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The trimmed Anderson model at strong disorder: localisation and its breakup

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Abstract. We explore the properties of discrete random Schrödinger operators in which the random part of the potential is supported on a sub-lattice (the trimmed Anderson model). In this setting, Anderson localisation at strong disorder does not always occur; alternatives include anomalous localisation and, possibly, delocalisation. We establish two new sufficient conditions for localisation at strong disorder as well as a sufficient condition for its absence, and provide examples for both situations. The main technical ingredient is a pair of Wegner-type estimates which are applicable when the covering condition does not hold. Finally, we discuss a coupling between random operators at weak and strong disorder. This coupling is used in an heuristic discussion of the properties of the trimmed Anderson model for sparse sub-lattices, and also in a new rigorous proof of a result of Aizenman pertaining to weak disorder localisation for the usual Anderson model.

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1. Introduction

In this paper we collect several observations pertaining to the spectral properties of random Schrödinger operators in the absence of the so-called covering condition, which stipulates that the random potential is supported on the entire lattice. Let $\mathbf{\Lambda}$ be a lattice of bounded connectivity $\leq \kappa$, and let

$$H(g) = -\Delta + V_0 + gV \tag{1.1}$$

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be the operator acting on $\ell_2(\mathbf{\Lambda})$ by

$$[H(g)\psi](x) = \sum_{y\sim x}(\psi(x) - \psi(y)) + (V_0(x) + gV(x))\psi(x), \quad x \in \mathbf{\Lambda}. \quad (1.2)$$

Here we assume that $V_0 : \mathbf{\Lambda} \to \mathbb{R}$ is a deterministic background potential, $V : (\Omega\times)\mathbf{\Lambda} \to \mathbb{R}$ is a random potential that assumes independent identically distributed entries with distribution μ on a sublattice $\Gamma \subset \mathbf{\Lambda}$, and $g \geq 0$ is a coupling constant. Following [8], we call (1.1) a Γ-trimmed random Schrödinger operator on $\mathbf{\Lambda}$. The usual Anderson model is recovered when $\Gamma = \mathbf{\Lambda} = \mathbb{Z}^d$.

Recall that the Anderson model exhibits localisation at strong disorder: for $g \gg 1$, the spectrum is pure point and the eigenfunctions are exponentially localised. Two strategies of proof are available: the first one, called multi-scale analysis, was devised by Fröhlich and Spencer [10] and the second one, the fractional moment method,— by Aizenman and Molchanov [2]. Both have many variants and ramifications, too numerous to be listed here, and surveyed, for example, by Figotin and Pastur [17, Chapter 15C], Kirsch [14], and Stolz [20]. We also mention the work of Imbrie [11] in which an iterative scheme to diagonalise the random operator is suggested.

It is expected, on physical grounds [16], that, as the strength of the disorder decreases, the Anderson model undergoes a phase transition, and the absolutely continuous component of the spectrum emerges. From the mathematical physics perspective, the proof of such actuality remains one of the greatest challenges in the field.

The variant of the Anderson model which we consider in this work is characterized by two parameters: the strength g of the disorder as in the standard Anderson model, and the sublattice Γ of $\mathbb{Z}^d$ in which we insert the random potential.

For $\Gamma = \mathbb{Z}^d$ we recover the usual Anderson model with almost sure pure point spectrum for large g. We mainly consider the case when Γ is a periodic sublattice of $\mathbb{Z}^d$, and explore the dependence of the spectral properties at strong disorder $g \gg 1$ on the geometry of Γ: when Γ is sufficiently dense (in the sense defined in Theorem 2 below), the behaviour is similar to that of the usual Anderson model (Anderson localisation), whereas for a sparser Γ new phenomena appear, see discussion below.

Another direction (which we do not explore in depth here) is to choose Γ at random, according to the product probability measure (site percolation). Then the case $g = \infty$ is known as quantum percolation (see the paper of Veselić [21] for a survey of results). Finite $g > 0$ leads to a model which combines the features of the Anderson model with those of quantum percolation. Thus one may expect an interesting phase diagram as one varies both the strength of disorder g and the relative density[1] of Γ; the results of the current paper indicate how parts of this phase diagram should look. In particular, our results suggest that the delocalisation part of the phase diagram for such models may be more amenable to analysis than in the usual Anderson model.

Our initial interest in the trimmed Anderson model was triggered by the following question. The known proofs of localisation make use of a priori estimates on the resolvent (Wegner-type bounds), and these in turn require that the support of the potential is the entire lattice (covering condition). One may ask whether localisation at strong disorder still holds when the covering condition is violated.

In the continuum setting, an affirmative answer to this question was established at the bottom of the spectrum using the unique continuation principle (UCP), [15, 19] (Wegner bounds for such models were first established in [6]). Although UCP is not applicable for the lattice Schrödinger operators, Rojas-Molina [18] and Klein with the first author [8] developed Wegner estimates adjusted for the trimmed Anderson model. These estimates allowed to prove localisation in the strong disorder regime, at the bottom of the spectrum. In [18], the case of zero background $V_0 = 0$ was considered, whereas [8] handled arbitrary bounded background potentials.

We make a further contribution in this direction, and prove (Theorems 1 and 2) localisation at strong disorder in several additional situations (not necessarily at the bottom of the spectrum).

Further, we explore the possible alternatives to localisation which may occur at strong disorder.

In certain situations, we prove (Theorem 3) that sufficiently high moments associated with the Green function diverge. Although this phenomenon occurs only at a discrete set of special energies, it implies (Lemma 1.4) the divergence of high moments associated with the quantum dynamics, which is in turn incompatible with strong forms of Anderson localisation. This anomalous behaviour has previously been rigorously observed only in one-dimensional models, cf. Jitomirskaya, Schulz-Baldes, and Stolz [13].

One possibility is the emergence of an absolutely continuous component of the spectral measure about the special energies. While we currently can not rigorously rule out this possibility, we find the following alternative (anomalous localisation) more plausible: the spectral measure is pure point, however, the localisation length of the eigenvectors diverges at the special energies with a power-law singularity.

[1] e.g. $\limsup_{R\to\infty} |B(0,R)\cap\Gamma|/|B(0,R)|$, where $B(0,R)$ is a ball of radius R as in (1.7).

The quantum dynamics picks up the contribution from all eigenvectors, therefore the position of the quantum particle is a heavy-tailed random variable, and its high moments diverge as the time grows.

A naïve classical analogue of this phenomenon (ignoring the subtleties of quantum dynamics and also the presence of multiple channels) is the following: a particle moves along a circle of length L with unit velocity, where L is a heavy-tailed random variable. While this is a case of localisation in any possible sense, sufficiently high moments of the distance from the origin at time t diverge as t grows to infinity.

Finally, in certain spectral regions the trimmed Anderson model at strong disorder can be coupled to a weak disorder Anderson-type model, and this leads us to believe that in these regions the model exhibits delocalisation in dimension $d \geq 3$.

