

Rotations of eigenvectors under unbounded perturbations

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Abstract. Let A be an unbounded selfadjoint positive definite operator with a discrete spectrum in a separable Hilbert space, and \tilde{A} be a linear operator, such that $\|(A - \tilde{A})A^{-\nu}\| < \infty$ ($0 < \nu \leq 1$). It is assumed that A has a simple eigenvalue. Under certain conditions \tilde{A} also has a simple eigenvalue. We derive an estimate for $\|e(A) - e(\tilde{A})\|$, where $e(A)$ and $e(\tilde{A})$ are the normalized eigenvectors corresponding to these simple eigenvalues of A and \tilde{A} , respectively. Besides, the perturbed operator \tilde{A} can be non-selfadjoint. To illustrate that estimate we consider a non-selfadjoint differential operator. Our results can be applied in the case when A is a normal operator.

Mathematics Subject Classification (2010). 47A55, 47A75, 47A30, 47A56.

Keywords. Linear operator, eigenvector, perturbation, differential operator.

1. Introduction and statement of the main result

The literature devoted to perturbations and approximations of the eigenvectors of various operators is rather rich. Mainly, the perturbations are assumed bounded, and the given operator and perturbed one are selfadjoint or normal. In particular, in the paper [3] Davis and Kahan considered bounded perturbations of invariant projections and eigenvectors in the case when the given operator and perturbed one are selfadjoint. Besides the spectrum can be continuous and discrete. The results from [3] were extended to normal operators (see [2] and references therein). In the paper, [10] approximations of Schrödinger eigenfunctions are explored by canonical perturbation theory. In [5] the author investigates eigenvectors of Toeplitz matrices under higher order three term recurrence and circulant perturbations. The paper [8] deals with approximations of eigenfunctions of the

periodic Schrödinger operators. The paper [14] introduces an algorithm to numerically approximate eigenfunctions of Sturm-Liouville problems corresponding to eigenvalues in a given region. In the papers [1, 11, 12, 13], the authors investigate stability and approximation properties of the eigenfunctions of Neumann and Dirichlet Laplacians. In particular, the lowest nonzero eigenvalue and corresponding eigenfunction is studied. The papers [6] and [7] deal with bounded and unbounded operators of the form $A = S + K$, where K is a compact operator S is a normal one, having a compact resolvent. Approximations of the eigenvectors of A , corresponding to simple eigenvalues are considered. Certainly, we could not survey the whole subject here and refer the reader to the above listed publications and references given therein.

In the present paper we investigate rotations of eigenvectors of operators with a discrete spectrum under unbounded perturbations. Besides, the perturbed operators can be non-selfadjoint.

Let H be a separable Hilbert space with a scalar product (\cdot, \cdot) , the norm $\|\cdot\| = \sqrt{(\cdot, \cdot)}$ and the unit operator I . For a linear unbounded operator B in H , $\text{Dom}(B)$ is the domain, $\sigma(B)$ denotes the spectrum of B , $R_\lambda(B) = (B - I\lambda)^{-1}$ ($\lambda \notin \sigma(B)$) is the resolvent; if B is bounded, then $\|B\|$ means its operator norm. Denote also $\Omega(c, r) := \{z \in \mathbb{C}: |z - c| \leq r\}$ ($c \in \mathbb{C}, r > 0$).

Let A be a selfadjoint positive definite operator in H with a discrete spectrum, and \tilde{A} be a linear operator with $\text{Dom}(A) = \text{Dom}(\tilde{A})$, such that for a $\nu \in (0, 1]$, the condition

$$q_\nu := \|(A - \tilde{A})A^{-\nu}\| < \infty \quad (1.1)$$

holds. Let $\lambda_k(A)$ ($k = 1, 2, \dots$) be the eigenvalues of A enumerated with their multiplicities in the increasing order. Suppose that for some integer $m \geq 1$, $\lambda_m(A)$ is a simple eigenvalue:

$$d_m = \inf_{k \neq m} |\lambda_k(A) - \lambda_m(A)|/2 > 0. \quad (1.2)$$

That is,

$$d_1 = \lambda_2(A) - \lambda_1(A)/2 > 0$$

and

$$d_m = \frac{1}{2} \max\{\lambda_{m+1}(A) - \lambda_m(A), \lambda_m(A) - \lambda_{m-1}(A)\} \quad (m \geq 2).$$

Now we are in a position to formulate our main result.

