spatial dimension may be any even number $n \geq 6$, just as part I of this work, [13], considered odd dimensions $n \geq 5$. This operator is the propagator of the Schrödinger equation

$$iu_t + Hu = 0, \quad u(x, 0) = f(x), \quad (1)$$

as formally, one can write the solution to (1) as $u(x, t) = e^{itH} f(x)$.

When $V = 0$, one has the dispersive estimate $\|e^{itH}\|_{L^1 \rightarrow L^\infty} \lesssim |t|^{-\frac{n}{2}}$. This can be easily seen by the representation

$$e^{-it\Delta} f(x) = \frac{1}{(4\pi it)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{i|x-y|^2/4t} f(y) dy,$$

which one obtains through elementary properties of the Fourier transform. The stability of dispersive estimates under perturbation by a short range potential, that is for a Schrödinger operator of the form $H = -\Delta + V$, where V is real-valued and decays at spatial infinity, is a well-studied problem. Where possible, the estimate is presented in the form

$$\|e^{itH} P_{ac}(H)\|_{L^1(\mathbb{R}^n) \rightarrow L^\infty(\mathbb{R}^n)} \lesssim |t|^{-n/2}. \quad (2)$$

Projection onto the continuous spectrum is needed as the perturbed Schrödinger operator H may possess pure point spectrum that experiences no decay at large times. Under relatively mild assumptions on the potential one has an L^2 conservation law for the operator e^{itH} . In addition, if $|V(x)| \leq C(1 + |x|)^{-\beta}$ for some $\beta > 1$ and is real-valued, the spectrum of H is composed of a finite number of non-positive eigenvalues and purely absolutely continuous spectrum on $(0, \infty)$, see [25].

The history of this problem is more thoroughly discussed in part I [13]. We recall briefly that the first results in the direction of (2), Rauch [24], Jensen and Kato [19], Jensen [17, 18], and Murata [23], studied mappings between weighted $L^2(\mathbb{R}^n)$ in place of $L^1(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$. Estimates precisely of the form in (2) are studied in [22, 29, 26, 14, 27, 15, 4, 6, 3, 16] by a number of authors in various dimensions, and with different characterizations of the potential $V(x)$ respectively. The first result on these global, $L^1 \rightarrow L^\infty$, dispersive estimates was the work of Journé, Soffer and Sogge [22]. Much of the more recent work has its roots in the work of Rodnianski and Schlag [26]. For a more detailed history, see the survey paper [28].

Our main concern is the effect of obstructions at zero energy on the time decay of the evolution. Jensen and Kato [19] showed that in three dimensions, if there is a resonance at zero energy then the propagator $e^{itH} P_{ac}(H)$ (as an operator between polynomially weighted $L^2(\mathbb{R}^3)$ spaces) has leading order decay of $|t|^{-1/2}$ instead of $|t|^{-3/2}$. In general the same effect occurs if zero is an eigenvalue, even though $P_{ac}(H)$ explicitly projects away from the associated eigenfunction. Global $L^1 \rightarrow L^\infty$ dispersive estimates are known in all lower dimensions when zero is not a regular point of the spectrum, due to Yajima, Erdoğan, Schlag and the authors in various combinations, see [14, 10, 30, 9, 12, 7, 5]. The goal of this work is to extend these studies to all higher dimension $n > 3$.

In dimensions five and higher resonances at zero do not occur. In [17] Jensen obtained leading order decay at the rate $|t|^{-\frac{n}{2}}$ as an operator on weighted $L^2(\mathbb{R}^n)$ spaces if zero is an eigenvalue. For $n \geq 5$, the subsequent terms of the asymptotic expansion have decay rates $|t|^{1-\frac{n}{2}}$ and $|t|^{-\frac{n}{2}}$ and map between more heavily weighted $L^2(\mathbb{R}^n)$ spaces. We are able to recover the same structure of time decay with respect to mappings from $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$, with a finite-rank leading order term and a remainder that belongs to weighted spaces. In fact, our results imply Jensen's results on weighted $L^2(\mathbb{R}^n)$ spaces with reduced weights. Perhaps the most surprising result we prove is the full dispersive estimate (2) holds without any spatial weights if the zero-energy eigenfunctions satisfy two orthogonality conditions, see Theorem 1.2 part (3) below.

In addition we note that there has been much study of the wave operators, which are defined by strong limits on $L^2(\mathbb{R}^n)$,

$$W_{\pm} = s\text{-}\lim_{t \rightarrow \pm\infty} e^{itH} e^{it\Delta}.$$

The L^p boundedness of the wave operators, see [31, 11, 21], relates to dispersive estimates by way of the ‘intertwining property,’ which allows us to translate certain

mapping properties of the free propagator to the perturbed operator,

$$f(H)P_{ac} = W_{\pm}f(-\Delta)W_{\pm}^*.$$

The identity is valid for Borel functions f . In dimensions $n \geq 5$, boundedness of the wave operators on L^p for $\frac{n}{n-2} < p < \frac{n}{2}$ in the presence of an eigenvalue at zero was established by Yajima [31] in odd dimensions, and Finco and Yajima [11] in even dimensions. In particular, with p' the conjugate exponent satisfying $\frac{1}{p} + \frac{1}{p'} = 1$, the boundedness of the wave operators imply the mapping estimate

$$\|e^{itH}P_{ac}(H)\|_{L^p \rightarrow L^{p'}} \lesssim |t|^{-\frac{n}{2} + \frac{n}{p}}.$$

Roughly speaking, the range of p in the wave operator results yield a time decay rate of $|t|^{-\frac{n}{2} + 2+}$. Similar results in lower dimensions can be found in [30, 21].

The main results in this paper mirror the ones obtained in odd dimensions [13] and we will use the same notation and conventions where possible. Our work here is mostly self-contained; we have omitted proofs that are proved verbatim, or those that require only minor modifications of those in [13]. To state our main results, define a smooth cut-off function $\chi(\lambda)$ with $\chi(\lambda) = 1$ if $\lambda < \lambda_1/2$ and $\chi(\lambda) = 0$ if $\lambda > \lambda_1$, for a sufficiently small $0 < \lambda_1 \ll 1$. Further define $\langle x \rangle := (1 + |x|)$, then we use the notation for weighted L^p spaces

$$\|f\|_{L^{p,\sigma}} := \|(1 + |x|)^{\sigma} f\|_p$$

and the abbreviations $a- := a - \epsilon$ and $a+ := a + \epsilon$ for a small, but fixed, $\epsilon > 0$. We prove the following low energy bounds.

Theorem 1.1. *Assume that $n \geq 6$ is even, $|V(x)| \lesssim \langle x \rangle^{-\beta}$, for some $\beta > n$ and that zero is not an eigenvalue of $H = -\Delta + V$ on \mathbb{R}^n . Then,*

$$\|e^{itH}\chi(H)P_{ac}(H)\|_{L^1 \rightarrow L^\infty} \lesssim |t|^{-\frac{n}{2}}.$$

Theorem 1.2. *Assume that $n \geq 6$ is even, $|V(x)| \lesssim \langle x \rangle^{-\beta}$, and that zero is an eigenvalue of $H = -\Delta + V$ on \mathbb{R}^n . The low energy Schrödinger propagator $e^{itH}\chi(H)P_{ac}(H)$ possesses the following structure.*

- (1) *Suppose that there exists $\psi \in \text{Null } H$ such that $\int_{\mathbb{R}^n} V\psi \, dx \neq 0$. Then there is a rank-one time dependent operator $\|F_t\|_{L^1 \rightarrow L^\infty} \lesssim |t|^{2-\frac{n}{2}}$ such that for $|t| > 1$,*

$$e^{itH}\chi(H)P_{ac}(H) - F_t = \mathcal{E}_1(t).$$

Where, $\|\mathcal{E}_1\|_{L^1 \rightarrow L^\infty} = o(|t|^{2-\frac{n}{2}})$ if $\beta > n$ and $\|\mathcal{E}_1\|_{L^1 \rightarrow L^\infty} = O(|t|^{1-\frac{n}{2}})$ if $\beta > n + 4$.

- (2) Suppose that $\int_{\mathbb{R}^n} V\psi \, dx = 0$ for each $\psi \in \text{Null } H$ but $\int_{\mathbb{R}^n} x_j V\psi \, dx \neq 0$ for some ψ and some $j \in [1, \dots, n]$. Then there exists a finite-rank time dependent operator G_t satisfying $\|G_t\|_{L^1 \rightarrow L^\infty} \lesssim |t|^{1-\frac{n}{2}}$ such that for $|t| > 1$,

$$e^{itH} \chi(H) P_{ac}(H) - G_t = \mathcal{E}_2(t).$$

Where, $\|\mathcal{E}_2\|_{L^1 \rightarrow L^\infty} = O(|t|^{1-\frac{n}{2}})$ and $\|\mathcal{E}_2\|_{L^{1,0+} \rightarrow L^{\infty,0-}} = o(|t|^{1-\frac{n}{2}})$ if $\beta > n + 4$ and $\|\mathcal{E}_2\|_{L^{1,1} \rightarrow L^{\infty,-1}} = O(|t|^{-\frac{n}{2}})$ if $\beta > n + 8$.

- (3) Suppose $\beta > n + 8$ and that $\int_{\mathbb{R}^n} V\psi \, dx = 0$ and $\int_{\mathbb{R}^n} x_j V\psi \, dx = 0$ for all $\psi \in \text{Null } H$ and all $j \in [1, \dots, n]$. Then

$$\|e^{itH} \chi(H) P_{ac}(H)\|_{L^1 \rightarrow L^\infty} \lesssim |t|^{-\frac{n}{2}}$$

We note that the assumption that $\int_{\mathbb{R}^n} V\psi \, dx = 0$ for each $\psi \in \text{Null } H$ is equivalent to assuming that the operator $P_e V 1 = 0$ with P_e the projection onto the zero-energy eigenspace. Further, $\int_{\mathbb{R}^n} x_j V\psi \, dx = 0$ for each $j = 1, 2, \dots, n$ is equivalent to assuming the operator $P_e V x = 0$.

These results are fashioned similarly to the asymptotic expansions in [17], with particular emphasis on the behavior of the resolvent of H at low energy. If one assumes greater decay of the potential, then it becomes possible to carry out the resolvent expansion to a greater number of terms, which permits a more detailed description of the time decay of $e^{itH} \chi(H) P_{ac}(H)$. We note that while F_t and G_t above have a concise construction, expressions for higher order terms in the expansion are unwieldy enough to discourage writing out an exact formula.

The extension to the main theorem is as follows.

Corollary 1.3. *If $|V(x)| \lesssim \langle x \rangle^{-n-8^-}$, and there is an eigenvalue of H at zero energy, then we have the operator-valued expansion*

$$e^{itH} \chi(H) P_{ac}(H) = c|t|^{2-\frac{n}{2}} P_e V 1 V P_e + |t|^{1-\frac{n}{2}} A_{-1} + |t|^{-\frac{n}{2}} A_0(t). \quad (3)$$

There exist uniform bounds for $P_e V 1 V P_e : L^1 \rightarrow L^\infty$, $A_{-1} : L^1 \rightarrow L^\infty$, and $A_0(t) : L^{1,2} \rightarrow L^{\infty,-2}$. The operator $P_e V 1 V P_e$ is a rank one operator and A_{-1} is finite rank. Furthermore, if $P_e V 1 = 0$, then $A_0(t) : L^{1,1} \rightarrow L^{\infty,-1}$. If $P_e V 1 = 0$ and $P_e V x = 0$ then A_{-1} vanishes and $A_0(t) : L^1 \rightarrow L^\infty$ uniformly in t .

We note that this expansion could continue indefinitely in powers of $|t|^{-\frac{n}{2}-k}$, $k \in \mathbb{N}$. The operators would be finite rank between successively more heavily weighted spaces and it would require more decay on the potential V . We do not pursue this issue.

High energy dispersive bounds in dimension $n \geq 4$ require more assumptions on the smoothness of the potential, which was shown in the counterexample constructed by the first author and Visan in [15]. In contrast the present work is concerned with the effect of zero energy eigenvalues, which is strictly a low energy issue. Accordingly our theorems stated above use the low-energy cut-off $\chi(H)$ so that no differentiability on the potential is required.

As in odd dimensions, we note that the estimates we prove can be combined with the large energy estimates in, for example, [31, 11] to prove analogous statements for the full evolution $e^{itH} P_{ac}(H)$ without the low-energy cut-off. The work cited above assumes that the polynomially weighted Fourier transform of V satisfies

$$\mathcal{F}((x)^{2\sigma} V) \in L^{n*}(\mathbb{R}^n) \quad \text{for } \sigma > \frac{1}{n_*} = \frac{n-2}{n-1}.$$

Roughly speaking, this corresponds to having more than $\frac{n-3}{2} + \frac{n-3}{n-2}$ derivatives of V in L^2 .

The statements of our main results are identical to those given in the companion paper, [13] for odd dimensions $n \geq 5$. The analysis for even dimensions in this paper proceeds along similar lines, but is technically more challenging. One reason for this is the appearance of the logarithms in the expansions and the inability to write a closed-form expression for the resolvents, see (7) below.

The limiting resolvent operators are defined as

$$R_V^\pm(\lambda^2) = \lim_{\epsilon \rightarrow 0^+} (-\Delta + V - (\lambda^2 \pm i\epsilon))^{-1}.$$

These operators are well-defined on certain weighted $L^2(\mathbb{R}^n)$ spaces, see [2]. In fact, there is a zero energy eigenvalue precisely when this operator becomes unbounded as $\lambda \rightarrow 0$. While the number of spatial dimensions does not appear explicitly in the expression above, the behavior of resolvents for small λ is strongly shaped by whether n is odd or even. When odd dimensional resolvents are expanded in powers of λ , one has the operator-valued expansion

$$R_V^+(\lambda^2) = \frac{A}{\lambda^2} + \frac{B}{\lambda} + O(1), \quad 0 < \lambda < \lambda_1 \ll 1.$$

In even dimensions one has expansions in terms of $\lambda^k (\log \lambda)^\ell$. For instance, in [7] it was shown that in \mathbb{R}^2 if there is a zero energy eigenvalue that one has the operator-valued expansion (for $0 < \lambda < \lambda_1$)

$$R_V^+(\lambda^2) = \frac{A}{\lambda^2} + \frac{B}{\lambda^2(a \log \lambda + z)} + O(\lambda^{-2} (\log \lambda)^{-2}), \quad a \in \mathbb{R} \setminus \{0\}, z \in \mathbb{C} \setminus \mathbb{R}.$$

If, in addition, one assumes that there are no zero-energy resonances (solutions to $H\psi = 0$ with $\psi \notin L^2(\mathbb{R}^2)$ but $\psi \in L^\infty(\mathbb{R}^2)$), one has the expansion

$$R_V^+(\lambda^2) = \frac{A}{\lambda^2} + (a \log \lambda + z)B + O((\log \lambda)^{-1}),$$

with different constants a, z and a different operator B . We give only results for R_V^+ since $R_V^-(\lambda^2) = \overline{R_V^+(\lambda^2)}$. In [5] it was shown that the resolvents in four-spatial dimensions have similar, though not identical, expansions as those written above for two dimensions. In these lower dimensions it is known that, whether zero is an eigenvalue or not, time decay of the Schrödinger evolution is faster if there is not a resonance at zero, see [23, 10, 30, 28, 7, 8, 5] for example.

As usual (cf. [26, 14, 27]), the dispersive estimates follow by considering the operator $e^{itH} \chi(H) P_{ac}(H)$ as an element of the functional calculus of H . Using the Stone formula, and the standard change of variables $\lambda \mapsto \lambda^2$, we have

$$e^{itH} \chi(H) P_{ac}(H) f(x) = \frac{1}{2\pi i} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) [R_V^+(\lambda^2) - R_V^-(\lambda^2)] f(x) d\lambda,$$

with the difference of resolvents $R_V^\pm(\lambda^2)$ providing the absolutely continuous spectral measure. For $\lambda > 0$ (and if also at $\lambda = 0$ if zero is a regular point of the spectrum) the resolvents are well-defined on certain weighted L^2 spaces. The key issue when zero energy is not regular is to control the singularities in the spectral measure as $\lambda \rightarrow 0$.

Here $R_V^\pm(\lambda^2)$ are operators whose integral kernel we write as $R_V^\pm(\lambda^2)(x, y)$. That is, the action of the operator is defined by

$$R_V^\pm(\lambda^2) f(x) = \int_{\mathbb{R}^n} R_V^\pm(\lambda^2)(x, y) f(y) dy.$$

The analysis in this paper focuses on bounding the oscillatory integral

$$\int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) [R_V^+(\lambda^2) - R_V^-(\lambda^2)](x, y) d\lambda \quad (4)$$

in terms of x, y and t . A uniform bound of the form $\sup_{x,y} |(4)| \lesssim |t|^{-\alpha}$ would give us an estimate on $e^{itH} P_{ac}(H)$ as an operator from $L^1 \rightarrow L^\infty$. We leave open the option of dependence on x and y to allow for estimates between weighted L^1 and weighted L^∞ spaces. That is, an estimate of the form $|(4)| \lesssim |t|^{-\alpha} \langle x \rangle^{\sigma'} \langle y \rangle^\sigma$ implies an estimate for $e^{itH} P_{ac}(H)$ as an operator from $L^{1,\sigma}$ to $L^{\infty,-\sigma'}$.

The paper is organized as follows. We begin in Section 2 by developing expansions for the free resolvent and develop necessary machinery to understand the spectral measure $E'(\lambda) = \frac{1}{2\pi i} [R_V^+(\lambda^2) - R_V^-(\lambda^2)]$. In Section 3, we prove dispersive estimates for the finite Born series series (44), which is the portion of the

low energy evolution that is unaffected by zero-energy eigenvalues. Each of these terms experiences time decay of order $|t|^{-\frac{n}{2}}$, consistent with the generic dispersive estimate (2). Next, in Section 4 we prove dispersive estimates for the tail of the Born series (45), which is the portion of the evolution that is sensitive to the existence of zero-energy eigenvalues and to the eigenspace orthogonality conditions specified in Theorem 1.2. Finally, in Section 5 we provide a characterization of the spectral subspaces of L^2 related to the zero energy eigenspace and provide technical integral estimates required to establish the dispersive bounds.

2. Resolvent Expansions

In this section we first develop expansions for the integral kernels of the free resolvents $R_0^\pm(\lambda^2) := (-\Delta - (\lambda^2 \pm i0))^{-1}$ to understand the perturbed resolvent operators $R_V^\pm(\lambda^2) := (-\Delta + V - (\lambda^2 \pm i0))^{-1}$ with the aim of understanding the spectral measure in (4).

In developing these expansions we employ the following notation used in [13] when considering odd spatial dimensions. We write

$$f(\lambda) = \tilde{O}(g(\lambda))$$

to indicate that

$$\frac{d^j}{d\lambda^j} f(\lambda) = O\left(\frac{d^j}{d\lambda^j} g(\lambda)\right).$$

If the relationship holds only for the first k derivatives, we use the notation $f(\lambda) = \tilde{O}_k(g(\lambda))$. With a slight abuse of notation, we may write $f(\lambda) = \tilde{O}(\lambda^k)$ for an integer k , to indicate that $\frac{d^j}{d\lambda^j} f(\lambda) = O(\lambda^{k-j})$. This distinction is particularly important for when $k \geq 0$ and $j > k$.

Writing the free resolvent in terms of the Hankel functions we have

$$R_0(z)(x, y) = \frac{i}{4} \left(\frac{z^{1/2}}{2\pi|x-y|} \right)^{\frac{n}{2}-1} H_{\frac{n}{2}-1}^{(1)}(z^{1/2}|x-y|). \quad (5)$$

Here $H_{\frac{n}{2}-1}^{(1)}(\cdot)$ is the Hankel function of the first kind. When n is even we have the Hankel function of integer order, which cannot be expressed in closed form. This stands in contrast to the odd dimensional free resolvents which possess a closed form expansion composed of finitely many terms, see for example [17]. That difference, along with the appearance of the logarithm in the expansion (7) often makes the even dimensional case more technically difficult.