Now let us state the results in more detail. Throughout the paper, we make the following three

Assumptions

- **Inv)** Λ is the d-dimensional lattice $\mathbb{Z}^d$; the sublattice Γ and the background potential V_0 are invariant under a cofinite subgroup $\mathcal{G} \subset \mathbb{Z}^d$.
- **Reg1)** The distribution μ is α-regular for some $\alpha > 0$, meaning that, for any $\epsilon > 0$ and $t \in \mathbb{R}$, $\mu[t - \epsilon, t + \epsilon] \leq C\epsilon^\alpha$.
- **Reg2)** μ has a finite q-moment for some $q > 0$, meaning that

$$M_q = \int |t|^q d\mu(t) < \infty.$$

The invariance assumption **Inv)** is introduced mainly for convenience, and to inscribe the problem into the familiar setting of ergodic (metrically transitive) random operators; it can be mostly omitted or relaxed. The regularity assumptions **Reg1**) and **Reg2**) are essentially used in the arguments.

1.1. Anderson localisation. Denote by $G_z[H] = (H - z)^{-1}$ the resolvent of a self-adjoint operator H acting on $\ell^2(\mathbb{Z}^d)$. If the fractional moment bound

$$\sup_{\epsilon>0} \sup_{x\in\mathbb{Z}^d} \sum_{y\in\mathbb{Z}^d} \mathbb{E}|G_{\lambda+i\epsilon}[H](x, y)|^s e^{\eta\|x-y\|} < \infty \qquad (1.3)$$

holds for some $0 < s < 1$ and $\eta > 0$, we say that H exhibits Anderson localisation at $\lambda \in \mathbb{R}$. Here $\|\cdot\|$ stands for the graph distance (i.e. the ℓ^1 distance) on $\mathbb{Z}^d$.

The methods developed by Aizenman [1] (see further [3]) show that if (1.3) holds for all values of λ in an interval $I \subset \mathbb{R}$, then one has the following more physical dynamical localisation for the spectral restriction $H|_I = \mathbf{P}_I[H]\, H\, \mathbf{P}_I[H]$ of the operator H to I:

$$\sup_{x\in\mathbb{Z}^d} \mathbb{E} \sup_{t\geq 0} \sum_{y\in\mathbb{Z}^d} |e^{itH|_I}(x,y)|^2 e^{\tilde{\eta}\|x-y\|} < \infty. \tag{1.4}$$

These methods do not require major modification in the context of the current paper, therefore we focus on single-energy bounds (1.3).

Following the previous work [8], we are interested in the following question: under which conditions on Γ and λ does Anderson localisation hold at strong disorder, $g \gg 1$? As observed in [8], the restriction

$$H_\Gamma = P_{\Gamma^c} H P^*_{\Gamma^c}$$

of H to the complement of Γ plays an important rôle (here $P_{\Gamma^c}\colon \ell_2(\mathbb{Z}^d) \to \ell_2(\Gamma^c)$ denotes coordinate projection).

Theorem 1. *Let $H(g)$ be a Γ-trimmed random Schrödinger operator on $\mathbb{Z}^d$ satisfying the Assumptions. Suppose $\lambda \notin \sigma(H_\Gamma)$. Then there exist $0 < s < 1$ and $g_0 > 0$ so that (1.3) holds for all $g \geq g_0$.*

Remark 1.1. It was shown in [8] that $\inf_{E\in\sigma(H_\Gamma)} E > \inf_{E\in\sigma(H(g))} E$ almost surely, which implies that the statement above is non empty.

In section 4.1, we prove the more general Proposition 4.1, and deduce Theorem 1. The proof is a relatively straightforward application of the fractional moment method of [2].

The condition $\lambda \notin \sigma(H_\Gamma)$ is however not necessary for Anderson localisation. To illustrate this, consider the case when the complement of Γ is a union of finite connected components. The following theorem implies that, if the connected components are separated by a double layer of of sites in Γ (“double insulation”), Anderson localisation holds at all energies, including the eigenvalues of H_Γ.

Theorem 2. *Let $H(g)$ be a Γ-trimmed random Schrödinger operator on $\mathbb{Z}^d$ satisfying the Assumptions. If Γ^c is the union of finite connected components B_j such that $\operatorname{dist}(B_i, B_j) \geq 3$ for $i \neq j$, then there exist $0 < s < 1$ and $g_0 > 0$ such that (1.3) holds for all $g \geq g_0$ and all $\lambda \in \mathbb{R}$.*

The proof of Theorem 2 appears in Section 4.2; it is also based on the fractional moment method, and makes use of a Wegner-type estimate which we prove in Section 3.1.

The reason due to which double insulation forces localisation has to do with the fact that it rules out the existence of non-trivial formal solutions ψ for H which are supported on Γ^c. Therefore, in the case when the complement of Γ is a union of finite connected components, the following conjecture would be a generalisation of both Theorem 1 and Theorem 2.

Conjecture 1.2. *Suppose that the complement of Γ is a union of finite connected components, and that $\lambda \in \mathbb{R}$ is such that the eigenvalue equation*

$$H(0)\psi = \lambda\psi \tag{1.5}$$

has no non-trivial formal solution ψ supported on Γ^c. Then (1.3) holds for sufficiently large g.

1.2. Anomalous localisation. The situation is different when the eigenvalue equation (1.5) has a solution supported on Γ^c. Let us first consider the case when all the connected components of Γ^c are finite. We believe that, generically, in this situation

$$\lim_{\epsilon\to+0} \sum_{y\in\mathbb{Z}^d} \epsilon^2 \mathbb{E}|G_{\lambda+i\epsilon}[H](x,y)|^2 \|x-y\|^p \tag{1.6}$$

is infinite for sufficiently large $p > 0$. The following theorem confirms this belief under additional hypotheses.

Theorem 3. *Let $H(g)$ be a trimmed random Schrödinger operator satisfying the Assumptions, with arbitrary $g > 0$, so that all the connected components of $\mathbb{Z}^d \setminus \Gamma$ are finite. Fix $x \in \mathbb{Z}^d$, and suppose that there exist a sequence of connected finite subgraphs $B_n \subset \mathbb{Z}^d$ and a pair of constants $C, c > 0$ such that*

(1) *$B(x, R_n) \subset B_n \subset B(x, (R_n)^C)$, where $R_n < R_{n+1} < (R_n)^C$ and*

$$B(x,R) = \{y \in \mathbb{Z}^d \colon \|y-x\| \leq R\}; \tag{1.7}$$

(2) *there exists $y \in B_n$ such that $\|x-y\| \geq (R_n)^c$, and the spectral projection $\mathbf{P}_{\{\lambda\}}[H_n(0)]$ onto the eigenspace of the restriction $H_n(0) = P_{B_n} H(0) P_{B_n}^*$ corresponding to λ satisfies*

$$\left|\mathbf{P}_{\{\lambda\}}[H_n(0)](x,y)\right| \geq (R_n)^{-C} ; \tag{1.8}$$

(3) Range $\mathbf{P}_{\{\lambda\}}[H_n(0)] \subset \ell^2(\Gamma^c)$;

(4) $\min\{|\lambda' - \lambda| \mid \lambda' \in \sigma(H_n(0)) \setminus \{\lambda\}\} \geq (R_n)^{-C}$.

Then (1.6) $= \infty$ *for sufficiently large* p.

Remark 1.3. The second and third assumptions state that there exist non-trivial formal solutions of (1.5) on large boxes, and that all these solutions are supported on Γ^c. These conditions imply in particular the existence of a non-trivial formal solution on the entire lattice. The condition (1.8) implies that these formal solutions exhibit at most power-law spatial growth, i.e. they are generalized eigenfunctions of $H(0)$, whereas the last assumption of the theorem asserts that the spectral gap between λ and the rest of the spectrum decreases as a power of the size of the system, which is a generic condition for a periodic Schrödinger operator. Finally, the first assumption is a mild regularity condition on the growth of the boxes.