Theorem 1.1. For some integer $m \geq 1$, let conditions (1.1), (1.2), and

$$2q_\nu \lambda_{m+1}^\nu(A) < d_m \quad (1.3)$$

be satisfied. Then \tilde{A} has in $\Omega(\lambda_m(A), d_m)$ a simple eigenvalue, say $\lambda(\tilde{A})$. Moreover, the eigenvector $e(A)$ of A corresponding to $\lambda_m(A)$ and the eigenvector $e(\tilde{A})$ of \tilde{A} , corresponding to $\lambda(\tilde{A})$ with $\|e(A)\| = \|e(\tilde{A})\| = 1$ satisfy the inequality

$$\|e(A) - e(\tilde{A})\| \leq \frac{2q_\nu \lambda_{m+1}^\nu(A)}{d_m - 2q_\nu \lambda_{m+1}^\nu(A)}.$$

This theorem is proved in the next two sections.

If A is a non positive definite selfadjoint operator, with $-\infty < \inf \sigma(A) = -c_0 < 0$, then Theorem 1.1 can be applied to the operator $A + cI$ for any $c > c_0$. If A is a normal operator, then Theorem 1.1 can be applied to the operator $A_R = (A + A^*)/2$, since A_R commutes with A and therefore A_R and A have joint eigenvectors.

2. Preliminaries

Lemma 2.1. Let T_1 be a normal invertible operator in H and T_2 be a linear operator in H with $\text{Dom}(T_2) = \text{Dom}(T_1)$, such that for a $\nu \in (0, 1]$, the condition

$$\hat{q}_\nu = \|(T_1 - T_2)T_1^{-\nu}\| < \infty \quad (2.1)$$

holds. Assume, in addition, that for a $\lambda \notin \sigma(T_1)$, the inequality

$$\hat{q}_\nu \|T_1^\nu R_\lambda(T_1)\| < 1, \quad (2.2)$$

is fulfilled. Then $\lambda \notin \sigma(T_2)$,

$$\|R_\lambda(T_2)\| \leq \frac{\|R_\lambda(T_1)\|}{1 - \hat{q}_\nu \|T_1^\nu R_\lambda(T_1)\|} \quad (2.3)$$

and

$$\|R_\lambda(T_2) - R_\lambda(T_1)\| \leq \frac{\hat{q}_\nu \|T_1^\nu R_\lambda(T_1)\| \|R_\lambda(T_1)\|}{1 - \hat{q}_\nu \|T_1^\nu R_\lambda(T_1)\|}. \quad (2.4)$$

Proof. Indeed,

$$\begin{aligned} R_\lambda(T_1) - R_\lambda(T_2) &= R_\lambda(T_2)(T_2 - T_1)R_\lambda(T_1) \\ &= R_\lambda(T_2)(T_2 - T_1)T_1^{-\nu}T_1^\nu R_\lambda(T_1). \end{aligned}$$

Hence it follows that

$$\|R_\lambda(T_2) - R_\lambda(T_1)\| \leq \|R_\lambda(T_2)\|\hat{q}_\nu\|T_1^\nu R_\lambda(T_1)\|. \quad (2.5)$$

Now (2.2) implies (2.3). So $\lambda \notin \sigma(T_2)$. Moreover, (2.5) and (2.3) imply (2.4), as claimed. \square

It is simple to show that the previous lemma is valid for operators in a Banach space, provided T_1 is a sectorial invertible operator.

Let P be a projection in H . Denote by $e(P)$ the eigenvector of P with $\|e(P)\| = 1$. We need the following result.

Lemma 2.2. *Let P_1, P_2 be two projections in H satisfying*

$$\|P_1 - P_2\| \leq \delta \text{ with } \delta < 1. \quad (2.6)$$

In addition, let P_1 be one-dimensional. Then P_2 is also one-dimensional (due to [9, p.156, Problem III.2.1]). Moreover,

$$\|e(P_2) - e(P_1)\| \leq \frac{2\delta}{1-\delta}.$$

Proof. For simplicity put $e(P_k) = e_k$ ($k = 1, 2$). Due to (2.6) and the equality $\|e(P_1)\| = 1$ we can write $P_2e_1 \neq 0$, since $P_1e_1 = e_1$. Thanks to the relation $P_2e_1 = P_2(P_2e_1)$, P_2e_1 is an eigenvector of P_2 . Put $\eta = \|P_2e_1\|$. Then $e_2 = \frac{1}{\eta}P_2e_1$ is a normalized eigenvector of P_2 . So

$$e_1 - e_2 = P_1e_1 - \frac{1}{\eta}P_2e_1 = e_1 - \frac{1}{\eta}e_1 + \frac{1}{\eta}(P_1 - P_2)e_1.$$

