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Superlevel sets and nodal extrema of Laplace–Beltrami eigenfunctions

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Abstract. We estimate the volume of superlevel sets of Laplace–Beltrami eigenfunctions on a compact Riemannian manifold. The proof uses the Green's function representation and the Bathtub principle. As an application, we obtain upper bounds on the distribution of the extrema of a Laplace–Beltrami eigenfunction over its nodal domains. Such bounds have been previously proved by L. Polterovich and M. Sodin in the case of compact surfaces. Our techniques allow to generalize these results to arbitrary dimensions. We also discuss a different approach to the problem based on reverse Hölder inequalities due to G. Chiti.

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1. Introduction and main results

1.1. Notation. Let (M^n, g) be a compact, connected n−dimensional Riemannian manifold with or without boundary. Let $\Delta_g\colon C^\infty(M) \to C^\infty(M)$ denote the negative Laplace–Beltrami operator on M. In local coordinates $\{x_i\}_{i=1}^n$, we write

$$\Delta_g = \frac{-1}{\sqrt{\det(g)}} \sum \frac{\partial}{\partial x_i}\Big(\sqrt{\det(g)}\, g^{ij} \frac{\partial}{\partial x_j}\Big), \tag{1.1.1}$$

where the matrix (g^{ij}) is the inverse matrix of $g = (g_{ij})$.

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We consider the closed eigenvalue problem,

$$\Delta_g u_\lambda = \lambda u_\lambda, \tag{1.1.2}$$

and when M has a boundary, we impose Dirichlet eigenvalue problem,

$$\begin{cases} \Delta_g u = \lambda u & \text{in } M, \\ u = 0 & \text{on } \partial M. \end{cases} \tag{1.1.3}$$

In both settings, Δ_g has a discrete spectrum,

$$0 \leq \lambda_1(M,g) \leq \lambda_2(M,g) \leq \ldots \nearrow +\infty,$$

where $\lambda_1(M,g) > 0$ if $\partial M \neq \emptyset$. Let $\|.\|_p$ be the usual $\|.\|_{L^p(M)}$ norm and let σ be the Riemannian volume form on M and let $\mathrm{Vol}_g(M)$ denote the Riemannian volume of M. We normalize u in such a way that $\|u\|_2^2 = 1$. If M has no boundary, we require that $\int_M u d\sigma = 0$.

1.2. Volume of superlevel sets. We define a nodal domain A of an eigenfunction u_λ on M as a maximal connected open subset of $\{u_\lambda \neq 0\}$. We denote by $\mathcal{A}(u_\lambda)$ the collection of all its nodal domains.

Let us first consider the Euclidean case. It is known that nodal domains can not be too small. For instance, this can be seen by the Faber–Krahn inequality, stating that given $A_i \in \mathcal{A}(u_\lambda)$,

$$\mathrm{Vol}(A_i) \geq \lambda_1(B)^{n/2} \, \mathrm{Vol}(B) \lambda^{-n/2}, \tag{1.2.1}$$

where B denotes a n-dimensional ball. Denote by $V_\delta^i = \{x \in A_i \colon |u_\lambda(x)| \geq \delta \|u_\lambda\|_{L^\infty(A_i)}\}$ the δ-superlevel sets of the restriction of an eigenfunction to one of its nodal domain. The next result can be seen as a refinement of that observation. Indeed, each δ-superlevel set of an eigenfunction can not be too small:

Lemma 1.2.2. *Let $n \geq 3$. For all $\delta \in (0,1)$, we have that*

$$\mathrm{Vol}(V_\delta^i) \geq (1-\delta)^{n/2} (2(n-2))^{n/2} \alpha_n \lambda^{-n/2}, \tag{1.2.3}$$

where α_n stands for the volume of the n-dimensional unit ball.

The preceding lemma and its proof were suggested by F. Nazarov and M. Sodin, see [16].

Letting $\delta \to 0$ in (1.2.3) yields that

$$\mathrm{Vol}(V_0^i) = \mathrm{Vol}(A_i) \geq C_n \lambda^{-n/2},$$

which is an inequality *à la Faber–Krahn* comparable to (1.2.1). However, the constant is not optimal when compared to Faber–Krahn inequality since $C_{n,\delta}$ tends to $C_n = (2(n-2))^{n/2}\alpha_n$ as $\delta \to 0$.

The proof of Lemma 1.2.2 is based on the maximum principle, applied to a precise linear combination of the eigenfunction u_λ and of a certain function w. The function w is defined as the solution of the following Poisson problem:

$$\Delta w = -\lambda \chi_{V_\delta^i} u_{\lambda,i} \quad \text{in } \mathbb{R}^n,$$

where $\chi_{V_\delta^i}$ denotes the characteristic function associated to V_δ^i and $u_{\lambda,i}$ denotes the restriction of u_λ to A_i. An upper bound on the function w is required to apply the maximum principle. The bound is proved using decreasing rearrangement of functions, as done in [26, p. 185]. The next result is a generalization of Lemma 1.2.2, adapted to manifolds of arbitrary dimension:

Theorem 1.2.4. *Let $\delta \in (0,1)$ and $n \geq 2$. There exist $\lambda_0 > 0$ and $k_{g,\delta,\lambda_0} > 0$ such that for all $\lambda \geq \lambda_0$, we have that*

$$\mathrm{Vol}_g(V_\delta^i) \geq k_{g,\delta,\lambda_0} \lambda^{-n/2}, \quad \textit{for all } i. \tag{1.2.5}$$

The proof of Theorem 1.2.4 for $n \geq 3$ is similar to the proof of its $\mathbb{R}^n$ counterpart. The key idea is to choose a specific linear combination involving $u_{\lambda,i}$ and the solution of the following Poisson problem,

$$\Delta w = -\lambda \chi_{V_\delta^i} u_{\lambda,i} \quad \text{in } M.$$

In order to apply the maximum principle, it is required to bound the function w in terms of λ and of the volume of V_δ^i. The method used to do so differs from the one used in $\mathbb{R}^n$ since decreasing rearrangement of functions no longer works on arbitrary manifolds. Instead, we use an upper bound for Green functions on M in conjunction with a certain form of the bathtub principle (see [12, Theorem 1.14]), that is an upper bound for the integral of a non-negative decreasing radial function:

Lemma 1.2.6. *Let* $x_0 \in M$. *Let* $r(x) = d_g(x_0, x)$ *the Riemannian distance between* x *and* x_0. *Let* $f(r)$ *denote a non-negative strictly decreasing function. Given fixed positive constant* $C > 0$, *then*

$$\sup_{\Omega \subset M,\ \mathrm{Vol}_g(\Omega)=C} \int_{\Omega} f(r)d\sigma = \int_{\Omega^*} f(r)d\sigma,$$

where Ω^* *is the geodesic ball centered at* x_0 *of radius* R, *where* R *is such that* $\mathrm{Vol}(\Omega) = \mathrm{Vol}(\Omega^*)$.

Lemma 1.2.6 can also be seen as a weaker form of decreasing rearrangement that has the advantage of being applicable in a more general setting.