The proof of Theorem 3 appears in Section 5.

Let us present a couple of examples for the case of zero background potential $V_0 = 0$ in two dimensions $d = 2$. One can also construct examples in higher dimension along the same lines.

The first example is

$$\Gamma_1(k, m) = \left\{x \in \mathbb{Z}^2 \mid x_1 \in k\mathbb{Z} \ \text{ or } \ x_2 \in m\mathbb{Z}\right\},$$

where $k, m \geq 2$ is a pair of fixed natural numbers. In this case, any eigenfunction of the Dirichlet Laplacian in the rectangular fundamental cell $\{x_1 \in \{1, \dots, k-1\}, x_2 \in \{1, \dots, m-1\}\}$ can be extended (by reflection) to a periodic eigenfunction of the Laplacian on $\mathbb{Z}^2$ which vanishes on $\Gamma_1(k, m)$.

The same is true for

$$\Gamma_2(k) = \left\{x \in \mathbb{Z}^2 \mid x_1 \in k\mathbb{Z} \ \text{ or } \ x_2 - x_1 \in 2\mathbb{Z}\right\}$$

when $k \geq 2$. In this example the fundamental cell is a parallelogram.

Although Theorem 3 proves the divergence of (1.6) at a single energy only, this is sufficient to imply that sufficiently high moments

$$M_p(x, t) = \sum_{y \in \mathbb{Z}^d} \mathbb{E}|e^{itH}(x, y)|^2 \|x - y\|^p$$

associated with the unitary evolution (quantum dynamics) e^{itH} also diverge. Indeed, the following lemma holds (see Section 5 for the proof).

Lemma 1.4. *For any* $\lambda \in \mathbb{R}$, $\epsilon > 0$, *and* $x \in \mathbb{Z}^d$,

$$\int_0^\infty \epsilon e^{-\epsilon t} M_p(x,t) dt \geq \sum_{y \in \mathbb{Z}^d} \mathbb{E} \epsilon^2 |G_{\lambda + i\epsilon}[H](x,y)|^2 \|x - y\|^p. \tag{1.9}$$

Thus, in the setting of Theorem 3, the moments $M_p(x,t)$ are unbounded (as $t \to \infty$) for sufficiently large t. We emphasise that this behaviour is not necessarily a sign of delocalisation. If, as we assumed, all the components of Γ^c are finite, a solution ψ of (1.5) supported on Γ^c may exist only for a discrete set of energies λ. It is plausible that the operator $H(g)$ at strong disorder has pure point spectrum with exponentially decaying eigenfunctions, and that the anomalous behaviour (1.6) reflects the divergence of the localisation length at the special energies. If this is the case, it is an instance of a phenomenon sometimes referred to as anomalous localisation, cf. the survey of Izrailev, Krokhin, and Makarov [12].

To establish anomalous localisation (as opposed to, say, the presence of continuous spectrum in the vicinity of λ), one needs to complement Theorem 3 with an upper bound on (1.6) for small $p > 0$. We have not been able to accomplish this task. To the best of our knowledge, anomalous localisation has to date only been proved in several one-dimensional models; we refer in particular to the work of Jitomirskaya, Schulz-Baldes and Stolz [13].

1.3. Delocalisation. The third possibility that can occur in the invariant setting **Inv**) is that Γ^c is connected (or at least has an infinite connected component), and λ lies in a band of absolutely continuous spectrum of the periodic operator H_Γ.

Conjecture 1.5. *Let* $g \gg 1$, *and let* I *be an interval in the absolutely continuous spectrum of* H_Γ. *If* $d = 2$, $H(g)$ *exhibits localisation* (1.3); *when* $d \geq 3$, $H(g)$ *has absolutely continuous spectrum on* I.

To support this conjecture, we introduce in Section 6.1 a (rigorous) coupling between random operators at strong and weak disorder. Similar ideas have been applied in different context by Wang [22].

In Section 6.2 we provide an heuristic argument (making use of this coupling) in favour of Conjecture 1.5: a trimmed Anderson operator at strong disorder is coupled to an Anderson-type operator at weak disorder in the same dimension. If the Anderson-type operator exhibits localisation at $d = 2$ and delocalisation at $d = 3$ (as one may believe based on the conjectures for the usual Anderson model [16]), the same properties are inherited by the trimmed Anderson operator from which we started.

1.4. Other topics. The following topics are also discussed in this paper.

First, the proofs of Theorems 2 and 3 require somewhat non-standard Wegner-type estimates, which we prove in Sections 3.1 and 3.2.

Second, as an additional application of the strong-to-weak disorder coupling of Section 6.1, we provide a new proof of a theorem of Aizenman [1] (labelled here as Theorem 4) on localisation at the spectral edges at weak disorder.

2. Preliminaries

2.1. Notation. Two sites $x, y \in \mathbf{\Lambda}$ are adjacent, $x \sim y$, if they are connected by an edge.

If $B \subset \mathbf{\Lambda}$ is a subset of the lattice, the boundary ∂B is the set of edges (x, y) with $x \in B$ and $y \notin B$; denote by $\partial_{\mathrm{in}} B$ and $\partial_{\mathrm{out}} B$ its projections onto the x- and y-coordinate, respectively. P_B and P_{B^c} denote the coordinate projections onto B and its complement, respectively.

Denote by $\sigma(A)$ the spectrum of an operator A, and by $G_z[A] = (A - z)^{-1}$ the resolvent of A (defined for $z \notin \sigma(A)$). If A acts on $\ell^2(\mathbf{\Lambda})$, denote by

$$G_z[A](x, y) = \langle \delta_x, (A - z)^{-1} \delta_y \rangle, \quad x, y \in \mathbf{\Lambda},$$

the matrix elements of the resolvent (the Green function).

If A is self-adjoint and $J \subset \mathbb{R}$ is a Borel set, we denote by $\mathbf{P}_J[A]$ the spectral projection on J. Sometimes we use the notation

$$\mathbf{Q}_J[A] = \mathbf{P}_{J^c}[A] = \mathbb{1} - \mathbf{P}_J[A].$$

Finally, we denote by C a sufficiently large positive constant, and by c a sufficiently small positive constant; the values of C and c may change from line to line.