But

$$\eta \geq \|P_1e_1\| - \|(P_1 - P_2)e_1\| \geq 1 - \delta.$$

Hence $\frac{1}{\eta} \leq (1 - \delta)^{-1}$ and

$$\begin{aligned} \|e_1 - e_2\| &\leq \left(\frac{1}{\eta} - 1\right)\|e_1\| + \frac{1}{\eta}\|P_1 - P_2\|\|e_1\| \\ &\leq (1 - \delta)^{-1} - 1 + (1 - \delta)^{-1}\delta \\ &= 2\delta(1 - \delta)^{-1}, \end{aligned}$$

as claimed. \square

3. Proof of Theorem 1.1

For simplicity put $\lambda_k(A) = \lambda_k$. Denote

$$C := \partial\Omega(\lambda_m, d_m) = \{z \in \mathbb{C} : |z - \lambda_m| = d_m\},$$

and

$$P(A) := -\frac{1}{2\pi i} \int_C R_\lambda(A) d\lambda \text{ and } P(\tilde{A}) = -\frac{1}{2\pi i} \int_C R_\lambda(\tilde{A}) d\lambda.$$

That is, $P(A)$ and $P(\tilde{A})$ are the Riesz projections onto the eigenspaces of A and \tilde{A} , respectively, corresponding to the points of the spectra, which belong to $\Omega(\lambda_m, d_m)$. We have

$$\|P(A) - P(\tilde{A})\| \leq \frac{1}{2\pi} \int_C \|R_z(\tilde{A}) - R_z(A)\| |dz| \leq d_m \sup_{z \in C} \|R_z(\tilde{A}) - R_z(A)\|.$$

Since A is selfadjoint, one has

$$\|R_z(A)\| = \rho^{-1}(A, \lambda), \quad (3.1)$$

where $\rho(A, \lambda) = \inf_{s \in \sigma(A)} |\lambda - s|$ - the distance between $\lambda \in \mathbb{C}$ and $\sigma(A)$. Inequality (2.5) implies

$$\|P(A) - P(\tilde{A})\| \leq q_\nu d_m \sup_{z \in C} \|R_z(\tilde{A})\| \|A^\nu R_z(A)\| \leq d_m q_\nu l_0 b_\nu, \quad (3.2)$$

where

$$l_0 = \sup_{z \in C} \|R_z(\tilde{A})\|, b_\nu = \sup_{z \in C} \|A^\nu R_z(A)\| = \sup_{t \in [0, 2\pi]} \|A^\nu R_{\lambda_m + d_m e^{it}}(A)\|.$$

Since A is selfadjoint, we have

$$b_\nu = \sup_{t \in [0, 2\pi]} \sup_j \frac{\lambda_j^\nu}{|\lambda_j - \lambda_m - d_m e^{it}|} \leq \sup_j \frac{\lambda_j^\nu}{||\lambda_j - \lambda_m| - d_m|}.$$

Recall that $|\lambda_j - \lambda_m| \geq 2d_m$ ($j \neq m$). Put

$$s_j = \frac{\lambda_j^\nu}{||\lambda_j - \lambda_m| - d_m|}.$$

So $s_m = \lambda_m^\nu / d_m$. Let $j \leq m - 1$. Then

$$s_j = \frac{\lambda_j^\nu}{\lambda_m - \lambda_j - d_m} \leq \frac{\lambda_{m-1}^\nu}{\lambda_m - \lambda_{m-1} - d_m} \leq s_m.$$

Now let $j \geq m + 1$. Then

$$\begin{aligned}
 s_j &= \frac{\lambda_j^v}{\lambda_j - \lambda_m - d_m} \\
 &= \frac{1}{\lambda_j^{1-v} - (\lambda_m + d_m)\lambda_j^{-v}} \\
 &\leq \frac{1}{\lambda_j^{1-v} - (\lambda_m + d_m)\lambda_{m+1}^{-v}} \\
 &\leq \frac{1}{\lambda_{m+1}^{1-v} - (\lambda_m + d_m)\lambda_{m+1}^{-v}} \\
 &= \frac{\lambda_{m+1}^v}{\lambda_{m+1} - \lambda_m - d_m} \\
 &\leq \frac{\lambda_{m+1}^v}{d_m}.
 \end{aligned}$$