For compact surfaces, using a slight adaptation of the result proved in [14, Section 3], it is possible to obtain a lower bound on the density of the δ-superlevel set V_δ^i of an eigenfunction u_λ:

Proposition 1.2.7. *Let* (M, g) *be a Riemannian surface and let* $\delta \in (0, 1)$. *For any* p *such that* $u_\lambda(p) = m_{A_i}$, *there exists a positive constant* $k_{g,\delta}$ *such that the ball* $B_p(k_{g,\delta}\lambda^{-1/2})$ *is included in* V_δ^i. *In particular, this implies that*

$$\rho_\lambda(V_\delta^i) \geq k_{g,\delta}\lambda^{-1/2} \quad \text{for all } i,$$

where $\rho_\lambda(V_\delta^i)$ *denotes the inner radius of the* δ*-superlevel set* V_δ^i *of the eigenfunction* u_λ.

Proposition 1.2.7 implies Theorem 1.2.4 in the two dimensional case.

1.3. Nodal extrema on closed manifolds.

The second objective of the paper is to study the distribution of so called nodal extrema, defined as follows:

$$m_{A_i} := \max_{x \in A_i} |u_\lambda(x)|,$$

where $A_i \in \mathcal{A}(u_\lambda)$. Nodal extrema on compact surfaces were previously studied in [22]. We consider the more general case of compact Riemannian manifolds of arbitrary dimension. Since the proofs given in [22] rely on the classification of surfaces and the existence of conformal coordinates, no direct generalization of their results is possible.

Our first main result in that direction is the following:

Theorem 1.3.1. *Let* (M^n, g) *be a compact closed manifold with* $n \geq 2$. *If* λ *is large enough, then there exists* $k_g > 0$ *such that*

$$\sum_{i=1}^{|\mathcal{A}(u_\lambda)|} m_{A_i}^p \leq k_g \lambda^{\frac{n}{2} + p\delta(p)}, \tag{1.3.2}$$

holds for any $p \geq 2$. *Here,* $\delta(p)$ *corresponds to*

$$\delta(p) = \begin{cases} \frac{n-1}{4}\Big(\frac{1}{2} - \frac{1}{p}\Big), & 2 \leq p \leq \frac{2(n+1)}{n-1}, \\ \frac{n}{2}\Big(\frac{1}{2} - \frac{1}{p}\Big) - \frac{1}{4}, & \frac{2(n+1)}{n-1} \leq p \leq +\infty. \end{cases} \tag{1.3.3}$$

Note that $\delta(p)$ is C. Sogge's classical L^p bounds, $\|u\|_p \leq C\lambda^{\delta(p)}\|u\|_2$ ([17, Chapter 5]). The proof of Theorem 1.3.1 is an application of Theorem 1.2.4.

As an immediate corollary of Theorem 1.3.1, we have the following:

Corollary 1.3.4. *Let* (M^n, g) *be a compact closed manifold. If* λ *is large enough, then there exists* $k_g > 0$ *such that*

$$\sum_{i=1}^{|\mathcal{A}(u_\lambda)|} m_{A_i} \leq k_g \lambda^{n/2}. \tag{1.3.5}$$

Indeed, a consequence of Weyl's law and Courant's theorem is that the number of nodal domains $|\mathcal{A}(u_\lambda)|$ is bounded by $k_g\lambda^{n/2}$ (see for instance [7, 3]). Using the latter fact and then applying Cauchy-Schwartz inequality yield

$$\sum_{i=1}^{|\mathcal{A}(u_\lambda)|} m_{A_i} \leq \Bigg(\sum_{i=1}^{|\mathcal{A}(u_\lambda)|} m_{A_i}^2 \cdot \sum_{i=1}^{|\mathcal{A}(u_\lambda)|} 1\Bigg)^{1/2} \leq k_g \lambda^{n/4} |\mathcal{A}(u_\lambda)|^{1/2} \leq k_g \lambda^{n/2},$$

which is the desired result.

Remark 1.3.6. For $p = 1, 2$, it is easy to see that the inequalities are sharp on $\mathbb{T}^n$ ($\prod \sin(nx_i), \lambda = n^2$). For $p > 2(n+1)/(n-1)$, extremals are zonal spherical harmonics. Otherwise, the extremals are highest weight spherical harmonics.

One can visualise inequalities expressed in Theorem 1.3.1 and in Corollary 1.3.4 by considering "fine" dust particles on a vibrating membrane. Indeed, where the membrane's velocity is high, Bernoulli's equation tells us that the air pressure is low. Since the dust particles are most influenced by air pressure, they are swept by the pressure gradient near nodal extrema (see [6] for some figures illustrating nodal extrema and for more information on such experiments).

Remark 1.3.7. One can easily obtain bounds on m_{A_i} using the classical Hormander–Levitan–Avakumovic L^∞ bound (see for instance [17]). Indeed, it implies that there exists a constant $k_g > 0$ such that $\|u_\lambda\|_{L^\infty(A_i)} \leq k_g \lambda^{(n-1)/4}$. Therefore, we have that

$$\sum_{i=1}^{|\mathcal{A}(u_\lambda)|} \|u_\lambda\|_{L^\infty(A_i)} \leq k_g |\mathcal{A}(u_\lambda)| \lambda^{(n-1)/4} \leq k_g \lambda^{\frac{3n-1}{4}},$$

which is not optimal when compared to the sharp inequality given in Corollary 1.3.4.

We also obtain a generalization of [22, Corollary 1.7]. The result is the following:

Corollary 1.3.8. *Given $a > 0$, consider nodal domains such that $m_{A_i} \geq a\lambda^{(n-1)/4}$. If λ is large enough, then there exists $k_g > 0$ such that the number of such nodal domains does not exceed $k_g a^{-2(n+1)/(n-1)}$. In particular, for fixed a, it remains bounded as $\lambda \to \infty$.*

Indeed, letting N_λ denote the number of such nodal domains, using (1.3.2) with $p = \frac{2(n+1)}{n-1}$, we have that

$$N_\lambda (a\lambda^{(n-1)/4})^{2(n+1)/(n-1)} \leq \sum_{i=1}^{N_\lambda} m_{A_i}^{2(n+1)/(n-1)} \leq k_g \lambda^{(n+1)/2},$$

yielding the conclusion.

1.4. Elliptic operators on Euclidean domains.

We obtain analogous results to Theorem 1.3.1. More precisely, we obtain bounds on the distribution of nodal extrema of eigenfunctions associated to the Dirichlet problem of general second order elliptic operators in the divergence form on an Euclidean bounded domain Ω.

Consider the following Dirichlet eigenvalue problem:

$$\begin{cases} L(u) = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4.1}$$

where we consider a general elliptic operator L defined as

$$L(u) := -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) + cu.$$

Here, the coefficients $a_{ij}(x)$ are real measurable functions such that $a_{ij} = a_{ji}$, for all $1 \leq i, j \leq n$. We assume that $c(x)$ is a bounded measurable function such that $c(x) \geq 0$. Note that the non negativity of c can be assumed without loss of generality (see [9, Remark 1.1.3, p. 3]). For convenience, we normalize the coefficients in such a way that 1 is the lower ellipticity constant. Thus, the assumption reads

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq |\xi|^2, \quad \text{for all } \xi \in \mathbb{R}^n. \tag{1.4.2}$$

We are ready to state the result:

Theorem 1.4.3. *Consider u_λ an eigenvalue of (1.4.1) associated to the eigenvalue λ, then*

$$\sum_{i=1}^{|\mathcal{A}(u_\lambda)|} m_{A_i} \leq K_{n,1}\,\mathrm{Vol}(\Omega)^{1/2}\lambda^{n/2}, \tag{1.4.4}$$

and

$$\sum_{i=1}^{|\mathcal{A}(u_\lambda)|} m_{A_i}^2 \leq K_{n,2}^2\lambda^{n/2}. \tag{1.4.5}$$

The constant $K_{n,p}$ depends on n and on p and is given by

$$K_{n,p} = \frac{2^{1-n/2}(n\alpha_n)^{-1/p}}{\Gamma\left(\frac{n}{2}\right)\left(\int_0^{j_{n/2-1}} r^{p-np/2+n-1} J_{n/2-1}^{p}(r)\,dr\right)^{1/p}}. \tag{1.4.6}$$

The main tool to prove Theorem 1.4.3 is Chiti's reverse Hölder inequality satisfied by any elliptic operator in divergence form with Dirichlet boundary conditions.

