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Efficient Anderson localization bounds for large multi-particle systems

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Abstract. We study multi-particle interactive quantum disordered systems on a polynomially growing countable connected graph $(\mathcal{Z}, \mathcal{E})$. The main novelty is to give localization bounds uniform in finite volumes (subgraphs) in $\mathcal{Z}^N$ as well as for the whole of $\mathcal{Z}^N$. Such bounds are proved here by means of a comprehensive fixed-energy multi-particle multi-scale analysis. We consider – for the first time in the literature – a discrete N-particle model with an infinite-range, sub-exponentially decaying interaction, and establish (1) exponential spectral localization, and (2) strong dynamical localization with sub-exponential rate of decay of the eigenfunction correlators with respect to the natural symmetrized distance in the multi-particle configuration space.

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Contents

1. Introduction. The model and results

Until recently, the rigorous Anderson localization theory focused on single-particle models. (In the physical community, notable papers on multi-particle systems with interaction appeared as early as in 2005–2006; see [25, 6].)

Initial rigorous results on multi-particle lattice localization for a finite-range two-body interaction potential were presented in [16, 17, 18] and [4, 5]; continuous models in the Euclidean space have been considered in [8, 15]. In these papers, both Spectral Localization (SL) and Dynamical Localization (DL) have been established. A considerable progress was made in [27, 28], with the help of an adapted bootstrap variant of the Multi-Scale Analysis (MSA) developed in the earlier works [23, 24]. The resulting bootstrap multi-particle MSA (MPMSA) was applied in [27, 28] to multi-particle systems in the lattice and in the Euclidean space, respectively. More recently, a very important step was made in the paper [22] which extended the multi-particle Fractional-Moment Method (MPFMM) from the lattice case [4, 5] to the continuous one, with an infinite-range two-body interaction potential. As usual, (MP)FMM provides exponential[1] decay bounds upon the eigenfunction correlators (EFCs), while the bootstrap MPMSA achieves only a sub-exponential decay of the EFCs at large distances.

The main motivation for the present work comes from the fact that in all above-mentioned mathematical works the decay bounds on the eigenfunctions and EFCs were proved in the so-called *Hausdorff distance* (HD) which is actually a pseudo-distance in the multi-particle configuration space. In the context of the multi-particle Anderson localization, the HD appears explicitly in [4, 5] (as well as in [27, 28]), while in [16, 17, 18] it was used implicitly, through the notion of separated cubes. The point is that there are arbitrarily distant loci in the multi-particle space which might support quantum tunneling between them, and the HD bounds do not reflect this possibility. As a result, the SL and DL have been proved so far in an infinitely extended physical configuration space, but some tunneling processes could not be completely ruled out in arbitrarily large, yet bounded domains. The existence of efficient multi-particle localization estimates – even for a bounded number of particles $N \geq 3$ – remained a challenging open question. These aspects of the rigorous multi-particle localization theory were analyzed by Aizenman and Warzel in [4, 5]; a partial solution was given in [10, 13, 14]. The mathematical core of the problem is an eigenvalue concentration (EVC) bound for two distant

[1] In Ref. [22], exponential decay of the EFC is proved for the models in $\mathbb{R}^d$ with exponential decay of the interaction potential $U(r)$; sub-exponential decay of $U(r)$ results in sub-exponential decay of the EFC.

loci in the multi-particle space. In the current paper we employ a probabilistic result from [14] and prove a suitable EVC bound (cf. Theorem 2.2) for a class of sufficiently regular marginal probability distributions of IID external random potentials. It has to be emphasized that the problem in question appears only for the number of particles $N \geq 3$, and the proof of localization for two-particle systems given in [17] operates with the (symmetrized) norm-distance in the two-particle space. As a result, the two-particle localization holds also in finite (but arbitrarily large) regions of the physical configuration space, under mild regularity conditions upon the random potential; see [19].

In the present paper we focus on an interactive N-particle Anderson model, on a countable connected graph $(\mathcal{Z}; \mathcal{E})$ with a polynomially growing size of a ball when the radius increases to infinity. The main method used is a new variant of the MPMSA. The results are summarized as follows.

- We prove uniform localization bounds, in terms of decay of eigenfunctions (EFs) and eigenfunction correlators (EFCs) valid for finite or infinite subgraphs of $\mathcal{Z}^N$, including the whole $\mathcal{Z}^N$. The uniform decay is established with respect to the natural symmetrized max-distance in the multi-particle configuration space, and this makes the new bounds suitable for applications to physical models where quantum particles evolve in a sample of disordered media of finite size. Previously published results provided less efficient bounds in finite volumes (for $N \geq 3$).
- As in [22], we treat systems with infinite-range interaction potentials (but in a countable graph instead of $\mathbb{R}^d$). Specifically, we consider a two-body potential decaying at a large distance r as e^{-r^ζ} where $\zeta > 0$. Surprisingly, the SL holds here with an exponential rate (e^{-mr}, $m > 0$) even if $0 < \zeta < 1$. Note that an exponential decay of EFs was proved in [11] under the assumption of decay of the interaction with rate e^{-r^ζ}, but only for $\zeta \in (0; 1]$ sufficiently close or equal to 1. Paper [22] treats the EFC decay (in continuous models) in several cases, including the following:
  - the interaction potential decays at an exponential rate e^{-ar}; in this case the EFCs also decay exponentially fast;
  - the interaction potential decays sub-exponentially, as e^{-r^ζ} with $\zeta \in (0, 1)$; in this case the EFCs also decay sub-exponentially.

We present competing bounds for the EFCs (in the discrete model). As was said, in contrast to Ref. [22], we also establish *exponential* decay of the EFs, thanks to the logical independence of the (exponential) decay analysis of the EFs from that of the EFCs.

The rest of the paper is devoted to the proof of Theorem 1.1. In particular, in Section 2 we establish the crucial ingredient of the proof: eigenvalue concentration (EVC) bounds. The bulk of the work is about the proof of assertion (A): it is carried in Section 3. The main strategy here is the induction on the number of particles N, initially developed in [17, 18]. Each step $N-1 \rightsquigarrow N$, $N = 2, \dots, N^*$, employs the multi-scale analysis of multi-particle Hamiltonians. Unlike Ref. [18], in Section 3, we make use of a more efficient scaling technique, essentially going back to the work [23] and recently adapted in Ref. [27] to multi-particle systems. However, the scaling scheme used in Ref. [18] (and going back to [20]) is required in Section 4 to prove *exponential* decay of the EFs.

At several points, our strategy deviates from the bootstrap MPMSA developed in [27]. In particular, we avoid the actual energy-interval MSA stage of the bootstrap and employ some general functional-analytic results to derive the energy-interval bounds (more difficult to prove directly) from their simpler, fixed-energy counterparts. Speaking informally, we carry out (independently) two separate scaling analyses analogous (but not identical) to two of the four phases of the bootstrap method. We do not perform, either, several geometrical optimizations making the bootstrap possible. This results in a shorter proof of sub-exponential decay of the EFCs, although we have to emphasize that the complete bootstrap scheme would undoubtedly result in stronger probabilistic estimates. Note also that the base of induction ($N = 1$) requires a proof of localization for single-particle systems on graphs with a polynomial growth of the size of a ball with the radius. The required estimates were proved in Ref. [12] following the techniques from Ref. [23].

1.1. The multi-particle Hamiltonian. Consider a finite or locally finite, connected, non-oriented graph $(\mathcal{Z}, \mathcal{E})$, with the vertex set $\mathcal{Z}$ and the edge set $\mathcal{E}$. (For brevity, we often refer to $\mathcal{Z}$ only.) We assume that $\mathcal{E}$ does not include cyclic edges $x \leftrightarrow x$ and denote by $\mathrm{d}(\cdot,\cdot)$ the graph distance on $\mathcal{Z}$: $\mathrm{d}(x,y)$ equals the length of the shortest path $x \leftrightsquigarrow y$ over the edges. (By definition, $\mathrm{d}(x,x) = 0$.) We assume that graph $(\mathcal{Z}, \mathcal{E})$ belongs to a class $\mathfrak{G}(d, C_d)$ for some $d, C_d > 0$, meaning that the size of a ball $\mathrm{B}(x,L) := \{y \colon \mathrm{d}(x,y) \le L\}$ is polynomially bounded:

$$\sup_{x \in \mathcal{Z}} \sharp \mathrm{B}(x,L) \le C_d L^d, \quad L \ge 1. \tag{1.1}$$

Notice that the property (1.1) is inheritable by inclusion, so any connected subgraph of $\mathcal{Z} \in \mathfrak{G}(d, C_d)$, finite or infinite, also belongs to the class $\mathfrak{G}(d, C_d)$.

Physically, $\mathcal{Z}$ represents the configuration space of a single quantum particle.

The configuration space of N distinguishable particles is the graph $(\mathcal{Z}^N, \mathcal{E}_N)$, where $\mathcal{Z}^N$ is the Cartesian power, and the edge set $\mathcal{E}_N$ is defined as follows. Given two vertices, $\mathbf{x} = (x_1, \dots, x_N)$, $\mathbf{y} = (y_1, \dots, y_N) \in \mathcal{Z}^N$, the edge $\mathbf{x} \leftrightarrow \mathbf{y}$ exists if, for some $j = 1, \dots, N$, there exists an edge $x_j \leftrightarrow y_j$ in $\mathcal{E}$ while for $i \neq j$ we have $x_i = y_i$. We refer to $\mathbf{x}$ and $\mathbf{y}$ as N-particle configurations (briefly, configurations) on $\mathcal{Z}$ and use the same notation $\mathrm{d}(\mathbf{x}, \mathbf{y})$ for the graph distance on $\mathcal{Z}^N$.

Apart from the distance $\mathrm{d}(\cdot\,,\cdot)$ on $\mathcal{Z}^N$, it will be convenient to use the max-distance $\boldsymbol{\rho}$ and the symmetrized max-distance $\boldsymbol{\rho}_\mathrm{S}$, defined as follows:

$$\boldsymbol{\rho}(\mathbf{x}, \mathbf{y}) = \max_{1 \le j \le N} \mathrm{d}(x_j, y_j); \quad \boldsymbol{\rho}_\mathrm{S}(\mathbf{x}, \mathbf{y}) = \min_{\pi \in \mathfrak{S}_N} \boldsymbol{\rho}(\mathbf{x}, \pi(\mathbf{y})). \tag{1.2}$$

Here the symmetric group $\mathfrak{S}_N$ acts on $\mathcal{Z}^N$ by permutations of the coordinates.

In the case where $\mathcal{Z} = \mathbb{Z}^d$, one obtains $\mathcal{Z}^N \cong (\mathbb{Z}^d)^N$, $\boldsymbol{\rho}(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|_\infty = \max_i \mathrm{d}(x_i, y_i)$, $\mathrm{d}(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|_1 = \sum_i |x_i - y_i|$.

Next, $\mathbf{B}^{(N)}(\mathbf{x}, L)$ denotes the ball of radius L in $\mathcal{Z}^N$ centered at $\mathbf{x} = (x_1, \dots, x_N)$ in the metric $\boldsymbol{\rho}$ (below often called an *N-particle ball*):

$$\mathbf{B}^{(N)}(\mathbf{x}, L) := \{\mathbf{y} \colon \boldsymbol{\rho}(\mathbf{x}, \mathbf{y}) \le L\} = \mathop{\times}_{j=1}^{N} \mathrm{B}(x_j, L). \tag{1.3}$$

It will be often convenient to omit the superscript N and use the boldface notation: $\boldsymbol{\mathcal{Z}} = \mathcal{Z}^N$, $\boldsymbol{\mathcal{E}} = \mathcal{E}_N$, $\mathbf{B}(\mathbf{x}, L) = \mathbf{B}^{(N)}(\mathbf{x}, L)$, etc.

