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A solvability condition for a tokamak problem

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Abstract. By using the matrix representation of the Laplacian, under Dirichlet and Robin boundary conditions, we recast a boundary inversion problem for the tokamak into a simple algebraic system. The solution is then obtained explicitly in terms of the Fourier coefficients of the observation.

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1. Introduction

We are concerned with a boundary inversion problem related to the tokamak problem. As treated by Demidov and Moussaoui in [6], the question is to evaluate, if possible, the constants a and b in

$$\begin{cases} \Delta u = au + b \geq 0 & \text{on } \Omega \subset \mathbb{R}^d,\ d \geq 2, \\ u = 0 \text{ and } (\partial_n u = \Phi) & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

from a single reading of the outer normal derivative $\partial_n u = \Phi \in L^1(\partial\Omega)$ of the solution. Here Ω is an open bounded connected Lipschitz domain in $\mathbb{R}^d$. We shall refer to $\Phi \neq 0$ as an observation, as it comes from a nontrivial solution generated by the real constants a and b. By using conformal mappings and asymptotics of the solution near a singularity, such as a corner, Demidov and Moussaoui show that partial data of Φ on one side of a corner is sufficient to compute the values a and b. Although this seems to be a simple question, it is related to the Pompeiu problem and subsequently to the geometry of Ω, see [1, 3, 4, 5, 7, 13, 14].

One of the open questions raised in [6] is the computation of the values $\{a, b\}$ in the case of a smooth boundary with no corners. In this note, we answer this question by making use of pseudo-spectral methods. Note that although this boundary inversion problem originated from the physics of plasma, magnetohydrodynamics also has interesting applications in medical imaging (cf. [11]).

2. Preliminaries

Let Ω is be an open bounded connected Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$, whose boundary $\partial\Omega$ satisfies the strong local Lipschitz condition [12]. Denote by (φ_n) and (ψ_n) the eigenbases generated by the Dirichlet and Robin Laplacian, with norms $\|\varphi_n\| = \|\psi_n\| = 1$,

$$\text{(D)} \quad \begin{cases} -\Delta\varphi_n = \lambda_n\varphi_n, \\ \varphi_n(x) = 0, \qquad x \in \partial\Omega, \end{cases}$$

and

$$\text{(R)} \quad \begin{cases} -\Delta\psi_n = \mu_n\psi_n, \\ \partial_n\psi_n(x) + \rho(x)\psi_n(x) = 0, \quad x \in \partial\Omega, \end{cases}$$

where $0 \leq \rho(x) \in L^1(\partial\Omega)$. Since the solution satisfies $u \in H_0^1(\Omega) \subset H^1(\Omega)$, its expansion in both eigenbases reads as

$$u = \sum_{n\geq 1} c_n\varphi_n = \sum_{n\geq 1} d_n\psi_n \text{ in } L^2(\Omega), \tag{2.1}$$

and since $\{\varphi_n\}$ and $\{\psi_n\}$ are also bases in $H^1(\Omega)$, we deduce that

$$\sum_{n\geq 1} \lambda_n|c_n|^2 < \infty \quad \text{and} \quad \sum_{n\geq 1} \mu_n|d_n|^2 < \infty.$$

Multiplying (1.1) by φ_n and ψ_n, and integrating by parts leads to

$$\begin{cases} (\mathbb{L} + aI_d)\boldsymbol{c} + b\mathbf{1}_D = 0, \\ (\mathbb{M} + aI_d)\boldsymbol{d} + b\mathbf{1}_R = \boldsymbol{\tau}, \end{cases} \tag{2.2}$$

where $\boldsymbol{c} = (c_n)$, $\boldsymbol{d} = (d_n) \in \ell^2$ are defined by (2.1), $\mathbb{L} = \operatorname{diag}(\lambda_n)$, $\mathbb{M} = \operatorname{diag}(\mu_n)$, and

$$\boldsymbol{\tau} = \left(\int_{\partial\Omega} \Phi(x)\psi_n(x)ds\right), \quad \mathbf{1}_D = \left(\int_\Omega \varphi_n(x)dx\right), \quad \mathbf{1}_R = \left(\int_\Omega \psi_n(x)dx\right). \tag{2.3}$$

Note that $\varphi_1(x) \neq 0$ for any $x \in \Omega$, as it is the principal Dirichlet eigenfunction [9].