We note that

$$H_{\frac{n}{2}-1}^{(1)}(z) = J_{\frac{n}{2}-1}(z) + iY_{\frac{n}{2}-1}(z),$$

where $J_{\frac{n}{2}-1}$ and $Y_{\frac{n}{2}-1}$ are the Bessel functions of integer order. We note the small $|z| \ll 1$ expansions for the Bessel functions (c.f. [1])

$$J_{\frac{n}{2}-1}(z) = \left(\frac{z}{2}\right)^{\frac{n}{2}-1} \sum_{k=0}^{\infty} \frac{(-\frac{z^2}{4})^k}{k! \Gamma(\frac{n}{2} + k)} \quad (6)$$

and

$$\begin{aligned} Y_{\frac{n}{2}-1}(z) &= \frac{-1}{\pi(2z)^{\frac{n}{2}-1}} \sum_{k=0}^{\frac{n}{2}-2} \frac{(\frac{n}{2} - k - 2)!}{k!} \left(\frac{z^2}{4}\right)^k + \frac{2}{\pi} \log(z/2) J_{\frac{n}{2}-1}(z) \\ &\quad - \frac{z^{\frac{n}{2}-1}}{\pi 2^{\frac{n}{2}-1}} \sum_{k=0}^{\infty} \left\{ \psi(k+1) + \psi\left(\frac{n}{2} + k + 2\right) \right\} \frac{(-\frac{1}{4}z^2)^k}{k! (\frac{n}{2} - 1 + k)!} \end{aligned} \quad (7)$$

In addition, one has the large $|z| \gtrsim 1$ expansion

$$J_{\frac{n}{2}-1}(z) = e^{iz} \omega_+(z) + e^{-iz} \omega_-(z), \quad \omega_{\pm}(z) = \tilde{O}(z^{-\frac{1}{2}}). \quad (8)$$

A similar expansion is valid for $Y_{\frac{n}{2}-1}(z)$ with different functions $\omega_{\pm}(z)$ that satisfy the same bounds. In fact, such an expansion is valid for any Bessel function of integer or half-integer order for $|z| \gtrsim 1$.

Recall that $R_0^-(\lambda^2) = R_0^+(\lambda^2)$. In particular, using the expansions of the Bessel functions (6) and (7) in (5) with $z = \lambda|x - y|$, we use the following explicit representation for the kernel of the limiting resolvent operators $R_0^{\pm}(\lambda^2)$ (see, e.g., [17]). In particular,

$$R_0^+(\lambda^2)(x, y) = \sum_{j=0}^{\infty} \sum_{k=0}^1 \lambda^{2j} (\log \lambda)^k G_j^k, \quad (9)$$

which is valid when $\lambda|x - y| \ll 1$ for operators G_j^k which are defined by

$$G_j^0 = \begin{cases} c_j |x - y|^{2+2j-n} & 0 \leq j \leq \frac{n}{2} - 2, \\ (a_j + ib_j) |x - y|^{2+2j-n} & j \geq \frac{n}{2} - 1, \\ +c_j |x - y|^{2+2j-n} \log |x - y|, \end{cases} \quad (10)$$

and

$$G_j^1 = \begin{cases} 0 & 0 \leq j \leq \frac{n}{2} - 2, \\ b_j |x - y|^{2+2j-n} & j \geq \frac{n}{2} - 1, \end{cases} \quad (11)$$

where $a_j, b_j, c_j \in \mathbb{R}$ and $b_j \neq 0$.

It is worth noting that $G_0^0 = (-\Delta)^{-1}$. To make the expansions more usable for the purposes of this paper, when $j \geq \frac{n}{2} - 1$, we break the operators into real and imaginary parts. We define

$$G_j^r = a_j |x - y|^{2+j-n} + c_j |x - y|^{2+j-n} \log |x - y|, \quad (12)$$

$$G_j^c = b_j |x - y|^{2+j-n}. \quad (13)$$

We choose to use this representation since it allows us to separate operators by the size of its λ dependence as $\lambda \rightarrow 0$ and explicitly identify the imaginary parts of the expansion.

In addition, the following functions of λ occur naturally in the expansion.

$$g_1^+(\lambda) = \lambda^{n-2}(a_1 \log \lambda + z_1),$$

$$g_2^+(\lambda) = \lambda^n(a_2 \log \lambda + z_2),$$

$$g_3^+(\lambda) = \lambda^{n+2}(a_3 \log \lambda + z_3)$$

with $a_j \in \mathbb{R} \setminus \{0\}$ and $z_j \in \mathbb{C} \setminus \mathbb{R}$. In addition, we have that

$$g_j^-(\lambda) = \overline{g_j^+(\lambda)},$$

and

$$g_j^+(\lambda) - g_j^-(\lambda) = 2\Im(z_j)\lambda^{n-4+2j}, \quad j = 1, 2, 3. \quad (14)$$

It is worth noting that from the expansions of the Bessel functions, (7), we have

$$g_1^\pm(\lambda)G_{n-2}^c + \lambda^{n-2}G_{n-2}^r = \lambda^{n-2}(A_1^\pm + A_2 \log(\lambda|x - y|)), \quad (15)$$

$$g_2^\pm(\lambda)G_n^c + \lambda^n G_n^r = \lambda^{n-2}(\lambda|x - y|)^2(B_1^\pm + B_2 \log(\lambda|x - y|)) \quad (16)$$

$$g_3^\pm(\lambda)G_{n+2}^c + \lambda^{n+2}G_{n+2}^r = \lambda^{n-2}(\lambda|x - y|)^4(C_1^\pm + C_2 \log(\lambda|x - y|)), \quad (17)$$

for some constants $A_1^\pm, A_2, B_1^\pm, B_2, C_1^\pm, C_2$. This follows from (5) and the expansions (6), (7). In particular, we note that the logarithmic factors occur from the $\log(z/2)J_{\frac{n}{2}-1}(z)$ terms, which naturally factor to this form.

Define the function $\log^-(z) := -\chi_{\{0 < z < \frac{1}{2}\}} \log(z)$. Here we note that

$$|(1 + \log(\lambda|x - y|))\chi(\lambda|x - y|)\chi(\lambda)| \lesssim 1 + |\log \lambda| + \log^-(|x - y|). \quad (18)$$

This can be seen by considering the cases of $|x - y| < 1$ and $|x - y| > 1$ separately.

Lemma 2.1. *For $\lambda \leq \lambda_1$, we have the expansion(s) for the free resolvent,*

$$R_0^\pm(\lambda^2)(x, y) = G_0^0 + \lambda^2 G_1^0 + \cdots + \lambda^{n-4} G_{\frac{n}{2}-2}^0 + E_0^\pm(\lambda),$$

where

$$E_0^\pm(\lambda) = (1 + \log^-(|x - y|)) \tilde{O}_{\frac{n}{2}-1}(\lambda^{n-2}(1 + \log \lambda)).$$

Further, for $0 < \ell < 2$,

$$E_0^\pm(\lambda) = g_1^\pm(\lambda) G_{n-2}^c + \lambda^{n-2} G_{n-2}^r + E_1^\pm(\lambda),$$

with

$$E_1^\pm(\lambda) = |x - y|^\ell \tilde{O}_{\frac{n}{2}-1}(\lambda^{n-2+\ell}).$$

Further,

$$E_1^\pm(\lambda) = g_2^\pm(\lambda) G_n^c + \lambda^n G_n^r + E_2^\pm(\lambda),$$

with

$$E_2^\pm(\lambda) = |x - y|^{2+\ell} \tilde{O}_{\frac{n}{2}+1}(\lambda^{n+\ell}),$$

Finally,

$$E_2^\pm(\lambda) = g_3^\pm(\lambda) G_{n+2}^c + \lambda^{n+2} G_{n+2}^r + E_3^\pm(\lambda),$$

with

$$E_3^\pm(\lambda) = |x - y|^{4+\ell} \tilde{O}_{\frac{n}{2}+3}(\lambda^{n+2+\ell}).$$

Proof. Using the expansion (9) when $\lambda|x - y| \ll 1$, one has

$$\begin{aligned} R_0^\pm(\lambda^2) &= G_0^0 + \sum_{j=1}^{\frac{n-4}{2}} \lambda^{2j} G_j^0 + g_1^\pm(\lambda) G_{n-2}^c + \lambda^{n-2} G_{n-2}^r + g_2^\pm(\lambda) G_n^c \\ &\quad + \lambda^n G_n^r + g_3^\pm(\lambda) G_{n+2}^c + \lambda^{n+2} G_{n+2}^r \\ &\quad + \tilde{O}(\lambda^{n-2}(\lambda|x - y|)^6 \log(\lambda|x - y|)) \end{aligned} \quad (19)$$

This can, of course, be truncated earlier. For $E_0^\pm(\lambda)$ we note that for $\lambda|x - y| \ll 1$,

$$E_0^\pm(\lambda^2) = -g_1^\pm(\lambda) G_{n-2}^c - \lambda^{n-2} G_{n-2}^r + \tilde{O}(\lambda^{n-2}(\lambda|x - y|)^2 \log(\lambda|x - y|))$$

For the first two terms, using (15) and (18), we note that

$$\begin{aligned} \lambda^{n-2} G_{n-2}^r + g_1^\pm(\lambda) G_{n-2}^c &= \lambda^{n-2} (A_1^\pm + A_2 \log(\lambda|x - y|)) \\ &= (1 + \log^- |x - y|) \tilde{O}_{\frac{n}{2}-1}(\lambda^{n-2}(1 + \log \lambda)). \end{aligned}$$

The remaining error bounds for $\lambda|x-y| \ll 1$ are clear from (19), noting that

$$\tilde{O}(\lambda^{n-2}(\lambda|x-y|)^2 \log(\lambda|x-y|)) = \tilde{O}(\lambda^{n-2}(\lambda|x-y|)^\ell)$$

for any $0 \leq \ell < 2$.

On the other hand, if $\lambda|x-y| \gtrsim 1$ then the asymptotic expansion of the Hankel functions in (5), see (8) or [1], yield

$$R_0^\pm(\lambda^2) = e^{\pm i\lambda|x-y|} \frac{\lambda^{\frac{n-2}{2}}}{|x-y|^{\frac{n-2}{2}}} \omega_\pm(\lambda|x-y|) \quad (20)$$

where $\omega_\pm(z) = \tilde{O}(z^{-\frac{1}{2}})$. Here, differentiation in λ is comparable to either division by λ or multiplication by $|x-y|$. So that for $0 \leq k \leq \frac{n}{2} - 1$,

$$\begin{aligned} |\partial_\lambda^k R_0^\pm(\lambda^2)(x, y)| &\lesssim \frac{\lambda^{\frac{n-3}{2}}}{|x-y|^{\frac{n-1}{2}}} (\lambda^{-k} + |x-y|^k) \\ &\lesssim \lambda^{\frac{n-3}{2}} |x-y|^{k+\frac{1-n}{2}} \lesssim \lambda^{n-2-k}. \end{aligned} \quad (21)$$

Where we used $|x-y|^{-1} \lesssim \lambda$. If $k \geq \frac{n}{2}$, we note that multiplication by $|x-y|$ dominates division by λ in (21), and we have

$$|\partial_\lambda^k R_0^\pm(\lambda^2)(x, y)| \lesssim \lambda^{\frac{n-3}{2}} |x-y|^{k+\frac{1-n}{2}} \lesssim \lambda^{\frac{n-3}{2}} |x-y|^{\frac{1}{2}+k}. \quad (22)$$

The bound for $E_0^\pm(\lambda)$ follows from the bounds here and the fact that

$$E_0^\pm(\lambda) = R_0^\pm(\lambda^2) - G_0^0 - \lambda^2 G_1^0 - \dots - \lambda^{n-4} G_{\frac{n}{2}-2}^0.$$

For these terms, we note that for $\lambda|x-y| \gtrsim 1$ and $j \leq \frac{n}{2} - 2$ we have

$$|\partial_\lambda^k \lambda^{2j} G_j^0| \lesssim \begin{cases} \lambda^{2j-k} |x-y|^{2-n-2j} & k < 2j, \\ 0 & k \geq 2j \end{cases} \lesssim \lambda^{n-2-k}. \quad (23)$$

For the other error terms, we note that

$$E_1^\pm(\lambda) = E_0^\pm(\lambda) + g_1^\pm(\lambda) G_{n-2}^c + \lambda^{n-2} G_{n-2}^r,$$

$$E_2^\pm(\lambda) = E_1^\pm(\lambda) + g_2^\pm(\lambda) G_n^c + \lambda^n G_n^r,$$

$$E_3^\pm(\lambda) = E_2^\pm(\lambda) + g_3^\pm(\lambda) G_{n+2}^c + \lambda^{n+2} G_{n+2}^r,$$

For these terms, using (15), we note that when $\lambda|x-y| \gtrsim 1$,

$$\lambda^{n-2} G_{n-2}^r + g_1^\pm(\lambda) G_{n-2}^c = \lambda^{n-2} (A_1^\pm + A_2 \log(\lambda|x-y|)) = |x-y|^{0+} \tilde{O}(\lambda^{n-2+}).$$

Similarly, using (16),

$$\lambda^n G_n^r + g_2^\pm(\lambda) G_n^c = \lambda^n |x - y|^2 (B_1^\pm + B_2 \log(\lambda |x - y|)) = |x - y|^{2+} \tilde{O}(\lambda^{n+}),$$

and using (17)

$$\begin{aligned} \lambda^{n+2} G_{n+2}^r + g_3^\pm(\lambda) G_{n+2}^c &= \lambda^{n+2} |x - y|^4 (C_1^\pm + C_2 \log(\lambda |x - y|)) \\ &= |x - y|^{4+} \tilde{O}(\lambda^{n+2+}). \end{aligned}$$

Finally, we note that for $\lambda |x - y| \gtrsim 1$, it is acceptable to multiply upper bounds by powers of $\lambda |x - y|$. For $E_j^\pm(\lambda)$, $j = 1, 2$, we note that for $\alpha \geq 0$ we have,

$$|\partial_\lambda^k R_0^\pm(\lambda^2)(x, y)| \lesssim (\lambda |x - y|)^\alpha \begin{cases} \lambda^{n-2-k} & k \leq \frac{n}{2} - 1, \\ \lambda^{\frac{n-3}{2}} |x - y|^{\frac{1}{2}+k} & k \geq \frac{n}{2}. \end{cases} \quad (24)$$

The bounds then follow from selecting different values of α . \square

Corollary 2.2. *We have the expansion*

$$\begin{aligned} R_0^\pm(\lambda^2)(x, y) &= G_0^0 + \lambda^2 G_1^0 + \cdots + \lambda^{n-4} G_{\frac{n}{2}-2}^0 + g_1^\pm(\lambda) G_{n-2}^c + \lambda^{n-2} G_{n-2}^r \\ &\quad + |x - y|^{\frac{1}{2}+\alpha} \tilde{O}_\frac{n}{2}(\lambda^{n-\frac{3}{2}+\alpha}) \end{aligned}$$

for $0 \leq \alpha < \frac{3}{2}$.

The hypotheses of the lemma below are not optimal, but suffice for our purposes.

Lemma 2.3. *If $|V(x)| \lesssim \langle x \rangle^{-\frac{n+1}{2}-}$, $\sigma > \frac{1}{2}$ and $\kappa \geq \frac{n-3}{4}$, then*

$$\|(R_0^\pm(\lambda)^2 V)^{\kappa-1}(y, \cdot) R_0(\cdot, x)\|_{L^2_{y, -\sigma}} \lesssim \langle \lambda \rangle^\kappa \frac{n-3}{2}.$$

uniformly in x .

Proof. We note the bound

$$|R_0^\pm(\lambda^2)(x, y)| \lesssim \frac{1}{|x - y|^{n-2}} + \frac{\lambda^{\frac{n-3}{2}}}{|x - y|^{\frac{n-1}{2}}},$$

which follows from the asymptotic expansion (20) when $\lambda |x - y| \gtrsim 1$ and the fact that $|R_0^\pm| \lesssim |G_0^0| \lesssim |x - y|^{2-n}$ for $\lambda |x - y| \ll 1$. The proof follows as in Lemma 2.2 in the odd dimensional case, [13], by repeated use of Lemma 5.10. \square

We use the symmetric resolvent identity, which is valid for $\Im(\lambda) > 0$,

$$R_V^\pm(\lambda^2) = R_0^\pm(\lambda^2) - R_0^\pm(\lambda^2)vM^\pm(\lambda)^{-1}vR_0^\pm(\lambda^2), \quad (25)$$

with U the sign of V , $v = |V|^{1/2}$, and $w = Uv$. We need to invert

$$M^\pm(\lambda) = U + vR_0^\pm(\lambda^2)v$$

as an operator on $L^2(\mathbb{R}^n)$.

Lemma 2.3 allows us to make sense of the symmetric resolvent identity, provided $|V(x)| \lesssim \langle x \rangle^{-\frac{n-1}{2}}$, by iterating the standard resolvent identity

$$R_V^\pm(\lambda^2) = R_0^\pm(\lambda^2) - R_0^\pm(\lambda^2)VR_V^\pm(\lambda^2) = R_0^\pm(\lambda^2) - R_V^\pm(\lambda^2)VR_0^\pm(\lambda^2)$$

at least $\frac{n-3}{4}$ times on both sides of $M^\pm(\lambda)^{-1}$ in (25) to get to a polynomially weighted L^2 space, which multiplication by v then maps into L^2 .

In contrast to the odd dimensional case, [13], the expansions for the free resolvent in Lemma 2.1 are useful for understanding the operators $M^\pm(\lambda)^{-1}$, but more care is required for the dispersive estimates. The logarithmic nature of the resolvent causes certain technical difficulties, see Sections 3 and 4.

Our main tool used to invert $M^\pm(\lambda) = U + vR_0^\pm(\lambda^2)v$ for small λ is the following lemma (see Lemma 2.1 in [20]).

Lemma 2.4. *Let A be a closed operator on a Hilbert space \mathcal{H} and S a projection. Suppose $A + S$ has a bounded inverse. Then A has a bounded inverse if and only if*

$$B := S - S(A + S)^{-1}S$$

has a bounded inverse in $S\mathcal{H}$, and in this case

$$A^{-1} = (A + S)^{-1} + (A + S)^{-1}SB^{-1}S(A + S)^{-1}.$$

We use the following terminology.

Definition 2.5. We say an operator $K : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ with kernel $K(\cdot, \cdot)$ is absolutely bounded if the operator with kernel $|K(\cdot, \cdot)|$ is bounded from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.

We recall the definition of the Hilbert–Schmidt norm of an operator K with integral kernel $K(x, y)$,

$$\|K\|_{HS} = \left(\iint_{\mathbb{R}^{2n}} |K(x, y)|^2 dx dy \right)^{\frac{1}{2}}.$$

We note that Hilbert–Schmidt and finite rank operators are immediately absolutely bounded.