The graph Laplacian $\Delta_\mathcal{Z}$ on $\mathcal{Z}$ is given by

$$(\Delta_\mathcal{Z} f)(x) = \sum_{\langle x,y \rangle} (f(y) - f(x)) = -n_\mathcal{Z}(x) f(x) + \sum_{\langle x,y \rangle} f(y), \quad x \in \mathcal{Z}; \tag{1.4}$$

here $n_\mathcal{Z}(x) = \sharp\{y \colon \mathrm{d}(x, y) = 1\}$, and $\langle x, y \rangle$ stands for a pair $(x, y) \in \mathcal{Z} \times \mathcal{Z}$ with $\mathrm{d}(x, y) = 1$. Similarly, the Laplacian on $\boldsymbol{\mathcal{Z}}$ is defined by

$$\begin{aligned}(\boldsymbol{\Delta}_{\boldsymbol{\mathcal{Z}}} f)(\mathbf{x}) &= \sum_{1 \le j \le N} (\Delta_\mathcal{Z}^{(j)} f)(\mathbf{x}) \\ &= \sum_{\langle \mathbf{x},\mathbf{y} \rangle} (f(\mathbf{y}) - f(\mathbf{x})) \\ &= -n_{\boldsymbol{\mathcal{Z}}}(\mathbf{x}) f(\mathbf{x}) + \sum_{\langle \mathbf{x},\mathbf{y} \rangle} f(\mathbf{y}), \quad \mathbf{x} \in \boldsymbol{\mathcal{Z}}.\end{aligned} \tag{1.5}$$

Here $\Delta_\mathcal{Z}^{(j)}$ denotes the Laplacian acting on the jth component of $\mathbf{x}$, $\langle \mathbf{x}, \mathbf{y} \rangle$ stands for a pair $(\mathbf{x}, \mathbf{y}) \in \boldsymbol{\mathcal{Z}} \times \boldsymbol{\mathcal{Z}}$ with $\mathrm{d}(\mathbf{x}, \mathbf{y}) = 1$, and $n_{\boldsymbol{\mathcal{Z}}}(\mathbf{x}) = \sharp\,\{\mathbf{y} \colon \mathrm{d}(\mathbf{x}, \mathbf{y}) = 1\}$.

For $\Lambda \subset \mathbb{Z}$, $\mathbf{\Lambda} \subset \mathbf{Z}$, the Laplacians Δ_Λ and $\mathbf{\Delta}_{\mathbf{\Lambda}}$ (with Dirichlet's boundary condition) are introduced as follows:

$$\Delta_\Lambda = \mathbf{1}_\Lambda \Delta_{\mathbb{Z}} \mathbf{1}_\Lambda \upharpoonright \ell^2(\Lambda), \quad \mathbf{\Delta}_{\mathbf{\Lambda}} = \mathbf{1}_{\mathbf{\Lambda}} \mathbf{\Delta}_{\mathbf{Z}} \mathbf{1}_{\mathbf{\Lambda}} \upharpoonright \ell^2(\mathbf{\Lambda}).$$

The N-particle Hamiltonian $\mathbf{H}^{(N)}_{\mathbf{\Lambda}} = \mathbf{H}^{(N)}_{\mathbf{\Lambda}}(\omega)$ in a domain $\mathbf{\Lambda} \subseteq \mathbf{Z}$ acts as

$$\begin{aligned}(\mathbf{H}^{(N)}_{\mathbf{\Lambda}}\Psi)(\mathbf{x}) = (-\mathbf{\Delta}_{\mathbf{\Lambda}}\Psi)(\mathbf{x}) + g \sum_{1\le j\le N} V(x_j;\omega)\Psi(\mathbf{x}) \\ + \sum_{1\le i<j\le N} U(\mathrm{d}(x_i,x_j))\Psi(\mathbf{x}), \quad \mathbf{x} = (x_1,\dots,x_N) \in \mathbf{\Lambda}.\end{aligned} \tag{1.6}$$

Here V represents an external potential given by a random field $V\colon \mathbb{Z}\times\Omega \to \mathbb{R}$ relative to a probability space $(\Omega, \mathfrak{B}, \mathbb{P})$, and U a two-body interaction; see below. The constant $g \in \mathbb{R}$ is referred to as the coupling amplitude. Under the imposed conditions, with probability one, the operator $\mathbf{H}^{(N)}_{\mathbf{\Lambda}}(\omega)$ is bounded and self-adjoint in $\ell^2(\mathbf{\Lambda})$.

As usual, the complement of a subset $A \subset A'$, with A' clearly identified from the context, will be often denoted by $A^{\complement}$.

1.2. The assumptions and results. Our goal is to prove that, for sufficiently large values of the disorder amplitude $|g|$, the (random) eigenfunctions of $\mathbf{H}^{(N)}_{\mathbf{\Lambda}}$ in $\ell_2(\mathbf{\Lambda})$ feature strong decay properties, stated in appropriate terms. We stress that we establish the threshold for $|g|$ uniformly in $\mathbf{\Lambda}$ for a bounded range of values of N. Formal statements are given in Theorem 1.1 below.

First, introduce the following condition for a probability measure μ on $\mathbb{R}$.

(V1) *The probability measure μ has bounded support,* $\mathrm{supp}\,\mu = [a_V, b_V]$*, and admits smooth probability density p_μ satisfying the following conditions*:

$$0 < \underline{p} \le p_\mu(t) \le \bar{p} < \infty, \quad |p'_\mu(t)| \le C_p < \infty, \quad \textit{for all } t \in (a_V, b_V). \tag{1.7}$$

Our condition upon V is the following one.

(V2) *The random field $V\colon \mathbb{Z}\times\Omega \to \mathbb{R}$, relative to a probability space* $(\Omega, \mathfrak{B}, \mathbb{P})$*, is* IID*, with marginal probability measure of the form $\mu = \mu_1 * \mu_2$, where* $\mathrm{supp}\,\mu_2 \subset \mathbb{R}$ *is bounded and μ_1 fulfills the condition* (V1).

A prototypical example is the uniform distribution in $[0,1]$; here $\mu_1 = \mathrm{Unif}([0,1])$, $\mu_2 = \delta_0$. One can also take the convolution powers $\mu = \mu_1^{*n}$ of $\mu_1 = \mathrm{Unif}([0,1])$; for $n \ge 2$, the density of μ vanishes at the edges: $p(t) = \mathrm{O}(t^{n-1})$, $p(t) = \mathrm{O}((n-t)^{n-1})$.

Remark 1.1. Our methods apply also to the Gaussian distribution μ_1 and its convolutions $\mu = \mu_1 * \mu_2$ with arbitrary probability measures μ_2; in this particular case, some auxiliary estimates become slightly simpler. The basis for such an extension is the main result of Ref. [9]. On the other hand, the spectral reduction presented in Sect. 3.3 requires some additional arguments in the case of unbounded random potentials; this explains the condition of boundedness of supp μ_2 in (V2). We omit the adaptation to unbounded potentials only for the sake of brevity.[2]

The expectation relative to $(\Omega, \mathfrak{B}, \mathbb{P})$ is denoted by $\mathbb{E}[\,\cdot\,]$.

We assume the following condition upon U.

(U) *There exist* $\zeta, C = C_U \in (0, +\infty)$ *such that*

$$|U(r)| \le C\,\mathrm{e}^{-r^{\zeta}}, \quad r \ge 1. \tag{1.8}$$

The values $\zeta > 1$ do not give rise to a qualitatively faster decay of EFCs and EFs than for $\zeta = 1$. Indeed, one cannot achieve the decay of the EFCs faster than exponential, established by Aizenman and Warzel [4] for compactly supported interaction potentials.

Let $\mathcal{B}_1$ denote the set of all continuous functions $f : \mathbb{R} \to \mathbb{C}$ with $\|f\|_\infty \le 1$. The results of this paper are presented in Theorem 1.1.

Theorem 1.1. *Assume conditions* (V2) *and* (U) *and fix an integer* $N^* \ge 2$. *There exists* $\kappa = \kappa(\zeta, N^*) \in (0, \zeta)$ *with the following property. For any* $\nu > 0$, *there is a value* $g_0 = g_0(N^*, \nu) > 0$ *such that for all* $N = 1, \dots, N^*$ *and* $|g| \ge g_0(N^*, \nu)$ *the following conditions are satisfied*

(A) *There exist a constant* $C = C_{\mathrm{EFC}} > 0$ *such that for all* $\mathbf{x}, \mathbf{y} \in \mathcal{Z}$,

$$\mathbb{E}\big[\sup_{f \in \mathcal{B}_1} |\langle \mathbf{1}_{\mathbf{y}} \mid f(\mathbf{H}^{(N)}_{\mathcal{Z}}) \mid \mathbf{1}_{\mathbf{x}} \rangle| \big] \le C\,\mathrm{e}^{-\nu(\boldsymbol{\rho}_S(\mathbf{x},\mathbf{y}))^{\kappa}}. \tag{1.9}$$

(B) *With probability one,* $\mathbf{H}^{(N)}_{\mathcal{Z}}(\omega)$ *has pure point spectrum,*[3] *and all its eigenfunctions* $\boldsymbol{\Psi}_j(x;\omega)$ *decay exponentially fast: there exists a nonrandom number* $m = m_N > 0$ *such that for all* $\boldsymbol{\Psi}_j$ *there exist a constant* $C_j = C_j(\omega)$ *and a site* $\hat{\mathbf{x}}_j = \hat{\mathbf{x}}_j(\omega) \in \mathcal{Z}$ *(a localization center) such that*

$$|\boldsymbol{\Psi}_j(\mathbf{x}, \omega)| \le C_j(\omega)\,\mathrm{e}^{-m\boldsymbol{\rho}_S(\mathbf{x}, \hat{\mathbf{x}}_j)}, \quad \mathbf{x} \in \mathcal{Z}. \tag{1.10}$$

[2] An example of such adaptation, in a slightly different situation, is given in our book [19, Sect. 2.5.3], where we assumed a power-law decay of the the tail probabilities: $\mathbb{P}\{|V(0;\omega)| > t\} \le C t^{-a}$, $a > 0$.

[3] Naturally, this statement is trivial for a finite graph $\mathcal{Z}$. The assertion **(B)** addresses essentially the case of an infinite graph, albeit our probabilistic bounds on the EFs are useful for finite graphs, too.

The conventional form of strong dynamical localization follows from (1.9) by taking the functions $f = f_t : \lambda \mapsto \mathrm{e}^{-\mathrm{i}t\lambda}$, $t \in \mathbb{R}$.

Compared with [4, 18, 27], the equations (1.9)–(1.10) show the decay in a more suitable form involving the metric $\boldsymbol{\rho}_{\mathrm{S}}$ rather than the Hausdorff distance. In (1.10), $\boldsymbol{\rho}_{\mathrm{S}}$ can be replaced by $\boldsymbol{\rho}$, thanks to the random factor $C_j(\omega)$. In (1.9), using $\boldsymbol{\rho}$ instead of $\boldsymbol{\rho}_{\mathrm{S}}$ would result in a factor $C = C(\mathbf{x})$. We stress that $\boldsymbol{\rho}_{\mathrm{S}}$ is a physically *natural* distance in $\mathcal{Z}^N$.

► **Some comments** are in order here. The bootstrap MPMSA developed by Klein and Nguyen [27] allows one to prove the bound of the form

$$\mathbb{E}[\sup_{f \in \mathcal{B}_1} |\langle \mathbf{1}_{\mathbf{y}} \,|\, f(\mathbf{H}^{(N)}_{\mathcal{Z}}) \,|\, \mathbf{1}_{\mathbf{x}} \rangle|] \le C\, \mathrm{e}^{-\nu (\mathrm{d}_{\mathcal{H}}(\mathbf{x},\mathbf{y}))^{\kappa}}, \tag{1.11}$$

where $\mathrm{d}_{\mathcal{H}}$ is the Hausdorff distance in $\mathcal{Z}^N$ defined as follows: with $\mathbf{x} = (x_1, \dots, x_N)$, $\mathbf{y} = (y_1, \dots, y_N)$, let $\Pi\mathbf{x} := \{x_1, \dots, x_N\}$, $\Pi\mathbf{y} := \{y_1, \dots, y_N\}$, and

$$\mathrm{d}_{\mathcal{H}}(\mathbf{x}, \mathbf{y}) = \max[\max_i \mathrm{dist}(x_i, \Pi\mathbf{y}), \max_i \mathrm{dist}(y_i, \Pi\mathbf{x})].$$

In other words, this is the usual Hausdorff distance between the sets $\Pi\mathbf{x}$ and $\Pi\mathbf{y}$ in the metric space $(\mathcal{Z}, \mathrm{d})$. For the finite-range interactions, Aizenman and Warzel [4] proved even exponential decay bounds, also in the Hausdorff distance. But the problem with this pseudo-distance can be seen in the formula

$$\mathrm{d}_{\mathcal{H}}(\mathbf{x}, \mathbf{y}) \ge \boldsymbol{\rho}_{\mathrm{S}}(\mathbf{x}, \mathbf{y}) - \min[\mathrm{diam}\, \Pi\mathbf{x},\ \mathrm{diam}\, \Pi\mathbf{y}],$$

which suggests that $\mathrm{d}_{\mathcal{H}}(\mathbf{x}, \mathbf{y})$ could be small even for the points $\mathbf{x}, \mathbf{y}$ with large values of $\boldsymbol{\rho}_{\mathrm{S}}(\mathbf{x}, \mathbf{y})$. Indeed, let $N = 3$, $d = 1$, $\mathbf{x} = (0, 0, L)$ and $\mathbf{y} = (0, L, L)$. Then $\boldsymbol{\rho}_{\mathrm{S}}(\mathbf{x}, \mathbf{y}) \to +\infty$ as $L \to \infty$, while $\mathrm{d}_{\mathcal{H}}(\mathbf{x}, \mathbf{y}) \equiv 0$. Furthermore, taking in this example $\mathcal{Z} = \mathcal{Z}_L = [0, L] \cap \mathbb{Z}^1$, we see that the localization bounds established in terms of the Hausdorff distance cannot rule out the quantum particle transfer processes between the points 0 and L in *arbitrarily large* but finite domains $\mathcal{Z}_L$ in the physical space. ◄

The rest of the paper is devoted to the proof of Theorem 1.1. In particular, in Section 2 we establish an important ingredient of the proof: the eigenvalue concentration (EVC) bounds. The bulk of the work is about the proof of assertion **(A)**: it is carried in Section 3. The main strategy here is the induction on the number of particles N, initially developed in [17, 18]. Each step $N - 1 \rightsquigarrow N$, $N = 2, \dots, N^*$, employs the multi-scale analysis of multi-particle Hamiltonians. Unlike Ref. [18], we make use of a more efficient scaling technique, essentially going back to the deep work by Germinet and Klein [23] and recently adapted by Klein and Nguyen [27] to the multi-particle systems.