Remark 1.4.7. Since Theorem 1.4.3 can be applied to general elliptic operators such as the Laplace–Beltrami operator in local coordinates as defined in (1.1.1), it can also be used with a Laplacian eigenfunction on compact Riemannian manifolds provided that all its nodal domains can always be included in a single chart of M.

Remark 1.4.8. A notable feature of [22, Theorem 1.3] is that the bounds on the distribution of the nodal extrema hold for a larger class of functions defined on compact surfaces, including eigenfunctions associated to the bi-laplacian clamped plate problem. Both approaches can not be extended to the bi-laplacian case since they rely on the maximum principle, which is known not to hold for such operators.

1.5. Neumann boundary conditions in the planar case. Let Ω be a bounded planar domain with piecewise analytic boundary. We consider the Neumann eigenvalue problem on Ω, namely

$$\begin{cases} \Delta u = \mu u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5.1}$$

Using an argument of [23] based on a result of [27], it is possible to bound the number of nodal domains touching the boundary of Ω by $C_\Omega\sqrt{\mu}$. By doing so, it is an easy matter to obtain the following:

Theorem 1.5.2. *Let Ω be a bounded planar domain with piecewise analytic boundary, then there exists $C_\Omega > 0$ and $K_\Omega > 0$ such that*

$$\sum_{i=1}^{|\mathcal{A}(u_\mu)|} m_{A_i} \leq C_\Omega \mu, \tag{1.5.3}$$

and

$$\sum_{i=1}^{|\mathcal{A}(u_\mu)|} m_{A_i}^2 \leq K_\Omega \mu. \tag{1.5.4}$$

1.6. Manifolds with Dirichlet boundary conditions. In order to obtain similar results for manifolds with boundary conditions, one has to use Sogge-Smith's adapted bounds for such setting (see [18]). For the sake of clarity, we recall these results here.

Let (M^n, g) be a compact Riemannian manifold with boundary. Let u_λ denote a Dirichlet eigenfunction associated to λ, then there exists $k_g > 0$ such that

$$\|u_\lambda\|_p \leq k_g \lambda^{n(1/2-1/p)/2-1/4} \|u_\lambda\|_2, \tag{1.6.1}$$

for $p \geq 4$ if $n \geq 4$, and $p \geq 5$ if $n = 3$. One can easily adapt the proof of Theorem 1.3.1 using Sogge-Smith results to get:

Theorem 1.6.2. *Let (M^n, g) be a compact Riemannian manifold with boundary. If λ is large enough, there exists $k_g > 0$ such that*

$$\sum_{i=1}^{|\mathcal{A}(u_\lambda)|} m_{A_i} \leq k_g \lambda^{n/2}, \tag{1.6.3}$$

and

$$\sum_{i=1}^{|\mathcal{A}(u_\lambda)|} m_{A_i}^2 \leq k_g \lambda^{n/2}. \tag{1.6.4}$$

Moreover, we have the following

$$\sum_{i=1}^{|\mathcal{A}(u_\lambda)|} m_{A_i}^p \leq k_g \lambda^{n/2+np(1/2-1/p)/2-p/4}, \tag{1.6.5}$$

for any $p \geq 4$ *if* $n \geq 4$, *and* $p \geq 5$ *if* $n = 3$.

In [18], it is conjectured that the following bound holds:

$$\|u\|_p \leq C\lambda^{\alpha(p)}\|u\|_2,$$

where

$$\alpha(p) = \begin{cases} \left(\frac{2}{3} + \frac{n-2}{2}\right)\left(\frac{1}{4} - \frac{1}{2p}\right), & 2 \leq p \leq \frac{6n+4}{3n-4}, \\ \frac{n}{2}\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{4}, & \frac{6n+4}{3n-4} \leq p \leq +\infty. \end{cases} \tag{1.6.6}$$

Hence, a version of Theorem 1.6.2 without the restrictions could be obtained if one showed these latter bounds:

Conjecture 1.6.7. *Let* (M^n, g) *be a manifold with boundary. If* λ *is large enough, then there exists* $k_g > 0$ *such that*

$$\sum_{i=1}^{|\mathcal{A}(u_\lambda)|} m_{A_i}^p \leq k_g \lambda^{n/2+p\alpha(p)}, \tag{1.6.8}$$

for any $p \geq 2$.

We also obtain a generalization of [22, Corollary 1.7] in the case of manifolds with boundary. Using (1.6.5) with $p = \frac{6n+4}{3n-4}$, we get the following:

Corollary 1.6.9. *Given* $a > 0$, *consider nodal domains such that* $m_{A_i} \geq a\lambda^{(n-1)/4}$. *If* λ *is large enough, then there exists* $k_g > 0$ *such that the number of such nodal domains does not exceed* $k_g a^{-(6n+4)/(3n-4)}$. *In particular, for fixed* a, *it remains bounded as* $\lambda \to \infty$.

1.7. Bounds for the *p*-Laplacian. For $1 < p < \infty$, the p–Laplacian of a function f on an open bounded Euclidean domain Ω is defined by

$$\Delta_p f = \operatorname{div}(|\nabla f|^{p-2}\nabla f).$$

We consider the following eigenvalue problem:

$$\Delta_p u + \lambda |u|^{p-2} u = 0 \text{ in } \Omega, \tag{1.7.1}$$

where we impose the Dirichlet boundary conditions. We say that λ is an eigenvalue of $-\Delta_p$ if (1.7.1) has a nontrivial weak solution $u_{\lambda,p} \in W_0^{1,p}(\Omega)$. That is, for any $v \in C_0^\infty(\Omega)$,

$$\int_\Omega |\nabla u_\lambda|^{p-2} \nabla u_\lambda \cdot \nabla v - \lambda \int_\Omega |u_\lambda|^{p-2} u_\lambda v = 0. \tag{1.7.2}$$

The function u_λ is then called an eigenfunction of $-\Delta_p$ associated to the eigenvalue λ. The function u_λ is then called an eigenfunction of $-\Delta_p$ associated to λ. Note that if $p = 2$, the p-Laplacian corresponds to the usual Laplacian and is linear. Otherwise, we say that the p-Laplacian is "half-linear" in the sense that it is $(p-1)$-homogeneous but not additive.