Lemma 2.6. *Assuming that $v(x) \lesssim \langle x \rangle^{-\beta}$. If $\beta > \frac{n}{2} + \ell$ for any $0 < \ell < 2$, then we have*

$$\begin{aligned} M^\pm(\lambda) &= U + vG_0^0v + \sum_{j=1}^{\frac{n-4}{2}} \lambda^{2j} vG_j^0v + g_1^\pm(\lambda)vG_{n-2}^c v \\ &\quad + \lambda^{n-2}vG_{n-2}^r v + M_0^\pm(\lambda), \end{aligned} \quad (26)$$

Where the operators G_j^0 , G_j^r and G_j^c are absolutely bounded with real-valued kernels. Further,

$$\sum_{j=0}^{\frac{n}{2}-1} \left\| \sup_{0 < \lambda < \lambda_1} \lambda^{j+2-n-\ell} \partial_\lambda^j M_0^\pm(\lambda) \right\|_{HS} \lesssim 1. \quad (27)$$

If $\beta > \frac{n}{2} + 2 + \ell$, for $0 < \ell < 2$, then

$$M_0^\pm(\lambda) = g_2^\pm(\lambda)vG_n^c v + \lambda^n vG_n^r v + M_1^\pm(\lambda), \quad (28)$$

with

$$\sum_{j=0}^{\frac{n}{2}} \left\| \sup_{0 < \lambda < \lambda_1} \lambda^{j-n-\ell} \partial_\lambda^j M_1^\pm(\lambda) \right\|_{HS} \lesssim 1. \quad (29)$$

If $\beta > \frac{n}{2} + 4 + \ell$, then for $0 < \ell < 2$

$$M_1^\pm(\lambda) = g_3^\pm(\lambda)vG_{n+2}^c v + \lambda^{n+2}vG_{n+2}^r v + M_2^\pm(\lambda) \quad (30)$$

with

$$\sum_{j=0}^{\frac{n}{2}} \left\| \sup_{0 < \lambda < \lambda_1} \lambda^{j-2-n-\ell} \partial_\lambda^j M_2^\pm(\lambda) \right\|_{HS} \lesssim 1. \quad (31)$$

Proof. The proof follows from the definition of the operators $M^\pm(\lambda)$ and the expansion for the free resolvent in Lemma 2.1. The bound on the error terms follows from the fact that if $k > -\frac{n}{2}$ then $\langle x \rangle^{-\beta} |x - y|^k (1 + \log |x - y|) \langle y \rangle^{-\beta}$ is bounded in Hilbert–Schmidt norm. To see this we note that the kernel is bounded by the sum $\langle x \rangle^{-\beta} |x - y|^{k+} \langle y \rangle^{-\beta} + \langle x \rangle^{-\beta} |x - y|^{k-} \langle y \rangle^{-\beta}$ which are Hilbert–Schmidt provided $\beta > \frac{n}{2} + k$. \square

Remark 2.7. The error estimates here can be more compactly summarized as

$$M_0^\pm(\lambda) = \tilde{O}_{\frac{n}{2}-1}(\lambda^{n-2+\ell}), \quad M_1^\pm(\lambda) = \tilde{O}_{\frac{n}{2}}(\lambda^{n+\ell}), \quad M_2^\pm(\lambda) = \tilde{O}_{\frac{n}{2}}(\lambda^{n+2+\ell})$$

as absolutely bounded operators on $L^2(\mathbb{R}^n)$, for $0 < \lambda < \lambda_1$.

We note that $U + vG_0^0v$ is not invertible if there is an eigenvalue at zero, see Lemma 5.1. Define S_1 to be the Riesz projection onto the kernel of $U + vG_0^0v$ as an operator on $L^2(\mathbb{R}^n)$. Then the operator $U + vG_0^0v + S_1$ is invertible on L^2 , and we may define

$$D_0 := (U + vG_0^0v + S_1)^{-1}. \quad (32)$$

We note that $U + vG_0^0v$ is a compact perturbation of the invertible operator U , hence S_1 is finite rank by the Fredholm alternative. This operator can be seen to be absolutely bounded exactly as in the odd dimensional case, see Lemma 2.7 in [13].

Lemma 2.8. *If $v(x) \lesssim \langle x \rangle^{-\frac{n+1}{2}-}$, then the operator D_0 is absolutely bounded in $L^2(\mathbb{R}^n)$.*

We will apply Lemma 2.4 with $A = M^\pm(\lambda)$ and $S = S_1$, the Riesz projection onto the kernel of $U + vG_0^0v$. Thus, we need to show that $M^\pm(\lambda) + S_1$ has a bounded inverse in $L^2(\mathbb{R}^n)$ and

$$B_\pm(\lambda) = S_1 - S_1(M^\pm(\lambda) + S_1)^{-1}S_1 \quad (33)$$

has a bounded inverse in $S_1L^2(\mathbb{R}^n)$.

Lemma 2.9. *Suppose that zero is not a regular point of the spectrum of $H = -\Delta + V$, and let S_1 be the corresponding Riesz projection on the zero energy eigenspace. The for sufficiently small $\lambda_1 > 0$, the operators $M^\pm(\lambda) + S_1$ are invertible for all $0 < \lambda < \lambda_1$ as bounded operators on $L^2(\mathbb{R}^n)$. Further, for any $0 < \ell < 2$, if $\beta > \frac{n}{2} + \ell$ then we have the following expansions:*

$$\begin{aligned} (M^\pm(\lambda) + S_1)^{-1} &= D_0 + \sum_{j=1}^{\frac{n-4}{2}} \lambda^{2j} C_{2j} - g_1^\pm(\lambda) D_0 v G_{n-2}^c v D_0 \\ &\quad + \lambda^{n-2} C_{n-2} + \tilde{M}_0^\pm(\lambda) \end{aligned}$$

where $\tilde{M}_0^\pm(\lambda)$ satisfies the same bounds as $M_0^\pm(\lambda)$ and the operators C_k are absolutely bounded on L^2 with real-valued kernels. Further, if $\beta > \frac{n}{2} + 2 + \ell$ then

$$\tilde{M}_0^\pm(\lambda) = -g_2^\pm(\lambda) D_0 v G_n^c v D_0 + \lambda^2 g_1^\pm(\lambda) C_n^1 + \lambda^n C_n + \tilde{M}_1^\pm(\lambda),$$

where $C_n^1 = D_0 v G_{n-2}^c v D_0 v G_1^0 v D_0 + D_0 v G_1^0 v D_0 v G_{n-2}^c v D_0$, and $\tilde{M}_1^\pm(\lambda)$ satisfies the same bounds as $M_1^\pm(\lambda)$. Finally, if $\beta > \frac{n}{2} + 4 + \ell$ then

$$\begin{aligned} \tilde{M}_1^\pm(\lambda) &= -g_3^\pm(\lambda) D_0 v G_{n+2}^c v D_0 + \lambda^2 g_2^\pm(\lambda) C_{n+2}^1 + \lambda^4 g_1^\pm(\lambda) C_{n+2}^2 \\ &\quad + \lambda^{n+2} C_{n+2} + \tilde{M}_2^\pm(\lambda), \end{aligned}$$

with $C_{n+2}, C_{n+2}^1, C_{n+2}^2$ absolutely bounded operators with real-valued kernels and $\tilde{M}_2^\pm(\lambda)$ satisfies the same bounds as $M_2^\pm(\lambda)$.

Proof. We use a Neumann series expansion. We show the case of M^+ and omit the superscript, the ‘-’ case follows similarly. Using (26) we have

$$\begin{aligned}
& [M(\lambda) + S_1]^{-1} \\
&= \left[U + vG_0v + S_1 + \sum_{j=1}^{\frac{n-4}{2}} \lambda^{2j} vG_j^0v + g_1(\lambda)vG_{n-2}^c v \right. \\
&\quad \left. + \lambda^{n-2}vG_{n-2}^r v + M_0(\lambda) \right]^{-1} \\
&= D_0 \left[\mathbb{1} + \sum_{j=1}^{\frac{n-4}{2}} \lambda^{2j} vG_j^0v D_0 + g_1(\lambda)vG_{n-2}^c v D_0 \right. \\
&\quad \left. + \lambda^{n-2}vG_{n-2}^r v D_0 + M_0(\lambda)D_0 \right]^{-1} \\
&= D_0 - \lambda^2 D_0 vG_1^0v D_0 + \sum_{j=2}^{\frac{n-4}{2}} \lambda^{2j} C_{2j} - g_1(\lambda)vG_{n-2}^c v D_0 - \lambda^{n-2}vG_{n-2}^r v D_0 \\
&\quad - D_0 M_0(\lambda) D_0 \\
&\quad + \lambda^2 [D_0 vG_1^0v D_0 [g_1(\lambda)vG_{n-2}^c v + \lambda^{n-2}vG_{n-2}^r v + M_0(\lambda)] D_0 \\
&\quad + D_0 [g_1(\lambda)vG_{n-2}^c v + \lambda^{n-2}vG_{n-2}^r v + M_0(\lambda)] D_0 vG_1^0v D_0] + \tilde{M}_2(\lambda).
\end{aligned}$$

One can find explicitly the operators C_k in terms of D_0 and the operators G_k^0 , but this is not worth the effort. The operator $C_2 = D_0 vG_1^0v D_0$ is important due to its relationship with the projection onto the zero energy eigenspace, see Lemma 5.3.

What is important in our analysis in Section 4 are the imaginary parts, that is the terms that arise with the functions $g_1(\lambda)$, $g_2(\lambda)$ or $g_3(\lambda)$. The first of these occurs from

$$D_0 [g_1(\lambda)vG_{n-2}^c v + \lambda^{n-2}vG_{n-2}^r v + M_0(\lambda)] D_0.$$

This provides an most singular term of size $\lambda^{n-2} \log \lambda$ as $\lambda \rightarrow 0$. The next $\lambda^n \log \lambda$ term arises from the contribution of the $D_0 vM_0(\lambda)v D_0$ term or the ‘ x^2 ’ term in the Neumann series, that is the term with both G_1^0 and G_{n-2}^c . The error bounds follow from the bounds in Lemma 2.6 and the Neumann series expansion above.

For the longer expansions, one needs to use more terms in the Neumann series and take more care with ‘ x^2 ’ and ‘ x^3 ’ terms that arise. \square

Remark 2.10. We note here that is zero is regular the above Lemma suffices to establish the dispersive estimates using the techniques in Sections 3 and 4. In this case, $S_1 = 0$, $D_0 = (U + vG_0v)^{-1}$ is still absolutely bounded and we have the expansion

$$M^\pm(\lambda)^{-1} = D_0 + \sum_{j=1}^{\frac{n-4}{2}} \lambda^{2j} C_{2j} - g_1^\pm(\lambda) D_0 v G_{n-2}^c v D_0 + \lambda^{n-2} C_{n-2} + \tilde{M}_0^\pm(\lambda),$$

with C_{2j} real-valued, absolutely bounded operators.

Now we turn to the operators $B_\pm(\lambda)$ for use in Lemma 2.4. Recall that

$$B_\pm(\lambda) = S_1 - S_1(M^\pm(\lambda) + S_1)^{-1}S_1,$$

and that $S_1 D_0 = D_0 S_1 = S_1$. Thus

$$\begin{aligned} B_\pm(\lambda) &= S_1 - S_1 \left[D_0 + \sum_{j=1}^{\frac{n-4}{2}} \lambda^{2j} C_{2j} - g_1^\pm(\lambda) D_0 v G_{n-2}^c v D_0 \right. \\ &\quad \left. + \lambda^{n-2} C_{n-2} + \tilde{M}_0^\pm(\lambda) \right] S_1 \\ &= - \sum_{j=1}^{\frac{n-4}{2}} \lambda^{2j} S_1 C_{2j} S_1 + g_1^\pm(\lambda) S_1 v G_{n-2}^c v S_1 \\ &\quad - \lambda^{n-2} S_1 C_{n-2} S_1 - S_1 \tilde{M}_0^\pm(\lambda) S_1 \\ &= -\lambda^2 S_1 v G_1^0 v S_1 - \sum_{j=2}^{\frac{n-4}{2}} \lambda^{2j} S_1 C_{2j} S_1 + g_1^\pm(\lambda) S_1 v G_{n-2}^c v S_1 \\ &\quad - \lambda^{n-2} S_1 C_{n-2} S_1 - S_1 \tilde{M}_0^\pm(\lambda) S_1. \end{aligned} \tag{34}$$

So that the invertibility of $B_\pm(\lambda)$ hinges upon the invertibility of the operator $S_1 v G_1^0 v S_1$, which is established in Lemma 5.2 below. Accordingly, we define $D_1 := (S_1 v G_1^0 v S_1)^{-1}$ as an operator on $S_1 L^2$. Noting that $D_1 = S_1 D_1 S_1$, it is clear that D_1 is absolutely bounded.

Lemma 2.11. *We have the following expansions, if $\beta > \frac{n}{2} + \ell$ for $0 < \ell < 2$ then*

$$\begin{aligned} B_\pm(\lambda)^{-1} &= -\frac{D_1}{\lambda^2} + \sum_{j=2}^{\frac{n-4}{2}} \lambda^{2j-4} B_{2j} + \frac{g_1^\pm(\lambda)}{\lambda^4} D_1 v G_{n-2}^c v D_1 \\ &\quad + \lambda^{n-6} B_{n-2} + \tilde{B}_0^\pm(\lambda), \end{aligned}$$

where $\tilde{B}_0^\pm(\lambda)$ satisfies the same bounds as $\lambda^{-4}M_0^\pm(\lambda)$ and the operators B_k are absolutely bounded on L^2 with real-valued kernels. Further, if $\beta > \frac{n}{2} + 2 + \ell$ then

$$\tilde{B}_0^\pm(\lambda) = \frac{g_2^\pm(\lambda)}{\lambda^4} D_1 v G_n^c v D_1 + \frac{g_1^\pm(\lambda)}{\lambda^2} B_n^1 + \lambda^{n-4} B_n + \tilde{B}_1^\pm(\lambda),$$

where

$$\begin{aligned} B_n^1 &= D_1 v G_{n-2}^c v D_0 v G_1^0 v D_1 + D_1 v G_1^0 v D_0 v G_{n-2}^c v D_1 \\ &\quad + D_1 C_4 D_0 v G_{n-2}^c v D_1 + D_1 v G_{n-2}^c v D_0 C_4 D_1, \end{aligned}$$

and $\tilde{B}_1^\pm(\lambda)$ satisfies the same bounds as $\lambda^{-4}M_1^\pm(\lambda)$. Finally, if $\beta > \frac{n}{2} + 4 + \ell$

$$\tilde{B}_1^\pm(\lambda) = \frac{g_3^\pm(\lambda)}{\lambda^4} B_{n+2}^1 + \frac{g_2^\pm(\lambda)}{\lambda^2} B_{n+2}^2 + g_1^\pm(\lambda) B_{n+2}^3 + \lambda^{n-2} B_{n+2}^4 + \tilde{B}_2^\pm(\lambda),$$

with B_{n+2}^j absolutely bounded operators with real-valued kernels, and $\tilde{B}_2^\pm(\lambda)$ satisfies the same bounds as $\lambda^{-4}M_2^\pm(\lambda)$.

Proof. As usual we consider the ‘+’ case and omit subscripts, the ‘-’ case follows similarly. We begin by noting that

$$\begin{aligned} B(\lambda)^{-1} &= \left[-\lambda^2 S_1 v G_1^0 v S_1 - \sum_{j=2}^{\frac{n-4}{2}} \lambda^{2j} S_1 C_{2j} S_1 - g_1^\pm(\lambda) S_1 v G_{n-2}^c v S_1 \right. \\ &\quad \left. + \lambda^{n-2} S_1 C_{n-2} S_1 - S_1 \tilde{M}_0^\pm(\lambda) S_1 \right]^{-1} \\ &= -\frac{D_1}{\lambda^2} \left[\mathbb{1} + \sum_{j=2}^{\frac{n-4}{2}} \lambda^{2j-2} S_1 C_{2j} S_1 D_1 - g_1^\pm(\lambda) S_1 v G_{n-2}^c v S_1 D_1 \right. \\ &\quad \left. + \lambda^{n-2} S_1 C_{n-2} S_1 D_1 - \lambda^{-2} S_1 \tilde{M}_0^\pm(\lambda) S_1 D_1 \right]^{-1}, \end{aligned}$$

where $D_1 := (S_1 v G_1^0 v S_1)^{-1}$ is an absolutely bounded operator on $S_1 L^2(\mathbb{R}^n)$ by Lemma 5.2 below.

We again only concern ourselves with explicitly finding the operators for the first few occurrences of the functions $g_1(\lambda)$, $g_2(\lambda)$ and $g_3(\lambda)$. The terms that arise with only powers of the spectral parameter λ come with only real-valued, absolutely bounded operators which are easier to control. This again follows by a careful analysis of the various terms that arise in the Neumann series expansion. \square

Remark 2.12. The error estimates here can be more compactly summarized as

$$\tilde{B}_0^\pm(\lambda) = \tilde{O}_{\frac{n}{2}-1}(\lambda^{n-6+\ell}), \quad \tilde{B}_1^\pm(\lambda) = \tilde{O}_{\frac{n}{2}}(\lambda^{n-4+\ell}), \quad \tilde{B}_2^\pm(\lambda) = \tilde{O}_{\frac{n}{2}}(\lambda^{n-2+\ell})$$

as absolutely bounded operators on $L^2(\mathbb{R}^n)$, for $0 < \lambda < \lambda_1$. The leading λ^2 term in $B_\pm(\lambda)$, (34), causes an effective loss of four powers of λ in the expansion for $B_\pm(\lambda)^{-1}$ and hence later for $M^\pm(\lambda)^{-1}$ and the perturbed resolvents $R_V^\pm(\lambda^2)$. Heuristically speaking, this corresponds to being able to integrate by parts only $\frac{n}{2} - 2$ times in (4) before the integral is too singular as $\lambda \rightarrow 0$, which is why a generic eigenfunction at zero causes a two power loss of time decay. This loss in the spectral parameter in the expansions, necessitates going out to size λ^{n+2+} in the expansions for $R_0^\pm(\lambda^2)$ to obtain the desired $|t|^{-\frac{n}{2}}$ time decay in Section 4.

To prove parts (2) and (3) of Theorem 1.2, we need the following corollary.

Corollary 2.13. *Under the hypotheses of Lemma 2.11, if $P_e V 1 = 0$ then,*

$$\begin{aligned} B_\pm(\lambda)^{-1} &= -\frac{D_1}{\lambda^2} + \sum_{j=2}^{\frac{n-4}{2}} \lambda^{2j-4} B_{2j} + \lambda^{n-6} B_{n-2} + \frac{g_2^\pm(\lambda)}{\lambda^4} D_1 v G_n^c v D_1 \\ &\quad + \lambda^{n-4} B_n + \tilde{B}_1^\pm(\lambda). \end{aligned}$$

If, in addition, $P_e V x = 0$ then

$$\begin{aligned} B_\pm(\lambda)^{-1} &= -\frac{D_1}{\lambda^2} + \sum_{j=1}^{\frac{n-4}{2}} \lambda^{2j-4} B_{2j} + \lambda^{n-6} B_{n-2} + \lambda^{n-4} B_n \\ &\quad + \frac{g_3^\pm(\lambda)}{\lambda^4} B_{n+2}^1 + \lambda^{n-2} B_{n+2}^4 + \tilde{B}_2^\pm(\lambda). \end{aligned}$$

Proof. We note that $D_1 = S_1 D_1 S_1$, along with the identities

$$S_1 = -w G_0^0 v S_1 = -S_1 v G_0^0 w. \quad (35)$$

So that, using $P_e = G_0^0 v D_1 v G_0^0$ by (66),

$$D_1 = S_1 D_1 S_1 = w G_0^0 v D_1 v G_0^0 w = w P_e w. \quad (36)$$

As a consequence, we have

$$D_1 v G_{n-2}^c = c_{n-2} w P_e V 1. \quad (37)$$

The first claim follows clearly from Lemma 2.11 since the coefficient of λ^{n-6} is a scalar multiple of the operator $P_e V1$. Further,

$$\begin{aligned} c_n^{-1} D_1 v G_n^c v D_1 &= w P_e V [x^2 - 2x \cdot y + y^2] V P_e w \\ &= w P_e V x^2 V P_e w - 2w P_e V x \cdot y V P_e w + w P_e V y^2 V P_e w. \end{aligned}$$

We see that when $P_e V1 = 0$ and $P_e Vx = 0$, the operator $D_1 v G_n^c v D_1 = 0$. We also note that it is now clear that when $P_e V1, P_e Vx = 0$, one has

$$\begin{aligned} B_n^1 &= D_1 v G_{n-2}^c v D_0 v G_1^0 v D_1 + D_1 v G_1^0 v D_0 v G_{n-2}^c v D_1 \\ &\quad + D_1 C_4 D_0 v G_{n-2}^c v D_1 + D_1 v G_{n-2}^c v D_0 C_4 D_1 = 0 \end{aligned}$$

as well. \square

Effectively, all terms that have the function $g_1^\pm(\lambda)$ become zero if $P_e V1 = 0$ and all terms with the function $g_2^\pm(\lambda)$ become zero if $P_e Vx = 0$ as well.