As was already said, the base of induction ($N = 1$) requires a proof of localization bounds for single-particle systems on graphs of polynomial growth of balls, and we cannot simply refer to [27] where, formally speaking, only the lattice systems (on $\mathbb{Z}^d$, $d \geq 1$) were studied (this is of course a pure formality). The required estimates for the 1-particle Anderson models on graphs were proved in Ref. [12], where it was emphasized that the main scaling technique was due to Germinet and Klein [23]. In fact, our arguments also cover the case $N = 1$ (and become simpler in this situation).

The proof of assertion **(B)** is contained in Section 4; it makes use of a number of facts established in other sections. This proof is based on a modified version of the MPMSA presented in [19]. In particular, the case of the two-body potential U satisfying (U) is treated as a small perturbation of a finite range interaction (obtained by a suitable truncation of U). Some technical proofs are presented in the Appendix. Others repeat arguments published elsewhere (sometimes with minor changes) and are omitted.

From here on, we fix a positive integer $N^* \geq 2$ and consider $N = 1, \dots, N^*$ without stressing it every time again. The dependence of various quantities upon N^* is not emphasized but of course is crucial throughout the whole construction. The analysis (which we omit from our paper) of the formulae for the key exponents m_N, ν_N and $P(N)$ given in (3.2) and (4.1) shows that in the strong disorder regime, the disorder amplitude $|g(N)|$ required for the N-particle localization is to be of order of e^{ab^N}, with some $a > 0, b > 1$.

2. Eigenvalue concentration bounds

2.1. The resolvent inequalities. Singular and resonant sets. The main tool in the proof of Theorem 1.1 are the decay properties of the Green functions (GFs) of the operator $\mathbf{H}^{(N)}_{\mathbf{\Lambda}}$, $(\mathbf{x}, \mathbf{y}) \in \mathbf{\Lambda} \times \mathbf{\Lambda} \mapsto \mathbf{G}_{\mathbf{\Lambda}}(\mathbf{x}, \mathbf{y}; E)$, for a finite set $\mathbf{\Lambda} \subset \mathcal{Z}$ and $E \notin \Sigma(\mathbf{H}^{(N)}_{\mathbf{\Lambda}})$; here $\Sigma(\mathbf{H}^{(N)}_{\mathbf{\Lambda}})$ is the spectrum of the finite-dimensional operator $\mathbf{H}^{(N)}_{\mathbf{\Lambda}}$, i.e., the collection of its eigenvalues (EVs) counting multiplicity. As usual, $\mathbf{G}_{\mathbf{\Lambda}}(\mathbf{x}, \mathbf{y}; E)$ denotes the matrix entry of the resolvent $\mathbf{G}_{\mathbf{\Lambda}}(E) = \mathbf{G}^{(N)}_{\mathbf{\Lambda}}(E; \omega)$ in the delta-basis:

$$\mathbf{G}_{\mathbf{\Lambda}}(\mathbf{x}, \mathbf{y}; E) = \langle \mathbf{1}_{\mathbf{x}}, (\mathbf{H}^{(N)}_{\mathbf{\Lambda}} - E\mathbf{I})^{-1} \mathbf{1}_{\mathbf{y}} \rangle,$$

(here $\mathbf{I}$ is the identity operator and $\langle \cdot , \cdot \rangle$ stands for the canonical scalar product in $\ell^2(\mathbf{\Lambda})$), and the base for the argument is the Geometric Resolvent Inequality (GRI): for any subset $\mathbf{\Lambda}' \subset \mathbf{\Lambda}$ and configurations $\mathbf{x} \in \mathbf{\Lambda}'$, $\mathbf{y} \in \mathbf{\Lambda} \setminus \mathbf{\Lambda}'$,

$$|\mathbf{G}_{\mathbf{\Lambda}}(\mathbf{x}, \mathbf{y}; E)| \le \sum_{\langle \mathbf{u}, \mathbf{v}\rangle \in \partial_{\mathbf{\Lambda}} \mathbf{\Lambda}'} |\mathbf{G}_{\mathbf{\Lambda}'}(\mathbf{x}, \mathbf{u}; E)| \, |\mathbf{G}_{\mathbf{\Lambda}}(\mathbf{v}, \mathbf{y}; E)| \,. \tag{2.1}$$

Here $\partial_{\mathbf{\Lambda}} \mathbf{\Lambda}'$ stands for the edge-boundary of $\mathbf{\Lambda}'$ relative to $\mathbf{\Lambda}$:

$$\partial_{\mathbf{\Lambda}} \mathbf{\Lambda}' = \{(\mathbf{u}, \mathbf{v})\colon \mathbf{u} \in \mathbf{\Lambda}', \ \mathbf{v} \in \mathbf{\Lambda} \setminus \mathbf{\Lambda}', \ \mathrm{d}(\mathbf{u}, \mathbf{v}) = 1\}.$$

Note that for $\mathcal{Z} \in \mathfrak{G}(d, C_d)$, one has a crude upper bound $\sharp\, \partial \mathbf{B}^{(N)}(\mathbf{x}, L) \le C_{\mathcal{Z},N} L^{Nd}$, with some $C_{\mathcal{Z},N}$ depending upon d, C_d, N. The inner boundary $\partial^- \mathbf{\Lambda}$ is determined by

$$\partial^- \mathbf{\Lambda} = \{\mathbf{u} \in \mathbf{\Lambda}\colon \boldsymbol{\rho}(\mathbf{u}, \boldsymbol{\mathcal{Z}} \setminus \mathbf{\Lambda}) = 1\}.$$

The distance $\mathrm{dist}(\cdot\,, \cdot)$ below refers to the standard metric on the line $\mathbb{R}$.

Definition 2.1. Given $E \in \mathbb{R}$, $\beta \in (0, 1)$, $\delta \in (0, 1]$ and $m > 0$, an N-particle ball $\mathbf{B}^{(N)}(\mathbf{x}, L) \subset \boldsymbol{\mathcal{Z}}$ is called

- (E, β)-*resonant* ((E, β)-R, in short), if

$$\mathrm{dist}(\Sigma(\mathbf{H}^{(N)}_{\mathbf{B}^{(N)}(\mathbf{x},L)}), E) < 2\mathrm{e}^{-L^\beta}, \tag{2.2}$$

  and (E, β)-*nonresonant* ((E, β)-NR), otherwise;

- (E, δ, m)-*nonsingular* ((E, δ, m)-NS), if

$$\max_{\mathbf{y} \in \partial^- \mathbf{B}^{(N)}(\mathbf{x},L)} |\mathbf{G}^{(N)}_{\mathbf{B}^{(N)}(\mathbf{x},L)}(\mathbf{x}, \mathbf{y}; E)| \le (C_{\mathcal{Z},N} L^{Nd})^{-1} \mathrm{e}^{-mL^\delta}, \tag{2.3}$$

  and (E, δ, m)-*singular* ((E, δ, m)-S), otherwise.

Here and below, the implicit constant $C_{\mathcal{Z},N}$ is used in the quantities like $C_{\mathcal{Z},N} L^{Nd}$ to avoid more cumbersome expressions of the form

$$\sup_{\mathbf{x} \in \boldsymbol{\mathcal{Z}}^N} \sharp \partial \mathbf{B}^{(N)}(\mathbf{x}, L).$$

Typically, the properties (E, δ, m)-NS and (E, δ, m)-S will be used with $m = m_N$ where $N \mapsto m_N$ varies in a certain specified manner (see (3.2) and (4.1)).

2.2. One- and two-volume EVC bounds. We start with a one-volume EVC bound that is an analog of the well-known Wegner estimate [30], used in the proof of Theorems 3.1 and 4.1:

Theorem 2.1. *Fix $\beta \in (0,1)$. Under the assumption* (V2)*, there exists a constant $C = C(\beta, V)$ such that for all $E \in \mathbb{R}$, $1 \le N \le N^*$, $\mathbf{x} \in \mathcal{Z}$ and integer $L > 1$,*

$$\sup_{E \in \mathbb{R}} \mathbb{P}\{ \mathbf{B}^{(N)}(\mathbf{x}, L) \text{ is } (E,\beta)\text{-R} \} \le C \mathrm{e}^{-L^{\beta/2}}. \tag{2.4}$$

The proof is omitted: it repeats the one given in [19, Theorem 3.4.1]. Note also that under the assumption of existence and boundedness of the marginal probability density, the one-volume multi-particle EVC bound with *optimal* volume dependence, as in the original Wegner bound, was proved by Kirsch [26]. Under a more restrictive assumption of analyticity of the probability density, it was established in our earlier works; cf. [19] and references therein. In [19, Theorem 3.4.1], the assertion of Theorem 2.1 is actually inferred from a much more general EVC bound valid for arbitrary continuous (not necessarily absolutely continuous) marginal probability distributions, but having a non-optimal volume dependence.

A suitable one-volume EVC estimate is sufficient for the proof of fast decay of the Green functions which, owing to a result by Martinelli and Scoppola [29], implies a.s. absence of a.c. spectrum.

A (new) two-volume EVC bound is the subject of Theorem 2.2 below; this is the crucial statement making possible the efficient, uniform decay bounds on the EF correlators with respect to the symmetrized max-distance.

Given an integer $R \ge 0$, we will say that two balls $\mathbf{B}^{(N)}(\mathbf{x}, L)$, $\mathbf{B}^{(N)}(\mathbf{y}, L)$ are R-distant iff

$$\boldsymbol{\rho}_{\mathrm{S}}(\mathbf{x}, \mathbf{y}) \ge R. \tag{2.5}$$

Theorem 2.2. *Under the assumption* (V2)*, there exist some constants $A = A(N,d)$, $C(V,N,d) \in (0,+\infty)$ such that for all $s > 0$, integer $L > 1$ and any pair of $4NL$-distant balls $\mathbf{B}^{(N)}(\mathbf{x}, L)$, $\mathbf{B}^{(N)}(\mathbf{y}, L)$, the spectra $\Sigma_{\mathbf{x}} := \Sigma(\mathbf{H}^{(N)}_{\mathbf{B}(\mathbf{x},L)})$, $\Sigma_{\mathbf{y}} := \Sigma(\mathbf{H}^{(N)}_{\mathbf{B}(\mathbf{y},L)})$ obey*

$$\mathbb{P}\{\mathrm{dist}(\Sigma_{\mathbf{x}}, \Sigma_{\mathbf{y}}) \le s\} \le C\, L^{A} s^{2/3}. \tag{2.6}$$

Remark 2.1. For the Gaussian distribution, the results of Ref. [10] imply a stronger version of Theorem 2.2, with the RHS of (2.6) of the form $CL^{A}s$, i.e., with optimal s-dependence.

Remark 2.2. While the "one-volume" bound of the form (2.4) can indeed be qualified as an eigenvalue *concentration* bound, extending the celebrated Wegner estimate [30], its "two-volume" counterpart (2.6) is actually an eigenvalue *comparison* bound, measuring the distance between the spectra of two possibly correlated random operators. The non-triviality of the bound (2.6) for the pairs of balls $\mathbf{B}_L(\mathbf{x}), \mathbf{B}_L(\mathbf{y})$, distant in $\boldsymbol{\rho}_S$ but not in $\mathrm{d}_{\mathcal{H}}$, resides in the fact that the respective operators $\mathbf{H}_{\mathbf{B}_L(\mathbf{x})}$ and $\mathbf{H}_{\mathbf{B}_L(\mathbf{y})}$ can be stochastically correlated in a very strong way: every value of the random potential V affecting $\mathbf{H}_{\mathbf{B}_L(\mathbf{x})}$ affects also $\mathbf{H}_{\mathbf{B}_L(\mathbf{y})}$, and vice versa. For the pairs $\mathbf{B}_L(\mathbf{x}), \mathbf{B}_L(\mathbf{y})$ sufficiently distant in the Hausdorff distance $\mathrm{d}_{\mathcal{H}}$, one of the operators $\mathbf{H}_{\mathbf{B}_L(\mathbf{x})}, \mathbf{H}_{\mathbf{B}_L(\mathbf{y})}$ is not affected by some sub-sample of the random potential, affecting the other one (cf. [17, 18], [27, 28]).