It is known that the first eigenvalue of the Dirichlet eigenvalue problem of the p-Laplace operator, denoted by $\lambda_{1,p}$, is characterized as

$$\lambda_{1,p} = \min_{0 \neq u \in C_0^\infty(\Omega)} \left\{ \frac{\displaystyle\int_\Omega |\nabla u|^p \, dx}{\displaystyle\int_\Omega |u|^p \, dx} \right\}. \tag{1.7.3}$$

The infimum is attained for a function $u_{1,p} \in W_0^{1,p}(\Omega)$. In addition, $\lambda_{1,p}$ is simple and isolated. Moreover, the eigenfunction u_1 associated to $\lambda_{1,p}$ does not change sign, and it is the only such eigenfunction.

Via, for instance, the Lyusternick–Schnirelmann maximum principle, it is possible to construct $\lambda_{k,p}$ for $k \geq 2$ and hence obtain an increasing sequence of so-called variational eigenvalues of (1.7.1) tending to $+\infty$. There exist other variational characterizations of these eigenvalues. However, no matter which variational characterization one chooses, it always remains to show that all the eigenvalues obtained that way exhaust the whole spectrum of Δ_p.

Less is known about nodal geometry of eigenfunctions for the p-Laplace operator. For instance, it is not clear if the the interior of the set $\{x \in \Omega : u_\lambda(x) = 0\}$ is empty or not for p-Laplacian eigenfunctions. For more details on nodal geometry of the p-Laplace operator, see for instance [11, 20, 21].

Nevertheless, using a L^∞ bound obtained in [13, Lemma 4.1], one can still obtain an extension of (1.4.4) for the p-Laplace operator.

Theorem 1.7.4. *Let Ω be a smooth bounded open set in $\mathbb{R}^n$. Consider $u_{p,\lambda}$ an eigenfunction of the Dirichlet p-Laplacian eigenvalue problem associated to the eigenvalue λ. Let $\|u_{p,\lambda}\|_{p,\Omega} = 1$, then we have the following*:

$$\sum_{i=1}^{|\mathcal{A}(u_\lambda)|} m_{A_i} \leq 4^n \operatorname{Vol}(\Omega)^{1-1/p} \lambda^{n/p}. \tag{1.7.5}$$

Notice that if $p = 2$, this result corresponds to what we expect in the case of the usual Laplace operator.

The Courant nodal theorem combined with the Weyl Law yield that the number of nodal domains of a Dirichlet eigenfunction associated to an elliptic operator L does not exceed $C\lambda^{n/2}$. For the p-Laplacian case, the number of nodal domains N_λ associated to an arbitrary eigenfunction is known to be bounded, see [8]. It is also shown in [8] that the number of nodal domains of an eigenfunction u_k associated to a variational eigenvalue is bounded by $2k - 2$. Moreover, it is known that there exists two positive constants depending on Ω such that $ck^{p/n} \leq \lambda_{k,p} \leq Ck^{p/n}$ (see [2]). Combining both results yields that $N_\lambda \leq C\lambda^{n/p}$ if λ is a variational eigenvalue. We show that a similar result holds even for non-variational eigenvalue:

Corollary 1.7.6. *For any eigenfunction of* (1.7.1) *and any $a > 0$, there exists a positive constant $C > 0$ such that the number of nodal domains $A \in \mathcal{A}(f)$ with $m_A \geq a$ does not exceed $Ca^{-1}\lambda^{n/p}$.*

Indeed, letting N_λ denote the number of such nodal domains, using (1.7.5), we have that

$$N_\lambda a \leq \sum_{i=1}^{N_\lambda} m_{A_i} \leq C\lambda^{n/p},$$

yielding the conclusion.

1.8. Structure of the paper.

In Section 2, we prove the main results, namely we start with Lemma 1.2.2 in $\mathbb{R}^n$ and then we prove Theorem 1.2.4 for arbitrary compact Riemannian manifolds. This leads to the proof of Theorem 1.3.1 which is an application of Theorem 1.2.4. In Section 3, we prove Theorems 1.4.3, 1.5.2 and 1.7.4.

2. Proofs of main results

2.1. Proof of Lemma 1.2.2. Before proving Theorem 1.2.4 that holds for compact Riemannian manifolds, we give a proof of such result in the Euclidean case to give the intuition behind the proof more clearly.

In order to prove Lemma 1.2.2, we need a technical result concerning Poisson equation. Let $\Omega \subset \mathbb{R}^n, n \geq 3$, denote a bounded domain of $\mathbb{R}^n$ and consider the following problem:

$$\Delta w = f\chi_\Omega \quad \text{in } \mathbb{R}^n, \tag{2.1.1}$$

where χ_Ω is the characteristic function of Ω and $\|f(x)\|_{L^\infty(\Omega)} = 1$. It is well known that the solution of such problem is given by $w(x) = (f\chi_\Omega * \Phi)(x)$, where

$$\Phi(x-y) = \frac{1}{n(n-2)\alpha_n}|x-y|^{2-n}$$

is the fundamental solution of the Laplace operator.

Proposition 2.1.2. *Let $\Omega \subset \mathbb{R}^n, n \geq 3$ and $\|f(x)\|_{L^\infty(\Omega)} = 1$. Then, we have that*

$$\|w\|_{L^\infty(\Omega)} \leq \frac{1}{2(n-2)\alpha_n^{2/n}} \operatorname{Vol}(\Omega)^{2/n}.$$

Moreover, equality holds if $f \equiv 1$ and if Ω is a ball.

Before we give a proof, we give a quick overview of classical rearrangements of functions. Let u be a measurable function defined on an open set Ω. We can form the distribution function of u, denoted by $\mu(t)$, the decreasing rearrangement of u, $u^*(s)$ into $[0, +\infty]$ and the spherically symmetric rearrangement of u, $u^\star$. The distribution function of u

$$\mu(t) = \operatorname{meas}\{x \in \Omega : |u(x)| > t\},$$

is a right-continuous function of t, decreasing from $\mu(0) = |\operatorname{supp}(u)|$ to $\mu(+\infty) = 0$ as t increases. The decreasing rearrangement of u, a positive, left continuous function into $[0, +\infty]$, is defined as

$$u^*(s) = \inf\{t \geq 0 : \mu(t) < s\}.$$

The spherically symmetric rearrangement of u is a function $u^\star$ from $\mathbb{R}^n$ into $[0, +\infty]$ whose level sets $\{x \in \mathbb{R}^n : u^\star(x) > t\}$ are concentric balls with the same measure as the level sets $\{x \in \Omega : |u(x)| > t\}$. More precisely, $u^\star$ is defined as

$$u^\star(x) = u^*(\alpha_n|x|^n) = \inf\{t \geq 0 : \mu(t) < \alpha_n|x|^n\}.$$

Note that $\|u\|_\infty = u^*(0) = u^\star(0)$. We refer to [25] for more details on rearrangements of functions.