We are now ready to give a full expansion for the operators $M^\pm(\lambda)^{-1}$. We state several versions of the expansions for $M^\pm(\lambda)^{-1}$. These different expansions allow us to account for cancellation properties of the eigenfunctions and have finer control on the time decay rate of the error terms of the evolution given in Theorem 1.2 at the cost of more decay on the potential.

Lemma 2.14. *Assume $|V(x)| \lesssim \langle x \rangle^{-\beta}$ for some $\beta > n + 8$, then*

$$\begin{aligned} M^\pm(\lambda)^{-1} &= -\frac{D_1}{\lambda^2} + \sum_{j=0}^{\frac{n-8}{2}} \lambda^{2j} M_{2j} + \frac{g_1^\pm(\lambda)}{\lambda^4} M_{n-6}^L + \lambda^{n-6} M_{n-6} \\ &\quad + \frac{g_1^\pm(\lambda)}{\lambda^2} M_{n-4}^{L1} + \frac{g_2^\pm(\lambda)}{\lambda^4} M_{n-4}^{L2} + \lambda^{n-4} M_{n-4} \\ &\quad + g_1^\pm(\lambda) M_{n-2}^{L1} + \frac{g_2^\pm(\lambda)}{\lambda^2} M_{n-2}^{L2} + \frac{g_3^\pm(\lambda)}{\lambda^4} M_{n-2}^{L3} \\ &\quad + \lambda^{n-2} M_{n-2} + \tilde{O}_{\frac{n}{2}}(\lambda^{n-2+}) \end{aligned} \tag{38}$$

for sufficiently small λ , with all operators M_k and M_k^{Lj} real-valued and absolutely bounded.

Proof. This follows from the expansions in Lemmas 2.9 and 2.11, and the inversion lemma, Lemma 2.4. \square

Later on it will be important to explicitly identify the form of the operator M_{n-6}^L . We use Lemma 2.9 to see that

$$(M^\pm(\lambda) + S_1)^{-1} = D_0 + O(\lambda^2).$$

Pairing this with the $g_1^\pm(\lambda)$ term in Lemma 2.11, the smallest λ contribution that is not strictly real-valued is

$$\frac{g_1^\pm(\lambda)}{\lambda^4} D_0 D_1 v G_{n-2}^c v D_1 D_0.$$

Since $D_1 D_0 = D_0 D_1 = D_1$, we have

$$M_{n-6}^L = D_1 v G_{n-2}^c v D_1 = w P_e V 1 V P_e w. \quad (39)$$

The expansion (38) can be truncated to require less decay on the potential by using less of the expansions in Lemmas 2.1 and 2.11. Specifically, stopping with the error terms $\tilde{M}_0^\pm(\lambda)$ and $\tilde{B}_0^\pm(\lambda)$ respectively with $\ell = 0+$.

Corollary 2.15. *Assume $|V(x)| \lesssim \langle x \rangle^{-n-}$, then*

$$\begin{aligned} M^\pm(\lambda)^{-1} &= -\frac{D_1}{\lambda^2} + \sum_{j=0}^{\frac{n-8}{2}} \lambda^{2j} M_{2j} + \frac{g_1^\pm(\lambda)}{\lambda^4} M_{n-6}^L + \lambda^{n-6} M_{n-6} \\ &\quad + \tilde{O}_{\frac{n}{2}-1}(\lambda^{n-6+}). \end{aligned} \quad (40)$$

If $|V(x)| \lesssim \langle x \rangle^{-n-4-}$, then

$$\begin{aligned} M^\pm(\lambda)^{-1} &= -\frac{D_1}{\lambda^2} + \sum_{j=0}^{\frac{n-8}{2}} \lambda^{2j} M_{2j} + \frac{g_1^\pm(\lambda)}{\lambda^4} M_{n-6}^L + \lambda^{n-6} M_{n-6} \\ &\quad + \frac{g_1^\pm(\lambda)}{\lambda^2} M_{n-4}^{L1} + \frac{g_2^\pm(\lambda)}{\lambda^4} M_{n-4}^{L2} \\ &\quad + \lambda^{n-4} M_{n-4} + \tilde{O}_{\frac{n}{2}}(\lambda^{n-4+}). \end{aligned} \quad (41)$$

with the operators M_{2j} and M_{2j}^{Lk} all real-valued and absolutely bounded.

The lemma can also be modified to better account for cancellation properties of the projection onto the zero-energy eigenspace.

Corollary 2.16. *Under the hypotheses of Lemma 2.14, if $P_e V 1 = 0$ and $|V(x)| \lesssim \langle x \rangle^{-n-4-}$, then*

$$\begin{aligned} M^\pm(\lambda)^{-1} &= -\frac{D_1}{\lambda^2} + \sum_{j=0}^{\frac{n-8}{2}} \lambda^{2j} M_{2j} + \lambda^{n-6} M_{n-6} + \frac{g_2^\pm(\lambda)}{\lambda^4} M_{n-4}^{L2} \\ &\quad + \lambda^{n-4} M_{n-4} + \tilde{O}_{\frac{n}{2}}(\lambda^{n-4+}). \end{aligned} \quad (42)$$

If $|V(x)| \lesssim \langle x \rangle^{-n-8-}$, then

$$\begin{aligned} M^\pm(\lambda)^{-1} &= -\frac{D_1}{\lambda^2} + \sum_{j=0}^{\frac{n-8}{2}} \lambda^{2j} M_{2j} + \lambda^{n-6} M_{n-6} + \frac{g_2^\pm(\lambda)}{\lambda^4} M_{n-4}^{L2} \\ &\quad + \lambda^{n-4} M_{n-4} + \frac{g_2^\pm(\lambda)}{\lambda^2} M_{n-2}^{L2} + \frac{g_3^\pm(\lambda)}{\lambda^4} M_{n-2}^{L3} + \lambda^{n-2} M_{n-2} \\ &\quad + \tilde{O}_{\frac{n}{2}}(\lambda^{n-2+}). \end{aligned} \quad (43)$$

If in addition, $P_e V x = 0$, and $|V(x)| \lesssim \langle x \rangle^{-n-8-}$, then

$$\begin{aligned} M^\pm(\lambda)^{-1} &= -\frac{D_1}{\lambda^2} + \sum_{j=0}^{\frac{n-8}{2}} \lambda^{2j} M_{2j} + \lambda^{n-6} M_{n-6} + \lambda^{n-4} M_{n-4} \\ &\quad + \frac{g_3^\pm(\lambda)}{\lambda^4} M_{n-2}^{L3} + \lambda^{n-2} M_{n-2} + \tilde{O}_{\frac{n}{2}}(\lambda^{n-2+}). \end{aligned}$$

Proof. The proof follows as in the proof of Lemma 2.14 using Corollary 2.13 in place of Lemma 2.11. \square

3. The finite Born series terms

In this section we estimate the contribution of the finite Born series, (44) showing that it can be bounded by $|t|^{-\frac{n}{2}}$ uniformly in x and y . These terms in the expansion of the spectral measure contain only the free resolvent $R_0^\pm(\lambda^2)$ and therefore are not sensitive to the existence of zero energy eigenvalues or their cancellation properties. In even dimensions the lack of a closed form representation for $R_0^\pm(\lambda^2)$ causes much more technical difficulties in these calculations as compared to the corresponding section in [13]. Many of the techniques we develop here to overcome these difficulties are vital in controlling the more singular terms considered in Section 4.

Iterating the standard resolvent identity

$$R_V^\pm(\lambda^2) = R_0^\pm(\lambda^2) - R_0^\pm(\lambda^2) V R_V^\pm(\lambda^2) = R_0^\pm(\lambda^2) - R_V^\pm(\lambda^2) V R_0^\pm(\lambda^2),$$

we form the identity

$$R_V^\pm(\lambda^2) = \sum_{k=0}^{2m+1} (-1)^k R_0^\pm(\lambda^2) [V R_0^\pm(\lambda^2)]^k \quad (44)$$

$$+ [R_0^\pm(\lambda^2) V]^m R_0^\pm(\lambda^2) v M^\pm(\lambda)^{-1} v R_0^\pm(\lambda^2) [V R_0^\pm(\lambda^2)]^m. \quad (45)$$

In light of Lemma 2.3 the identity holds for $m + 1 \geq \frac{n-3}{4}$ and $|V(x)| \lesssim \langle x \rangle^{-\frac{n+1}{2}-}$ as an identity from $L^{2, \frac{1}{2}+} \rightarrow L^{2, -\frac{1}{2}-}$, as in the limiting absorption principle.

Proposition 3.1. *The contribution of (44) to (4) is bounded by $|t|^{-\frac{n}{2}}$ uniformly in x and y . That is,*

$$\sup_{x, y \in \mathbb{R}^n} \left| \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \left[\sum_{k=0}^{2m+1} (-1)^k \{R_0^+(VR_0^+)^k - R_0^-(VR_0^-)^k\} \right] (\lambda^2)(x, y) d\lambda \right| \lesssim |t|^{-\frac{n}{2}}.$$

We prove this claim with series of Lemmas. The following corollary to Lemma 2.1 is useful.

Lemma 3.2. *We have the expansion*

$$(R_0^\pm(\lambda^2)V)^k R_0^\pm(\lambda^2)(x, y) = K_0 + \lambda^2 K_2 + \cdots + \lambda^{n-4} K_{n-4} + \tilde{E}_0^\pm(\lambda)(x, y),$$

here the operators K_j have real-valued kernels. Furthermore, the error term $\tilde{E}_0^\pm(\lambda)$ satisfies

$$\tilde{E}_0^\pm(\lambda)(x, y) = (1 + \log^- |x - \cdot| + \log^- |\cdot - y|) \tilde{O}_{\frac{n}{2}-1}(\lambda^{n-2-}).$$

Furthermore, if one wishes to have $\frac{n}{2}$ derivatives, the extended expansion

$$\tilde{E}_0^\pm(\lambda)(x, y) = g_1^\pm(\lambda) K_{n-2}^c + \lambda^{n-2} K_{n-2}^r + \tilde{E}_1^\pm(\lambda)(x, y),$$

satisfies the bound

$$\tilde{E}_1^\pm(\lambda)(x, y) = \langle x \rangle^{\frac{1}{2}} \langle y \rangle^{\frac{1}{2}} \tilde{O}_{\frac{n}{2}}(\lambda^{n-\frac{3}{2}}).$$

Proof. This follows from the expansions for $R_0^\pm(\lambda^2)$ in Lemma 2.1 for $\tilde{E}_0^\pm(\lambda)(x, y)$ or Corollary 2.2 for $\tilde{E}_1^\pm(\lambda)(x, y)$.

For the iterated resolvents, the desired bounds come from simply multiplying out the terms. It is easy to see that

$$K_0 = (G_0^0 V)^k G_0^0$$

and

$$K_2 = \sum_{j=0}^k (G_0^0 V)^j G_1^0 (V G_0^0)^{k-j}$$

one can obtain similar expressions for the other operators, but they are not needed. \square

Remark 3.3. The spatially weighted bound $|\partial_\lambda^{\frac{n}{2}} \tilde{E}_1^\pm(\lambda)(x, y)| \lesssim \langle x \rangle^{\frac{1}{2}} \lambda^{\frac{n-3}{2}}$ is only needed if all $\frac{n}{2}$ derivatives act on the leading resolvent, $R_0^\pm(\lambda^2)(x, z_1)$, in the product. Similarly, the upper bound $\langle y \rangle^{\frac{1}{2}} \lambda^{\frac{n-3}{2}}$ is only needed if all derivatives act on the lagging resolvent, $R_0^\pm(\lambda^2)(z_k, y)$, in the product. All other expressions that arise would be consistent with $\tilde{E}_1^\pm(\lambda)$ belonging to the class $\tilde{O}_{\frac{n}{2}}(\lambda^{n-2-})$.

The desired time decay follows from taking the difference and noting that

$$\begin{aligned} & [(R_0^+(\lambda^2)V)^k R_0^+(\lambda^2) - (R_0^-(\lambda^2)V)^k R_0^-(\lambda^2)](x, y) \\ &= [g_1^+(\lambda) - g_1^-(\lambda)]K_{n-2}^c + \tilde{E}_1^+(\lambda)(x, y) - \tilde{E}_1^-(\lambda)(x, y) \\ &= c_1 \lambda^{n-2} K_{n-2}^c + \langle x \rangle^{\frac{1}{2}} \langle y \rangle^{\frac{1}{2}} \tilde{O}_{\frac{n}{2}}(\lambda^{n-\frac{3}{2}}). \end{aligned}$$

The first term contributes $|t|^{-\frac{n}{2}}$ by Lemma 5.6 as an operator from $L^1 \rightarrow L^\infty$, whereas the second term can be bounded by $|t|^{-\frac{n}{2}}$ (from Corollary 5.9), but maps $L^{1, \frac{1}{2}} \rightarrow L^{\infty, -\frac{1}{2}}$. This method fails to obtain an unweighted $L^1 \rightarrow L^\infty$ only when all the λ derivatives act on either a leading or lagging free resolvent. In the following Lemmas, we show how the unweighted bound can be achieved.

The following variation of stationary phase from [27] will be useful in the analysis.

Lemma 3.4. *Let $\phi'(\lambda_0) = 0$ and $1 \leq \phi'' \leq C$. Then,*

$$\begin{aligned} \left| \int_{-\infty}^{\infty} e^{it\phi(\lambda)} a(\lambda) d\lambda \right| &\lesssim \int_{|\lambda-\lambda_0| < |t|^{-\frac{1}{2}}} |a(\lambda)| d\lambda \\ &+ |t|^{-1} \int_{|\lambda-\lambda_0| > |t|^{-\frac{1}{2}}} \left(\frac{|a(\lambda)|}{|\lambda-\lambda_0|^2} + \frac{|a'(\lambda)|}{|\lambda-\lambda_0|} \right) d\lambda. \end{aligned}$$

Rather than use the expansions of Lemma 2.1, we need to utilize finer cancellation properties of the free resolvents than can be captured in these expansions.

We note that by (5) and the definition of the Hankel functions, we have

$$[R_0^+ - R_0^-](\lambda^2)(x, y) = \frac{i}{2} \left(\frac{\lambda}{2\pi|x-y|} \right)^{\frac{n}{2}-1} J_{\frac{n}{2}-1}(\lambda|x-y|). \quad (46)$$

Noting (6), for $\lambda|x-y| \ll 1$, we have

$$\begin{aligned} & [R_0^+ - R_0^-](\lambda^2)(x, y) \\ &= \frac{i}{2} \left(\frac{\lambda}{2\pi|x-y|} \right)^{\frac{n}{2}-1} \left(\frac{\lambda|x-y|}{2} \right)^{\frac{n}{2}-1} \sum_{k=0}^{\infty} c_k (\lambda|x-y|)^{2k} \\ &= \lambda^{n-2} G_{n-2}^c + \tilde{O}(\lambda^{n-2} (\lambda|x-y|)^\epsilon), \quad 0 \leq \epsilon < 2. \end{aligned} \quad (47)$$

In particular, we note that there are no logarithms in this expansion. On the other hand, if $\lambda|x - y| \gtrsim 1$, using (8), we have

$$\begin{aligned} & [R_0^+ - R_0^-](\lambda^2)(x, y) \\ &= \frac{\lambda^{\frac{n}{2}-1}}{|x - y|^{\frac{n}{2}-1}} (e^{i\lambda|x-y|}\omega_+(\lambda|x - y|) + e^{-i\lambda|x-y|}\omega_-(\lambda|x - y|)). \end{aligned} \quad (48)$$

Lemma 3.5. *We have the expansion*

$$[R_0^+ - R_0^-](\lambda^2)(x, y) = \tilde{O}_{\frac{n}{2}-1}(\lambda^{n-2})$$

Proof. This follows from (47) with $\epsilon = 0$, (48) and (21) in the proof of Lemma 2.1. \square

To best utilize certain cancellations between the difference of the iterated resolvents, we note the following algebraic fact,

$$\prod_{k=0}^M A_k^+ - \prod_{k=0}^M A_k^- = \sum_{\ell=0}^M \left(\prod_{k=0}^{\ell-1} A_k^- \right) (A_\ell^+ - A_\ell^-) \left(\prod_{k=\ell+1}^M A_k^+ \right). \quad (49)$$

When applied to the summand in Proposition 3.1 it yields operators of the form $(R_0^- V)^j (R_0^+ - R_0^-) (V R_0^+)^{\ell}$, with $j + \ell = k$. We separate them further into cases where the difference $R_0^+ - R_0^-$ occurs on the leading resolvent of the product (i.e. $j = 0$), the lagging resolvent ($\ell = 0$), or a generic position in the interior.

The first case of the difference occurring on a leading or lagging resolvent is the most delicate. If the difference acts on an inner resolvent, we obtain an extra λ^{n-2} smallness from Lemma 3.5. This extra smallness, along with using some recurrence relationships for the free resolvents in Lemma 3.7 allow us to avoid using expansions for the leading and lagging resolvents to more easily obtain the time decay. This is done in detail in Lemma 3.8 and follows quickly from the arguments in the more delicate case considered in Lemma 3.6.

With respect to avoiding spatial weights Remark 3.3 explains that we need only consider when the first $\frac{n}{2} - 1$ derivatives when integrating by parts act on a leading (respectively lagging) resolvent. Instead of integrating by parts the final time, we use a modification of stationary phase from Lemma 3.4 to attain the time decay and avoid the spatial weights.

Lemma 3.6. *If $|V(x)| \lesssim \langle x \rangle^{-\frac{n+2}{2}-}$, we have the bound*

$$\sup_{x, y \in \mathbb{R}^n} \left| \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \{ [R_0^+ - R_0^-](\lambda^2) (V R_0^+)^k(\lambda^2) \}(x, y) d\lambda \right| \lesssim |t|^{-\frac{n}{2}}.$$

Proof. By Lemma 3.2, Remark 3.3 and the discussion following it, we need only consider the contribution when, upon integrating by parts, all of the derivatives act on the leading or lagging free resolvent. In the proof we consider when all derivatives act on the leading difference of free resolvents, which we regard as the most delicate case. As the remaining operator $(VR_0^+)^k \chi(\lambda)$ is left undisturbed, it suffices to note that it has a bounded kernel, uniformly in λ . The case where all derivatives act on the lagging free resolvent is somewhat delicate as well; this term fits best in the framework of Lemma 3.8 below.

For all other placement of derivatives, we note that if any derivatives act on ‘inner resolvents’ or the cut-off, an error bound with polynomial weights suffices as growth in these variables is controlled by the decay of the surrounding potentials. Meanwhile, at most $\frac{n}{2} - 1$ derivatives would act on a leading or lagging resolvent so that they too can be bounded without weights.