Klein and Nguyen [27, 28] proved both the one-volume and two-volume multi-particle EVC bound with optimal volume dependence, but their analog of Theorem 2.2 is established for the balls separated in Hausdorff distance and not just in the symmetrized distance $\boldsymbol{\rho}_S$. This is why we need Theorem 2.2 replacing [17, Theorem 2]. Both $\mathrm{d}_{\mathcal{H}}$ and $\boldsymbol{\rho}_S$ are pseudo-distances in the configuration space of $N > 1$ distinguishable particles (considered in the present paper), but $\boldsymbol{\rho}_S$ becomes a *bona fide* metric in the physically more relevant case of indistinguishable quantum particles. Notice that the notions of the Hausdorff and symmetrized distance become equivalent in the particular case where $N = 2$. The crucial difference between them appears only for $N \geq 3$, thus settings apart the class of two-particle Anderson models.

Proof. The proof of Theorem 2.2 will be obtained by collecting the assertions of Lemmas 2.1, 2.3, and 2.4 (in the latter, $\vartheta' = \vartheta'' = 2/3$ by Lemma 2.1) and Corollary 2.2. □

Given a random field $V(x;\omega)$, $x \in \mathcal{Z}$, and a finite subset $\mathcal{Q} \subset \mathcal{Z}$, consider the sample mean $\xi_{\mathcal{Q}}$ and the fluctuations of $V|_{\mathcal{Q}}$ relative to $\xi_{\mathcal{Q}}$:

$$\xi_{\mathcal{Q}}(\omega) := (\sharp\, \mathcal{Q})^{-1} \sum_{x \in \mathcal{Q}} V(x;\omega), \quad \eta_x(\omega) = \eta_{x,\mathcal{Q}}(\omega) = V(x;\omega) - \xi_{\mathcal{Q}}(\omega), \qquad (2.7)$$

and the sigma-algebra $\mathfrak{F}_{\mathcal{Q}} \subset \mathfrak{B}$ generated by the fluctuations $\{\eta_x, x \in \mathcal{Q}\}$ and by $\{V(y;\omega), y \notin \mathcal{Q}\}$.

We use the following property reflecting regularity of the conditional mean.

(RCM) *There exist constants* $C', C'', A', A'', \vartheta', \vartheta'' \in (0, +\infty)$ *such that for any finite subset* $\mathcal{Q} \subset \mathcal{Z}$, *the (random) continuity modulus* $\mathfrak{s}_{\mathcal{Q}}(\cdot)$ *of the conditional distribution function* $F_{\xi}(\cdot\,|\mathfrak{F}_{\mathcal{Q}})$ *of the sample mean* $\xi_{\mathcal{Q}}$, *defined by*

$$\mathfrak{s}_{\mathcal{Q}}(s) := \inf_{b \in \mathbb{Q}: b > s} \sup_{a \in \mathbb{Q}} (F_{\xi}(a + b\,|\mathfrak{F}_{\mathcal{Q}}) - F_{\xi}(a\,|\mathfrak{F}_{\mathcal{Q}})), \qquad (2.8)$$

satisfies for all $s \in (0, 1]$

$$\mathbb{P}\{\mathfrak{s}_{\mathcal{Q}}(s;\omega) \geq C'(\sharp\, \mathcal{Q})^{A'} s^{\vartheta'}\} \leq C''(\sharp\, \mathcal{Q})^{A''} s^{\vartheta''}. \qquad (2.9)$$

The property (RCM) is shaped to be used in the context of IID random fields V, but it can be reformulated in terms of diam $\mathcal{Q}$ instead of $|\mathcal{Q}|$. In any case, the independence of values of V is not formally required, but, clearly, for an IID random field $V: \mathcal{Z} \times \Omega \to \mathbb{R}$ the condition (RCM) is simply a property of the single-site marginal distribution. Bearing this in mind, below we sometimes refer to (RCM) as a property of, or a condition for, a probability measure on $\mathbb{R}$. It is fulfilled – deterministically, i.e., with $C'' = 0$ – for an IID Gaussian field, e.g., with zero mean and a unit variance; in this case the sample mean is independent of the fluctuations $\eta_\bullet$ and has a normal distribution with variance $\sigma^2 = (\sharp\,\mathcal{Q})^{-1}$.

The form of the property (RCM) actually used in the key Lemma 2.4 stated below is as follows; it does not refer directly to the conditional continuity modulus $\mathfrak{s}_{\mathcal{Q}}$.

*Let be given $n > 1$ IID r.v. $X_1, \dots, X_n$; introduce [**as before**] the sample mean ξ, the fluctuations $\eta_i = X_i - \xi$, and the sigma-algebra $\mathfrak{F}_\eta$ generated by the fluctuations [**the direct analog of $\mathfrak{F}_{\mathcal{Q}}$ when $X_\bullet$ are labeled by the points of a finite set $\mathcal{Q}$, $|\mathcal{Q}| = n$**]. Then for any $s \in (0,1]$ there is a measurable subset $\widetilde{\mathcal{S}}_s \subset \Omega$ with $\mathbb{P}\{\widetilde{\mathcal{S}}_s\} \le C'' n^{A''} s^{\vartheta''}$ such that for any $\mathfrak{F}_\eta$-measurable real-valued r.v. ζ, setting $I(s;\omega) := [\zeta(\omega), \zeta(\omega) + s] \subset \mathbb{R}$, one has*

$$\mathbb{P}\{\mathbf{1}_{\xi(\omega) \in I(s;\omega)} \mathbf{1}_{\Omega \setminus \widetilde{\mathcal{S}}_s} \,|\, \mathfrak{F}_\eta\} \le C' n^{A'} s^{\vartheta'}, \tag{2.10}$$

and consequently,

$$\mathbb{P}\{\xi(\omega) \in I(s;\omega)\} \le C' n^{A'} s^{\vartheta'} + C'' n^{A''} s^{\vartheta''}. \tag{2.11}$$

Lemma 2.1 ([14, Theorem 4]). *If the marginal probability measure μ of an IID random field $V: \mathcal{Z} \times \Omega \to \mathbb{R}$ obeys* (V1), *then V fulfills property* (RCM) *with $\vartheta' = \vartheta'' = 2/3$.*

The values of the constants A', A'', C', C'' figuring in (RCM) are of minor, if any, importance for our proofs. For the reader's convenience, we summarize in Appendix D the proof of Lemma 2.1.

Corollary 2.2. *Let the probability measure $\mathbb{P}$ of an IID random field $V: \mathcal{Z} \times \Omega \to \mathbb{R}$ have the single-site marginal measure $\mu = \mu_1 * \mu_2$, where μ_1 satisfies (V1). Then $\mathbb{P}$ fulfills the condition* (RCM).

Proof. First, notice that adding a non-random background potential, $V(x;\omega) \rightsquigarrow V(x;\omega) + W(x)$, preserves the property (RCM). Further, the representation $\mu = \mu_1 * \mu_2$ implies that V is the sum of two independent, IID random fields V_i,

$i = 1, 2$, where the marginal measure of V_i is μ_i. Conditioning on V_2 renders the latter non-random. Therefore, the required probability bound is obtained by conditioning on V_2, since V_1 satisfies (RCM). ☐

Before we move further, let us introduce some notation. In (2.12) we define the support (or projection) $\Pi\mathbf{x}$ of the configuration $\mathbf{x} = (x_1, \dots, x_N) \in \mathcal{Z}^N$, the support $\Pi\mathbf{B}$ of the ball $\mathbf{B} = \mathbf{B}^{(N)}(\mathbf{x}, L)$, and – given a subset $\mathcal{J} \subset \{1, N\}$ – the partial supports (projections) $\Pi_{\mathcal{J}}\mathbf{x}$ and $\Pi_{\mathcal{J}}\mathbf{B}$:

$$\Pi\mathbf{x} = \bigcup_{1\le i\le N}\{x_i\} \subset \mathcal{Z}, \qquad \Pi\mathbf{B} = \bigcup_{1\le j\le N} \mathrm{B}(x_j, L) \subset \mathcal{Z}, \tag{2.12a}$$

$$\Pi_{\mathcal{J}}\mathbf{x} = \bigcup_{j\in\mathcal{J}}\{x_j\} \subset \mathcal{Z}, \quad \Pi_{\mathcal{J}}\mathbf{B} = \bigcup_{j\in\mathcal{J}} \mathrm{B}(x_j, L) \subset \mathcal{Z}, \tag{2.12b}$$

with $\Pi_{\varnothing}\mathbf{x} = \varnothing$ (for $\mathcal{J} = \varnothing$).

Definition 2.2. A ball $\mathbf{B}^{(N)}(\mathbf{x}, L)$ is called *weakly separated from* $\mathbf{B}^{(N)}(\mathbf{y}, L)$ if there exists a bounded subset $\mathcal{Q} \subset \mathcal{Z}$ of the single-particle configuration space, with $\operatorname{diam} \mathcal{Q} \le 2NL$, and subsets $\mathcal{J}_1, \mathcal{J}_2 \subset \{1, \dots, N\}$ such that $\sharp\mathcal{J}_1 > \sharp\mathcal{J}_2$ (possibly, with $\mathcal{J}_2 = \varnothing$) and

$$(\Pi_{\mathcal{J}_1}\mathbf{B}^{(N)}(\mathbf{x}, L) \cup \Pi_{\mathcal{J}_2}\mathbf{B}^{(N)}(\mathbf{y}, L)) \subset \mathcal{Q}, \tag{2.13a}$$

$$(\Pi_{\mathcal{J}_1^c}\mathbf{B}^{(N)}(\mathbf{x}, L) \cup \Pi_{\mathcal{J}_2^c}\mathbf{B}^{(N)}(\mathbf{y}, L)) \subset \mathcal{Z} \setminus \mathcal{Q}. \tag{2.13b}$$

A pair of balls $\mathbf{B}^{(N)}(\mathbf{x}, L)$, $\mathbf{B}^{(N)}(\mathbf{y}, L)$ is called *weakly separated* if at least one of the balls is weakly separated from the other.

To stress the role of the set $\mathcal{Q}$, we will say, where appropriate, that $\mathbf{B}^{(N)}(\mathbf{x}, L)$ and $\mathbf{B}^{(N)}(\mathbf{y}, L)$ are weakly $\mathcal{Q}$-separated.

► The notion of weak separation and its role can be explained as follows. If $\mathbf{B}^{(N)}(\mathbf{x}, L)$ is weakly $\mathcal{Q}$-separated from $\mathbf{B}^{(N)}(\mathbf{y}, L)$, then there are more particles in $\mathcal{Q}$ from the configurations $\mathbf{z} \in \mathbf{B}^{(N)}(\mathbf{x}, L)$ than from $\mathbf{z} \in \mathbf{B}^{(N)}(\mathbf{y}, L)$, and this makes the EVs of $\mathbf{H}_{\mathbf{B}^{(N)}(\mathbf{x},L)}$ more sensitive to the fluctuations of the random potential in $\mathcal{Q}$ than the EVs of $\mathbf{H}_{\mathbf{B}^{(N)}(\mathbf{y},L)}$. However, even a weakly separated pair of balls can have $\Pi\mathbf{B}^{(N)}(\mathbf{x}, L) = \Pi\mathbf{B}^{(N)}(\mathbf{y}, L)$ (for $N \ge 3$), and then *every* $V(u;\omega)$ affecting $\mathbf{H}_{\mathbf{B}^{(N)}(\mathbf{x},L)}$ affects also $\mathbf{H}_{\mathbf{B}^{(N)}(\mathbf{y},L)}$, and vice versa. (*This is why we qualify the above form of separation" as "weak."*) ◄

Lemma 2.3 ([13, Lemma 8]). *Any pair of $4NL$-distant balls $\mathbf{B}^{(N)}(\mathbf{x}, L)$, $\mathbf{B}^{(N)}(\mathbf{y}, L)$ is weakly separated.*

Lemma 2.4. *Let* $V : \mathbb{Z} \times \Omega \to \mathbb{R}$ *be a random field satisfying the condition* (RCM). *Assume that the balls* $\mathbf{B}^{(N)}(\mathbf{x}, L)$, $\mathbf{B}^{(N)}(\mathbf{y}, L)$ *are weakly separated. Then for any* $s > 0$ *the following bound holds for the spectra* $\Sigma_{\mathbf{x}} := \Sigma(\mathbf{H}^{(N)}_{\mathbf{B}(\mathbf{x},L)})$ *and* $\Sigma_{\mathbf{y}} := \Sigma(\mathbf{H}^{(N)}_{\mathbf{B}(\mathbf{y},L)})$:

$$\mathbb{P}\{\mathrm{dist}(\Sigma_{\mathbf{x}}, \Sigma_{\mathbf{y}}) \le s\} \le \widetilde{C}' L^{a'} s^{\vartheta'} + \widetilde{C}'' L^{a''} s^{\vartheta''} \tag{2.14}$$

for some $a', a'', \widetilde{C}', \widetilde{C}''$ *and the same* $\vartheta', \vartheta'' \in (0, \infty)$ *as in* (2.9).