2.2. Properties of the resolvent. The following two formulæ are especially useful for computing the Green function. The first one is the Schur–Banachiewicz formula: if A is an invertible operator acting on $\ell^2(\mathbf{\Lambda})$, $X \subset \mathbf{\Lambda}$, then

$$P_X A^{-1} P_X^* = \left(P_X A P_X^* - P_X A P_{X^c}^* \frac{1}{P_{X^c} A P_{X^c}^*} P_{X^c} A P_X^* \right)^{-1}. \tag{2.1}$$

The second one is the resolvent identity, valid when A is an operator of the form $A = -\Delta + U$ (the potential U need not be real):

$$G_z[A](x,y) = \begin{cases} \sum_{X \ni u' \sim u \in X^c} G_z[A](x,u')G_z[A_X](u,y), & x \in X,\ y \notin X, \\ \sum_{X^c \ni u \sim u' \in X} G_z[A_X](x,u)G_z[A](u',y), & x \notin X,\ y \in X, \\ \sum_{\substack{X^c \ni u \sim u' \in X, \\ X \ni v' \sim v \in X^c}} G_z[A_X](x,u)G_z[A](u',v')G_z[A_X](v,y), & x, y \notin X. \end{cases} \tag{2.2}$$

Next, we shall make use of the Combes–Thomas estimate [7], which states that if $A = -\Delta + U$ is a Schrödinger operator (U is now real) on a lattice $\mathbf{\Lambda}$ of bounded connectivity, and $z \notin \sigma(H)$, then $|G_z[A](x,y)|$ decays exponentially in $\operatorname{dist}(x,y)$:

$$|G_z[A](x,y)| \leq C \exp(-c \operatorname{dist}(x,y)) \quad (z \notin \sigma(A)), \tag{2.3}$$

where the constants $C, c > 0$ depend only on the distance from z to the spectrum of A and on the connectivity of the lattice. A version with a sharp dependence of c on the distance from the spectrum was proved by Barbaroux, Combes, and Hislop [4].

2.3. Fractional moments: auxiliary estimates. Here we cite two estimates which commonly appear in the applications of the fractional moment method, and go back to the original work of Aizenman and Molchanov [2].

The first one is a decoupling inequality for rational functions. We cite it in the form of [9, Proposition 3.1], which is slightly more general than the original one of [2, Appendix III] (where fractional-linear functions were considered).

Lemma 2.1. *Let μ be a probability measure on $\mathbb{R}$ satisfying the assumptions* **Reg1**) *and* **Reg2**). *Let $a_1, \dots, a_l, b_1, \dots, b_m \in \mathbb{C}$, and let $s, r > 0$ be such that $rm < \alpha$ and $q \geq (sl + rm)\frac{\alpha}{\alpha - rm}$. Then*

$$\int \frac{\prod_{j=1}^{l} |v - a_j|^s}{\prod_{i=1}^{m} |v - b_i|^r} \, d\mu(v) \asymp \frac{\prod_{j=1}^{l} (1 + |a_j|)^s}{\prod_{i=1}^{m} (1 + |b_i|)^r},$$

where the "$\asymp$" sign means that LHS $\leq C$ RHS $\leq C'$ LHS, *and the numbers $C, C' > 0$ do not depend on the a_j and b_i.*

The following Wegner-type estimates are a restatement of those in [2, Appendix II]:

Lemma 2.2. *Let A be a random self-adjoint operator acting on $\ell^2(\mathbf{\Lambda})$, and let $x, y \in \mathbf{\Lambda}$.*

(1) *If $A(x,x)$ is sampled from a measure μ obeying* **Reg1)** *independently of all the other entries of A, then*

$$\mathbb{E}|G_z[A](x,x)|^s < C_s < \infty$$

for any $s < \alpha$, uniformly in $z \notin \mathbb{R}$.

(2) *If both $A(x,x)$ and $A(y,y)$ are sampled from a measure μ obeying* **Reg1)** *independently of each other and of the other entries of A, then also*

$$\mathbb{E}|G_z[A](x,y)|^s < C_s < \infty.$$

2.4. Fractional moments: decay of the resolvent. It is convenient to express the decay of the off-diagonal elements of the resolvent, and, more generally, of a kernel $A : X \times X \to \mathbb{C}$, in terms of the following quantity χ, which was introduced by Aizenman [1], and which quantifies the exponential decay of a kernel with respect to a metric. If ρ is a metric on $X \subset \mathbf{\Lambda}$, set

$$\chi_\rho(A) = \sup_{x \in X} \sum_{y \in X} e^{\rho(y,x)} |A(y,x)|.$$

The expression $|A|^s$ will denote a point-wise power of the point-wise absolute value of the kernel A, thus

$$\chi_\rho(|A|^s) = \sup_{x \in X} \sum_{y \in X} e^{\rho(y,x)} |A(y,x)|^s.$$

We denote

$$\|\rho\| = \sup_{x \sim y} \rho(x,y)$$

and assume (here and forth) that this quantity is finite.

The resolvent identity (2.2) implies the bounds

$$\chi_\rho(P_{X^c} G_z[A] P_{X^c}^*) \le \kappa^2 e^{2\|\rho\|} \chi_\rho^2(G_z[A_X]) \chi_\rho(P_X G_z[A] P_X^*) \tag{2.4}$$

and

$$\chi_\rho(G_z[A]) \le \kappa e^{\|\rho\|} \chi_\rho(G_z[A_X])(1 + \kappa e^{\|\rho\|} \chi_\rho(G_z[A_X]) \chi_\rho(P_X G_z[A] P_X^*)). \tag{2.5}$$

The next statement is a translation of [2, Lemma 2.1] to the χ-notation of [1]. We now set $X = \mathbf{\Lambda}$, and let $A^{\text{off-diag}}$ denote the off-diagonal part of a kernel A.

Lemma 2.3 (Aizenman–Molchanov). *Let A be an operator acting on $\ell^2(\mathbf{\Lambda})$, and let $V : \mathbf{\Lambda} \to \mathbb{R}$ be an independent identically distributed random potential satisfying the decoupling inequality*

$$\mathbb{E}\frac{|V(y)-a|^s}{|V(y)-b|^s} \geq C_s^{-1}\mathbb{E}\frac{1}{|V(y)-b|^s}, \quad a,b \in \mathbb{C},\ b \notin \mathbb{R}. \tag{2.6}$$

for some $0 < s < 1$ and $C_s > 0$. Let ρ be a metric on $\mathbf{\Lambda}$ such that $\chi_\rho(|A^{\text{off-diag}}|^s)$ is finite. Then, for

$$g^s > C_s \chi_\rho(|A^{\text{off-diag}}|^s),$$

one has

$$\chi_\rho(\mathbb{E}|G_z[A+gV]|^s) \leq \frac{C_s}{g^s - C_s\chi_\rho(|A|^s)}.$$

Proof of Lemma 2.3. Let $x, y \in \mathbf{\Lambda}$. According to the definition of the resolvent, $G_z[A+gV](gV + A - z\mathbb{1}) = \mathbb{1}$, which can be written as

$$\begin{aligned} &G_z[A+gV](x,y)(gV(y) + A(y,y)) \\ &\quad = -\sum_{u\neq y} G_z[A+gV](x,u)A(u,y) + \delta(x-y). \end{aligned}$$

Taking expectation of the s-moment and applying the inequality $|a+b|^s \leq |a|^s + |b|^s$, we obtain

$$\begin{aligned} &\mathbb{E}|G_z[A+gV](x,y)|^s|gV(y) + A(y,y)|^s \\ &\quad \leq \sum_{u\neq y} \mathbb{E}|G_z[A+gV](x,u)|^s|A(u,y)|^s + \delta(x-y). \end{aligned}$$

As a function of $V(y)$, the expression $G_z[A+gV](x,y)$ has the form

$$G_z[A+gV](x,y) = a(V(y)-b)^{-1},$$

where a, b may be random but do not depend on $V(y)$. Therefore the decoupling estimate (2.6) yields the inequality

$$\mathbb{E}|G_z[A+gV](x,y)|^s|gV(y) + A(y,y)|^s \geq C_s^{-1} g^s \mathbb{E}|G_z[A+gV](x,y)|^s,$$

which implies

$$\mathbb{E}|G_z[A+gV](x,y)|^s \leq \frac{C_s}{g^s}\Big\{\sum_{u\neq y}\mathbb{E}|G_z[A+gV](x,u)|^s|A(u,y)|^s + \delta(x-y)\Big\}.$$

Multiplying both sides by $e^{\rho(x,y)}$ and summing over $y \in \mathbf{\Lambda}$, we obtain

$$\chi_\rho(\mathbb{E}|G_z[A+gV]|^s) \leq \frac{C_s}{g^s}\{\chi_\rho(\mathbb{E}|G_z[A+gV]|^s)\,\chi_\rho(|A^{\text{off-diag}}|^s) + 1\}. \qquad \square$$

Remark 2.4. Lemma 2.1 provides examples of distributions satisfying (2.6), here and in Theorem 4 below.