Proof of Proposition 2.1.2. Let us consider first the case where $f \equiv 1$ and if Ω is a ball centered at x of radius R. Straightforward computation shows that

$$\begin{aligned}\left|\int_{\mathbb{R}^n} \chi_\Omega(y)\Phi(x-y)dy\right| &= \frac{1}{n(n-2)\alpha_n}\int_\Omega |x-y|^{2-n}dy\\ &= \frac{1}{(n-2)}\int_0^R r^{2-n}r^{n-1}dr\\ &= \frac{1}{2(n-2)}R^2\\ &= \frac{1}{2(n-2)\alpha_n^{2/n}}\operatorname{Vol}(B_R)^{2/n}.\end{aligned}$$

Now, for the general case, notice that

$$\begin{aligned}|w(x)| &= \left|\int_{\mathbb{R}^n} f(y)\chi_\Omega(y)\Phi(x-y)dy\right|\\ &\le \frac{1}{n(n-2)\alpha_n}\int_{\mathbb{R}^n}|f(y)|\chi_\Omega(y)|x-y|^{2-n}dy\\ &\le \frac{1}{n(n-2)\alpha_n}\int_{\mathbb{R}^n}\chi_\Omega(y)|x-y|^{2-n}dy.\end{aligned}$$

The following is a classical result of Hardy and Littlewood that can be found in [10]:

$$\int_{\mathbb{R}^n} u(x)v(x)dx \le \int_{\mathbb{R}^n} u^\star(x)v^\star(x)dx.$$

Therefore, since $\Phi = \Phi^\star$, we get that

$$\begin{aligned}|w(x)| &\le \int_{\mathbb{R}^n}\chi_\Omega(y)\Phi(x-y)dy\\ &\le \int_{\mathbb{R}^n}\chi_{\Omega^\star}(y)\Phi^\star(x-y)dy\\ &= \frac{1}{n(n-2)\alpha_n}\int_{\Omega^\star}|x-y|^{2-n}dy,\end{aligned}$$

where $\Omega^\star$ denotes a ball centered at x of same volume of Ω. By the previous case, one gets the desired result. □

Remark 2.1.3. The last step of the proof of Proposition 2.1.2 is to show that

$$\int_{\Omega} \Phi(x-y)dy \leq \int_{\Omega^\star} \Phi(x-y)dy. \tag{2.1.4}$$

A generalization of (2.1.4) is given by Lemma 1.2.6.

That being done, we can start the main proof of this section.

Proof of Lemma 1.2.2. Renormalize u_λ such that $\|u_\lambda\|_\infty = 1$. Consider $\delta \in (0,1)$. We want to show that there exists a constant $C_{n,\delta} > 0$ such that

$$\mathrm{Vol}(V_\delta^i) \geq C_{n,\delta} \lambda^{-n/2}.$$

Let $g = u - \delta$. We have that $\Delta g = \Delta u_{\lambda,i} = \lambda u_{\lambda,i}$ in V_δ^i. By Proposition 2.1.2, there exists $w(x)$ satisfying (2.1.1) with $f = -\lambda u_{\lambda,i}$ and $\Omega = V_\delta^i$ such that

$$\|w\|_\infty \leq \frac{1}{2(n-2)\alpha_n^{2/n}} \lambda \,\mathrm{Vol}(V_\delta^i)^{2/n}.$$

Consider the function $g + w$ on V_δ^i. On the boundary, we have that

$$g + w \leq \frac{1}{2(n-2)\alpha_n^{2/n}} \lambda \,\mathrm{Vol}(V_\delta^i)^{2/n}.$$

Consider x_0 in V_δ^i such that $u_{\lambda,i}(x_0) = 1 = \|u_\lambda\|_\infty$. Thus, we have that

$$(g+w)(x_0) \geq (1-\delta) - \frac{1}{2(n-2)\alpha_n^{2/n}} \lambda \,\mathrm{Vol}(V_\delta^i)^{2/n}.$$

Moreover, since $\Delta(g+w) = \lambda u_{\lambda,i} - \lambda u_{\lambda,i} = 0$, we can use the maximum principle on $g + w$. This implies that

$$(1-\delta) - \frac{1}{2(n-2)\alpha_n^{2/n}} \lambda \,\mathrm{Vol}(V_\delta^i)^{2/n} \leq \frac{1}{2(n-2)\alpha_n^{2/n}} \lambda \,\mathrm{Vol}(V_\delta^i)^{2/n}$$
$$\iff \mathrm{Vol}(V_\delta^i)^{2/n} \geq \frac{1}{2}(1-\delta)\Big(\frac{\lambda}{2(n-2)\alpha_n^{2/n}}\Big)^{-1},$$

yielding that $\mathrm{Vol}(V_\delta^i) \geq (1-\delta)^{n/2}(2(n-2))^{n/2}\alpha_n \lambda^{-n/2}$. $\square$

2.2. Proof of Theorem 1.2.4. The proof of Theorem 1.2.4 for manifolds of dimension $n \geq 3$ is in the same spirit as the proof for $\mathbb{R}^n$. The main difference is that we can not use Proposition 2.1.2 which relies on the fundamental solution of the Laplace operator on $\mathbb{R}^n$. We consider instead the Green's representation of the solution to the Poisson problem on M.

Let Ω be a compact smooth domain of (M^n, g) where $n \geq 3$. It is known that there exists a Green's function (see for instance [19]), namely a smooth function G defined on $\Omega \times \Omega \setminus \{(x, x) \colon x \in \Omega\}$ such that

- $G(x, y) = G(y, x)$, for all $x \neq y$;
- for fixed y, $\Delta_x G(x, y) = 0$, for all $x \neq y$;
- $G(x, y) \geq 0$ and G vanishes on the boundary of Ω;
- as $x \to y$ for fixed y, $G(x, y) \leq \rho(x, y)^{2-n}(1 + o(1)), n \geq 3$, where $\rho(x, y)$ is the geodesic distance between x and y (see [19, p. 81]).

Moreover, if we consider the following problem,

$$\begin{cases} \Delta_g w = f & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2.1}$$

its unique solution is given by

$$w(y) = \int_\Omega G(x, y) f(x) d\sigma.$$

Proposition 2.2.2. *Let $n \geq 3$, $\|u_\lambda\|_\infty = 1$ and $\delta \in (0, 1)$. Let A_i denote a nodal domain of u_λ and $V_\delta^i = \{x \in A_i \colon |u_\lambda(x)| \geq \delta m_{A_i}\}$. There exist λ_0 and $k_{g,\lambda_0} > 0$ such that for all $\lambda > \lambda_0$ and for any $x_0 \in V_\delta^i$, we have that*

$$|w(x_0)| \leq k_{g,\lambda_0} \lambda \operatorname{Vol}_g(V_\delta^i)^{2/n},$$

where w is the solution of problem (2.2.1) with $\Omega = V_\delta^i$ and $f = -\lambda u_\lambda$.

We want to prove an analogous result to Proposition 2.1.2. To do so, we treat split the argument into two cases depending on if the volume of V_δ^i is "large" or "small." We define "small V_δ^i" in such a way that we can apply normal coordinates. This becomes handy since Green functions on M behaves roughly like the fundamental solution of the Laplace operator on $\mathbb{R}^n$. Using Lemma 1.2.6, it is then possible to bound w like claimed.

Proof of Proposition 2.2.2. Let A_i a nodal domain of u_λ and let x_0 be any point such that $u_\lambda(x_0) = m_{A_i}$.

Let $B_{x_0}(r) := \exp_{x_0}(B_0(r))$ denote the geodesic ball of radius r centered at x. It is known that for r small enough, we have that

$$\mathrm{Vol}_g(B_{x_0}(r)) = r^n\,\mathrm{Vol}(B_0(1))\Big(1 - \frac{\mathrm{scal}_g(x_0))}{6(n+2)}r^2 + o(r^2)\Big),$$

where $\mathrm{scal}_g(x_0)$ denotes the scalar curvature at x_0. Therefore, there exists $\epsilon \in (0,1)$ such that for all $0 < r \le \epsilon \le \mathrm{injrad}(M,g)$, there exist $A_g > 0$ and $B_g > 0$ such that

$$A_g r^n \le \mathrm{Vol}_g(B_{x_0}(r)) \le B_g r^n. \tag{2.2.3}$$

Renormalize u_λ such that $\|u_\lambda\|_\infty = 1$. Fix a nodal domain A_i and $x_0 \in A_i$.