Proof. Let $\mathcal{Q}$ be a set satisfying the conditions (2.13) for some $\mathcal{J}_1, \mathcal{J}_2 \subset \{1, \dots, N\}$ with $\sharp \mathcal{J}_1 = n_1 > n_2 = \sharp \mathcal{J}_2$. Introduce the sample mean $\xi = \xi_{\mathcal{Q}}$ of V over $\mathcal{Q}$, and the conditional continuity modulus $\mathfrak{s}_{\mathcal{Q}}(\cdot)$ as in (2.8). We have $\sharp\mathcal{Q} \le C_d(2NL)^d \le \mathrm{const}\, L^d$, thus with $a' = a'(A', N, d)$, $a'' = a''(A'', N, d)$ and some $\widetilde{C}', \widetilde{C}''$, we have (cf. (2.9) and (1.1))

$$\sharp\, \mathbf{B}^{(N)}(\mathbf{x}, L)\, C'(\sharp\mathcal{Q})^{A'} \le \widetilde{C}' L^{a'}, \tag{2.15a}$$

$$\sharp\, \mathbf{B}^{(N)}(\mathbf{y}, L) C''(\sharp\mathcal{Q})^{A''} \le \widetilde{C}'' L^{a''}. \tag{2.15b}$$

It follows from $V(x;\omega) = \xi_{\mathcal{Q}}(\omega) + \eta_x(\omega)$ (cf. (2.7)) that $\mathbf{H}^{(N)}_{\mathbf{B}(\mathbf{x},L)}(\omega)$ and $\mathbf{H}^{(N)}_{\mathbf{B}(\mathbf{y},L)}(\omega)$ admit the representations

$$\mathbf{H}^{(N)}_{\mathbf{B}(\mathbf{x},L)}(\omega) = n_1 \xi(\omega)\, \mathbf{I} + \mathbf{A}'(\omega), \tag{2.16a}$$

$$\mathbf{H}^{(N)}_{\mathbf{B}(\mathbf{y},L)}(\omega) = n_2 \xi(\omega)\, \mathbf{I} + \mathbf{A}''(\omega), \tag{2.16b}$$

where the operators $\mathbf{A}'(\omega)$ and $\mathbf{A}''(\omega)$ are $\mathfrak{F}_{\mathcal{Q}}$-measurable. Let $\Sigma_{\mathbf{x}} = \{\lambda_1, \dots, \lambda_{K'}\}$ and $\Sigma_{\mathbf{y}} = \{\mu_1, \dots, \mu_{K''}\}$, be the spectra of $\mathbf{H}^{(N)}_{\mathbf{B}(\mathbf{x},L)}$ and $\mathbf{H}^{(N)}_{\mathbf{B}(\mathbf{y},L)}$, counting multiplicity, with $K' = \sharp\, \mathbf{B}(\mathbf{x}, L)$ and $K'' = \sharp\mathbf{B}(\mathbf{y}, L)$.

Owing to (2.16), we have $\lambda_i(\omega) = n_1\xi(\omega) + \lambda_i^{(0)}(\omega)$, $\mu_j(\omega) = n_2\xi(\omega) + \mu_j^{(0)}(\omega)$, where the random variables $\lambda_i^{(0)}(\omega)$ and $\mu_j^{(0)}(\omega)$ are $\mathfrak{F}_{\mathcal{Q}}$-measurable. Therefore,

$$\lambda_i(\omega) - \mu_j(\omega) = (n_1 - n_2)\xi(\omega) + (\lambda_i^{(0)}(\omega) - \mu_j^{(0)}(\omega)),$$

with $n_1 - n_2 \ge 1$, owing to our assumption. Further, we can write

$$\mathbb{P}\{\mathrm{dist}(\Sigma_{\mathbf{x}}, \Sigma_{\mathbf{y}}) \le s\} \le \sum_{1 \le i \le K'} \sum_{1 \le j \le K''} \mathbb{E}[\mathbb{P}\{|\lambda_i - \mu_j| \le s\, |\mathfrak{F}_{\mathcal{Q}}\}].$$

Note that for all i and j we have (cf. (2.8))

$$\begin{aligned}\mathbb{P}\{|\lambda_i - \mu_j| \le s \,|\, \mathfrak{F}_{\mathcal{Q}}\} &= \mathbb{P}\{|(n_1 - n_2)\xi + \lambda_i^{(0)} - \mu_j^{(0)}| \le s \,|\mathfrak{F}_{\mathcal{Q}}\} \\ &\le \mathfrak{s}_{\mathcal{Q}}(2|n_1 - n_2|^{-1} s \,|\, \mathfrak{F}_{\mathcal{Q}}) \\ &\le \mathfrak{s}_{\mathcal{Q}}(2s \,|\mathfrak{F}_{\mathcal{Q}}).\end{aligned}$$

Set

$$\mathcal{D}_L = \{\omega\colon \sup_{t\in\mathbb{R}} |F_\xi(t + s \,|\mathfrak{F}_{\mathcal{Q}}) - F_\xi(t \,|\mathfrak{F}_{\mathcal{Q}})| \ge \tilde{C}' L^{a'} s^{\vartheta'}\}.$$

By (RCM) (cf. also (2.15)), $\mathbb{P}\{\mathcal{D}_L\} \le \tilde{C}'' L^{a''} s^{\vartheta''}\}$. Thus,

$$\begin{aligned}\mathbb{P}\{\mathrm{dist}(\Sigma_{\mathbf{x}}, \Sigma_{\mathbf{y}}) \le s\} &\le \mathbb{E}[\mathbf{1}_{\Omega\setminus\mathcal{D}_L} \mathbb{P}\{\mathrm{dist}(\Sigma_{\mathbf{x}}, \Sigma_{\mathbf{y}}) \le s \,|\mathfrak{F}_{\mathcal{Q}}\}] + \mathbb{P}\{\mathcal{D}_L\} \\ &\le \tilde{C}' L^{a'} s^{\vartheta'} + \tilde{C}'' L^{a''} s^{\vartheta''},\end{aligned} \tag{2.17}$$

as claimed in (2.14). This finishes the proof of Lemma 2.4. □

► By (V2), Lemma 2.1 and Corollary 2.2, (RCM) is fulfilled with $\vartheta' = \vartheta'' = 2/3$, thus the RHS in (2.17) is bounded by $CL^A s^{2/3}$ (cf. Lemma 2.4), for any pair of $4NL$-distant balls of radius L (cf. Lemma 2.3). This completes the proof of Theorem 2.2. ◄

Theorem 2.2 is crucial in the proof of Lemma 3.7. Namely, it allows us to infer from the fixed-energy decay bounds (simpler to establish) their energy-interval counterparts, required for the proof of spectral and dynamical localization, without an additional energy-interval scaling analysis employed in the bootstrap multi-scale approach (cf. [27] and [23]).

2.3. Weakly interactive balls. In presence of a decaying interaction, the balls in the N-particle configuration space can be classified in terms of their distance to the sites where the interaction potential takes the largest values. For our purposes, the following simple classification suffices.

Definition 2.3. An N-particle ball $\mathbf{B}^{(N)}(\mathbf{u}, L)$, with $N \ge 2$, $\mathbf{u} = (u_1, \dots, u_N)$, is called *weakly interactive* (WI) if

$$\mathrm{diam}(\Pi\mathbf{u}) := \max_{1\le i<j\le N} \mathrm{d}(u_i, u_j) > 3NL, \tag{2.18}$$

and strongly interactive (SI), otherwise.

The meaning of Definition 2.3 is that a particle system in a WI ball can be decomposed into distant subsystems that interact "weakly" with each other, while for an SI ball such a decomposition may be impossible. See Lemma 2.5.

Lemma 2.5. *For any* WI *ball* $\mathbf{B}^{(N)}(\mathbf{u}, L)$ *there exists a decomposition* $\{1, \dots, N\} = \mathcal{J} \sqcup \mathcal{J}^{\complement}$, *such that,*

$$d(\Pi_{\mathcal{J}} \mathbf{B}^{(N)}(\mathbf{u}, L), \Pi_{\mathcal{J}^{\complement}} \mathbf{B}^{(N)}(\mathbf{u}, L)) > L. \tag{2.19}$$

Proof. Suppose that $\operatorname{diam}(\Pi \mathbf{u}) > 3NL$; we have to show that the projection $\Pi \mathbf{B}(\mathbf{u}, 3L/2)$ is a disconnected subset of $\mathbb{Z}$.

Assume otherwise; then every nontrivial partial projection $\Pi_{\mathcal{J}} \mathbf{u}$, $\varnothing \subset \mathcal{J} \subset \{1, \dots, N\}$, is at distance $\leq 2 \cdot \frac{3L}{2} = 3L$ from $\Pi_{\mathcal{J}^{\complement}} \mathbf{u}$. Then a straightforward induction in $N \geq 2$ shows that $\operatorname{diam} \; \Pi \mathbf{u} \leq (N-1) \cdot 3L < 3NL$, contrary to our hypothesis.

Now, as $\Pi \mathbf{B}(\mathbf{u}, 3L/2)$ is disconnected, there is a nontrivial decomposition $\{1, \dots, N\} = \mathcal{J} \sqcup \mathcal{J}^{\complement}$ for which

$$\begin{aligned} &d(\Pi_{\mathcal{J}} \mathbf{B}(\mathbf{u}, 3L/2), \Pi_{\mathcal{J}^{\complement}} \mathbf{B}(\mathbf{u}, 3L/2)) > 0 \\ &\qquad \Longrightarrow \; d(\Pi_{\mathcal{J}} \mathbf{B}(\mathbf{u}, L), \Pi_{\mathcal{J}^{\complement}} \mathbf{B}(\mathbf{u}, L)) > \frac{1}{2} L + \frac{1}{2} L = L, \end{aligned}$$

as asserted in (2.19). □

The decomposition $(\mathcal{J}, \mathcal{J}^{\complement})$ figuring in Lemma 2.5 may be not unique. We will assume that such a decomposition (referred to as the canonical one) is associated in some unique way with every N-particle WI ball. Accordingly, we fix the notation $N' = \sharp \mathcal{J}$, $N'' = \sharp \mathcal{J}^{\complement} = N - N'$, and further, for $\mathbf{x} = (x_1, \dots, x_N) \in \mathbf{B}(\mathbf{u}, L)$,

$$\mathbf{x}_{\mathcal{J}} = (x_{i_1}, \dots, x_{i_{N'}}), \; \mathbf{x}_{\mathcal{J}^{\complement}} = (x_{j_1}, \dots, x_{j_{N''}}) \tag{2.20}$$

where $\mathcal{J} = \{i_1, \dots, i_{N'}\}$ and $\mathcal{J}^{\complement} = \{j_1, \dots, j_{N''}\}$, with $1 \leq i_1 < \dots < i_{N'} \leq N$ and $1 \leq j_1 < \dots < j_{N''} \leq N$. This gives rise to the Cartesian product representation

$$\mathbf{B}^{(N)}(\mathbf{u}, L) = \mathbf{B}' \times \mathbf{B}'', \tag{2.21}$$

with $\mathbf{B}' = \mathbf{B}^{(N')}(\mathbf{u}_{\mathcal{J}}, L)$ and $\mathbf{B}'' = \mathbf{B}^{(N'')}(\mathbf{u}_{\mathcal{J}^{\complement}}, L)$, which we call the *canonical factorization.* Consequently, the operator $\mathbf{H}^{(N)}_{\mathbf{B}}$ in a WI ball $\mathbf{B} = \mathbf{B}(\mathbf{u}, L)$ can be represented in the following way:

$$\mathbf{H}^{(N)}_{\mathbf{B}} = \mathbf{H}^{(N')}_{\mathbf{B}'} \otimes \mathbf{I}^{(N'')} + \mathbf{I}^{(N')} \otimes \mathbf{H}^{(N'')}_{\mathbf{B}''} + \mathbf{U}_{\mathbf{B}', \mathbf{B}''}. \tag{2.22}$$

Here the summand $\mathbf{U}_{\mathbf{B}', \mathbf{B}''}$ takes into account only the interaction between subsystems in the balls $\mathbf{B}'$ and $\mathbf{B}''$ and has a small norm for L large.