3. Wegner estimates

In this section we prove two Wegner-type estimates.

3.1. First Wegner estimate.

We start from a general property of discrete Schrödinger operators, cf. Bourgain and Klein [5, §2.2].

Lemma 3.1. *Let $B \subset \mathbb{Z}^d$ be a finite box, and let $\partial_{in} B \subset B' \subset B$. If ψ is an eigenvector of a random Schrödinger operator $-\Delta|_B + U$ on B with eigenvalue λ, and $x \in B$ is a site with first coordinate*

$$x_1 = \max_{y \in B} y_1 - n,$$

then

$$|\psi(x)| \leq \sum_{y \in B'} |\psi(y)| \sum_{S \in \mathfrak{S}_{xy}} \prod_{u \in S} |U(u) + 2d - \lambda|,$$

where

$$\mathfrak{S}_{xy} \subset \{u \in B \mid u_1 > x_1\}, \quad \sum_y \#\mathfrak{S}_{xy} \leq (2d)^n,$$

every $S \in \mathfrak{S}_{xy}$ is of cardinality $\#S \leq n$, and $\#S \cap B' \leq 1$.

Proof. With the convention that an empty product is equal to one, the estimate holds for $n = 0$ and, more generally, for $x \in B'$. If $x \notin B'$, we proceed by induction. The eigenvalue equation at $x' = x + e_1$ yields

$$|\psi(x)| \leq |2d + U(x') - \lambda||\psi(x')| + \sum_{w \sim x', w \neq x} |\psi(w)|,$$

whence the claim follows with

$$\mathfrak{S}_{xy} = \{S \cup \{x'\} \mid S \in \mathfrak{S}_{x'y}\} \cup \bigcup_{w \sim x', w \neq x} \mathfrak{S}_{wy}. \qquad \square$$

In the context of trimmed random Schrödinger operators, Lemma 3.1 implies:

Corollary 3.2. *Let $B \subset \mathbb{Z}^d$ be a finite box such that $\partial_{in} B \subset \Gamma$. Then, for sufficiently small s, the restriction $H(g)|_B = P_B H(g) P_B^*$ of an operator $H(g)$ satisfying the assumptions* **Reg1**) *and* **Reg2**) *admits the estimate*

$$\mathbb{E}\|G_z[H(g)|_B]\|^s \leq C(B, g), \quad z \notin \mathbb{R}.$$

Proof. Let $\{\psi\}_{\psi\in\Psi}$ be the eigenfunctions of $H(g)|_B$. Then, for every $x \in B$,

$$1 = \sum_{\psi\in\Psi} |\psi(x)|^2 \le C'_B \sum_{y\in\partial_{\text{in}}B} |\psi(y)|^2 \sum_{S\in\mathfrak{S}_{xy}} \prod_{u\in S} |U(u) + 2d - \lambda|^2,$$

where $U = V_0 + gV$, hence for $\Re w = \Re z = \lambda$

$$\Im \operatorname{tr} G_w[H(g)|_B]$$
$$\le C'_B \sum_{y\in\partial_{\text{in}}B} |\psi(y)|^2 \sum_{S\in\mathfrak{S}_{xy}} \prod_{u\in S} |U(u) + 2d - \lambda|^2 \Im \operatorname{tr} P_{\partial_{\text{in}}B} G_w[H(g)|_B] P^*_{\partial_{\text{in}}B}.$$

The proof is concluded by taking the s-th moment with sufficiently small $s > 0$ (note that it is sufficient to establish the bound for λ in a compact interval depending on B and g). ☐

Corollary 3.2 yields the bound

$$\mathbb{E}|G_z[H(g)|_B](x, y)|^s \le C(B, g). \tag{3.1}$$

The constant $C(B, g)$ grows exponentially in the diameter of B, and as $1 + g^s$ in g. Note that, for $x, y \in \Gamma \cap B$, Lemma 2.2 yields the better estimate

$$\mathbb{E}\, |G_z[H(g)|_B](x, y)|^s \le C g^{-s} \tag{3.2}$$

(note that the factor g^{-s} is due to the normalisation, which is different from that of Lemma 2.2).

3.2. Second Wegner estimate.

The next deterministic lemma holds for any $H(g) = H(0) + gV$ with V supported on Γ.

Let $B \subset \mathbb{Z}^d$ be a finite box, and let $H(g)|_B = P_B H(g) P^*_B$ be the restriction of $H(g)$ to B. Denote by mult_λ the multiplicity of λ in the spectrum of $H(0)|_B$, and by gap_λ the distance from λ to $\sigma(H(0)|_B) \setminus \{\lambda\}$.

Lemma 3.3. *Suppose all the eigenvectors of $H(0)|_B$ corresponding to λ are supported on $B \setminus \Gamma$. If ϕ is a normalised eigenvector of $H(g)|_B$ corresponding to λ', where $|\lambda - \lambda'| \le \text{gap}_\lambda/3$ such that $\phi \perp \operatorname{Ker}(H(0)|_B - \lambda)$, then*

$$\|\phi|_{B\cap\Gamma}\| \ge \frac{\text{gap}_\lambda}{3g\|V|_B\|_\infty}. \tag{3.3}$$

Proof of Lemma 3.3. If (3.3) fails,

$$\|(H(0)|_B - \lambda)\phi\| \le \|(\lambda - \lambda')\phi\| + \|(H(0)|_B - H(g)|_B)\phi\|$$
$$\le \text{gap}_\lambda/3 + \text{gap}_\lambda/3 < \text{gap}_\lambda,$$

in contradiction with the assumption. Thus (3.3) is proved. ☐

Lemma 3.4. *Assume that $H(g)$ satisfies* **Reg1**) *and* **Reg2**)*, and that all the eigenvectors of $H(0)|_B$ corresponding to λ are supported on $B \setminus \Gamma$. Then, for all $\epsilon \le \operatorname{gap}_\lambda /3$ and sufficiently small $s > 0$,*

$$\mathbb{P}\left\{H(g)|_B \text{ has} > \operatorname{mult}_\lambda \text{ eigenvalues in } (\lambda-\epsilon, \lambda+\epsilon)\right\} \le \frac{C\epsilon^s g^s}{\operatorname{gap}_\lambda^{2s}} \left(\# B \cap \Gamma\right)^2 .$$