Unlike in the odd dimensional case, one must consider the small and large $\lambda|x - z_1|$ regimes separately. Using (47), the small $\lambda|x - z_1|$ regime requires bounding

$$\left| \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) [\lambda^{n-2} + \chi(\lambda|x - z_1|)|x - z_1|^\epsilon \tilde{O}(\lambda^{n-2+\epsilon})] d\lambda \right| \lesssim |t|^{-\frac{n}{2}}. \quad (50)$$

The contribution of the first term follows from Lemma 5.6. The second term is bounded by using a slight modification of Lemma 5.7. In particular, we can safely integrate by parts $\frac{n}{2} - 1$ times without boundary terms to get

$$|t|^{1-\frac{n}{2}} \int_0^\infty e^{it\lambda^2} \chi(\lambda) \chi(\lambda|x - z_1|)|x - z_1|^\epsilon \tilde{O}(\lambda^{1+\epsilon}) d\lambda.$$

The integral can be broken up into two pieces, on $0 < \lambda < |t|^{-\frac{1}{2}}$ we take $\epsilon = 0$ and integrate to gain the extra power of $|t|^{-1}$. On $|t|^{-\frac{1}{2}} < \lambda$, we wish to gain another $|t|^{-1}$. First, if no derivatives act on the cut-off $\chi(\lambda|x - z_1|)$ we see that

$$\begin{aligned} & \int_{|t|^{-\frac{1}{2}}}^\infty e^{it\lambda^2} \chi(\lambda) \chi(\lambda|x - z_1|)|x - z_1|^\epsilon \tilde{O}(\lambda^{1+\epsilon}) d\lambda \\ & \lesssim \frac{|x - z_1|^\epsilon \lambda^\epsilon}{|t|} \Big|_{\lambda=|t|^{-\frac{1}{2}}} \\ & \quad + \frac{1}{|t|} \int_{|t|^{-\frac{1}{2}}}^\infty e^{it\lambda^2} \chi(\lambda) \chi(\lambda|x - z_1|)|x - z_1|^\epsilon \tilde{O}(\lambda^{\epsilon-1}) d\lambda. \end{aligned}$$

Integrating by parts again on the second term and taking $\epsilon > 0$ small enough (say $\epsilon = \frac{1}{2}$), we can bound with

$$\begin{aligned}
& \frac{|x - z_1|^\epsilon \lambda^\epsilon}{|t|} \Big|_{\lambda=|t|^{-\frac{1}{2}}} + \frac{|x - z_1|^\epsilon \lambda^{\epsilon-2}}{|t|^2} \Big|_{\lambda=|t|^{-\frac{1}{2}}} \\
& + \frac{1}{|t|^2} \int_{|t|^{-\frac{1}{2}}}^{\infty} \chi(\lambda) \chi(\lambda|x - z_1|) |x - z_1|^\epsilon \tilde{O}(\lambda^{\epsilon-3}) d\lambda \\
& \lesssim \frac{|x - z_1|^\epsilon \lambda^\epsilon}{|t|} \Big|_{\lambda=|t|^{-\frac{1}{2}}} + \frac{|x - z_1|^\epsilon \lambda^{\epsilon-2}}{|t|^2} \Big|_{\lambda=|t|^{-\frac{1}{2}}} + \frac{1}{|t|^2} \\
& \lesssim |x - z_1|^\epsilon |t|^{-1-\frac{\epsilon}{2}} + |t|^{-2} \\
& \lesssim |t|^{-1}.
\end{aligned} \tag{51}$$

The last inequality follows from $1 \gtrsim \lambda|x - z_1| > |t|^{-\frac{1}{2}}|x - z_1|$, which implies $|x - z_1| \lesssim |t|^{\frac{1}{2}}$. We also used that $\chi'(\lambda)$ is supported on $\lambda \approx 1$, so the bound $|\chi'(\lambda)| \lesssim \lambda^{-1}$ is true.

If, when integrating by parts, the derivative acts on the cut-off $\chi(\lambda|x - z_1|)$ we can bound by

$$\begin{aligned}
& \frac{|x - z_1|^\epsilon \lambda^\epsilon}{t} \Big|_{\lambda=|t|^{-\frac{1}{2}}} + \frac{1}{t} \int_{|t|^{-\frac{1}{2}}}^{\infty} |x - z_1|^{1+\epsilon} \chi'(\lambda|x - z_1|) \lambda^\epsilon d\lambda \\
& \lesssim t^{-1} + \frac{|x - z_1|^{1+\epsilon}}{t} \int_{\lambda \sim |x - z_1|^{-1}} \lambda^\epsilon d\lambda \lesssim t^{-1}.
\end{aligned}$$

Here the boundary term is bounded by $|t|^{-1}$ as before, and the support of $\chi'(\lambda|x - z_1|)$ implies that $\lambda \approx |x - z_1|^{-1}$. A similar argument covers the case when the derivative acts on $\chi(\lambda|x - z_1|)$ in the second integration by parts in (51).

For the $\lambda|x - z_1| \gtrsim 1$ regime, we still consider only the most delicate term arises when all the derivatives act on the leading difference of free resolvents. Without loss of generality, we take $t > 0$. We note that the most difficult term from the contribution of (48) occurs with the negative phase. Here, one has to bound

$$\int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) e^{-i\lambda|x - z_1|} \frac{\lambda^{\frac{n}{2}-1}}{|x - z_1|^{\frac{n}{2}-1}} \omega_-(\lambda|x - z_1|) d\lambda.$$

We note that the λ smallness and the support of the cut-off $\chi(\lambda)$ allow us to integrate by parts $\frac{n}{2} - 1$ times without boundary terms, noting the second to last bound in (21) with $k = \frac{n}{2} - 1$, we need to control

$$\frac{1}{|t|^{\frac{n}{2}-1}} \int_0^\infty e^{it\lambda^2 - i\lambda|x - z_1|} \chi(\lambda) a(\lambda) d\lambda \tag{52}$$

where by (8),

$$|a(\lambda)| \lesssim \frac{\lambda^{\frac{1}{2}}}{|x - z_1|^{\frac{1}{2}}} \quad \text{and} \quad |a'(\lambda)| \lesssim \frac{1}{\lambda^{\frac{1}{2}}|x - z_1|^{\frac{1}{2}}}. \quad (53)$$

The stationary point of the phase occurs at $\lambda_0 = \frac{|x - z_1|}{2t}$. By Lemma 3.4, we need to bound three integrals,

$$\begin{aligned} & \int_{|\lambda - \lambda_0| < t^{-\frac{1}{2}}} |a(\lambda)| d\lambda + |t|^{-1} \int_{|\lambda - \lambda_0| > t^{-\frac{1}{2}}} \left(\frac{|a(\lambda)|}{|\lambda - \lambda_0|^2} + \frac{|a'(\lambda)|}{|\lambda - \lambda_0|} \right) d\lambda \\ & := A + |t|^{-1}(B + C). \end{aligned} \quad (54)$$

We begin by showing that $A \lesssim |t|^{-1}$. There are two cases to consider. First, if $\lambda_0 \gtrsim t^{-\frac{1}{2}}$, we have $\lambda \lesssim \lambda_0$, so that

$$\begin{aligned} A & \lesssim \int_{|\lambda - \lambda_0| < t^{-\frac{1}{2}}} \frac{\lambda_0^{\frac{1}{2}}}{|x - z_1|^{\frac{1}{2}}} d\lambda \\ & \lesssim t^{-\frac{1}{2}} \lambda_0^{\frac{1}{2}} |x - z_1|^{-\frac{1}{2}} \\ & \lesssim t^{-1}. \end{aligned}$$

Here we used that $\lambda_0 = |x - z_1|/2t$ in the last inequality.

In the second case one has $\lambda_0 \lesssim t^{-\frac{1}{2}}$, then $\lambda \lesssim t^{-\frac{1}{2}}$, so that

$$\begin{aligned} A & \lesssim \int_0^{t^{-\frac{1}{2}}} \frac{\lambda^{\frac{1}{2}}}{|x - z_1|^{\frac{1}{2}}} \tilde{\chi}(\lambda|x - z_1|) d\lambda \\ & \lesssim t^{-\frac{3}{4}} |x - z_1|^{-\frac{1}{2}}. \end{aligned}$$

Here $\tilde{\chi} = 1 - \chi$ is a cut-off away from zero which we employ to emphasize the support condition that $\lambda|x - z_1| \gtrsim 1$. For this integral to have a non-zero contribution, one must have $|x - z_1|^{-1} \lesssim \lambda \lesssim t^{-\frac{1}{2}}$, which then yields $A \lesssim t^{-1}$ as desired.

We now move to bounding B , the first integral supported on $|\lambda - \lambda_0| \gtrsim t^{-\frac{1}{2}}$. By Lemma 3.4, we need only show that $B \lesssim 1$. Again we consider two cases. First, if $\lambda_0 \ll t^{-\frac{1}{2}}$, one sees that $|\lambda - \lambda_0| \approx \lambda$. So that

$$\begin{aligned} B & \lesssim \int_{\mathbb{R}} \frac{\tilde{\chi}(\lambda|x - z_1|)}{|x - z_1|^{\frac{1}{2}} \lambda^{\frac{3}{2}}} d\lambda \\ & \lesssim |x - z_1|^{-\frac{1}{2}} \int_{|x - z_1|^{-1}}^{\infty} \lambda^{-\frac{3}{2}} d\lambda \\ & \lesssim 1. \end{aligned}$$

In the second case one has $\lambda_0 \gtrsim t^{-\frac{1}{2}}$. In this case, we let $s = \lambda - \lambda_0$

$$\begin{aligned}
 B &\lesssim \int_{|s|>t^{-\frac{1}{2}}} \frac{(s + \lambda_0)^{\frac{1}{2}}}{|x - z_1|^{\frac{1}{2}} |s|^2} ds \\
 &\lesssim \frac{1}{|x - z_1|^{\frac{1}{2}}} \left(\int_{|s|>t^{-\frac{1}{2}}} s^{-\frac{3}{2}} + \lambda_0^{\frac{1}{2}} s^{-2} ds \right) \\
 &\lesssim \frac{t^{\frac{1}{4}}}{|x - z_1|^{\frac{1}{2}}} + \frac{t^{\frac{1}{2}} \lambda_0^{\frac{1}{2}}}{|x - z_1|^{\frac{1}{2}}} \\
 &\lesssim 1.
 \end{aligned}$$

The last inequality follows since $t^{-\frac{1}{2}} \lesssim \lambda_0 = |x - z_1|/2t$ implies that $t^{\frac{1}{2}} \lesssim |x - z_1|$.

We now turn to the final term C , we need only show $C \lesssim 1$. The first case is again when $\lambda_0 \ll t^{-\frac{1}{2}}$, in which case $|\lambda - \lambda_0| \approx \lambda$, and

$$C \lesssim \int_{\mathbb{R}} \frac{\tilde{\chi}(\lambda|x - z_1|)}{\lambda^{\frac{3}{2}} |x - z_1|^{\frac{1}{2}}} d\lambda \lesssim 1.$$

In the second case $\lambda_0 \gtrsim t^{-\frac{1}{2}}$, which yields that $|x - z_1| \gtrsim t^{\frac{1}{2}}$. In this case,

$$\begin{aligned}
 C &\lesssim \int_{|\lambda - \lambda_0|>t^{-\frac{1}{2}}} \frac{\tilde{\chi}(\lambda|x - z_1|)}{|x - z_1|^{\frac{1}{2}} \lambda^{\frac{1}{2}} |\lambda - \lambda_0|} d\lambda \\
 &\lesssim |x - z_1|^{-\frac{1}{2}} \left(\int_{|\lambda - \lambda_0|>t^{-\frac{1}{2}}} \frac{d\lambda}{|\lambda - \lambda_0|^{\frac{3}{2}}} + \int_{\mathbb{R}} \frac{\tilde{\chi}(\lambda|x - z_1|)}{\lambda^{\frac{3}{2}}} d\lambda \right) \\
 &\lesssim t^{\frac{1}{4}} |x - z_1|^{-\frac{1}{2}} + 1 \\
 &\lesssim 1.
 \end{aligned}$$

We note that if the ‘+’ phase is encountered instead of the ‘-’, in place of (52), after again integrating by parts $\frac{n}{2} - 1$ times, one needs to bound

$$\frac{1}{|t|^{\frac{n}{2}-1}} \int_0^\infty e^{it\lambda^2 + i\lambda|x-z_1|} \chi(\lambda) a(\lambda) d\lambda, \quad (55)$$

in which case, one can simply use that

$$\frac{d}{d\lambda} (e^{it\lambda^2 + i\lambda|x-z_1|}) = (2it\lambda + i|x-z_1|) e^{it\lambda^2 + i\lambda|x-z_1|}$$

and integrate by parts. The bound on $a(\lambda)$ shows that the boundary terms are zero, so that

$$\begin{aligned} |(55)| &\lesssim |t|^{1-\frac{n}{2}} \int_0^\infty \left| \frac{\lambda^{-\frac{1}{2}} |x-z_1|^{-\frac{1}{2}} \tilde{\chi}(\lambda|x-z_1|)}{2t\lambda + |x-z_1|} \right| d\lambda \\ &\lesssim |t|^{-\frac{n}{2}} \int_{\mathbb{R}} \frac{\tilde{\chi}(\lambda|x-z_1|)}{\lambda^{\frac{3}{2}} |x-z_1|^{\frac{1}{2}}} d\lambda \\ &\lesssim |t|^{-\frac{n}{2}}. \end{aligned}$$

The assumed decay rate on the potential is chosen so that all spatial integrals are absolutely convergent. The analysis here is essentially the same as in the odd dimensional case. We note that

$$|\partial_\lambda^j R_0^\pm(\lambda^2)(x, y)| \lesssim |x-y|^{j+2-n} + \lambda^{\frac{n-3}{2}} |x-y|^{j+\frac{1-n}{2}}, \quad (56)$$

as developed in the proof of Lemma 2.1. The second term decays more slowly for large x, y , so it dictates the decay requirements for the potential. In the iterated resolvent, differentiated $\frac{n}{2}$ times, we need to control integrals of the form

$$\int_{\mathbb{R}^{kn}} \frac{1}{|x-z_1|^{\frac{n-1}{2}-\alpha_0}} \prod_{j=1}^k \frac{V(z_j)}{|z_j-z_{j+1}|^{\frac{n-1}{2}-\alpha_j}} d\vec{z},$$

where $\alpha_j \in \mathbb{N}_0$ and $\sum \alpha_j = \frac{n}{2}$, $z_{k+1} = y$ and $d\vec{z} = dz_1 dz_2 \cdots dz_k$. (There is a caveat that if $\alpha_0 = \frac{n}{2}$ then the last derivative is applied as in the stationary phase argument (52) and does not yield a factor of $|x-z_1|^{\frac{1}{2}}$ in the numerator. Similarly if $\alpha_k = \frac{n}{2}$, the value of $\frac{n-1}{2} - \alpha_k$ should be treated as zero rather than $-\frac{1}{2}$.) Using arithmetic-geometric mean inequalities, any integral we need to control is dominated by the sum

$$\begin{aligned} \int_{\mathbb{R}^{kn}} \frac{1}{|x-z_1|^{\frac{n-1}{2}}} \prod_{j=1}^k \frac{V(z_j)}{|z_j-z_{j+1}|^{\frac{n-1}{2}}} &\left(|x-z_1|^{\frac{n-1}{2}} \right. \\ &\left. + \sum_{\ell=2}^{k-1} |z_\ell - z_{\ell+1}|^{\frac{n}{2}} + |z_k - y|^{\frac{n-1}{2}} \right) d\vec{z}. \end{aligned}$$

Choose a representative element from the summation over ℓ . This negates a factor of $|z_\ell - z_{\ell+1}|^{(1-n)/2}$ in the product and replaces it with $|z_\ell - z_{\ell+1}|^{\frac{1}{2}} \lesssim \langle z_\ell \rangle^{\frac{1}{2}} \langle z_{\ell+1} \rangle^{\frac{1}{2}}$. With $|V(z_j)| \lesssim \langle z_j \rangle^{-\beta}$, we have to control an integral of the form

$$\int_{\mathbb{R}^{kn}} \frac{1}{|x-z_1|^{\frac{n-1}{2}}} \left(\prod_{j=1}^{\ell-1} \frac{\langle z_j \rangle^{-\beta}}{|z_j-z_{j+1}|^{\frac{n-1}{2}}} \langle z_\ell \rangle^{\frac{1}{2}} \right) \left(\langle z_{\ell+1} \rangle^{\frac{1}{2}} \prod_{j=\ell}^k \frac{\langle z_j \rangle^{-\beta}}{|z_j-z_{j+1}|^{\frac{n-1}{2}}} \right) d\vec{z}$$

with $y = z_{k+1}$. Assuming that $\beta > \frac{n+2}{2}$, this is bounded uniformly in x, y by iterating the single integral estimate

$$\sup_{z_{j-1} \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\langle z_j \rangle^{\frac{1}{2}-\beta}}{|z_{j-1} - z_j|^{\frac{n-1}{2}}} dz_j \lesssim 1, \quad (57)$$

starting with $j = \ell$ we can iterate the above bound and work outward the integrating in $z_{\ell+1}$ to z_k and $z_{\ell-1}$ to z_1 .

To make certain that the local singularities of the resolvent are integrable uniformly in x and y , cancellation in the first factor $(R_0^+(\lambda^2) - R_0^-(\lambda^2))(x, z_1)$ is crucial. By (46), this is a bounded function of the spatial variables. Differentiation of resolvents with respect to λ generally improves their local regularity, so for this purpose the worst case is when all derivatives act on the cut-off function $\chi(\lambda)$ instead. Then we are left to control an integral of the form

$$\int_{\mathbb{R}^{kn}} \left(\prod_{j=1}^{k-1} \frac{\langle z_j \rangle^{-\beta}}{|z_j - z_{j+1}|^{n-2}} \right) \frac{\langle z_k \rangle^{-\beta}}{|z_k - y|^{n-2}} d\vec{z},$$

which is bounded so long as $\beta > 2$, using an estimate analogous to (57). We note that the lack of the $|x - z_1|^{2-n}$ singular terms is vital to this iterated integral being bounded for any $k = 1, 2, \dots$. If the ‘+/-’ difference acts on an inner resolvent, say on $R_0^+(\lambda^2)(z_\ell, z_{\ell+1}) - R_0^-(\lambda^2)(z_\ell, z_{\ell+1})$ we are lead to bound

$$\int_{\mathbb{R}^{kn}} \frac{1}{|x - z_1|^{n-2}} \left(\prod_{j=1}^{k-1} \frac{\langle z_j \rangle^{-\beta}}{|z_j - z_{j+1}|^{n-2}} \right) \frac{\langle z_k \rangle^{-\beta} |z_\ell - z_{\ell+1}|^{n-2}}{|z_k - y|^{n-2}} d\vec{z},$$

Here, one simply integrates $d\vec{z}$ first in the z_ℓ variable and proceed outward through the rest of the product. \square

We still need to consider the case in which all derivatives act on the leading or lagging free resolvent and the ‘+/-’ difference affects a different free resolvent, that is we wish to control the contribution of

$$\lambda \left[\left(\frac{1}{\lambda} \frac{d}{d\lambda} \right)^{\frac{n}{2}-1} R_0^-(\lambda^2) \right] V(R_0^-(\lambda^2) V)^j (R_0^+(\lambda^2) - R_0^-(\lambda^2)) (V R_0^+(\lambda^2))^\ell, \quad (58)$$

for all $j, \ell \geq 0$. Here if we simply integrate by part the final time, we have polynomial weights in the spatial variables when the final derivative also acts on the leading free resolvent. As noted in the discussion preceding Lemma 3.6, this is somehow simpler than the previous case. In particular, the argument follows using the techniques of the previous lemma, and the resulting calculation is streamlined

using the following Lemma. We first define $\mathcal{G}_n(\lambda, |x - y|)$ to be the kernel of the n -dimensional free resolvent operator $R_0^+(\lambda^2)$, and hence $\mathcal{G}_n(-\lambda, |x - y|)$ is the kernel of $R_0^-(\lambda^2)$, then

Lemma 3.7. *For $n \geq 2$, the following recurrence relation holds.*

$$\left(\frac{1}{\lambda} \frac{d}{d\lambda}\right) \mathcal{G}_n(\lambda, r) = \frac{1}{2\pi} \mathcal{G}_{n-2}(\lambda, r).$$

Proof. The proof follows from the recurrence relations of the Hankel functions, found in [1] and the representation of the kernel given in (5). \square

This tells us that the action of $\frac{1}{\lambda} \frac{d}{d\lambda}$ takes an n -dimensional free resolvent to an $n - 2$ dimensional free resolvent. With this, we are now ready to prove

Lemma 3.8. *If $|V(x)| \lesssim \langle x \rangle^{-\frac{n+2}{2}}$ and $j, \ell \geq 0$, we have the bound*

$$\sup_{x, y \in \mathbb{R}^n} \left| \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) [R_0^-(\lambda^2) V(R_0^-(\lambda^2) V)^j [R_0^+ - R_0^-](\lambda^2) (VR_0^+(\lambda^2))^\ell](x, y) d\lambda \right| \lesssim |t|^{-\frac{n}{2}}.$$

Proof. As in the proof of Lemma 3.6, we need only consider the case when all the derivatives act on the leading resolvent. The other cases are less delicate and can be treated identically.