Lemma 2.6. *Let* $\mathbf{B}^{(N)}(\mathbf{x}, L)$, $\mathbf{B}^{(N)}(\mathbf{y}, L)$ *be a pair of* SI *balls with* $\boldsymbol{\rho}(\mathbf{x}, \mathbf{y}) > 8NL$. *Then*

$$\Pi\mathbf{B}^{(N)}(\mathbf{x}, L) \cap \Pi\mathbf{B}^{(N)}(\mathbf{y}, L) = \varnothing \tag{2.23}$$

and, therefore, the random operators $\mathbf{H}_{\mathbf{B}^{(N)}(\mathbf{x},L)}(\omega)$ *and* $\mathbf{H}_{\mathbf{B}^{(N)}(\mathbf{y},L)}(\omega)$ *are independent.*

Proof. By definition, for any SI balls $\mathbf{B}^{(N)}(\mathbf{x}, L)$, $\mathbf{B}^{(N)}(\mathbf{y}, L)$ we have

$$\max_{i,j} \mathrm{d}(x_i, x_j) \le 3NL, \quad \max_{i,j} \mathrm{d}(y_i, y_j) \le 3NL,$$

and it follows from the assumption $\boldsymbol{\rho}(\mathbf{x}, \mathbf{y}) > 8NL$ that for some $i', j' \in \{1, \dots, N\}$ $\mathrm{d}(x_{i'}, y_{j'}) > 8NL$, thus for any $i, j \in \{1, \dots, N\}$

$$\mathrm{d}(x_i, y_j) \ge \mathrm{d}(x_{i'}, y_{j'}) - \mathrm{d}(x_{i'}, x_i) - \mathrm{d}(y_{j'}, y_j) > 8NL - 6NL = 2NL.$$

Respectively, for the balls $\mathbf{B}_L(\mathbf{x})$, $\mathbf{B}_L(\mathbf{y})$ of radius L we have, with $N \ge 1$,

$$\mathrm{dist}(\Pi\mathbf{B}_L(\mathbf{x}), \Pi\mathbf{B}_L(\mathbf{y})) > 2(N-1)L \ge 0,$$

so $\Pi\mathbf{B}^{(N)}(\mathbf{x}, L) \cap \Pi\mathbf{B}^{(N)}(\mathbf{y}, L) = \varnothing$. Hence the samples of the random potential in $\mathbf{H}_{\mathbf{B}^{(N)}(\mathbf{x},L)}(\omega)$ and $\mathbf{H}_{\mathbf{B}^{(N)}(\mathbf{y},L)}(\omega)$ are independent. □

Throughout the paper we consider a sequence of integers $L_k > 1$ (length scales) of one of the two forms:

$$\text{(a)} \qquad L_k := L_0 Y^k, \quad k = 0, 1, \dots, \tag{2.24a}$$

or

$$(b) \quad L_{k+1} = \lfloor L_k^{\alpha} \rfloor, \quad k = 0, 1, \dots \tag{2.24b}$$

with given initial integer values $L_0 \ge 1$, $Y > 1$ and a scaling exponent $\alpha > 1$.

3. Fixed-energy and energy-interval estimates

The aim in this section is to prove the assertion **(A)** of Theorem 1.1. Until subsection 3.3 we work with much simpler fixed-energy estimates, preparing the grounds for the passage to the variable-energy (energy-interval) ones.

Throughout the section we use the conditions, listed in the table (3.2), imposed upon the key parameters of the inductive scheme involved. A few comments are in order:

(i) Y and L_0 are positive integers determining the length scales $L_k = L_0 Y^k$;

(ii) $\kappa \in (0, \zeta)$ measures the rate of sub-exponential decay of the EFCs;

(iii) $\beta \in (0, 1)$ is a resonance/nonresonance threshold value, emerging in (2.2);

(iv) $m^* \geq 1$ and $\delta \in (0, 1)$ are the parameters figuring in (2.3) and measuring the rate of sub-exponential decay of the GFs; we also use a sequence

$$m_N := m^* \left(1 + 3L_0^{-\delta+\beta}\right)^{N^*-N}; \tag{3.1}$$

(v) $\nu^* \geq 1$, used in (3.4) through the scaled value ν_N, plays a role similar to that of m^*, but controls the EFC (not GF) decay.

Recall that N ranges in $\{1, \dots, N^*\}$.

$m^*, \nu^* \geq 1$	$L_k = L_0 Y^k, \quad Y \geq 51N^*$	
$0 < \beta < \delta$	$\delta < \min\left(1 - \frac{\ln 12}{\ln Y}, \zeta\right)$, hence $\frac{1}{4} Y^{1-\delta} \geq 3$	(3.2)
$0 < \kappa < \min\left(\zeta, \frac{\beta}{2}, \frac{\ln \frac{4}{3}}{Y}\right)$	$\nu_N = \nu^* \left(2Y^\kappa\right)^{N^*-N}$, $m_N = m^* \left(1 + 3L_0^{-\delta+\beta}\right)^{N^*-N}$	

We assume (3.2) to be satisfied throughout the whole Section 3; this will not be reminded every time again. (Some technical statements remain valid under broader restrictions than those from (3.2).)

3.1. Scaling the GFs. Property $\mathbf{S}(N, k)$. Before we start the scaling procedure, we would like to stress that we consider from the beginning an underlying connected graph $\mathcal{Z} \in \mathfrak{G}(d, C_d)$, with some $d, C_d \in (0, +\infty)$, which can be finite or infinite. Observe that the growth property $\mathfrak{G}(d, C_d)$ is inheritable by inclusion, and as we shall see, all our bounds ultimately depend upon the graph $\mathcal{Z}$ only through the parameters d and C_d. Naturally, for a finite graph $\mathcal{Z}$, the induction in the length scales L_k is to be stopped as soon as $L_k \geq \operatorname{diam} \mathcal{Z}$. With this observation in mind, below we proceed, formally, as if $\mathcal{Z}$ were infinite.

Definition 3.1. Let be given a real number $E \in \mathbb{R}$, positive real numbers $\beta, \delta \in (0, 1)$, $m^* \geq 1$, and an integer $k \geq 0$. An N-particle ball $\mathbf{B} = \mathbf{B}^{(N)}(\mathbf{u}, L_{k+1})$ is called (E, m_N)-*good* if it contains no pair of $8NL_k$-distant (E, δ, m_N)-S balls of radius L_k.

The notion defined in the next definition appears in the formulation of Lemma 3.1 and in an important ingredient of its proof, Lemma C.2 (see the proof of Lemma C.2 in Appendix C).

Definition 3.2. Given $E \in \mathbb{R}$ and $\beta \in (0,1)$, a ball $\mathbf{B}^{(N)}(\mathbf{x}, L_k)$, $k \geq 1$, is called (E,β)*-completely non-resonant* ((E,β)-CNR) if for all $L_{k-1} \leq \ell \leq L_k$ one has

$$\operatorname{dist}(\Sigma(\mathbf{B}^{(N)}(\mathbf{x}, \ell)), E) \geq 2\mathrm{e}^{-L_k^\beta}.$$

The following statement will be used in the proof of Lemma 3.4 and 3.1. To keep clear the main flow of argument, we prove separately (in Appendix C) two key ingredients of its proof, Lemma C.1 and Lemma C.2. Another pre-requisite for the proof of Lemma 3.1 is Definition C.1.

Lemma 3.1. *Suppose that a ball* $\mathbf{B}^{(N)}(\mathbf{u}, L_{k+1})$ *is* (E,β)-CNR *and* (E, m_N)-*good. If* L_0 *is large enough, then* $\mathbf{B}^{(N)}(\mathbf{u}, L_{k+1})$ *is also* (E, δ, m_N)-NS.

Proof. Let $\mathbf{\Lambda} = \mathbf{B}^{(N)}(\mathbf{u}, L_{k+1})$, $\mathbf{B} = \mathbf{B}^{(N)}(\mathbf{u}, L_{k+1}-1)$ (cf. Definition C.1), and fix any point $\mathbf{y} \in \partial^- \mathbf{\Lambda}$, so that $\mathbf{y} \notin \mathbf{B}$. To prove the claim, we have to assess the Green functions $\mathbf{G}^{(N)}_{\mathbf{\Lambda}}(\mathbf{x}, \mathbf{y}; E) \equiv \mathbf{G}^{(N)}_{\mathbf{B}_{L_{k+1}(\mathbf{u})}}(\mathbf{x}, \mathbf{y}; E)$ for $\mathbf{x} \in \mathbf{B}$.

By hypothesis, either $\mathbf{\Lambda}$ contains no (E, δ, m_N)-S ball of radius L_k, or there is a ball $\mathbf{B}^{(N)}(\mathbf{w}, L_k) \subset \mathbf{B}$ such that any ball $\mathbf{B}^{(N)}(\mathbf{v}, L_k)$ with $\mathbf{v} \in \mathbf{B} \setminus \mathbf{B}^{(N)}(\mathbf{w}, 8NL_k)$ is (E, δ, m_N)-NS. Bearing in mind Lemma C.2, with $L = L_{k+1} - 1$, denote by $\mathcal{S}$ the union of all spherical layers $\mathcal{L}_r(\mathbf{u})$ such that $\mathcal{L}_r(\mathbf{u}) \cap \mathbf{B}^{(N)}(\mathbf{w}, 8NL_k) \neq \varnothing$. It follows from the relation $\beta < \delta$ (cf. (3.2)) that, for L_0 or m_N large enough,

$$m_N - 2L_k^{-\delta} L_{k+1}^\beta = m_N(1 - 2m_N^{-1} L_k^{-\delta+\beta} Y^\beta) \geq \frac{3}{4} m_N > 0. \tag{3.3}$$

By Lemma C.2, for any with a fixed $\mathbf{y} \in \partial^- \mathbf{\Lambda}$ (thus $\mathbf{y} \notin \mathbf{B}$), the function $f : \mathbf{x} \in \mathbf{B} \mapsto |\mathbf{G}^{(N)}_{\mathbf{\Lambda}}(\mathbf{x}, \mathbf{y}; E)|$ is $(L_k, q, \mathcal{S})$-dominated in $\mathbf{B}$, in the sense of Definition C.1, with $q \leq \mathrm{e}^{-\frac{3}{4} m_N L_k^\delta}$.

Owing to Lemma C.1, we can write, with the convention $-\ln 0 = +\infty$, that

$$-\ln f(\mathbf{x}) \geq -\ln \Big\{ \mathrm{e}^{L_{k+1}^\beta} \exp\Big[-\frac{3m_N}{4} L_k^\delta \\ \frac{(L_{k+1} - 1) - (2 \cdot 8NL_k + 1)L_k - L_k}{L_k + 1} \Big] \Big\},$$

thus by virtue of the conditions in the table (3.2) (viz. $L_0 \geq 3$, $\frac{1}{4}Y^{1-\delta} \geq 3$, $m_N \geq 1$, $\beta < \delta$, $Y \geq 51N^* \geq 51N$), one obtains

$$\begin{aligned} -\ln f(\mathbf{x}) &\geq \frac{3}{4} m_N L_k^\delta \frac{(Y-17N)L_k}{L_k+1} - L_{k+1}^\beta \\ &\geq \frac{3}{4} m_N L_{k+1}^\delta \Big(\frac{Y-17N}{Y^\delta}(1-L_k^{-1}) - \frac{4}{3} L_{k+1}^{\beta-\delta} \Big) \\ &\geq m_N L_{k+1}^\delta \, \frac{3}{4}\Big(\frac{1}{3}Y^{1-\delta} - 1\Big) \\ &\geq m_N L_{k+1}^\delta \, \Big(\frac{1}{4}Y^{1-\delta} - 1\Big) \\ &\geq 2 m_N L_{k+1}^\delta \\ &\geq m_N L_{k+1}^\delta + \ln(C_{\mathcal{Z},N} L_{k+1}^{Nd}), \end{aligned}$$

provided L_0 is large enough. □

Given $L_0, Y, \delta, \kappa, m^*, \nu^*$, consider the following property $\mathsf{S}(N,k)$ depending upon N and k (S stands for "singularity"):

$\mathsf{S}(N,k)$ *for all* $E \in \mathbb{R}$, $1 \leq n \leq N$ *and configuration* $\mathbf{u} \in \mathcal{Z}$

$$\mathbb{P}\{ \mathbf{B}^{(n)}(\mathbf{u}, L_k) \text{ is } (E,\delta,m_n)\text{-}S \} \leq \mathrm{e}^{-\nu_n L_k^\kappa}. \tag{3.4}$$

Recall (cf. (3.2)) that $\nu_n \equiv \nu^*(2Y^\kappa)^{N^*-n}$. The MPMSA inductive scheme consists in checking $\mathsf{S}(N,k)$ for all $N \in \{1,\dots,N^*\}$ and $k \geq 0$. The initial length scale bound $\mathsf{S}(N,0)$, for all $1 \leq N \leq N^*$ with a fixed $N^* \geq 2$, follows directly from Lemma 3.2.