Proof. Suppressing the dependence of the spectral projectors on the operator $H(g)|_B$, we have

$$\begin{aligned}
\operatorname{tr} \mathbf{P}_{[\lambda-\epsilon,\lambda+\epsilon]} &= \operatorname{tr} \mathbf{P}_{\{\lambda\}} \mathbf{P}_{[\lambda-\epsilon,\lambda+\epsilon]} + \operatorname{tr} \mathbf{Q}_{\{\lambda\}} \mathbf{P}_{[\lambda-\epsilon,\lambda+\epsilon]} \\
&\le \operatorname{tr} \mathbf{P}_{\{\lambda\}} + 2\epsilon \Im \operatorname{tr} \mathbf{Q}_{\{\lambda\}} \mathbf{P}_{[\lambda-\epsilon,\lambda+\epsilon]} G_{\lambda+i\epsilon}[H(g)|_B] \mathbf{P}_{[\lambda-\epsilon,\lambda+\epsilon]} \mathbf{Q}_{\{\lambda\}} \\
&\le \operatorname{mult}_\lambda + \frac{18\epsilon g^2 \|V|_B\|_\infty^2}{\operatorname{gap}_\lambda^2} \Im \operatorname{tr} P_\Gamma G_{\lambda+i\epsilon}[H(g)|_B] P_\Gamma^*,
\end{aligned}$$

where we used Lemma 3.3 and the expansion

$$\Im \operatorname{tr} G_{\lambda+i\epsilon}[H(g)|_B] = \sum_j \sum_{x \in B} \frac{\epsilon |\psi_j(x)|^2}{(\lambda_j - \lambda)^2 + \epsilon^2}$$

over eigenvectors $H(g)|_B \psi_j = \lambda_j \psi_j$.

Let

$$N = \operatorname{tr} \mathbf{P}_{[\lambda-\epsilon,\lambda+\epsilon]} - \operatorname{mult}_\lambda, \tag{3.4}$$

then

$$\begin{aligned}
N &\le \frac{18\epsilon g^2 \|V|_B\|_\infty^2}{\operatorname{gap}_\lambda^2} \Im \operatorname{tr} P_\Gamma G_{\lambda+i\epsilon}[H(g)|_B] P_\Gamma^* \\
&\le \frac{18\epsilon g^2 \|V|_B\|_2^2}{\operatorname{gap}_\lambda^2} \sum_{x \in B\cap\Gamma} |G_{\lambda+i\epsilon}[H(g)|_B](x,x)|.
\end{aligned}$$

Therefore

$$N^s \le \frac{18^s \epsilon^s g^{2s}}{\operatorname{gap}_\lambda^{2s}} \sum_{x,y \in B\cap\Gamma} |V(y)|^{2s} \left|G_{\lambda+i\epsilon}[H(g)|_B](x,x)\right|^s .$$

Integrating over the distribution of $V(x)$ and using the Cauchy–Schwarz inequality to decouple the potential from the Green function and then the Wegner-type estimate in the first item of Lemma 2.3 in conjunction with Lemma 2.1 to bound the latter, we conclude that

$$\mathbb{P}\{N \ge 1\} \le \mathbb{E} N^s \le \frac{C\epsilon^s g^s}{\operatorname{gap}_\lambda^{2s}} (\#\Gamma \cap B)^2 . \qquad \square$$

We furthermore obtain:

Corollary 3.5. *Assume that $H(g)$ satisfies* **Reg1**) *and* **Reg2**)*, and that all the eigenvectors of $H(0)|_B$ corresponding to λ are supported on $B \setminus \Gamma$. Then, for sufficiently small $s_0 > 0$ and any $0 < s < s_0$,*

$$\mathbb{E}\|\mathbf{Q}_{\{\lambda\}} G_{\lambda+i\epsilon}[H(g)|_B]\mathbf{Q}_{\{\lambda\}}\|^s \leq \frac{C g^{s_0} (\#\Gamma \cap B)^2}{\mathrm{gap}_\lambda^{2s_0}} + \frac{C}{\mathrm{gap}_\lambda^{s_0}}.$$

Proof. We have

$$\mathbb{E}\|\mathbf{Q}_{\{\lambda\}} G_{\lambda+i\epsilon}[H(g)|_B]\mathbf{Q}_{\{\lambda\}}\|^s = \int_0^\infty \mathbb{P}\{\|\mathbf{Q}_{\{\lambda\}} G_{\lambda+i\epsilon}[H(g)|_B]\mathbf{Q}_{\{\lambda\}}\|^s \geq t\} dt.$$

For $t \leq (3/\mathrm{gap}_\lambda)^{s_0}$ we bound the integrand by 1, and for larger t we use Lemma 3.4 (with s_0 in place of s). □

4. Localisation

4.1. Outside the spectrum of H_Γ. In this section we prove Theorem 1. It will be convenient to drop the assumption **Inv**), and to work on a general lattice $\mathbf{\Lambda}$ which we only assume to have bounded connectivity $\leq \kappa$.

If $X \subset \mathbf{\Lambda}$, let $T_X : \ell^2(X^c) \to \ell^2(X)$ be the adjacency operator,

$$T_X(x, y) = \begin{cases} 1 & \text{for } x \sim y, \\ 0 & \text{otherwise.} \end{cases}$$

The condition for localisation is expressed in terms of the kernel

$$K = (P_\Gamma \Delta P_\Gamma^* + T_\Gamma G_z[H_\Gamma] T_\Gamma^*)^{\text{off-diag}}. \tag{4.1}$$

Proposition 4.1. *Let $H(g)$ be a Γ-trimmed random Schrödinger operator satisfying* **Reg1**) *and* **Reg2**) *on a lattice $\mathbf{\Lambda}$ of connectivity $\leq \kappa$. For any $0 < s < \alpha(1 + 2\alpha q^{-1})^{-1}$ there exists $C_s > 0$ that may depend on s and the constants in* **Reg1**) *and* **Reg2**)*, such that the following holds: if*

$$g^s > C_s \chi_\rho(|K|^s),$$

then

$$\chi_\rho(\mathbb{E}|G_z[H(g)]|^s) \leq \frac{C_s \kappa e^{\|\rho\|}}{g^s - C_s \chi_\rho(|K|^s)} \chi_\rho(|G_z[H_\Gamma]|^s)\{1 + \kappa e^{\|\rho\|} \chi_\rho(|G_z[H_\Gamma]|^s)\}.$$

Let us show that Proposition 4.1 implies Theorem 1.

Proof of Theorem 1. Suppose $H(g)$ satisfies the Assumptions. According to the Combes–Thomas estimate (2.3), $G_z[H_\Gamma]$ decays exponentially for $\lambda \notin \sigma(H_\Gamma)$, therefore $\chi_\rho[|G_z[H_\Gamma]|^s]$ is finite when ρ is a small multiple of the graph metric on $\mathbb{Z}^d$, and hence so is

$$\chi_\rho(|K|^s) \le \kappa e^{\|\rho\|} + \kappa^2 e^{2\|\rho\|} \chi_\rho(|G_z[H_\Gamma]|^s).$$

According to Proposition 4.1, $\chi_\rho(\mathbb{E}|G_z[H(g)]|^s)$ is finite for sufficiently large g, therefore (1.3) holds. □

Now we prove Proposition 4.1.