So

$$b_v \leq s_{m+1} = \frac{\lambda_{m+1}^v}{d_m}.$$

Furthermore, condition (1.3) implies

$$q_v s_{m+1} = \frac{q_v \lambda_{m+1}^v}{d_m} < 1 \quad (3.3)$$

and therefore,

$$q_v \|A^v R_\lambda(A)\| \leq \frac{q_v \lambda_{m+1}^v}{d_m} < 1 \quad (\lambda \in C).$$

Consequently, by Lemma 2.1 and (3.1),

$$\begin{aligned}
 \|R_\lambda(\tilde{A})\| &\leq \frac{\|R_\lambda(A)\|}{1 - q_v b_v} \\
 &\leq \frac{1}{d_m(1 - q_v s_{m+1})} \\
 &= \frac{1}{d_m \left(1 - \frac{q_v \lambda_{m+1}^v}{d_m}\right)} \\
 &= \frac{1}{d_m - q_v \lambda_{m+1}^v} \quad (\lambda \in C).
 \end{aligned} \quad (3.4)$$

Due to (3.2) we thus have proved:

Lemma 3.1. *Under the hypothesis of Theorem 1.1 one has*

$$\|P(A) - P(\tilde{A})\| \leq \frac{q_v \lambda_{m+1}^v}{d_m - q_v \lambda_{m+1}^v} < 1.$$

The assertion of Theorem 1.1 follows from Lemmas 2.2 and 3.1.

4. Example

Consider in $L^2(0, 1)$ the problem

$$-u''(x) + a(x)u'(x) = \lambda u(x) \quad (\lambda \in \mathbb{C}; 0 < x < 1); \quad u(0) = u(1) = 0,$$

where $a(x)$ ($0 \leq x \leq 1$) is a bounded complex valued function. Take

$$A = -\frac{d^2}{dx^2}$$

with

$$\text{Dom}(A) = \{v \in L^2(0, 1): v'' \in L^2(0, 1), v(0) = v(1) = 0\},$$

and $\nu = 1/2$. Define \tilde{A} by

$$(\tilde{A}u)(x) = -u''(x) + a(x)u'(x) \tag{4.1}$$

with $\text{Dom}(\tilde{A}) = \text{Dom}(A)$.

Obviously, $\lambda_j(A) = \pi^2 j^2$ ($j = 1, 2, \dots$) and

$$q_{1/2} = \|(A - \tilde{A})A^{-1/2}\| = \sup_x |a(x)| \sup_{f \in \text{Dom}(A), \|f\|=1} \|(A^{-1/2}f)'_x\|.$$

But for $f \in \text{Dom}(A)$,

$$(A^{-1/2}f)(x) = \sum_{k=1}^{\infty} \frac{1}{\lambda_j^{1/2}(A)} (f, e_k) e_k(x) = \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{1}{j} (f, e_k) e_k(x),$$

where $e_k(x) = \sqrt{2} \sin \pi(kx)$. Thus

$$\begin{aligned} \frac{d}{dx}(A^{-1/2}f)(x) &= \frac{d}{dx} \sum_{k=1}^{\infty} \frac{1}{\lambda_j^{1/2}(A)} (f, e_k) e_k(x) \\ &= \sqrt{2} \sum_{k=1}^{\infty} (f, e_k) \cos \pi(kx). \end{aligned}$$

Hence,

$$\|(A^{-1/2}f)'_x\|^2 = \sum_{k=1}^{\infty} |(f, e_k)|^2 = \|f\|^2.$$

Therefore, $q_{1/2} = \sup_{0 \leq x \leq 1} |a(x)|$. In particular, for $m = 1$ we have $d_1 = 3\pi^2/2$. Condition (1.3) takes the form

$$2q_{1/2}\lambda_2^{1/2}(A) = 4q_{1/2}\pi < 3\pi^2/2.$$

Or

$$8q_{1/2} = 8 \sup_x |a(x)| < 3\pi. \quad (4.2)$$

Under this condition, by Theorem 1.1, the operator \tilde{A} defined by (4.1) has in the disc $|z - \pi^2| \leq 3\pi^2/2$ a simple eigenvalue and the corresponding normalized eigenvector satisfies the inequality

$$\|e(\tilde{A}) - \sqrt{2} \sin(\pi x)\| \leq \frac{8q_{1/2}}{3\pi - 8q_{1/2}},$$

provided (4.2) holds.

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Received October 13, 2014

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