Observe that (2.1) generates a transition operator, mapping Dirichlet to Robin Fourier coefficients

$$\boldsymbol{d} = \mathbb{A}\boldsymbol{c} \tag{2.4}$$

with entries $\mathbb{A} = (\varphi_k, \psi_n)_{nk}$. It follows from (2.1) that $\|u\|^2 = \sum_{n\geq 1} |c_n|^2 = \sum_{n\geq 1} |d_n|^2$ and so the operator $\mathbb{A}\colon \ell^2 \to \ell^2$ is an isometry, and unitary. Combining (2.2), (2.4), and the fact that $\mathbf{1}_R = \mathbb{A}\mathbf{1}_D$, we get a linear system with unknown $(a, b, \boldsymbol{c})$

$$\begin{cases} \mathbb{A}\mathbb{L}\boldsymbol{c} + a\mathbb{A}\boldsymbol{c} + b\mathbb{A}\mathbf{1}_D = 0, \\ \mathbb{M}\mathbb{A}\boldsymbol{c} + a\mathbb{A}\boldsymbol{c} + b\mathbb{A}\mathbf{1}_D = \boldsymbol{\tau}. \end{cases} \tag{2.5}$$

In order to reduce (2.5) to a system of only two unknowns a and b, we need to find $\mathbf{c}$, which is obtained by elimination,

$$\mathbb{T}\boldsymbol{c} = \boldsymbol{\tau}, \quad \text{where } \mathbb{T} = \mathbb{M}\mathbb{A} - \mathbb{A}\mathbb{L}. \tag{2.6}$$

To solve (2.6) we need to study the invertibility of $\mathbb{T}$ as an operator acting in ℓ^2.

Remark 1. In [2] it is shown that the spectrum of the map: $\mathbb{A} \longrightarrow \mathbb{T}$, see (2.6) is explicilty given by $\{\lambda_n\} - \{\mu_k\}$, and so has an inverse if and only if $\lambda_n \neq \mu_k$. However it does not tell us whether $\mathbb{T}$ is invertible, which we examine now.

For a fixed a and b, define the solutions set to be

$$S_{ab} = \{(c_n) \in \ell^2 \colon u = \sum c_n\varphi_n \in H_0^1(\Omega) \text{ is solution of (1.1)}\}.$$

We now show that $\mathbb{T}_S$, the restriction of $\mathbb{T}$ to S_{ab}, is always invertible.

Proposition 1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$, then $\mathbb{T}_S^{-1}$ exists.*

Proof. Otherwise, we would have two distinct solutions u_1 and $u_2 \in S_{ab}$, that would correspond to the same observation Φ. Obviously the difference $\omega = u_2 - u_1 \neq 0$ satisfies

$$\begin{cases} \Delta\omega = a\omega & \text{on } \Omega \subset \mathbb{R}^d,\ d \geq 2, \\ \omega = 0 \quad \text{and} \quad \partial_n\omega = 0 & \text{on } \partial\Omega. \end{cases}$$

This implies that $a < 0$ and by Green's formula, [1, (2.22)], we must have $\omega = 0$ in Ω, that is, the null space $N(\mathbb{T}_S) = \{0\}$, implying that $\mathbb{T}_S^{-1}$ exists. □

Thus, to solve (2.6) for $\boldsymbol{c}$, it is sufficient to confirm that $\boldsymbol{\tau} \in R(\mathbb{T}_S)$, since $\boldsymbol{\tau}$ is an observation with equations (2.6) and (2.5) being equivalent under $\mathbb{A}$. Thus, as $\boldsymbol{\tau} \in R(\mathbb{T}_S)$, we can find a unique $\boldsymbol{c} = (\mathbb{T}_S)^{-1}\boldsymbol{\tau}$ which reduces the first equation in (2.2) to two unknowns only, namely $a\boldsymbol{c} + b\mathbf{1}_D = -\mathbb{L}\boldsymbol{c}$. Next, we study the solvability of this infinite system in (a, b),

$$ac_n + b\gamma_n = -\lambda_n c_n \quad \text{for } n \geq 1, \tag{2.7}$$

where the sequences λ_n, $c_n = \int_\Omega u(x)\varphi_n(x)dx$ and $\gamma_n = \int_\Omega \varphi_n(x)dx$ are known for all $n \geq 1$. We use the following lemma:

Lemma 1. *If $\Phi \neq 0$, then the system (2.7) contains at least two independent equations.*

Proof. To find a and b, we need two independent equations, that is, their 2×2 determinant is nonzero, implying that the corresponding rank equals 2. If all possible 2×2 determinants vanish, it would mean that the vectors $\mathbf{c} = (c_n)$ and $\mathbf{1}_D = (\gamma_n)$ are proportional, implying (2.7) has rank one. In other words, there exists $\delta \neq 0$ such that

$$\int_\Omega u(x)\varphi_n(x)dx = \delta \int_\Omega \varphi_n(x)dx \quad \text{for all } n \geq 1.$$

In this case, the solution u must be constant in Ω,

$$u(x) = \sum_{n\geq 1} c_n\varphi_n(x) = \delta \sum_{n\geq 1} \int_\Omega \varphi_n(\eta)d\eta \, \varphi_n(x) \;= \delta\, \mathbf{1}_D(x),$$

and so $u = \Delta u = 0$ from (1.1), which means that $b = 0$ and $\Phi = \partial_n u = 0$ almost everywhere on $\partial\Omega$. This contradicts the fact that u is not trivial and thus there is at least one nonzero 2×2 determinant, which would deliver a unique solution (a, b). □

Thus by Proposition 1 and the above lemma we have proved the following result:

Proposition 2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$, then a single measurement $\Phi \neq 0$ suffices to compute the values of a and b.*

Below we illustrate this idea with two simple examples.