Proof of Proposition 4.1. Using the Schur–Banachiewicz formula (2.1) we get

$$P_\Gamma G_z[H] P_\Gamma^* = G_z[gV|_\Gamma - D - K],$$

where $D + K$ is the decomposition of

$$P_\Gamma \Delta P_\Gamma^* - V_0|_\Gamma + T_\Gamma G_z[H_\Gamma] T_\Gamma^*$$

into diagonal and off-diagonal parts (this notation is consistent with (4.1)). According to the Aizenman–Molchanov estimate (Lemma 2.3), for

$$g^s > C_s \chi_\rho(|K^{\text{off-diag}}|^s)$$

we have

$$\chi_\rho(P_\Gamma G_z[H] P_\Gamma^*) \le \frac{C_s}{g^s - C_s \chi_\rho(|K^{\text{off-diag}}|^s)}$$

(the assumption (2.6) is satisfied according to Lemma 2.1.) The proposition now follows from the corollary (2.5) of the resolvent identity. □

4.2. Double insulation

Proof of Theorem 2. Let us partition the lattice $\mathbb{Z}^d$ into disjoint pieces B:

$$\mathbb{Z}^d = \biguplus_{B \in \mathfrak{B}} B,$$

such that diam $B \le$ const and $\partial_{\text{in}} B \subset \Gamma$ for every $B \in \mathfrak{B}$. We do not make any additional assumptions on the shape of $B \in \mathfrak{B}$.

Let $x, y \in \mathbb{Z}^d$. Applying the resolvent identity (2.2), we can represent $G_z[H](x, y)$ as a sum of terms of the form

$$G_z[H|_{B_1}](x, u_1) G_z[H|_{B_2}](u_1', u_2) G_z[H|_{B_3}](u_2', u_3) \ldots G_z[H|_{B_n}](u_{n-1}', y),$$

where $B_1 \ni \{x, u_1\}$, $B_2 \ni \{u'_1, u_2\}, \ldots, B_n \ni \{u'_{n-1}, y\}$ are distinct boxes, and u_j is adjacent to u'_j. In particular, $u_j, u'_{j-1} \in \partial_{\rm in} B_j \subset \Gamma$.

Taking fractional moments, we obtain:

$$\mathbb{E}|G_z[H](x,y)|^s$$
$$\leq \sum \mathbb{E}|G_z[H|_{B_1}](x,u_1)|^s \prod_{j=2}^{n-1} \mathbb{E}|G_z[H|_{B_j}](u'_{j-1},u_j)|^s \mathbb{E}|G_z[H|_{B_n}](u'_{n-1},y)|^s.$$

For small $s > 0$, we bound the first and last term by $C(B_j, g)$ using (3.1), and all the other terms by const g^{-s} using (3.2). For large $g \geq g_0$, the resulting expansion converges, and is exponentially decaying in $\|x - y\|$. □

5. Anomalous localisation

Proof of Theorem 3. Set $\mathbf{P}_{\{\lambda\}} = \mathbf{P}_{\{\lambda\}}[H_n(0)]$, and $\mathbf{Q}_{\{\lambda\}} = \mathbf{Q}_{\{\lambda\}}[H_n(0)]$, and let

$$G_n^{\pm} = G_{\lambda \pm i\epsilon}[H_n(g)], \quad G_n^{\Re} = \frac{G_n^+ + G_n^-}{2}, \quad G_n^{\Im} = \frac{G_n^+ - G_n^-}{2i}.$$

Suppose in contrapositive that the assertion of the theorem is false. Then, for any $p > 0$, one can find a sequence $\epsilon_j \searrow 0$ such that the inequality

$$\mathbb{E}|G_{\lambda+i\epsilon}[H(g)](x,v)|^s \leq M_p \epsilon^{-s} \|x - v\|^{-sp/2}$$

holds for all $\epsilon = \epsilon_j$ and all $x, v \in \mathbb{Z}^d$. For any $A > 0$ we can find $n = n_j$ so that $(R_n)^{-CA} \leq \epsilon \leq (R_n)^{-A}$. We shall choose the value of A in the sequel.

For a fixed $x \in \mathbb{Z}^d$, consider a site $y \in \mathbb{Z}^d$ that satisfies $\|x - y\| \geq (R_n)^c$. Using the first resolvent identity, we can estimate

$$\begin{aligned}
&\mathbb{E}|G_n^{\Im}(x,y)|^s \\
&\quad \leq \mathbb{E}|G_{\lambda+i\epsilon}[H(g)](x,y)|^s + \sum_{\langle v,u\rangle \in \partial B_n} \mathbb{E}|G_{\lambda+i\epsilon}[H(g)](x,v)|^s |G_n^+(u,y)|^s \\
&\quad \leq \mathbb{E}|G_{\lambda+i\epsilon}[H(g)](x,y)|^s + 2d\epsilon^{-s} \sum_{v \in \partial^{\rm in} B_n} \mathbb{E}|G_{\lambda+i\epsilon}[H(g)](x,v)|^s \\
&\quad \leq M_p\{\epsilon^{-s}(R_n)^{-csp/2} + 2d\epsilon^{-2s}(R_n)^{-sp/2}\}.
\end{aligned} \tag{5.1}$$

On the other hand, according to Assumption 2 of the theorem, there exists y which satisfies $\|x-y\| \geq (R_n)^c$ and

$$\begin{aligned}\mathbb{E}|G_n^{\Im}(x,y)|^s &= \mathbb{E}|\mathbf{P}_{\{\lambda\}}G_n^{\Im}\mathbf{P}_{\{\lambda\}}(x,y) + \mathbf{Q}_{\{\lambda\}}G_n^{\Im}\mathbf{Q}_{\{\lambda\}}(x,y)|^s \\ &\geq |\mathbf{P}_{\{\lambda\}}G_n^{\Im}\mathbf{P}_{\{\lambda\}}(x,y)|^s - \mathbb{E}|\mathbf{Q}_{\{\lambda\}}G_n^{\Im}\mathbf{Q}_{\{\lambda\}}(x,y)|^s.\end{aligned} \tag{5.2}$$

According to Assumption 3 of the theorem,

$$f(H_n(g))\mathbf{P}_{\{\lambda\}}[H_n(0)] = f(\lambda)\mathbf{P}_{\{\lambda\}}[H_n(0)]$$

for any function f, therefore the assumption (1.8) allows to bound the first term of (5.2) from below by $\epsilon^{-s}R_n^{-C}$. If A is chosen to be sufficiently large, Corollary 3.5 implies that the second term is bounded from above by one half of this quantity. Therefore

$$\mathbb{E}|G_n^{\Im}(x,y)|^s \geq \epsilon^{-s}(R_n)^{-C}/2.$$

For sufficiently large p, this lower bound is in contradiction with the upper bound (5.1). □

We conclude this section with the

Proof of Lemma 1.4. We start from the identity

$$\int_0^\infty e^{it(H-\lambda+i\epsilon)}dt = iG_{\lambda-i\epsilon}[H],$$

which implies:

$$|G_{\lambda-i\epsilon}[H](x,y)| \leq \int_0^\infty |e^{itH}(x,y)|e^{-\epsilon t}dt.$$

Taking the expectation and applying the Cauchy–Schwarz inequality, we obtain:

$$\epsilon^2\mathbb{E}|G_{\lambda-i\epsilon}[H](x,y)|^2 \leq \int_0^\infty \epsilon e^{-\epsilon t}\mathbb{E}|e^{itH}(x,y)|^2 dt.$$

This proves (1.9), since the sign of ϵ does not affect the absolute value. □

6. Strong-to-weak disorder coupling

In this section we construct a coupling between a random operator at strong disorder and another one at weak disorder. A similar coupling appears in the work of Wang [22], who used it to construct examples of long-range operators with exponentially decaying Green function.