Lemma 3.2. *Suppose a positive integer M and an $M \times M$ Hermitian matrix $\mathbf{A}$ are given, as well as real-valued random variables (not assumed to be independent) $W_1, \dots, W_M$, with continuous distribution functions F_{W_i}, $1 \leq i \leq M$. Let $\mathbf{W}(\omega)$ be the diagonal random matrix* $\operatorname{diag}(W_1(\omega), \dots, W_M(\omega))$. *For any $s > 0$ and $\epsilon \in (0,1)$, there exists $g^* < \infty$ such that if $|g| \geq g^*$ then*

$$\sup_{E \in \mathbb{R}} \mathbb{P}\{ \|(\mathbf{A} + g\mathbf{W} - E\mathbf{I})^{-1}\| > s \} \leq \epsilon.$$

Consequently, for all $\delta \in (0,1)$, $\kappa \in (0,\zeta)$ and $m^*, \nu^* \geq 1$, there exists $g_0 \in (0,\infty)$ such that for all $1 \leq N \leq N^*$ and positive integer L_0, property $\mathsf{S}(N,0)$ *holds true for $|g| \geq g_0$.*

The proof is omitted; it is based on a well-known argument employed in a number of papers on the MSA (cf., e.g., [20, Proposition A.1.2]) and is not contingent upon the single- or multi-particle structure of the random diagonal entries of the matrix $\mathbf{A}$.

3.2. The GFs in WI balls. The MPMSA induction

Lemma 3.3. *Fix $\beta, \delta \in (0,1)$, $m^* \geq 1$ and $E \in \mathbb{R}$. Suppose that a* WI *ball $\mathbf{B}^{(N)}(\mathbf{u}, L_k)$ is (E,β)-*NR *and satisfies the two following conditions*:

$$\text{for all } \lambda' \in \Sigma(\mathbf{H}^{(N')}_{\mathbf{B}'})\mathbf{B}'' \text{ is } (E-\lambda', \delta, m_{N''}) - \text{NS}, \tag{3.5}$$

$$\text{for all } \lambda'' \in \Sigma(\mathbf{H}^{(N'')}_{\mathbf{B}''})\mathbf{B}' \text{ is } (E-\lambda'', \delta, m_{N'}) - \text{NS}. \tag{3.6}$$

If L_0 is large enough then $\mathbf{B}^{(N)}(\mathbf{u}, L_k)$ is (E,δ,m_N)-NS.

Proof. See Appendix A. □

Lemma 3.3 is used in the proof of Lemma 3.4.

Lemma 3.4. *Assume property $\mathsf{S}(N-1,k)$ for some given $L_0, Y > 1$, $\delta \in (0,1)$, $\kappa \in (0,\zeta)$ and $m^*, \nu^* \geq 1$ (see eq. (3.4)). If L_0 is large enough then for any $E \in \mathbb{R}$ and* WI *ball $\mathbf{B}^{(N)}(\mathbf{u}, L_k)$,*

$$\mathbb{P}\{\,\mathbf{B}^{(N)}(\mathbf{u}, L_k) \text{ is } (E,\delta,m_N)\text{-S}\,\} \leq \mathrm{e}^{-\frac{3}{2}\nu_N L^{\kappa}_{k+1}}. \tag{3.7}$$

Consequently, for L_0 large enough, for all $\mathbf{x} \in \mathcal{Z}$,

$$\begin{aligned}&\mathbb{P}\{\mathbf{B}^{(N)}(\mathbf{x}, L_{k+1}) \text{ contains a WI } (E,\delta,m_N)\text{-S ball } \mathbf{B}^{(N)}(\mathbf{u}, L_k)\}\\ &\quad\leq C^N_{\mathcal{Z}} L^{Nd}_{k+1} \cdot \mathrm{e}^{-\frac{3}{2}\nu_N L^{\kappa}_{k+1}} \leq \frac{1}{4}\mathrm{e}^{-\nu_N L^{\kappa}_{k+1}}.\end{aligned} \tag{3.8}$$

Proof. Denote by $\mathcal{S}$ the event in the LHS of (3.7). Let $\mathbf{B} = \mathbf{B}^{(N)}(\mathbf{u}, L_k)$, with the canonical factorization $\mathbf{B} = \mathbf{B}' \times \mathbf{B}'$, and denote $\mathbf{H}' = \mathbf{H}^{(N')}_{\mathbf{B}'}$, $\mathbf{H}'' = \mathbf{H}^{(N'')}_{\mathbf{B}'}$ (cf. (2.21), (2.22)). We have

$$\mathbb{P}\{\mathcal{S}\} < \mathbb{P}\{\,\mathbf{B} \text{ is not } E\text{-NR}\,\} + \mathbb{P}\{\,\mathbf{B} \text{ is } E\text{-NRand } (E,\delta,m_N)\text{-S}\,\}. \tag{3.9}$$

The first term in the RHS is assessed in Theorem 2.1 and bounded by $\mathrm{e}^{-L^{\beta/2}_{k+1}} < \frac{1}{2}\mathrm{e}^{-\frac{3}{2}\nu_N L^{\kappa}_{k+1}}$, since $\kappa < \beta/2$ (cf. (3.2)), so we focus on the second summand.

Introduce the events

$$\mathcal{S}' := \{\omega\colon \text{(3.5) is wrong}\},$$
$$\mathcal{S}'' := \{\omega\colon \text{(3.6) is wrong}\}.$$

Then, with $m' := m_{N'}$, by virtue of Lemma 3.3,

$$\mathbb{P}\{\mathcal{S}'\} = \mathbb{E}[\mathbb{P}\{\text{there exists } \lambda'' \in \Sigma(\mathbf{H}'')\colon \mathbf{B}' \text{ is } (E-\lambda'', \delta, m')\text{-S} \,|\, \mathfrak{F}''\}],$$

where the sigma-algebra $\mathfrak{F}''$ is generated by the sample of the random potential V in $\Pi\mathbf{B}''$. Conditioning by $\mathfrak{F}''$ renders $\Sigma(\mathbf{H}'')$ nonrandom. By definition of the canonical decomposition, $\Pi\mathbf{B}' \cap \Pi\mathbf{B}'' = \varnothing$, and since the random field V is IID, we have

$$\operatorname{ess\,sup} \mathbb{P}\{\, \mathbf{B}' \text{ is } (E-\lambda'', \delta, m')\text{-S} \,|\, \mathfrak{F}''\,\} \le \sup_{E''\in\mathbb{R}} \mathbb{P}\{\, \mathbf{B}' \text{ is } (E'', \delta, m')\text{-S}\,\}.$$

Further, by the assumed property $\mathsf{S}(N-1,k)$, for $N' \le N-1$,

$$\mathbb{P}\{\, \mathbf{B}' \text{ is } (E'', \delta, m')\text{-S}\} \le \mathrm{e}^{-\nu_{N-1} L_k^{\kappa}} = \mathrm{e}^{-2\nu_N L_{k+1}^{\kappa}} \tag{3.10}$$

(cf. the definition of ν_N in (3.2)). Thus we obtain that

$$\begin{aligned} \mathbb{P}\{\mathcal{S}'\} &\le \sharp\, \mathbf{B}'' \sup_{E''\in\mathbb{R}} \mathbb{P}\{\mathbf{B}' \text{ is } (E'', \delta, m')\text{-S}\} \\ &\le C_{\mathcal{Z}}^{N} L_k^{Nd} \exp\{-2\nu_N L_{k+1}^{\kappa}\} \le \frac{1}{2}\exp\left\{-\frac{3}{2}\nu_N L_{k+1}^{\kappa}\right\}; \end{aligned} \tag{3.11}$$

here the last inequality holds for L_0 large enough. Similarly, with $m'' := m_{N''}$,

$$\mathbb{P}\{\mathcal{S}''\} \le \frac{1}{2}\exp\left\{-\frac{3}{2}\nu_N L_{k+1}^{\kappa}\right\}. \tag{3.12}$$

Collecting (2.4), (3.9), (3.11) and (3.12), the assertion (3.7) follows.

To prove (3.8), notice that the number of WI balls of radius L_k inside $\mathbf{B}^{(N)}(\mathbf{x}, L_{k+1})$ is bounded by $\sharp\mathbf{B}^{(N)}(\mathbf{x}, L_{k+1})$, and the probability for a WI ball to be (E, δ, m_N)-S satisfies (3.7), so the last inequality in (3.8) follows, again for L_0 large enough. □

Now, given $k = 0, 1, \ldots$, consider the following probabilities:

$$\begin{aligned} \mathrm{P}_k &= \sup_{\mathbf{u}\in\mathcal{Z}} \mathbb{P}\{\, \mathbf{B}^{(N)}(\mathbf{u}, L_k) \text{ is } (E, \delta, m_N)\text{-S}\}, \\ \mathrm{Q}_{k+1} &= \sup_{\mathbf{u}\in\mathcal{Z}} \mathbb{P}\{\mathbf{B}^{(N)}(\mathbf{u}, L_{k+1}) \text{ is not } (E, \beta)\text{-CNR}\}, \\ \mathrm{S}_{k+1} &= \sup_{\mathbf{x}\in\mathcal{Z}^N} \mathbb{P}\{\mathbf{B}^{N)}(\mathbf{x}, L_{k+1}) \text{ contains a WI } (E, \delta, m_N)\text{-S ball } \mathbf{B}^{(N)}(\mathbf{u}, L_k)\}. \end{aligned}$$

Note that

$$\text{for } N = 1,\ \mathrm{S}_{k+1} = 0 \text{ (1-particle balls cannot be WI).} \tag{3.13}$$

Theorem 3.1. *Suppose that, for some given $Y \geq 2$, $\delta \in (0,1)$, $\kappa \in (0,\zeta)$ and $m^*, \nu^* \geq 1$, the property $\mathsf{S}(N,0)$ holds true with L_0 large enough. Then $\mathsf{S}(N,k)$ holds true for all $k \geq 0$ with the same $L_0, Y, \delta, \kappa, m^*$ and ν^*.*

Consequently, by the second assertion of Lemma 3.2, $\mathsf{S}(N,k)$ holds true for arbitrarily large m^* and ν^*, provided that $|g| \geq g_0(m^*, \nu^*)$ with $g_0(m^*, \nu^*)$ large enough.

Proof. It suffices to derive $\mathsf{S}(N,k+1)$ from $\mathsf{S}(N,k)$, so assume the latter. By virtue of Lemma 3.1, if a ball $\mathbf{B}^{(N)}(\mathbf{x}, L_{k+1})$ is (E,δ,m_N)-S, then it is either not (E,β)-CNR(with probability $\leq Q_{k+1}$) or (E,m_N)-bad.

By (3.8), the probability of having at least one weakly interacting (E,m_N)-S ball $\mathbf{B}^{(N)}(\mathbf{u}, L_k) \subset \mathbf{B}^{(N)}(\mathbf{x}, L_{k+1})$ obeys: $\mathrm{S}_{k+1} \leq \frac{1}{4}\,\mathrm{e}^{-\nu_N L_{k+1}^{\kappa}}$.

Note that this is the only point where the inductive hypothesis $\mathsf{S}(N-1,k)$ is actually required, for $N \geq 2$, while $\mathrm{S}_{k+1} = 0$ for $N = 1$, because of (3.13).