At this point, by using Lemma 3.7 a total of $\frac{n}{2} - 1$ times the leading free resolvent is a constant multiple of the two-dimensional free resolvent. Thus, we can reduce the contribution of (58) to

$$t^{1-\frac{n}{2}} \int_0^\infty e^{it\lambda^2} \chi(\lambda) \lambda (iJ_0(\lambda|x - \cdot|) + Y_0(\lambda|x - \cdot|)) V \tilde{O}_1(\lambda^{n-2}) d\lambda.$$

The Bessel functions of order zero appear as the kernel of a two-dimensional resolvent. The $\tilde{O}_1(\lambda^{n-2})$ expression is much smaller than necessary ($\tilde{O}_1(\lambda^{0+})$ would be adequate), so it can absorb singularities of the Bessel functions with respect to λ .

Expansions for these Bessel functions, see [1], [27], or [7], show that for $\lambda|x - z_1| \ll 1$,

$$|iJ_0(\lambda|x - z_1|) + Y_0(\lambda|x - z_1|)| = 1 + \log(\lambda|x - z_1|) + \tilde{O}_1((\lambda|x - z_1|)^{2-}),$$

$$|\partial_\lambda [iJ_0(\lambda|x - z_1|) + Y_0(\lambda|x - z_1|)]| = \lambda^{-1} + \tilde{O}_1((\lambda|x - z_1|)^{1-}).$$

Recall that

$$|(1 + \log(\lambda|x - z_1|))\chi(\lambda|x - z_1|)\chi(\lambda)| \lesssim 1 + |\log \lambda| + \log^- |x - z_1|.$$

The $\log \lambda = \tilde{O}_1(\lambda^{0-})$ singularity is easily negated by $\tilde{O}_1(\lambda^{n-2})$ as mentioned above. The $\log^- |x - z_1|$ singularity is integrable, and is managed by the estimate

$$\sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} \log^- |x - z_1| \langle z_1 \rangle^{-\beta} dz_1 \lesssim 1$$

for any $\beta > n$.

For $\lambda|x - y| \gtrsim 1$, one has the description

$$\begin{aligned} & iJ_0(\lambda|x - z_1|) + Y_0(\lambda|x - z_1|) \\ &= e^{i\lambda|x - z_1|} \omega_+(\lambda|x - z_1|) + e^{-i\lambda|x - z_1|} \omega_-(\lambda|x - z_1|) \end{aligned}$$

similar in form to (8) but with different functions $\omega_{\pm}(z)$. Differentiating directly with respect to λ is not advised, as the resulting $|x - z_1| \omega_{\pm}(\lambda|x - z_1|)$ term grows like $\lambda^{-\frac{1}{2}} \langle x \rangle^{\frac{1}{2}}$ for large x .

However this issue was encountered once before while evaluating (52). The same argument from Lemma 3.6 applies here as well and yields the desired unweighted bound, again with more than enough λ smallness to ensure the argument runs through. \square

This provides all we need for the proof of the main proposition in this section.

Proof of Proposition 3.1. The proposition follows from Lemma 3.6, the discussion following this Lemma and finally from Lemma 3.8. \square

4. Dispersive estimates: the leading terms

In this section we prove dispersive bounds for the most singular λ terms of the expansion for $R_V^+(\lambda^2) - R_V^-(\lambda^2)$. These terms are sensitive to the existence of zero energy eigenvalues and are the slowest decaying in time. This behavior arises in the last term involving the operator $M^{\pm}(\lambda)^{-1}$ in (45).

From the ‘+/-’ cancellation, we need to control the contribution of

$$\begin{aligned} & (R_0^+(\lambda^2)V)^m R_0^+(\lambda^2)vM^+(\lambda)^{-1}vR_0^+(\lambda^2)(VR_0^+(\lambda^2))^m \\ & - (R_0^-(\lambda^2)V)^m R_0^-(\lambda^2)vM^-(\lambda)^{-1}vR_0^-(\lambda^2)(VR_0^-(\lambda^2))^m \end{aligned} \quad (59)$$

to the Stone formula, (4). Thanks to the algebraic fact (49), we need to consider three cases. The difference of ‘+’ and ‘-’ terms may act on the operators $M^\pm(\lambda)^{-1}$ or on the free resolvents. As in the treatment of the finite Born series terms in Section 3, if the difference acts on free resolvents we need to distinguish if they are ‘inner’ resolvents which require less care than the case of ‘leading’ or ‘lagging’ resolvents.

4.1. No cancellation. We first consider the case in which there are no cancellation properties to take advantage of, that is when $P_e V 1 \neq 0$.

Lemma 4.1. *If $P_e V 1 \neq 0$ and $|V(x)| \lesssim \langle x \rangle^{-n-}$, then*

$$(59) = \lambda^{n-6} P_e V 1 V P_e + \tilde{O}_{\frac{n}{2}-1}(\lambda^{n-6+})$$

which contributes $c|t|^{2-\frac{n}{2}} P_e V 1 V P_e + O(|t|^{2-\frac{n}{2}+})$ to (4).

If $P_e V 1 \neq 0$ and $|V(x)| \lesssim \langle x \rangle^{-n-4-}$, then

$$\begin{aligned} (59) &= \lambda^{n-6} P_e V 1 V P_e + \frac{R_0^+(\lambda^2) - R_0^-(\lambda^2)}{\lambda^2} V P_e \\ &+ P_e V \frac{R_0^+(\lambda^2) - R_0^-(\lambda^2)}{\lambda^2} + \mathcal{E}(\lambda) \end{aligned}$$

which contributes $c|t|^{2-\frac{n}{2}} P_e V 1 V P_e + O(|t|^{1-\frac{n}{2}})$ to (4).

Here we cannot write the final error term $\mathcal{E}(\lambda)$ accurately as $\tilde{O}_k(\lambda^\alpha)$, as there are too many fine properties of this error term that this notation fails to capture if one hopes to attain the faster $|t|^{1-\frac{n}{2}}$ decay rate. One can explicitly reconstruct $\mathcal{E}(\lambda)$ from our proof, though we do not think it worthwhile to do so.

We note that the terms

$$\frac{R_0^+(\lambda^2) - R_0^-(\lambda^2)}{\lambda^2} V P_e + P_e V \frac{R_0^+(\lambda^2) - R_0^-(\lambda^2)}{\lambda^2}$$

appear in the expansion in all cases, see the statements of Lemmas 4.1, 4.2, and 4.3. The different cancellation assumptions on $P_e V 1$ and $P_e V x$ allow us some flexibility on how to control their contribution to (4). To avoid presenting three proofs of how to bound these terms, which would have a certain amount of overlap, we control these terms separately in Lemma 4.4 and Corollary 4.6 below.

Proof. The first statement is a straightforward application of Lemma 2.1 and Corollary 2.15 in the context of applying (49) to (59). Lemmas 5.6 and 5.7 then control the respective integrals in (4) due to the leading term and the remainder.

More precisely, the leading term appears if the ‘+/-’ difference in (49) falls on $M^\pm(\lambda)^{-1}$. In that case Corollary 2.15 indicates that

$$\begin{aligned} M^+(\lambda)^{-1} - M^-(\lambda)^{-1} &= \frac{g_1^+(\lambda) - g_1^-(\lambda)}{\lambda^4} M_{n-6}^L + \tilde{O}_{\frac{n}{2}-1}(\lambda^{n-6+}) \\ &= 2\Im(z_1)\lambda^{n-6} M_{n-6}^L + \tilde{O}_{\frac{n}{2}-1}(\lambda^{n-6+}), \end{aligned}$$

where we used (14) in the last line. Meanwhile $R_0^\pm(\lambda^2) = G_0^0 + \tilde{O}_{\frac{n}{2}-1}(\lambda^{0+})$. Together with the fact that V is integrable, this establishes the remainder as $\tilde{O}_{\frac{n}{2}-1}(\lambda^{n-6+})$. The operator in the leading term is seen, using identities (35), (37), and (39), to be

$$\begin{aligned} (G_0^0 V)^m G_0^0 v M_{n-6}^L v G_0^0 (V G_0^0)^m &= (G_0^0 V)^m G_0^0 v D_1 v G_{n-2}^c v D_1 v G_0^0 (V G_0^0)^m \\ &= P_e V 1 V P_e. \end{aligned}$$

If the +/- difference acts on any one of the resolvents in (59), we see that

$$\begin{aligned} R_0^+(\lambda^2) - R_0^-(\lambda^2) &= \tilde{O}_{\frac{n}{2}-1}(\lambda^{n-2}), \\ R_0^\pm(\lambda^2)(z_j, z_{j+1}) &= (1 + \log^- |z_j - z_{j+1}|) \tilde{O}_{\frac{n}{2}-1}(1) \end{aligned}$$

and $M^\pm(\lambda)^{-1} = \tilde{O}_{\frac{n}{2}-1}(\lambda^{-2})$. Recall that the notation $\tilde{O}_{\frac{n}{2}-1}(1)$ indicates that differentiation in λ is comparable to division by λ . That more than suffices to place all of these terms in the remainder.

Now assume that $|V(x)| \lesssim \langle x \rangle^{-n-4-}$. Carrying out the power series expansion further in Corollary 2.15, one obtains

$$\begin{aligned} M^+(\lambda)^{-1} - M^-(\lambda)^{-1} &= \frac{g_1^+(\lambda) - g_1^-(\lambda)}{\lambda^4} M_{n-6}^L + \frac{g_1^+(\lambda) - g_1^-(\lambda)}{\lambda^2} M_{n-4}^{L1} \\ &\quad + \frac{g_2^+(\lambda) - g_2^-(\lambda)}{\lambda^4} M_{n-4}^{L2} + \tilde{O}_{\frac{n}{2}-1}(\lambda^{n-4+}) \\ &= 2\Im(z_1)\lambda^{n-6} M_{n-6}^L + 2\Im(z_1)\lambda^{n-4} M_{n-4}^{L1} + 2\Im(z_2)\lambda^{n-4} M_{n-4}^{L2} \\ &\quad + \tilde{O}_{\frac{n}{2}-1}(\lambda^{n-4+}). \end{aligned}$$

Similarly, we have

$$R_0^\pm(\lambda^2)(x, y) = G_0^0 + \lambda^2 G_1^0 + (1 + \log^- |x - y|) \tilde{O}_{\frac{n}{2}-1}(\lambda^4).$$

Thus the term featuring $M^+(\lambda)^{-1} - M^-(\lambda)^{-1}$ has the form

$$\begin{aligned}
& (G_0^0 V)^m G_0^0 v [M^+(\lambda)^{-1} - M^-(\lambda)^{-1}] v G_0^0 (V G_0^0)^m \\
& + [\lambda^2 \Gamma_1 + (1 + \log^- |x - \cdot|) \tilde{O}_{\frac{n}{2}-1}(\lambda^4)] [M^+(\lambda)^{-1} - M^-(\lambda)^{-1}] v G_0^0 (V G_0^0)^m \\
& + (G_0^0 V)^m G_0^0 v [M^+(\lambda)^{-1} - M^-(\lambda)^{-1}] [\lambda^2 \Gamma_1 + (1 + \log^- |x - \cdot|) \tilde{O}_{\frac{n}{2}-1}(\lambda^4)] \\
& + [\lambda^2 \Gamma_1 + (1 + \log^- |x - \cdot|) \tilde{O}_{\frac{n}{2}-1}(\lambda^4)] [M^+(\lambda)^{-1} - M^-(\lambda)^{-1}] \\
& \quad [\lambda^2 \Gamma_1 + (1 + \log^- |\cdot - y|) \tilde{O}_{\frac{n}{2}-1}(\lambda^4)] \\
& = \lambda^{n-6} P_e V_1 V P_e + \lambda^{n-4} K_1 + \tilde{O}_{\frac{n}{2}-1}(\lambda^{n-4+}) K_2,
\end{aligned} \tag{60}$$

with K_1, K_2 operators that map $L^1 \rightarrow L^\infty$.

If the +/- difference falls on a free resolvent in the interior of the product, we have

$$(R_0^+(\lambda^2) - R_0^-(\lambda^2))(z_j, z_{j+1}) = c \lambda^{n-2} G_{n-2}^c + |z_j - z_{j+1}|^{0+} \tilde{O}_{\frac{n}{2}-1}(\lambda^{n-2+})$$

and $M^\pm(\lambda)^{-1} = -\lambda^{-2} D_1 + \tilde{O}_{\frac{n}{2}-1}(1)$. The resulting term of (59) takes the form $\lambda^{n-4} K_3 + \tilde{O}_{\frac{n}{2}-1}(\lambda^{n-4+})$, with K_3 another operator from L^1 to L^∞ .

We note that the extra power of $|z_j - z_{j+1}|^{0+}$ that appears in the remainder term is acted on by $R_0^-(\lambda^2)V$ on the left and $VR_0^+(\lambda^2)$ on the right, so that the decay of the potentials ensures that the product remains bounded between unweighted spaces.

The terms in which the '+/-' difference acts on the first (or last) free resolvent are trickier because one cannot differentiate too many times, or go too far into the power series expansion of $R_0^+(\lambda^2) - R_0^-(\lambda^2)$ without introducing weights. Suppose the difference acts on the leading resolvent; the other case is identical up to symmetry. Once again we can use the expansions for $M^\pm(\lambda)^{-1}$ and $R_0^\pm(\lambda^2)$ along with Lemma 3.5 to express this term as

$$\begin{aligned}
& [R_0^+(\lambda^2) - R_0^-(\lambda^2)] (V G_0^0)^m v \left(-\frac{D_1}{\lambda^2} \right) v (G_0^0 V)^m G_0^0 + \tilde{O}_{\frac{n}{2}-1}(\lambda^{n-2}) \\
& = \frac{R_0^+(\lambda^2) - R_0^-(\lambda^2)}{\lambda^2} V P_e + \tilde{O}_{\frac{n}{2}-1}(\lambda^{n-2}).
\end{aligned}$$

One can quickly show using Lemma 5.7 that the remainder contributes at most $|t|^{1-\frac{n}{2}}$ to the Stone formula. In fact this contribution is of the order $|t|^{-\frac{n}{2}}$, seen by adopting the methods of Lemma 3.6. The contribution of $\lambda^{-2}(R_0^+(\lambda^2) - R_0^-(\lambda^2))V P_e$ to (4) is rather intricate, and is discussed fully as Lemma 4.4. For the purpose of this Lemma, we note that $\lambda^{-2}(R_0^+(\lambda^2) - R_0^-(\lambda^2))V P_e$ is bounded by $|t|^{1-\frac{n}{2}}$ as an operator from $L^1 \rightarrow L^\infty$ by Lemma 4.4, which finishes the proof. \square

The remaining terms in the Born series are smaller than these for large $|t|$ by Proposition 3.1. In fact using the identities for S_1 and Lemma 5.3, at this point we can write

$$e^{itH} P_{ac}(H) = c|t|^{2-\frac{n}{2}} P_e V1VP_e + O(|t|^{1-\frac{n}{2}}), \quad (61)$$

where the operator $P_e V1VP_e$ is rank one, and the error term is understood as mapping L^1 to L^∞ . The weaker claim, with error term of size $o(|t|^{2-\frac{n}{2}})$ follows by using the first statement of Lemma 4.2.

4.2. The case of $P_e V1 = 0$. Here we consider when the operator $P_e V1 = 0$. This cancellation makes the initial term in Lemma 4.1 vanish, clearing the way for time decay at the faster rate of $|t|^{1-\frac{n}{2}}$. Here we provide more detail on the behavior of the next term in the evolution.

Lemma 4.2. *If $P_e V1 = 0$ and $|V(x)| \lesssim \langle x \rangle^{-n-8-}$, then*

$$(59) = \lambda^{n-4} \Gamma_1 + \frac{R_0^+(\lambda^2) - R_0^-(\lambda^2)}{\lambda^2} VP_e + P_e V \frac{R_0^+(\lambda^2) - R_0^-(\lambda^2)}{\lambda^2} \\ + \lambda^{n-2} \Gamma_2 + \mathcal{E}(\lambda),$$

where

$$\Gamma_1, \Gamma_2: L^1 \longrightarrow L^\infty.$$

The error term belongs to the class $\langle x \rangle^{\frac{1}{2}} \langle y \rangle^{\frac{1}{2}} \tilde{O}_{\frac{n}{2}}(\lambda^{n-2+})$, however its contribution to (4) is $O(|t|^{-\frac{n}{2}})$ without spatial weights. Assuming the result of Lemma 4.4, the total contribution to (4) of all terms is $|t|^{1-\frac{n}{2}} + \langle x \rangle \langle y \rangle O(|t|^{-\frac{n}{2}})$.

We note that the error term $\mathcal{E}(\lambda)$ here is distinct from the error term in Lemma 4.1.

Proof. The structure of the argument is the same as in the preceding lemma. The extra decay permits us to evaluate more terms of each power series, or better control the remainder. The fact that $P_e V1 = 0$ causes some of the leading order expressions to vanish.

When the ‘+/-’ cancellation in (49) acts on $M^\pm(\lambda)^{-1}$, the first nonzero term has size λ^{n-4} . In detail, we note that by Corollary 2.16, specifically (43) we have

$$M^+(\lambda) - M^-(\lambda) = \frac{g_2^+(\lambda) - g_2^-(\lambda)}{\lambda^4} M_{n-4}^{L2} + \frac{g_2^+(\lambda) - g_2^-(\lambda)}{\lambda^2} M_{n-2}^{L2} + \tilde{O}_{\frac{n}{2}}(\lambda^{n-}) \\ = c_1 \lambda^{n-4} M_{n-4}^{L2} + c_2 \lambda^{n-2} M_{n-2}^{L2} + \tilde{O}_{\frac{n}{2}}(\lambda^{n-2+}).$$

Writing the resolvents as $R_0^\pm(\lambda^2)(x, y) = G_0^0 + \lambda^2 G_1^0 + \langle x \rangle^{\frac{1}{2}} \langle y \rangle^{\frac{1}{2}} \tilde{O}_{\frac{n}{2}}(\lambda^4)$, as suggested by Corollary 2.2, we can see that

$$\begin{aligned} & [R_0^-(\lambda^2)V]^m R_0^-(\lambda^2)v[M^+(\lambda) - M^-(\lambda)]vR_0^+(\lambda^2)[VR_0^+(\lambda^2)]^m(x, y) \\ &= c_1 \lambda^{n-4} (G_0^0 V)^m G_0^0 v M_{n-4}^{L^2} v G_0^0 (VG_0^0)^m + \lambda^{n-2} K_2 \\ &+ \langle x \rangle^{\frac{1}{2}} \langle y \rangle^{\frac{1}{2}} \tilde{O}_{\frac{n}{2}}(\lambda^{n-2+}). \end{aligned}$$

Here K_2 is a finite rank operator made out of G_0^0 's and $vM_{n-2}^{L^2}v$ along with all the combinations consisting of G_0^0 's, $vM_{n-4}^{L^2}v$ and exactly one instance of G_1^0 . Lemma 5.6 shows that the first term contributes $|t|^{1-\frac{n}{2}}$ to (4) and the second term contributes $|t|^{-\frac{n}{2}}$. Lemma 5.8 shows that the last term generates a map from $L^{1, \frac{1}{2}}$ to $L^{\infty, -\frac{1}{2}}$ with norm $|t|^{-\frac{n}{2}}$. The half-power weights only arise if one allows $\frac{n}{2}$ derivatives to fall on the first or the last free resolvent in the product. The argument in Lemma 3.8 of using the stationary phase bound of Lemma 3.4 in place of the last integration by parts shows how that situation can be prevented, so that all the expressions with time decay $|t|^{-\frac{n}{2}}$ are bounded operators from L^1 to L^∞ .