6.1. The hedgehog lattice. Let $\mathbf{\Lambda}$ be a lattice. Construct the hedgehog lattice $\mathbf{\Lambda}^{Ш} = \mathbf{\Lambda} \times \{0, 1\}$ with bonds defined by

$$(x, i) \sim (y, j) \iff \begin{cases} \text{either } x = y \text{ and } i = 1 - j \\ \text{or } x \sim y \text{ and } i = j = 0 \end{cases} .$$

Given an operator $H(0)$ on $\mathbf{\Lambda}$ and a potential $U : \mathbf{\Lambda} \to \mathbb{C}$, consider the operator $H^{Ш}$ on $\ell^2(\mathbf{\Lambda}^{Ш})$, defined by

$$H^{Ш}\begin{pmatrix} \psi_1 \\ \psi_0 \end{pmatrix} = \begin{pmatrix} U & -\mathbb{1} \\ -\mathbb{1} & H(0) \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_0 \end{pmatrix}. \tag{6.1}$$

Observe that $H^{Ш}$ (or rather, $H^{Ш} + \mathbb{1}$) is a $(\mathbf{\Lambda} \times \{1\})$-trimmed random Schrödinger operator on $\ell^2(\mathbf{\Lambda}^{Ш})$, if the values of U are independent and real.

The Schur–Banachiewicz formula (2.1) relates the resolvent of $H^{Ш}$ to the resolvents of $H = H(0) + U_z^{\#}$, $U_z^{\#} = (z - U)^{-1}$, on $\mathbf{\Lambda} \times \{0\}$, and $H' = -G_z[H(0)] + U$ on $\mathbf{\Lambda} \times \{1\}$. Namely, set $P_0 = P_{\mathbf{\Lambda}\times\{0\}}$, $P_1 = P_{\mathbf{\Lambda}\times\{1\}}$. Then

$$P_0 G_z[H^{Ш}] P_0^* = G_z[H(0) + U_z^{\#}], \tag{6.2a}$$

$$P_1 G_z[H^{Ш}] P_1^* = G_z[-G_z[H(0)] + U]. \tag{6.2b}$$

The first application of the relations (6.2) is a new derivation of a theorem of Aizenman [1] which provides a sufficient condition for localisation at weak disorder near the spectral edges.

Let $V : \mathbf{\Lambda} \to \mathbb{R}$ be a random potential, and consider the operator $H(g) = H(0) + gV$ on $\ell^2(\mathbf{\Lambda})$.

Theorem 4 (Aizenman). *Fix $0 < s < 1$, and suppose the random potential $V : \mathbf{\Lambda} \to \mathbb{R}$ satisfies the decoupling property*

$$\mathbb{E}\frac{|V(y) - a|^s}{|V(y) - b|^s} \geq C_s^{-1}\mathbb{E}\frac{1}{|V(y) - b|^s}, \quad a, b \in \mathbb{C},\ b \notin \mathbb{R}. \tag{6.3}$$

Let ρ be a metric on $\mathbf{\Lambda}$ so that

$$\chi = \limsup_{\epsilon \to +0} \chi_\rho(|G_{\lambda+i\epsilon}[H(0)]|^s) < \infty.$$

Then, for $g^{-s} > C_\mu \chi$, one has

$$\limsup_{\epsilon \to +0} \chi_\rho(\mathbb{E}|G_{\lambda+i\epsilon}[H(0) + gV]|^s) \leq \frac{C_\mu \kappa^2 e^{2\|\rho\|} \chi^2}{g^{-s} - C_\mu \chi}.$$

Proof. Let $U = \lambda + i\epsilon - \frac{1}{gV}$, and construct the operator H^{III} associated with U as in (6.1). We have: $U^{\#}_{\lambda+i\epsilon} = gV$, therefore the second half of (6.2) yields:

$$P_1 G_{\lambda+i\epsilon}[H^{\text{III}}_\epsilon] P_1^* = G_{\lambda+i\epsilon}[-G_{\lambda+i\epsilon}[H(0)] + U].$$

By Lemma 2.3, if

$$g^{-s} > C_\mu \chi_\rho(|G_{\lambda+i\epsilon}[H(0)]|^s),$$

we have:

$$\chi_\rho(\mathbb{E}|P_1 G_{\lambda+i\epsilon}[H^{\text{III}}_\epsilon] P_1^*|^s) \le \frac{C_\mu}{g^{-s} - C_\mu \chi_\rho(|G_{\lambda+i\epsilon}[H(0)]|^s)}.$$

According to the corollary (2.4) of the resolvent identity,

$$\begin{aligned}
&\chi_\rho(\mathbb{E}|P_0 G_{\lambda+i\epsilon}[H^{\text{III}}_\epsilon] P_0^*|^s) \\
&\quad \le \kappa^2 e^{2\|\rho\|} \chi^2_\rho(G_{\lambda+i\epsilon}[H(0)]) \chi_\rho(\mathbb{E}|P_1 G_{\lambda+i\epsilon}[H^{\text{III}}_\epsilon] P_1^*|^s) \\
&\quad \le \frac{C_\mu \kappa^2 e^{2\|\rho\|} \chi^2_\rho(G_{\lambda+i\epsilon}[H(0)])}{g^{-s} - C_\mu \chi_\rho(|G_{\lambda+i\epsilon}[H(0)]|^s)}.
\end{aligned}$$

Applying the first half of (6.2) and taking the upper limit as $\epsilon \to +0$, we conclude the proof. □

6.2. Trimmed random Schrödinger operators. In this short section, we use the strong-to-weak disorder coupling to provide non-rigorous support for Conjecture 1.5.

We apply the strong-to-weak disorder relations (6.2) in the direction opposite to that of Section 6.1. First consider the hedgehog lattice ($\mathbf{\Lambda} \times \{1\}$)-trimmed operator $H^{\text{III}} + \mathbb{1}$ corresponding to $U = gV$, $g \gg 1$. The first part of (6.2) relates the resolvent of H^{III} to the resolvent of the operator $H(0) + (gV)^{\#}_z$.

Now consider the operator $H(0) + (gV)^{\#}_\lambda$ for λ in the absolutely continuous spectrum of $H(0)$. It is an Anderson-type random operator at weak disorder, which is known to exhibit localisation in dimension $d = 1$ (see Figotin and Pastur [17, Chapter 15A]), and is conjectured (in fact universally accepted by physicists, cf. [16]) to exhibit localisation in dimension $d = 2$, and delocalisation in dimension $d \ge 3$. Thus the same properties should hold for the trimmed random Schrödinger operator $H^{\text{III}} + \mathbb{1}$.

Finally observe that the above reasoning is not limited to the hedgehog lattice, and can be extended to more realistic lattices (such as $\mathbb{Z}^d$). Indeed, the Schur–Banachiewicz formula can still be applied, relating the resolvent of $H(g)$, $g \gg 1$, to the resolvent of a more complicated Anderson-type operator at weak coupling, which should share the phenomenological properties of the usual Anderson model.

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