Let $\lambda_0 = B_g^{-2/n}\epsilon^{-2}$. Notice that if $\lambda \ge \lambda_0$ and if $\mathrm{Vol}_g(V_\delta^i) > \mathrm{Vol}_g(B_{x_0}(\epsilon))$, the result holds with $k_g = A_g/B_g$.

On the other hand, if $\lambda \ge \lambda_0$, but $\mathrm{Vol}_g(V_\delta^i) \le \mathrm{Vol}_g(B_{x_0}(\epsilon))$, it is always possible to pick R such that $\mathrm{Vol}_g(V_\delta^i) = \mathrm{Vol}_g(B_R(x_0))$ and $R \le \epsilon$ hold.

Let us now work to get an upper bound on $|w(x_0)|$. We have that

$$|w(x_0)| = \left|\lambda \int_{V_\delta^i} G(x,x_0)u_\lambda(x)d\sigma\right| \le \lambda \int_{V_\delta^i} G(x,x_0)d\sigma.$$

Using upper bounds on the Green function (see bounds proved in [24]), we have that there exists $C_g > 0$ such that

$$G(x,x_0) \le C_g\rho(x,x_0)^{2-n}, \quad \text{for all } x \ne x_0,$$

implying that

$$|w(x_0)| \le C_g\lambda \int_{V_\delta^j} \rho(x,x_0)^{2-n}d\sigma.$$

As it was done in $\mathbb{R}^n$, we need to integrate on a ball to obtain a straightforward computable integral. To do so, we use Lemma 1.2.6 whose proof can be found in Section 2.4. Applying Lemma 1.2.6, we get the following:

$$C_g\lambda \int_{V_\delta^j} \rho(x,x_0)^{2-n}d\sigma \le C_g\lambda \int_{(V_\delta^i)^*} \rho^{2-n}d\sigma,$$

where $(V_\delta^i)^* = B_{x_0}(R) = \exp_{x_0}(B_0(R))$.

Using Gauss's Lemma, we now have that

$$\begin{aligned}
|w(x_0)| &\leq C_g\lambda \int_{(V_\delta^i)^*} \rho^{2-n}\Big(1-\frac{1}{6}R_{kl}x^k x^l + O(|x|^3)\Big)dx^1dx^2\dots dx^n \\
&\leq C_g\lambda\Big(\frac{n\omega_n}{2}R^2 - \frac{n\omega_n \operatorname{Scal}_g(x_0)}{6}\frac{R^4}{4} + O(R^5)\Big) \\
&\leq C_g\lambda\frac{n\omega_n}{2}R^2\Big(1-\frac{\operatorname{Scal}_g(x_0)}{6}\frac{R^2}{2} + O(R^3)\Big) \\
&\leq C_g B_g E_g \lambda \operatorname{Vol}_g(B_{x_0}(R))^{2/n} \\
&= k_{g,\lambda_0}\lambda \operatorname{Vol}_g(V_\delta^i)^{2/n}.
\end{aligned}$$

□

The last step to prove Theorem 1.2.4 is very similar to the last step in the proof of Lemma 1.2.2.

Proof of Theorem 1.2.4. Renormalize u_λ such that $\|u_\lambda\|_\infty = 1$. Let $g = u-\delta+w$. On the boundary of V_δ^i, we have that $g = \delta-\delta = 0$. Consider any x_0 in V_δ^i such that $u_{\lambda,i}(x_0) = 1$. By Proposition 2.2.2, we have that

$$g(x_0) \geq (1-\delta) - C_{g,\lambda_0}\lambda \operatorname{Vol}(V_\delta^i)^{2/n}.$$

Moreover, since $\Delta g = \Delta u_{\lambda,i} + \Delta w = \lambda u_{\lambda,i} - \lambda u_{\lambda,i} = 0$ in V_δ^i, we can use the maximum principle on g. This implies that

$$(1-\delta) - C_{g,\lambda_0}\lambda \operatorname{Vol}_g(V_\delta^i)^{2/n} \leq 0 \iff \operatorname{Vol}_g(V_\delta^i) \geq k_{g,\lambda_0}(1-\delta)^{n/2}\lambda^{-n/2}.$$

□

We now prove Proposition 1.2.7 which implies Theorem 1.2.4 in the two dimensional case.

Proof of Proposition 1.2.7. The proof essentially follows [14, Section 3]. It is shown in [14] that given a nodal domain A_i, there exists a ball $B_p(k_g\lambda^{-1/2}) \subset A_i$ centered at any point p such that $u_\lambda(p) = m_{A_i}$. This implies that

$$\rho_\lambda(A_i) \geq k_g\lambda^{-1/2}.$$

The proof of this fact uses harmonic measure techniques to get a bound on the distance from a point of a set, namely the point p where $u_\lambda(p) = m_{A_i}$, to its boundary. Instead of working on a nodal set A_i of a given eigenfunction u_λ, one can run the argument on a connected component of the δ-superlevel set containing p. Such a modification will only influence the constants in the estimates obtained in [14, Section 3]. Thus, arguing in a similar way, one obtains

$$\rho_\lambda(V_i^\delta) \geq k'_{g,\delta}\lambda^{-1/2},$$

which completes the proof of the proposition. □

2.3. Proof of Theorem 1.3.1 . Let $\delta \in (0,1)$ and λ be large enough. Recall that $\mathcal{A}(u) = \{A_i\}_{i=1}^{|\mathcal{A}(u_\lambda)|}$ is the collection of the nodal domains of u_λ. Consider

$$u_\lambda = \sum_{i=1}^{|\mathcal{A}(u_\lambda)|} u_{\lambda,i} \quad \text{where } u_{\lambda,i} = \begin{cases} u_\lambda & \text{if } x \in \mathcal{A}_i, \\ 0 & \text{elsewhere.} \end{cases} \tag{2.3.1}$$

Observe that $\lambda = \lambda_1(A_i)$ since $u_{\lambda,i}$ does not vanish in A_i (see [3] or [9]). Apply Theorem 1.2.4 in order to get the following:

$$\int_{A_i} |u_{\lambda,i}|^p d\sigma \geq \int_{V_\delta^i} \delta^p m_{A_i}^p d\sigma = \delta^p m_{A_i}^p \operatorname{Vol}_g(V_\delta^i) \geq k_{g,\delta,\lambda_0} m_{A_i}^p \lambda^{-n/2}.$$

If we sum over all nodal domains, we get that

$$\int_M |u_\lambda|^p d\sigma = \sum_{i=1}^{|\mathcal{A}(u_\lambda)|} \int_{A_i} |u_{\lambda,i}|^p d\sigma \geq k_{g,\delta,\lambda_0} \lambda^{-n/2} \sum_{i=1}^{|\mathcal{A}(u_\lambda)|} m_{A_i}^p. \tag{2.3.2}$$

To obtain (1.3.2), simply use Sogge's L^p bounds $\|u_\lambda\|_p \leq \lambda^{\delta(p)} \|u_\lambda\|_2$ in (2.3.2).