Now suppose the '+/-' difference acts on a free resolvent in the interior of the product. We may write

$$\begin{aligned} [R_0^+(\lambda^2) - R_0^-(\lambda^2)](z_j, z_{j+1}) &= \lambda^{n-2} G_{n-2}^c + \lambda^n G_n^c + |z_j - z_{j+1}|^{2+} \tilde{O}_{\frac{n}{2}}(\lambda^{n+}), \\ R_0^\pm(\lambda^2)(z_j, z_{j+1}) &= G_0^0 + \lambda^2 G_1^0 + \langle z_j \rangle^{\frac{1}{2}} \langle z_{j+1} \rangle^{\frac{1}{2}} \tilde{O}_{\frac{n}{2}}(\lambda^{2+}), \\ M^\pm(\lambda)^{-1} &= -\lambda^{-2} D_1 + M_0 + \tilde{O}_{\frac{n}{2}}(\lambda^{0+}). \end{aligned}$$

Note that $P_e V 1 = 0$ causes the leading term ($\lambda^{n-4} K_3$ in the previous lemma) to vanish because $(VG_0^0)^{m-j} D_1 = VP_e w$ and $G_{n-2}^c(z_j, z_{j+1}) = c_{n-2} 1$ is a constant function. Thus $G_{n-2}^c (VG_0^0)^{m-j} D_1 = 0$.

Expressions with λ^{n-2} occur by replacing the leading term in exactly one of the above power series by its successor. That is when $\lambda^n G_n^c$ occurs in place of $\lambda^{n-2} G_{n-2}^c$, $\lambda^2 G_1^0$ in place of G_0^0 or M_0 in place of $-\lambda^2 D_1$. The operator G_n^c has spatial growth of $|z_j - z_{j+1}|^2$ but it is controlled by the decay of the potentials as it is multiplied on both sides by $V(z_j)$ and $V(z_{j+1})$.

Remainders in the class $\tilde{O}_{\frac{n}{2}}(\lambda^{n-2+})$ are mostly bounded from L^1 to L^∞ as well, except that once again weights of $\langle x \rangle^{\frac{1}{2}}$ or $\langle y \rangle^{\frac{1}{2}}$ arise if all $\frac{n}{2}$ derivatives fall on the first or the last free resolvent. Following the calculations in Lemma 3.8, one can see that the contribution of these remainder terms to (4) has time decay $|t|^{-\frac{n}{2}}$ as a map between unweighted L^1 and L^∞ .

Now suppose the difference of free resolvents occurs at the leading resolvent of the product (59). The expression where one approximates all other free resolvents by G_0^0 , and $M^+(\lambda)^{-1}$ by $-\lambda^{-2}D_1$, is considered separately in Lemma 4.4. Under the assumption $P_e V 1 = 0$, its contribution to (4) is an operator with kernel bounded by $\langle x \rangle |t|^{-\frac{n}{2}}$. The analogous expression when the +/- difference is applied to the very last resolvent in the product yields a bound of $\langle y \rangle |t|^{-\frac{n}{2}}$. Put together, these operators form a map from $L^{1,1}$ to $L^{\infty,-1}$ with time decay $|t|^{-\frac{n}{2}}$.

Finally there is an assortment of remainder terms found by applying (49) to

$$(R_0^+(\lambda^2) - R_0^-(\lambda^2)) \left[(VR_0^+(\lambda^2)V)^m v M^+(\lambda)^{-1} v R_0^+(\lambda^2) (VR_0^+(\lambda^2))^m - (VG_0^0)^m v \left(-\frac{D_1}{\lambda^2} \right) v G_0^0 (VG_0^0)^m \right].$$

Each one is headed by $(R_0^+(\lambda^2) - R_0^-(\lambda^2))$, concludes with either $R_0^+(\lambda^2)$ or G_0^0 , and is of order λ^{n-2} . Following the calculations in Lemma 3.6 one can show that they contribute $|t|^{-\frac{n}{2}}$ to (4). \square

Hence we have if $P_e V 1 = 0$,

$$e^{itH} P_{ac}(H) = |t|^{1-\frac{n}{2}} \Gamma + O(|t|^{-\frac{n}{2}}),$$

where Γ is a finite rank operator mapping L^1 to L^∞ , which we do not make explicit and the error term is understood as an operator between weighted spaces. Combining this with the analysis for when $P_e V 1 \neq 0$, we have the expansion

$$e^{itH} P_{ac}(H) = c|t|^{2-\frac{n}{2}} P_e V 1 V P_e + |t|^{1-\frac{n}{2}} \Gamma_2 + O(|t|^{-\frac{n}{2}}),$$

with $\Gamma_2: L^1 \rightarrow L^\infty$ a finite rank operators, which is valid whether or not $P_e V 1 = 0$.

4.3. The case of $P_e V 1 = 0$ and $P_e V x = 0$. Finally we consider the evolution when we have both cancellation conditions on the zero-energy eigenfunctions.

Lemma 4.3. *If $P_e V 1 = 0$, $P_e V x = 0$ and $|V(x)| \lesssim \langle x \rangle^{-n-8^-}$, then*

$$(59) = \frac{R_0^+(\lambda^2) - R_0^-(\lambda^2)}{\lambda^2} V P_e + P_e V \frac{R_0^+(\lambda^2) - R_0^-(\lambda^2)}{\lambda^2} + \lambda^{n-2} \Gamma_3 + \mathcal{E}(\lambda),$$

where $\Gamma_3: L^1 \rightarrow L^\infty$. The error term contributes $O(|t|^{-\frac{n}{2}})$ as an operator from $L^1 \rightarrow L^\infty$. Assuming the result of Lemma 4.4, the total contribution to (4) of all terms is $O(|t|^{-\frac{n}{2}})$.

Again the error term $\mathcal{E}(\lambda)$ is distinct from the previous lemmas.

Proof. As in the proofs of Lemmas 4.1 and 4.2 we have to consider when the ‘+/-’ difference in (49) acts on either a resolvent of $M^\pm(\lambda)^{-1}$. In the latter case, the same argument as above goes through, though we note (from Corollary 2.16) that the operator $M_{n-4}^{L^2} = 0$, so that

$$\begin{aligned} M^+(\lambda) - M^-(\lambda) &= \frac{g_3^+(\lambda) - g_3^-(\lambda)}{\lambda^4} M_{n-2}^{L^3} + \tilde{O}_{\frac{n}{2}}(\lambda^{n-2+}) \\ &= c_2 \lambda^{n-2} M_{n-2}^{L^3} + \tilde{O}_{\frac{n}{2}}(\lambda^{n-2+}). \end{aligned}$$

This easily gives us the bound of $|t|^{-\frac{n}{2}}$ when combined with the previous sections as an operator from L^1 to L^∞ .

When the ‘+/-’ difference acts on free resolvents, we can control the contribution by $|t|^{-\frac{n}{2}}$ as an operator from $L^1 \rightarrow L^\infty$ if the difference acts on an ‘inner’ resolvent as before. For the remaining two terms, when the ‘+/-’ acts on a leading or lagging free resolvent, we use the following estimates of Lemma 4.4. \square

Lemma 4.4. *The operator*

$$\frac{R_0^+(\lambda^2) - R_0^-(\lambda^2)}{\lambda^2} V P_e$$

contributes $|t|^{1-\frac{n}{2}}$ to (4) as an operator from L^1 to L^∞ . If $P_e V 1 = 0$, then it contributes $|t|^{-\frac{n}{2}}$ as an operator from L^1 to $L^{\infty,-1}$. If in addition $P_e V x = 0$, then the contribution still has size $|t|^{-\frac{n}{2}}$, but acts as an operator from L^1 to L^∞ .

Here we need to be careful with the spatial variables to see that the orthogonality conditions allow us to move the dependence on x or y into an inner spatial variable, which can be controlled by the decay of the potential. To make this clear, we note that we wish to bound the integral

$$\int_0^1 e^{it\lambda^2} \chi(\lambda) \lambda^{-1} (R_0^+(\lambda^2) - R_0^-(\lambda^2))(x, z_1) V(z_1) P_e(z_1, y) d\lambda \quad (62)$$

in terms of t, x and y .

To prove this lemma, we first need to following oscillatory integral estimate, whose proof is in Section 5.

Lemma 4.5. *Let m be any positive integer. Suppose $|\Omega^{(k)}(z)| \leq \langle z \rangle^{\frac{1-m}{2}-k}$ for each $k \geq 0$. Then*

$$\int_0^\infty e^{it\lambda^2} \lambda^{m-1} e^{\pm i\lambda r} \Omega(\lambda r) \chi(\lambda) d\lambda \lesssim |t|^{-\frac{m}{2}} \quad (63)$$

with a constant that does not depend on the value of $r > 0$.

We note that m in this lemma is an arbitrary integer, not that value chosen in (45) that ensures the iterated resolvents are locally L^2 .

Proof of Lemma 4.4. According to (46), the integral kernel of $R_0^+(\lambda^2) - R_0^-(\lambda^2)$ can be expressed (modulo constants) as

$$\begin{aligned} K(\lambda, |x - z_1|) &= \lambda^{n-2} \frac{J_{\frac{n}{2}-1}(\lambda|x - z_1|)}{(\lambda|x - z_1|)^{\frac{n}{2}-1}} \\ &= \lambda^{n-2} (e^{i\lambda|x-z_1|} \Omega_+(\lambda|x - z_1|) + e^{-i\lambda|x-z_1|} \Omega_-(\lambda|x - z_1|)), \end{aligned}$$

where the functions Ω_{\pm} and their derivatives satisfy $|\Omega_{\pm}^{(k)}(z)| \lesssim \langle z \rangle^{\frac{1-n}{2}-k}$. Derivatives with respect to the spatial variable $r = |x - z_1|$ are obtained by differentiating (6) and (8) according to whether λr is small or large. Since the expansion of $z^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(z)$ in (6) has only even powers of z , its first derivative is bounded by $|z|$ rather than a constant. Thus we can write

$$\partial_r K(\lambda, r) = \lambda^n r (e^{i\lambda r} \Omega_{1,+}(\lambda r) + e^{-i\lambda r} \Omega_{1,-}(\lambda r)), \quad (64a)$$

$$\partial_r^2 K(\lambda, r) = \lambda^n (e^{i\lambda r} \Omega_{2,+}(\lambda r) + e^{-i\lambda r} \Omega_{2,-}(\lambda r)), \quad (64b)$$

where $|\Omega_{j,\pm}^{(k)}(z)| \lesssim \langle z \rangle^{\frac{1-n}{2}-j-k}$ for $j = 1, 2$ and all $k \geq 0$.

Roughly speaking, the bound on $\partial_r K(\lambda, r)$ gains two powers of λ at the cost of one power of $r = |x - z_1|$. This gains us an extra power of time decay in the contribution to the Stone formula, (4), at the cost of one power spatial weight. The bound on $\partial_r^2 K(\lambda, r)$ allows us to gain the desired time decay with no spatial weights.

As an immediate consequence we can apply Lemma 4.5 with $m = n - 2$ to obtain

$$\int_0^\infty e^{it\lambda^2} \lambda^{n-3} e^{\pm i\lambda|x-z_1|} \Omega_{\pm}(\lambda|x - z_1|) \chi(\lambda) d\lambda \lesssim |t|^{1-\frac{n}{2}}$$

and therefore

$$\int_0^\infty e^{it\lambda^2} \lambda^{-1} \chi(\lambda) (R_0^+(\lambda^2) - R_0^-(\lambda^2)) V P_e d\lambda$$

maps L^1 to L^∞ with norm decay of $|t|^{1-\frac{n}{2}}$.

When $P_e V 1 = 0$, we can extract a leading-order term by replacing $K(\lambda, |x - z_1|)$ by $K(\lambda, |x - z_1|) - K(\lambda, |x|)$ each place that it occurs. From an operator perspective this amounts to approximating $R_0^+(\lambda^2) - R_0^-(\lambda^2)$ by $K(\lambda|x|)1$. This term vanishes from the Schrödinger evolution precisely when $P_e V 1 = 0$.

The remainder can be written using the expression

$$K(\lambda, |x - z_1|) - K(\lambda, |x|) = \int_0^1 \partial_r K(\lambda, |x - sz_1|) \frac{(-z_1) \cdot (x - sz_1)}{|x - sz_1|} ds.$$

Based on the decomposition in (64) and Lemma 4.5 with $m = n$, we have the bound

$$\left| \int_0^\infty e^{it\lambda^2} \lambda^{-1} \chi(\lambda) \partial_r K(\lambda, |x - sz_1|) \frac{(-z_1) \cdot (x - sz_1)}{|x - sz_1|} d\lambda \right| \lesssim |t|^{-\frac{n}{2}} |x - sz_1| |z_1|$$

for each s . If $s \in [0, 1]$ we also have $|x - sz_1| \leq |x| + |z_1| \leq \langle x \rangle \langle z_1 \rangle$. It follows that $\int_0^\infty e^{it\lambda^2} \lambda^{-1} \chi(\lambda) (R_0^+(\lambda^2) - R_0^-(\lambda^2) - K(\lambda, |x|) 1) VP_e d\lambda$ maps L^1 to $L^{\infty, -1}$ provided V has enough decay so that the range of VP_e belongs to $L^{1,2}$, which follows from the fact that $P_e : L^1 \rightarrow L^\infty$, see Corollary 5.5, and the decay of V .

Now if in addition $PVx = 0$ we can gain more by going to the second order expression

$$\begin{aligned} & K(\lambda, |x - z_1|) - K(\lambda, |x|) + \partial_r K(\lambda, |x|) \frac{z_1 \cdot x}{|x|} \\ &= \int_0^1 (1-s) \left[\partial_r^2 K(\lambda, |x - sz_1|) \frac{(z_1 \cdot (x - sz_1))^2}{|x - sz_1|^2} \right. \\ & \quad \left. + \partial_r K(\lambda, |x - sz_1|) \left(\frac{|z_1|^2}{|x - sz_1|} - \frac{(z_1 \cdot (x - sz_1))^2}{|x - sz_1|^3} \right) \right] ds. \end{aligned}$$

Thanks to the bounds in (64) and Lemma 4.5 with $m = n$, there is a uniform estimate

$$\left| \int_0^\infty e^{it\lambda^2} \lambda^{-1} \chi(\lambda) \partial_r^2 K(\lambda, |x - sz_1|) \frac{(z_1 \cdot (x - sz_1))^2}{|x - sz_1|^2} d\lambda \right| \lesssim |t|^{-\frac{n}{2}} \langle z_1 \rangle^2,$$

and similarly for each of the terms with $\partial_r K(\lambda, |x - sz_1|)$ using (64) repeatedly. Plugging this back into the original operator integral yields

$$\begin{aligned} & \left\| \int_0^\infty e^{it\lambda^2} \lambda^{-1} \chi(\lambda) (R_0^+(\lambda^2) - R_0^-(\lambda^2) - K(\lambda, |x|) 1 \right. \\ & \quad \left. + \partial_r K(\lambda, |x|) \frac{x}{|x|} \cdot z_1) VP_e d\lambda \right\|_{L^1 \rightarrow L^\infty} \lesssim |t|^{-\frac{n}{2}}, \end{aligned}$$

provided VP_e has range in $L^{1,2}$, which is ensured by Corollary 5.5 and the decay of V . \square

Corollary 4.6. *The operator*

$$P_e V \frac{R_0^+(\lambda^2) - R_0^-(\lambda^2)}{\lambda^2}$$

contributes $|t|^{1-\frac{n}{2}}$ to (4) as an operator from L^1 to L^∞ . If $P_e V 1 = 0$, it contributes $|t|^{-\frac{n}{2}}$ to (4) as an operator from $L^{1,1}$ to L^∞ . If in addition $P_e V x = 0$ the contribution is as an operator from L^1 to L^∞ .

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. We note that the Theorem is proven by bounding the oscillatory integral in the Stone formula (4),

$$\left| \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) [R_V^+(\lambda^2) - R_V^-(\lambda^2)](x, y) d\lambda \right| \lesssim_{x,y} |t|^{-\alpha}. \quad (65)$$

We begin by proving part (1), where there is no x, y dependence. The proof follows by expanding $R_V^\pm(\lambda^2)$ into the Born series expansion, (44) and (45). The contribution of (44) is bounded by $|t|^{-\frac{n}{2}}$ by Proposition 3.1, while the contribution of (45) is bounded by $|t|^{2-\frac{n}{2}} P_e V 1 V P_e + O(|t|^{1-\frac{n}{2}})$ by Lemma 4.1.

To prove part (2), one uses Lemma 4.2 in the place of Lemma 4.1 in the proof of part (1). Finally, part (3) is proven by using Lemma 4.3. \square

We note that the proof of Theorem 1.1 is actually simpler. If zero is regular, the expansion of $M^\pm(\lambda)^{-1}$ is of the same form with respect to the spectral variable λ as $(M^\pm(\lambda) + S_1)^{-1}$ given in Lemma 2.9 with different operators that are still absolutely bounded and real-valued, see Remark 2.10. The dispersive bounds follow as in the analysis when zero is not regular without the most singular terms that arise from $-D_1/\lambda^2$.

We note that we need one further estimate on the operator

$$\frac{R_0^+(\lambda^2) - R_0^-(\lambda^2)}{\lambda^2} V P_e$$

that is not contained in Lemma 4.4 to prove the Corollary 1.3 in the case that $P_e V 1 \neq 0$. To establish that the operator with the $|t|^{1-\frac{n}{2}}$ decay rate is indeed finite rank, and to see why the operator $A_0(t)$ must map $L^{1,2}$ to $L^{\infty,-2}$ if $P_e V 1 \neq 0$, we need the following lemma:

Lemma 4.7. *The operator*

$$\frac{R_0^+(\lambda^2) - R_0^-(\lambda^2)}{\lambda^2} V P_e$$

contributes $c|t|^{1-\frac{n}{2}} 1VP_e + O(|t|^{-\frac{n}{2}})$ to (4), where the error term is an operator from L^1 to $L^{\infty,-2}$.

Proof. The desired bound follows using (46) as in Lemma 3.6. We first concern ourselves with when $\lambda|x - z_1| \ll 1$, in this case we note that using (47) out to one further term, we have

$$\begin{aligned} & [R_0^+(\lambda^2) - R_0^-(\lambda^2)](x, z_1) \\ &= \lambda^{n-2} G_{n-2}^c + \lambda^n G_n^c + \tilde{O}(\lambda^{n-2}(\lambda|x - z_1|)^{2+\epsilon}), \quad 0 \leq \epsilon < 2. \end{aligned}$$

Recalling that $G_n^c(x, z_1) = c_n|x - z_1|^2$, we can now write (for $\lambda|x - z_1| \ll 1$)

$$\begin{aligned} & \frac{R_0^+ - R_0^-(\lambda^2)(x, z_1)}{\lambda^2} V(z_1) P_e(z_1, y) \\ &= c_{n-2} \lambda^{n-4} V(z_1) P_e(z_1, y) + \lambda^{n-2} |x - z_1|^2 V(z_1) P_e(z_1, y) \\ & \quad + \tilde{O}(\lambda^{n-4}(\lambda|x - z_1|)^{2+\epsilon}) V(z_1) P_e(z_1, y). \end{aligned}$$

The first λ^{n-4} term can be seen to contribute $c|t|^{1-\frac{n}{2}}$ to (4) by Lemma 5.6. Similarly the second term with λ^{n-2} is seen to contribute $\langle x \rangle^2 |t|^{-\frac{n}{2}}$ to (4) by Lemma 5.6. The final error term is controlled identically to how one bounds (50) in Lemma 3.6 (with an additional factor of $|x - z_1|^2$), from which one again has a contribution of size $\langle x \rangle^2 |t|^{-\frac{n}{2}}$ to (4).