3. Example on the square

We now provide a simple example where the condition $\lambda_n \neq \mu_k$ holds for a square $\Omega = [0, \pi] \times [0, \pi]$ in $\mathbb{R}^2$. It is easily seen that the eigenfunctions of the Dirichlet Laplacian are $\varphi_{lm}(x, y) = \frac{2}{\pi} \sin(lx) \sin(my)$ and the spectrum $\sigma_D = \{l^2 + m^2 : 1 \leq l, m \in \mathbb{N}\}$.

Example 1. For the Robin boundary condition we take

$$\partial_y u(x, 0) = 0, \quad \partial_y u(x, \pi) = 0, \quad \partial_x u(0, y) = 0, \quad \partial_x u(\pi, y) + u(\pi, y) = 0,$$

where $0 \leq x, y \leq \pi$, implying

$$\rho(\pi, y) = \begin{cases} 1 & \text{for } y \in (0, \pi) \text{ and } 0 \\ & \text{otherwise on } \partial\Omega. \end{cases}$$

Thus the Robin eigenfunctions are $\psi_{kj}(x, y) = \frac{2}{\pi} \cos(\xi_j x) \cos(ky)$, where ξ_j are roots of

$$\cos(\xi_j \pi) - \xi_j \sin(\xi_j \pi) = 0 \tag{3.1}$$

and the spectrum equals $\sigma_R = \{\xi_j^2 + k^2 : 1 \leq j, k \in \mathbb{N}\}$. To see that $\sigma_R \cap \sigma_D = \emptyset$, write $\xi_j = j - 1 + \varepsilon_j$ for $j \geq 1$ and so $0 < \varepsilon_j < 1/2$, and a closer look at equation (3.1), or $x \tan(\pi x) = 1$, reveals that

$$\varepsilon_j \downarrow 0, \quad \text{and} \quad \pi(j-1)\varepsilon_j < 1 \quad \text{for } j \geq 1. \tag{3.2}$$

Thus if $\sigma_R \cap \sigma_D \neq \emptyset$ then there is at least one $\xi_j^2 \in \mathbb{N}$, which is impossible given (3.2). We have

$$0 < \xi_1^2 < \frac{1}{4}$$

and

$$0 < \xi_j^2 - (j-1)^2 = 2(j-1)\varepsilon_j + \varepsilon_j^2 < \frac{2}{\pi} + \frac{1}{\pi^2} < 1 \quad \text{for } j \geq 2$$

and, by Proposition 1, $\mathbf{c} = \mathbb{T}^{-1}\tau$ is known. Thus (2.7) yields

$$ac_n + b\gamma_n = -\lambda_n c_n \quad \text{for } n \geq 1, \tag{3.3}$$

where, for $\lambda_n = l^2 + m^2$,

$$\begin{aligned}\gamma_n &= \int_\Omega \varphi_n(x)dx \\ &= \int_0^\pi \int_0^\pi \frac{2}{\pi} \sin(lx)\sin(my)dxdy \\ &= \begin{cases} \dfrac{8}{\pi l m} & \text{if both } l \text{ and } m \text{ are odd,} \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

By Lemma 1, we know there are two independent equations in (3.3), which can be used to compute the values a and b.

Example 2. Partial data. Since the square Ω has four corners, one might ask if our method can reproduce the result by Demidov and Moussaoui in a simpler way. In other words, is the knowledge of Φ on one side of the square only sufficient to compute a and b? For this case use the following mixed boundary conditions instead of the Robin boundary condition,

$$\partial_y u(x,0) = 0,\ u\,(x,\pi) = 0,\ u(0,y) = 0,\ u\,(\pi,y) = 0, \quad \text{where } \ 0 \le x,\ \ y \le \pi.$$

The new eigenfunctions are

$$\psi_{kj}(x,y) = \frac{2}{\pi} \sin(kx)\cos((j+1/2)y),$$

and the new spectrum is $\sigma_M = \{\mu_{jk} := k^2 + j^2 + j + 1/4 \colon k \ge 1,\ j \ge 0\}$. It is easily seen that the condition $\sigma_M \cap \sigma_D = \emptyset$ in Remark 1 holds since $\inf|\lambda_n - \mu_{jk}| \ge 1/4$. The fact that only partial data on the lower side of the square is used can be seen from

$$\boldsymbol{\tau} = \left(\int_{\partial\Omega} \Phi(x,y)\psi_{jk}(x,y)ds\right) = \left(\frac{2}{\pi}\int_0^\pi \Phi(x,0)\sin(kx)dx\right).$$

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