Notice that one can read off (1.3.5) using the latter argument. Indeed, since

$$\int_M |u_\lambda| d\sigma \leq \operatorname{Vol}_g(M)^{1/2} \left(\int_M |u_\lambda|^2 d\sigma \right)^{1/2} = \operatorname{Vol}_g(M)^{1/2},$$

if we take $p = 1$ in (2.3.2), we get

$$\operatorname{Vol}_g(M)^{1/2} \geq \int_M |u_\lambda| d\sigma \geq k_{g,\delta,\lambda_0} \lambda^{-n/2} \sum_{i=1}^{|\mathcal{A}(u_\lambda)|} m_{A_i},$$

yielding (1.3.5).

2.4. Proof of Lemma 1.2.6. The proof of Lemma 1.2.6 is an application of the bathtub principle [12, Theorem 1.14]:

Theorem 2.4.1 (bathtub principle). *Let f be a real-valued, measurable function on a sigma finite measure space (X, Σ, μ) such that $\mu(\{x : f(x) < t\})$ is finite for all $t \in \mathbb{R}$. Fix $G > 0$ and consider the class of measurable functions on X defined by*

$$\mathcal{C} = \left\{ 0 \leq g \leq 1 : \int_X g \, d\mu = G \right\}.$$

Then, the minimization problem

$$I = \inf_{g \in \mathcal{C}} \int_X f(x) g(x) d\mu(x)$$

is solved by $g = \chi_{\{f<s\}}(x) + cs\mu(\{x\colon f(x) = s\})$, *where* s *is such that*

$$s = \sup_t\{\mu(\{x\colon f(x) < t\}) \le G\},$$

and c *is such that*

$$c\mu(\{x\colon f(x) = s\}) = G - \mu(\{x\colon f(x) < s\}).$$

The minimizer is unique if I *is finite and if*

$$G = \mu(\{x\colon f(x) < s\}) \quad \text{or} \quad G = \mu(\{x\colon f(x) \le s\}).$$

Under the assumptions of Lemma 1.2.6, f is a smooth, non negative, strictly decreasing real valued radial function. We prove an equivalent version of Lemma 1.2.6 for strictly increasing functions. In order to obtain the statement for strictly decreasing functions as stated in Lemma 1.2.6, it suffices to replace f by $-f$.

Recall that $r(x) = d_g(x, x_0)$ is the Riemannian distance between x and some fixed point $x_0 \in M$. In that setting, notice that $\mu(\{x\colon f(r(x)) \le t\})$ is finite for all $t \in \mathbb{R}$. Moreover, the function $t \to \mathrm{Vol}_g(\{x\colon f(r(x)) \le t\})$ is continuous and strictly increasing on $[0, \infty)$. In particular, for all positive constants G, there exists $t > 0$ such that $\mathrm{Vol}_g(\{x\colon f(r(x)) \le t\}) = G$. Therefore, the solution of the minimization problem stated in the bathtub principle under these assumptions is given by $g = \chi_{\{f \le R\}}$, where R is such that $\mathrm{Vol}_g(\Omega) = \int \chi_{\{f(r(x)) \le R\}} d\sigma$. Notice that $\chi_{\{f(r(x)) \le R\}}$ is the characteristic function of the ball $B_R(x_0)$ of radius R centered at x_0 that has the same Riemannian volume as Ω. Thus,

$$I = \inf_{g\in\mathcal{C}} \int_\Omega f(r(x))g(x)d\sigma = \int_{B_R} f(r(x))d\sigma,$$

yielding the desired result.

3. Proof of Theorems 1.4.3, 1.5.1, and 1.7.4

3.1. Proof of Theorem 1.4.3. We present the background required to obtain Theorem 1.4.3. For any fixed positive λ, we consider the $n-$ball,

$$B_\lambda^n = \{x \in \mathbb{R}^n\colon |x| \le j_{n/2-1}\lambda^{-1/2}\}, \tag{3.1.1}$$

where $j_{n/2-1}$ is the first positive zero of the Bessel function $J_{n/2-1}$. It is easy to see that the following problem,

$$\begin{cases} \Delta z = \mu z & \text{in } B_\lambda^n, \\ z = 0 & \text{on } \partial B_\lambda^n, \end{cases}$$

has its first eigenvalue equal to λ, and that the corresponding eigenfunction is given by

$$z(x) = |x|^{1-n/2} J_{n/2-1}(\lambda^{1/2}|x|). \tag{3.1.2}$$

We use the following result, due to G. Chiti (see [4, 5]), in the proof:

Proposition 3.1.3 ([4, Theorem 2]). *Let u be a function satisfying (1.4.1) and consider z, the eigenfunction to the Dirichlet eigenvalue problem on B_λ^n defined above. Then, for any $p \geq 1$,*

$$\|u\|_\infty \left(\int_\Omega |u|^p \right)^{-1/p} \leq \|z\|_\infty \left(\int_{B_\lambda^n} z^p \right)^{-1/p}, \tag{3.1.4}$$

with equality if and only if Ω is a ball, $c = 0$, $a_{ij} = \delta_{ij}$, λ is equal to the first eigenvalue of the equality in (1.4.1) and $|\Omega| = |B_\lambda^n|$, where $|E|$ denotes the Lebesgue measure of the set E.

Remark that we can compute the right hand side of (3.1.4) to obtain the following isoperimetric inequality,

$$\|u\|_\infty \leq K_{n,p} \lambda^{n/(2p)} \|u\|_p, \tag{3.1.5}$$

where $K_{n,p}$ is the constant defined in (1.4.6). Indeed, start by computing $\|z\|_\infty$. The fact that $r^{n/2-1} J_{n/2-1}(r)$ attains its maximum at $r = 0$ follows from Poisson's integral (see [28, Section 3.3]). Thus, we have that

$$z(0) = \lim_{|x| \to 0} \frac{J_{n/2-1}(\lambda^{1/2}|x|)}{|x|^{n/2-1}} = \frac{\lambda^{n/4-1/2}}{2^{n/2-1}\Gamma(\frac{n}{2})}. \tag{3.1.6}$$

Since z is a radial function, we get that

$$\left(\int_{B_\lambda^n} z^p \right)^{-1/p} = (nC_n)^{-1/p} \lambda^{1/2-n/4+n/(2p)} \left(\int_0^{j_{n/2-1}} r^{p-np/2+n-1} J_{n/2-1}^p(r) dr \right)^{-1/p}. \tag{3.1.7}$$

Combine (3.1.6) and (3.1.7), and plug them into (3.1.4) to get (3.1.5).

Proof of Theorem 1.4.3. We start by obtaining (1.4.5). Let us decompose u_λ the following way,

$$u_\lambda = \sum_{i=1}^{|\mathcal{A}(u_\lambda)|} u_i \quad \text{where } u_i = \begin{cases} u_\lambda & \text{if } x \in \mathcal{A}_i, \\ 0 & \text{elsewhere.} \end{cases} \tag{3.1.8}$$

Since $\operatorname{supp}(u_i) \cap \operatorname{supp}(u_j) = \varnothing$ for $i \neq j$, we note that

$$1 = \|u_\lambda\|^2_{L^2(M)} = \int_\Omega \sum_{i=1}^{|\mathcal{A}(u_\lambda)|} u_i^2 = \sum_{i=1}^{|\mathcal{A}(u_\lambda)|} \int_{A_i} u_i^2 = \sum_{i=1}^{|\mathcal{A}(u_\lambda)|} \|u_i\|^2_{L^2(A_i)}. \quad (3.1.9)$$

Recall that each u_i corresponds to an eigenfunction of the Dirichlet problem on these nodal domains. Indeed, since u_i does not vanish in A_i, it corresponds to the first eigenfunction on $\mathcal{A}_i$ and $\lambda_1(\mathcal{A}_i) = \lambda$ by a corollary of Courant's theorem (see [9]).