On the other hand, if $\lambda|x - z_1| \gtrsim 1$, we can write

$$\begin{aligned} & [R_0^+ - R_0^-](\lambda^2)(x, z_1) \\ &= e^{i\lambda|x-z_1|} \tilde{O}(\lambda^{n-2}(\lambda|x - z_1|^{\frac{1}{2}+\alpha})) + e^{-i\lambda|x-z_1|} \tilde{O}(\lambda^{n-2}(\lambda|x - z_1|^{\frac{1}{2}+\alpha})). \end{aligned}$$

As usual, the most delicate term is the ‘-’ phase. We need to control the contribution of

$$\int_0^\infty e^{it\lambda^2} \lambda^{-1} \chi(\lambda) e^{-i\lambda|x-z_1|} \tilde{O}(\lambda^{n-2}(\lambda|x - z_1|^{\frac{1}{2}+\alpha})) d\lambda.$$

Upon integrating by parts $\frac{n}{2} - 1$ times against the imaginary Gaussian, we are left to bound an integral of the form

$$|t|^{1-\frac{n}{2}} \int_0^\infty e^{it\lambda^2 - i\lambda|x-z_1|} a(\lambda) d\lambda,$$

where

$$|a(\lambda)| \lesssim \lambda^{-1} (\lambda|x - z_1|)^{\frac{1}{2} + \alpha} \lesssim \lambda^{\frac{1}{2}} |x - z_1|^{\frac{3}{2}} \lesssim |x - z_1|^2 \left(\frac{\lambda^{\frac{1}{2}}}{|x - z_1|^{\frac{1}{2}}} \right),$$

where we took $\alpha = 1$ in the second to last line. Similarly,

$$|a'(\lambda)| \lesssim |x - z_1|^2 \left(\frac{1}{\lambda^{\frac{1}{2}} |x - z_1|^{\frac{1}{2}}} \right).$$

Now, one can employ Lemma 3.4 as in the proof of Lemma 3.6 (with an extra factor of $|x - z_1|^2$) to see that this term contributes at most $\langle x \rangle^2 |t|^{-\frac{n}{2}}$ to (4). The ‘+’ phase again follows more simply from another integration by parts, this time against $e^{it\lambda^2 + i\lambda|x - z_1|}$. \square

Corollary 4.8. *The operator*

$$P_e V \frac{R_0^+(\lambda^2) - R_0^-(\lambda^2)}{\lambda^2}$$

contributes $c|t|^{1-\frac{n}{2}} P_e V 1 + O(|t|^{-\frac{n}{2}})$ to (4), where the error term is an operator from $L^{1,2}$ to L^∞ .

The proof of the corollary is identical in form to the proof of Lemma 4.7 with the spatial variables x and y trading places.

5. Spectral characterization and integral estimates

We provide a characterization of the spectral subspaces of $L^2(\mathbb{R}^n)$ that are related to the invertibility of certain operators in our expansions. This characterization and its proofs are identical to those given in [13], as such we provide the statements and omit the proofs. As in the odd case, the lack of resonances in dimensions $n > 4$ simplifies these characterizations. In addition, we state several oscillatory integral estimates from [13] and provide proofs for new integral estimates that are required in this paper.

Lemma 5.1. *Assume that $|V(x)| \lesssim \langle x \rangle^{-2\beta}$ for some $\beta \geq 2$, $f \in S_1 L^2(\mathbb{R}^n) \setminus \{0\}$ for $n \geq 5$ iff $f = wg$ for $g \in L^2 \setminus \{0\}$ such that $-\Delta g + Vg = 0$ in S' .*

Lemma 5.2. *The kernel of $S_1 v G_1^0 v S_1$ is trivial in $S_1 L^2(\mathbb{R}^n)$ for $n \geq 5$.*

We note that the proof in the odd dimensional case involves the operator G_2 in place of the operator G_1^0 . This is a notational discrepancy only, both of these operators have integral kernel which is a scalar multiple of $|x - y|^{4-n}$.

Lemma 5.3. *The projection onto the eigenspace at zero is*

$$G_0^0 v S_1 [S_1 v G_1^0 v S_1]^{-1} S_1 v G_0^0.$$

That is,

$$P_e = G_0^0 v D_1 v G_0^0. \quad (66)$$

Lemma 5.4. *Assume that $|V(x)| \lesssim \langle x \rangle^{-\beta}$ for some $\beta > 2$. If $g \in L^2$ is a solution of $(-\Delta + V)g = 0$ then $g \in L^\infty$.*

Corollary 5.5. *P_e is bounded operator from L^1 to L^∞ .*

In addition we have the following oscillatory integral bounds which prove useful in the preceding analysis. Some of these Lemmas along with their proofs appear in Section 6 of [13], accordingly we state them without proof.

Lemma 5.6. *If $k \in \mathbb{N}_0$, we have the bound*

$$\left| \int_0^\infty e^{it\lambda^2} \chi(\lambda) \lambda^k d\lambda \right| \lesssim |t|^{-\frac{k+1}{2}}.$$

Lemma 5.7. *For a fixed $\alpha > -1$, let $f(\lambda) = \tilde{O}_{k+1}(\lambda^\alpha)$ be supported on the interval $[0, \lambda_1]$ for some $0 < \lambda_1 \lesssim 1$. Then, if k satisfies $-1 < \alpha - 2k < 1$ we have*

$$\left| \int_0^\infty e^{it\lambda^2} f(\lambda) d\lambda \right| \lesssim |t|^{-\frac{\alpha+1}{2}}.$$

The following two bounds take advantage of the fact that n is even and hence $\frac{n}{2}$ is an integer.

Lemma 5.8. *If $\alpha > n - 3$ and $f(\lambda) = \tilde{O}_{\frac{n}{2}-1}(\lambda^\alpha)$ supported on the interval $[0, \lambda_1]$ for some $0 < \lambda_1 \lesssim 1$. Then,*

$$\left| \int_0^\infty e^{it\lambda^2} f(\lambda) d\lambda \right| \lesssim |t|^{1-\frac{n}{2}}.$$

Proof. The powers of λ allow us to integrate by parts $\frac{n}{2}-1$ times with no boundary terms, we are left to bound

$$|t|^{1-\frac{n}{2}} \int_0^\infty e^{it\lambda^2} \tilde{O}(\lambda^{-1+}) d\lambda.$$

By the assumption that the integral is supported on $[0, \lambda_1]$ the integral is bounded. \square

Corollary 5.9. *If $\alpha > n-1$ and $f(\lambda) = \tilde{O}_{\frac{n}{2}}(\lambda^\alpha)$ supported on the interval $[0, \lambda_1]$ for some $0 < \lambda_1 \lesssim 1$. Then,*

$$\left| \int_0^\infty e^{it\lambda^2} f(\lambda) d\lambda \right| \lesssim |t|^{-\frac{n}{2}}.$$

The following proof completes the dispersive bounds proven in Section 4.

Proof of Lemma 4.5. Assume that $t > 0$. The proof for $t < 0$ is identical with the \pm signs reversed. Suppose the phase angle $e^{i\lambda r}$ carries a positive sign. In this case there is no stationary phase point of $e^{it\lambda^2+\lambda r}$ in the domain of integration. One can estimate trivially that

$$\left| \int_0^{t^{-1/2}} e^{i(t\lambda^2+\lambda r)} \lambda^{m-1} \Omega(\lambda r) \chi(\lambda) d\lambda \right| \lesssim t^{-\frac{m}{2}},$$

and repeated integration by parts against $e^{it(\lambda^2+\lambda\frac{r}{t})}$ ($\frac{m}{2}$ times if m is even, $\frac{m+1}{2}$ if m is odd) gives the result. It is convenient to note that $|(\frac{d}{d\lambda})^k \Omega(\lambda r)| \lesssim \max(r, \lambda^{-1})^k \langle \lambda r \rangle^{\frac{1-m}{2}}$, so differentiating this expression has a similar effect as when derivatives act on the monomial λ^{m-1} and is better behaved when λr is small.

All boundary terms of the repeated integration by parts can be controlled using the crude bound $|\lambda + \frac{r}{2t}| \geq |\lambda|$. Most of the integral terms are controlled this way as well, but if m is even this creates a few apparent terms of the form $\int_{t^{-1/2}}^\infty |\lambda^{-1} \Omega(\lambda r) \chi(\lambda)| d\lambda$ if all derivatives fall on powers of λ or $(\lambda + \frac{r}{2t})$. In fact no such terms occur, due to cancellation in the derivative $\frac{d}{d\lambda} (\frac{\lambda}{\lambda + \frac{r}{2t}}) = \frac{r}{2t(\lambda + r/2t)^2}$. That leads instead to integrals of the form

$$\frac{r}{2t} \int_{t^{-1/2}}^\infty |(\lambda + r/2t)^{-2} \Omega(\lambda r) \chi(\lambda)| d\lambda \lesssim \frac{r}{2t} \int_0^\infty \frac{1}{(\lambda + r/2t)^2} d\lambda \lesssim 1.$$

Now consider the phase angle $e^{-i\lambda r}$, which causes $e^{i(t\lambda^2 - \lambda r)}$ to have a stationary point $\lambda_0 = \frac{r}{2t}$. If $r < 4\sqrt{t}$, then $0 \leq \lambda_0 < 2t^{-\frac{1}{2}}$, and the integral can be estimated in the same manner as above, splitting the domain into the two pieces $(0, 4t^{-\frac{1}{2}})$ and $(4t^{-\frac{1}{2}}, \infty)$. On the first interval, the bound is clear. On the second interval, the comparison $|\lambda - \lambda_0| \approx |\lambda|$ controls all boundary terms and most of the integral terms as before. For the exceptional integrals, the last bound comes from estimating

$$\frac{r}{2t} \int_{4t^{-1/2}}^{\infty} |(\lambda - r/2t)^{-2} \Omega(\lambda r) \chi(\lambda)| d\lambda \lesssim \frac{r}{2t} \int_{\frac{r}{2t}}^{\infty} \frac{1}{(\lambda - r/2t)^2} d\lambda \lesssim 1.$$

If $r > 4\sqrt{t}$, then $\lambda_0 > 2t^{-\frac{1}{2}}$. Here we apply stationary phase estimates to the interval $(\lambda_0 - t^{-\frac{1}{2}}, \lambda_0 + t^{-\frac{1}{2}})$. On this interval one can approximate $\lambda \approx \lambda_0$, and consequently $|\lambda^{m-1} \Omega(\lambda r)| \approx |\lambda_0^{m-1} \Omega(\lambda_0 r)| \lesssim t^{\frac{1-m}{2}}$. So this integral over the interval $|\lambda - \lambda_0| < t^{-1/2}$ contributes no more than $t^{-m/2}$ as desired.

Noting that $\partial_\lambda e^{it(\lambda - \lambda_0)^2} = 2it(\lambda - \lambda_0)e^{it(\lambda - \lambda_0)^2}$, integration by parts on the interval $[\lambda_0 + t^{-1/2}, +\infty)$ is relatively straightforward. Since $\lambda > \lambda - \lambda_0 > t^{-\frac{1}{2}}$, the worst behavior occurs when all derivatives act on powers of $(\lambda - \lambda_0)$. For all boundary terms arising in this manner it suffices to observe that $\lambda - \lambda_0 = t^{-1/2}$ and $|\lambda^{m-1} \Omega(\lambda r)| \lesssim t^{\frac{1-m}{2}}$ at the left endpoint. The integral terms is controlled by the estimate

$$\begin{aligned} & t^{-k} \int_{\lambda_0 + t^{-1/2}}^{\infty} \frac{\lambda^{m-1} \Omega(\lambda r)}{(\lambda - \lambda_0)^{2k}} d\lambda \\ & \lesssim t^{-k} \int_{\lambda_0 + t^{-1/2}}^{2\lambda_0} \frac{|\lambda_0^{m-1} \Omega(\lambda_0 r)|}{(\lambda - \lambda_0)^{2k}} d\lambda + t^{-k} \int_{2\lambda_0}^{\infty} \frac{|\Omega(\lambda r)|}{\lambda^{2k+1-m}} d\lambda. \end{aligned}$$

We note that we still have $|\lambda_0^{m-1} \Omega(\lambda_0 r)| \lesssim t^{\frac{1-m}{2}}$, thus by a simple change of variables we can bound the first integral by

$$t^{\frac{1-m}{2}-k} \int_{t^{-\frac{1}{2}}}^{\infty} s^{-2k} ds \lesssim t^{-\frac{m}{2}},$$

provided $2k > 1$. For the second integral, we have that $|\Omega(\lambda r)| \lesssim (\lambda r)^{\frac{1-m}{2}}$, and $2\lambda_0 = r/t$, so we need to bound

$$r^{\frac{1-m}{2}} t^{-k} \int_{r/t}^{\infty} \lambda^{\frac{m-1}{2}-2k} d\lambda \lesssim r^{\frac{1-m}{2}} t^{-k} \left(\frac{r}{t}\right)^{\frac{m+1}{2}-2k} \lesssim r^{1-2k} t^{k-\frac{m+1}{2}} \lesssim t^{-\frac{m}{2}}$$

provided $2k > \max(1, \frac{m+1}{2})$. Here we used that $r > 4\sqrt{t}$ in the last inequality.

Integration by parts on the interval $[0, \lambda_0 - t^{-1/2}]$ is only slightly more complicated. For all $m > 2$ there are no boundary terms at $\lambda = 0$, and if $m = 1$ the boundary term has size $(\lambda_0 t)^{-1} \approx r^{-1} \lesssim t^{-\frac{1}{2}}$ since $r > 4\sqrt{t}$. The boundary terms at $\lambda_0 - t^{-\frac{1}{2}}$ are handled identically to the ones at $\lambda_0 + t^{-\frac{1}{2}}$ in the previous case.

When m is even, after integrating by parts $\frac{m}{2}$ times, the main integral consists of expressions with the form

$$t^{-\frac{m}{2}} \int_0^{\lambda_0 - t^{-\frac{1}{2}}} |\lambda^{m-1-j} (\lambda - \lambda_0)^{j+\ell-m} r^\ell \Omega^{(\ell)}(\lambda r)| d\lambda \quad (67)$$

with $j + \ell \leq \frac{m}{2}$. There are three regimes to consider: $\lambda \in (0, \frac{1}{r})$, $\lambda \in (\frac{1}{r}, \frac{\lambda_0}{2})$, and $\lambda \in (\frac{\lambda_0}{2}, \lambda_0 - t^{-\frac{1}{2}})$. In the first regime we use that $|\Omega^{(\ell)}(\lambda r)| \lesssim 1$ and $|\lambda - \lambda_0| \approx \lambda_0$, to see that this integral contributes at most $t^{-\frac{m}{2}} (\frac{t}{r^2})^{m-j-\ell} \lesssim t^{-\frac{m}{2}}$ to the (67). On the second regime, we again have $|\lambda - \lambda_0| \approx \lambda_0$ but now $|\Omega^{(\ell)}(\lambda r)| \lesssim (\lambda r)^{\frac{1-m}{2}-\ell}$. The contribution of this regime to the integral is now bounded by

$$\begin{aligned} t^{-\frac{m}{2}} \lambda_0^{j+\ell-m} r^{\frac{1-m}{2}} \int_0^{\lambda_0} \lambda^{\frac{m-1}{2}-j-\ell} d\lambda &\lesssim t^{-\frac{m}{2}} r^{\frac{1-m}{2}} \lambda_0^{\frac{1-m}{2}} \\ &= t^{-\frac{m}{2}} \left(\frac{\sqrt{t}}{r}\right)^{(m-1)} \lesssim t^{-\frac{m}{2}}. \end{aligned}$$

Since $\frac{m-1}{2} - j - \ell > -1$, we safely extended the lower limit of integration to zero.

On the last regime we note that $\lambda \approx \lambda_0$, so that if we use $s = \lambda_0 - \lambda$ we can bound the contribution by

$$t^{-\frac{m}{2}} \lambda_0^{m-1-j} \Omega^{(\ell)}(\lambda_0 r) r^\ell \int_{t^{-\frac{1}{2}}}^{\lambda_0} s^{j+\ell-m} ds.$$

We first consider the case in which $j + \ell - m < -1$, then we can bound this integral by

$$t^{-\frac{m}{2}} r^{\frac{1-m}{2}} \lambda_0^{\frac{m-1}{2}-j-\ell} \int_{t^{-\frac{1}{2}}}^{\infty} s^{j+\ell-m} ds \lesssim t^{-\frac{m}{2}} \left(\frac{t^{\frac{1}{2}}}{r}\right)^{j+\ell} \lesssim t^{-\frac{m}{2}}.$$

The one exception is if $m = 2$ and $j + \ell = 1$, then we cannot extend the region of integration off to infinity, but instead note that

$$\int_{t^{-\frac{1}{2}}}^{\lambda_0/2} s^{j+\ell-m} ds = \int_{t^{-\frac{1}{2}}}^{\lambda_0/2} s^{-1} ds = \log\left(\frac{\lambda_0}{2t^{-\frac{1}{2}}}\right).$$

So that in this case the third region instead contributes $t^{-\frac{m}{2}} \left(\frac{\sqrt{t}}{r}\right) |\log(\frac{4\sqrt{t}}{r})|$ which is still uniformly bounded by $t^{-\frac{m}{2}}$ since $\sqrt{t}/r < \frac{1}{4}$.

When m is odd the representative expressions are

$$t^{-\frac{m+1}{2}} \int_0^{\lambda_0-t^{-\frac{1}{2}}} |\lambda^{m-1-j} (\lambda - \lambda_0)^{j+\ell-m-1} r^\ell \Omega^{(\ell)}(\lambda r)| d\lambda$$

with $j + \ell \leq \frac{m+1}{2}$. After breaking the integral into the same three regimes, one can similarly show that the contribution of each one is bounded by $t^{-\frac{m}{2}}$ as above. There is again a logarithmic issue in the second regime if $j + \ell = \frac{m+1}{2}$ and in third regime if $m = 1$ and $j + \ell = 1$. Both are resolved by the fact that $(\frac{\sqrt{t}}{r})^m |\log(\frac{4\sqrt{t}}{r})|$ is uniformly bounded over $r > 4\sqrt{t}$. \square

Finally we note the non-oscillatory integral estimate which is proven in [7].

Lemma 5.10. Fix $u_1, u_2 \in \mathbb{R}^n$ and let $0 \leq k, \ell < n$, $\beta > 0$, $k + \ell + \beta \geq n$, $k + \ell \neq n$. We have

$$\int_{\mathbb{R}^n} \frac{\langle z \rangle^{-\beta-}}{|z - u_1|^k |z - u_2|^\ell} dz \lesssim \begin{cases} \left(\frac{1}{|u_1 - u_2|} \right)^{\max(0, k+\ell-n)} & |u_1 - u_2| \leq 1, \\ \left(\frac{1}{|u_1 - u_2|} \right)^{\min(k, \ell, k+\ell+\beta-n)} & |u_1 - u_2| > 1. \end{cases}$$

Furthermore,

$$\int_{\mathbb{R}^n} \frac{\langle z \rangle^{-\beta-}}{|z - u_1|^k |z - u_2|^\ell} dz \lesssim \left(\frac{1}{|u_1 - u_2|} \right)^\alpha,$$

where one can take $\alpha = \max(0, k + \ell - n)$ or $\alpha = \min(k, \ell, k + \ell + \beta - n)$.

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