Thus, we can apply (3.1.5) with $p=2$ to each u_i so that for all $1 \le i \le |\mathcal{A}(u_\lambda)|$, we obtain that

$$\|u_i\|_{L^\infty(\mathcal{A}_i)} \le K_{n,2}\lambda^{n/4}\|u_i\|_{L^2(\mathcal{A}_i)}.$$

Therefore, we get that

$$m_{A_i} = \sup_{x\in\mathcal{A}_i} |u_i(x)| \le K_{n,2}\lambda^{n/4}\|u_i\|_{L^2(\mathcal{A}_i)}.$$

Squaring each side and summing over all nodal domains yield that

$$\sum_{i=1}^{|\mathcal{A}(u_\lambda)|} m^2_{\mathcal{A}_i} \le K^2_{n,2}\lambda^{n/2} \sum_{i=1}^{|\mathcal{A}(u_\lambda)|} \|u_i\|^2_{L^2(\mathcal{A}_i)},$$

and we obtain (1.4.5) by applying (3.1.9) to the latter equation. In order to get (1.4.4), we use (3.1.5) with $p = 1$, to get

$$\|u_i\|_{L^\infty(\mathcal{A}_i)} \le K_{n,1}\lambda^{n/2}\|u_i\|_{L^1(\mathcal{A}_i)}.$$

If we sum over all nodal domains and keep in mind that $\operatorname{supp}(u_i) \cap \operatorname{supp}(u_j) = \varnothing$ for $i \neq j$, we then get

$$\begin{aligned}\sum_{i=1}^{|\mathcal{A}(u_\lambda)|} m_{\mathcal{A}_i} &\le K_{n,1}\lambda^{n/2} \sum_{i=1}^{|\mathcal{A}(u_\lambda)|} \|u_i\|_{L^1(\mathcal{A}_i)} \\ &= K_{n,1}\lambda^{n/2}\|u_\lambda\|_{L^1(\Omega)} \\ &\le K_{n,1}\lambda^{n/2}\|u_\lambda\|_{L^2(\Omega)} \operatorname{Vol}(\Omega)^{1/2}.\end{aligned}$$

The last line follows from Cauchy-Schwartz inequality. Since $\|u_\lambda\|_{L^2(\Omega)} = 1$, the proof is completed. □

3.2. Proof of Theorem 1.5.1. Let I_1 denote the family of indexes of nodal domains touching the boundary of Ω and let $I_2 = |\mathcal{A}(u_\mu)| \setminus I_1$. Let us start by obtaining (1.5.4)

Notice that nodal domains whose index is in I_2 are such that the eigenfunction u restricted to them corresponds to the first eigenfunction of the Dirichlet eigenvalue problem on such A_i, so that $\mu = \lambda_1(A_i)$. Therefore, it is possible to use (3.1.5) with $p = 2$ as done in the proof of Theorem 1.4.3 in order to get that

$$\sum_{i \in I_2} m_{A_i}^2 \leq C\mu.$$

As for nodal domains whose index is in I_1, since by the Hormander-Levitan-Avakumovic L^∞ bound, we have that $m_{A_i} \leq C\mu^{1/4}$, we get that

$$\sum_{i \in I_1} m_{A_i}^2 \leq C\sqrt{\mu} \cdot (\mu^{1/4})^2 = C\mu,$$

yielding (1.5.4).

The same reasoning can be applied to obtain (1.5.3), namely

$$\sum_{i \in I_1} m_{A_i} + \sum_{i \in I_2} m_{A_i} \leq C\sqrt{\mu} \cdot \mu^{1/4} + C\mu \leq C'\mu,$$

yielding (1.5.3).

3.3. Proof of Theorem 1.7.4.

The proof is based on the following result:

Lemma 3.3.1 (Lemma 4.1 in [13]). *Let $u_{p,1}$ denote the first eigenfunction of the Dirichlet p-Laplacian eigenvalue problem on a bounded Euclidean domain $\Omega \subset \mathbb{R}^n$, then*

$$\|u_{p,1}\|_{L^\infty(\Omega)} \leq 4^n \lambda^{n/p} \|u_{p,1}\|_{L^1(\Omega)}.$$

Note that the constant term 4^n is not sharp.

Remark 3.3.2. One difference between Chiti-type inequalities and the preceding lemma is that Chiti-type inequalities apply to any eigenfunction of the Dirichlet eigenvalue problem rather than only to the first one. However, the generalization of Chiti's results to the p-Laplace operator (see [1]) is of the form

$$\|u\|_r \leq K(r, q, p, n, \lambda)\|u\|_q,$$

where u is any eigenfunction associated to eigenvalue λ, $0 < q < r \leq +\infty$. It is important to notice that the constant $K(r, q, p, n, \lambda)$ is not explicit (since we can not compute the eigenfunctions of the ball explicitly). Thus, we cannot use it as it was done for the Laplace operator.

We are ready to prove Theorem 1.7.4.

Proof. Let $\|u_{p,\lambda}\|_p = 1$. Consider $A_i \subset \Omega$ a nodal domain of $u_{p,\lambda}$. Let us decompose $u_{p,\lambda}$ the following way,

$$u_{p,\lambda} = \sum_{i=1}^{|\mathcal{A}(u_\lambda)|} u_i \quad \text{where } u_i = \begin{cases} u_{p,\lambda} & \text{if } x \in \mathcal{A}_i, \\ 0 & \text{elsewhere.} \end{cases} \tag{3.3.3}$$

Since u_i corresponds to the first eigenfunction of the Dirichlet p-Laplacian eigenvalue problem on A_i, Lemma 3.3.1 yields that

$$\|u_i\|_{\infty,A_i} \leq 4^n \lambda^{n/p} \|u_i\|_{1,A_i}, \quad \text{for all } 1 \leq i \leq |\mathcal{A}(u_\lambda)|.$$

Therefore, after summing over all nodal domains, we get that

$$\begin{aligned} \sum_{i=1}^{|\mathcal{A}(u_\lambda)|} \|u_i\|_{L^\infty(A_i)} &= \sum_{i=1}^{|\mathcal{A}(u_\lambda)|} m_{A_i} \\ &\leq 4^n \lambda^{n/p} \sum_{i=1}^{|\mathcal{A}(u_\lambda)|} \|u_i\|_{L^1(A_i)} \\ &\leq 4^n \lambda^{n/p} \|u_{p,\lambda}\|_{L^1(\Omega)} \\ &\leq 4^n \operatorname{Vol}(\Omega)^{1-1/p} \lambda^{n/p} \|u_{p,\lambda}\|_{L^p(\Omega)} \\ &= 4^n \operatorname{Vol}(\Omega)^{1-1/p} \lambda^{n/p}. \end{aligned}$$

□

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