Therefore, it remains to assess the probability of having at least 2 balls of radius L_k inside $\mathbf{B}^{(N)}(\mathbf{u}, L_{k+1})$ which are SI, (E,δ,m_N)-S and $8NL$-distant. The number of such pairs is $\leq C_{\mathcal{Z}}^{2N} L_{k+1}^{2Nd}$, thus, owing to Lemma 2.6, we have[4]

$$\mathrm{P}_{k+1} \leq \frac{1}{2} C_{\mathcal{Z}}^{2N} L_{k+1}^{2Nd} \mathrm{P}_k^2 + \mathrm{S}_{k+1} + \mathrm{Q}_{k+1}.$$

It follows from Theorem 2.1 that $\mathrm{Q}_{k+1} \leq C\, L_{k+1}^{Nd+1} \mathrm{e}^{-L_{k+1}^{\beta/2}}$, $\beta/2 > \kappa$, so with L_0 large enough, $Q_k \leq \frac{1}{4}\mathrm{e}^{-\nu_N L_{k+1}^{\kappa}}$ for any $k \geq 0$. Finally,

$$\mathrm{P}_{k+1} \leq \frac{1}{2} C_{\mathcal{Z}}^{2N} L_{k+1}^{2Nd} \mathrm{P}_k^2 + \frac{1}{4}\mathrm{e}^{-\nu_N L_k^{\kappa}} + \frac{1}{4}\mathrm{e}^{-\nu_N L_k^{\kappa}}. \tag{3.14}$$

The assertion of the theorem will follow if we show that, for L_0 large enough, $\mathrm{P}_{k+1} \leq \mathrm{e}^{-\nu_N L_k^{\kappa}}$, i.e., $C_{\mathcal{Z}}^{2N} L_{k+1}^{2Nd} \mathrm{P}_k^2 \leq \mathrm{e}^{-\nu_N L_{k+1}^{\kappa}}$. The last fact follows from (3.2). Indeed, for any finite C, A and L_0 large enough, with $\kappa \leq Y^{-1} \ln(4/3)$ (cf. (3.2)),

$$\begin{aligned}
-\ln(C L_{k+1}^{A} \mathrm{P}_k^2) &\geq 2\nu_N L_k^{\kappa} - C' \ln L_{k+1} \\
&= \nu_N L_{k+1}^{\kappa} \Big(\frac{2}{Y^{\kappa}} - \frac{C' \ln L_{k+1}}{\nu_N L_k^{\kappa}} \Big) \\
&\geq \nu_N L_{k+1}^{\kappa} \Big(\frac{3}{2} - \frac{C' \ln L_{k+1}}{\nu_N L_k^{\kappa}} \Big) \\
&> \nu_N L_{k+1}^{\kappa}.
\end{aligned}$$

This completes the proof of Theorem 3.1. □

Theorem 3.1 allows us to complete the MPMSA inductive scheme.

[4] A statement like Lemma 2.6 is not required for $N = 1$, if the random potential is IID, since disjoint 1-particle balls give rise to independent Hamiltonians.

Indeed, owing to Lemma 3.2, for any given L_0, δ, κ, m^* and ν^*, the property $\mathsf{S}(N, 0)$ holds true for sufficiently large $|g|$ and all $N = 1, \dots, N^*$. The scale induction step $k \rightsquigarrow k+1$ is provided by Lemmas 3.1, 3.3 and 3.4 and Theorem 3.1. By induction in k, this proves $\mathsf{S}(N, k)$ for $1 < N \le N^*$ and all $k \ge 0$, provided that $\mathsf{S}(N-1, k)$ is proved for all $k \ge 0$. The base of induction in N is obtained in a similar (in fact, simpler) manner.

3.3. From fixed-energy to energy-interval estimates. The core of the technical argument in this section relies upon the two-volume EVC bound (Theorem 2.2).

Given a positive integer L and $\mathbf{u} \in \mathcal{Z}$, define the quantity $\mathbf{F_u}(E) = \mathbf{F}^{(N)}_{\mathbf{u},L}(E)$:

$$\mathbf{F_u}(E) = C_{\mathcal{Z},N} L^{Nd} \max_{\mathbf{z} \in \partial^- \mathbf{B}(\mathbf{u},L)} |\mathbf{G}^{(N)}_{\mathbf{B}(\mathbf{u},L)}(\mathbf{u}, \mathbf{z}; E)| \,. \tag{3.15}$$

For brevity, we denote by $\Sigma_\mathbf{u}$ the spectrum of the operator $\mathbf{H}^{(N)}_{\mathbf{B}(\mathbf{u},L)}$. Owing to assumption $(\mathsf{V2})$, for $N \le N^*$, the norms of the operators $\mathbf{H}^{(N)}_{\mathbf{B}(\mathbf{u},L)}$ are uniformly bounded, and so are their spectra. Therefore, the spectral analysis of these operators can be restricted, without loss of generality, to some finite interval $I \subset \mathbb{R}$ independent of $k \ge 0$.

Lemma 3.5 (as well as Lemma 3.6) encapsulates a probabilistic estimate essentially going back to the work [21] by Elgart et al. Here we follow closely the book [19].

Lemma 3.5 ([19, Theorems 2.5.1 and 4.3.11]). *Let be given an integer $L \ge 1$, balls $\mathbf{B}^{(N)}(L, \mathbf{x})$, $\mathbf{B}^{(N)}(L, \mathbf{y})$, an interval $I \subset \mathbb{R}$ of length $|I| < \infty$ and real numbers $a_L, b_L, c_L, q_L > 0$ satisfying*

$$b_L \le \min\{M^{-1} a_L c_L^2, c_L\}. \tag{3.16}$$

with $M := \max[\sharp\mathbf{B}^{(N)}(L, \mathbf{x}), \sharp\mathbf{B}^{(N)}(L, \mathbf{y})]$. Suppose in addition that

$$\max[\mathbb{P}\{\mathbf{F_x}(E) \ge a_L\},\ \mathbb{P}\{\mathbf{F_y}(E) \ge a_L\}] \le q_L. \tag{3.17}$$

Assume also that for some $A, C, > 0$ and $\theta \in (0, 1]$, for all $\epsilon > 0$

$$\mathbb{P}\left\{\operatorname{dist}(\Sigma_\mathbf{x}, \Sigma_\mathbf{y}) \le \epsilon\right\} \le C L^A \epsilon^\theta. \tag{3.18}$$

Then one has, with $A' = A + 2Nd$ and some $C' < \infty$,

$$\mathbb{P}\{\textit{there exists } E \in I\colon \min[\mathbf{F_x}(E), \mathbf{F_y}(E)] \ge a_L\} \le \frac{2|I| q_L}{b_L} + C' L^{A'} c_L^\theta. \tag{3.19}$$

The proof of Lemma 3.5 relies upon a more general result, Lemma 3.6. A similar assertion was proved earlier for the balls in $\mathbb{Z}^d$ (cf. [19, Theorem 2.5.1]) and adapted to more general graphs (cf. [12, Theorem 2]). The single- or multi-particle nature of the random potential is irrelevant here. Below we state it for a ball $\mathbf{B}(\mathbf{u}, L)$, mainly to set up a framework for the proof of Lemma 3.5.

Lemma 3.6 ([12, Theorem 2]). *Given a positive integer* L *and a ball* $\mathbf{B} = \mathbf{B}^{(N)}(\mathbf{u}, L)$, *set* $M = \sharp\mathbf{B}$ *and consider a random operator* $\mathbf{H}_\mathbf{B} = \mathbf{H}^{(N)}_\mathbf{B}(\omega)$ *of the form*

$$(\mathbf{H}_\mathbf{B}\Psi)(\mathbf{x}) = (-\mathbf{\Delta}_\mathbf{B}\Psi)(\mathbf{x}) + W(\mathbf{x};\omega)\Psi(\mathbf{x}), \quad \mathbf{x} \in \mathbf{B}, \tag{3.20}$$

where $(\mathbf{x},\omega) \mapsto W(\mathbf{x};\omega) \in \mathbb{R}$ *is a given random function. (No a priori condition is imposed upon the distribution of* $W(\mathbf{x};\omega)$.) *Let* $E_j = E_j(\omega)$, $1 \le j \le M$, *be the (random) eigenvalues of* $\mathbf{H}_\mathbf{B}$ *listed in some measurable way. Take a bounded interval* $I \subset \mathbb{R}$ *and let the numbers* $a_L, b_L, c_L, q_L > 0$ *satisfy*

$$b_L \le \min\{M^{-1}a_Lc_L^2,\, c_L\} \tag{3.21}$$

and for all $E \in I$

$$\mathbb{P}\{\mathbf{F}_\mathbf{u}(E) \ge a_L\} \le q_L, \tag{3.22}$$

where $\mathbf{F}_\mathbf{u}(E)$ *is as in* (3.15). *Then there is an event* $\mathcal{B}_\mathbf{u}$ *of probability* $\mathbb{P}\{\mathcal{B}_\mathbf{u}\} \le |I|b_L^{-1}q_L$ *such that for all* $\omega \notin \mathcal{B}_\mathbf{u}$

$$\mathcal{E}_\mathbf{u}(2a_L) := \{E \in I : \mathbf{F}_\mathbf{u}(E) \ge 2a_L\} \subset \cup_{j=1}^M I_j, \tag{3.23}$$

where $I_j := (E_j - 2c_L, E_j + 2c_L)$.

Proof of Lemma 3.5. Let $\mathcal{B}_\mathbf{x}$ and $\mathcal{B}_\mathbf{y}$ be the events introduced in Lemma 3.6 for the balls $\mathbf{B}^{(N)}(L,\mathbf{x})$ and $\mathbf{B}^{(N)}(L,\mathbf{y})$, and let $\mathcal{B} = \mathcal{B}_\mathbf{x} \cup \mathcal{B}_\mathbf{y}$. As in Lemma 3.6, denote by $\mathcal{E}_\mathbf{x}(2a_L)$, $\mathcal{E}_\mathbf{y}(2a_L)$ the energy sets related to $\mathbf{B}^{(N)}(L,\mathbf{x})$, $\mathbf{B}^{(N)}(L,\mathbf{y})$, and introduce the event $\mathcal{S}_{\mathbf{x},\mathbf{y}} = \{\omega : \mathcal{E}_\mathbf{x}(2a_L) \cap \mathcal{E}_\mathbf{y}(2a_L) \neq \varnothing\}$. Then we have

$$\mathbb{P}\{\mathcal{S}_{\mathbf{x},\mathbf{y}}\} \le \mathbb{P}\{\mathcal{B}\} + \mathbb{P}\{\mathcal{S}_{\mathbf{x},\mathbf{y}} \setminus \mathcal{B}\} \le 2b_L^{-1}q_L|I| + \mathbb{P}\{\mathcal{S}_{\mathbf{x},\mathbf{y}} \setminus \mathcal{B}\}. \tag{3.24}$$

For any $\omega \notin \mathcal{B}$, each of the energy subsets $\mathcal{E}_\mathbf{x}(2a)$, $\mathcal{E}_\mathbf{y}(2a)$ is covered by intervals of length $4c_L$ centered at the respective EVs of $\mathbf{H}^{(N)}_{\mathbf{B}(\mathbf{x},L)}$ and $\mathbf{H}^{(N)}_{\mathbf{B}(\mathbf{y},L)}$.

Recall that we have *assumed*[5] the two-volume EVC estimate (3.18) (with an exponent $\theta > 0$) for the pair $\mathbf{B}^{(N)}(\mathbf{x}, L)$, $\mathbf{B}^{(N)}(\mathbf{y}, L)$. Thus, with $\Sigma_\mathbf{x} :=$

[5] In the application of Lemma 3.5 to the proof of Lemma 3.7, we rely on Theorem 2.2 actually proving (3.18) with $\vartheta = 2/3$.

$\Sigma(\mathbf{H}^{(N)}_{\mathbf{B}(\mathbf{x},L)})$, $\Sigma_{\mathbf{y}} := \Sigma(\mathbf{H}^{(N)}_{\mathbf{B}(\mathbf{y},L)})$, and some $C', A' \in (0, +\infty)$,

$$\begin{aligned}\mathbb{P}\{\mathcal{S}_{\mathbf{x},\mathbf{y}} \setminus \mathcal{B}\} &\leq \mathbb{P}\{\operatorname{dist}(\Sigma_{\mathbf{x}}, \Sigma_{\mathbf{y}}) \leq 4c_L\} \\ &\leq CL^A(4c_L)^\theta \leq C'L^{A'}c_L^\theta.\end{aligned} \tag{3.25}$$

Collecting (3.24) and (3.25), the assertion of Lemma 3.5 follows. □

We use Lemma 3.5 in the proof of Lemma 3.7, with $q_L = \mathrm{e}^{-\nu_N L^\kappa}$, $L = L_k$.

Lemma 3.7. *Assuming L_0 large enough, for any $k \geq 0$ and any pair of $4NL_k$-distant balls $\mathbf{B}^{(N)}(\mathbf{x}, L_k)$, $\mathbf{B}^{(N)}(\mathbf{y}, L_k)$, the following bound holds true*:

$$\mathbb{P}\{\textit{there exists } E \in \mathbb{R}\colon \min[\mathbf{F}_{\mathbf{x}}(E), \mathbf{F}_{\mathbf{y}}(E)] > \mathrm{e}^{-\frac{\nu_N}{3}L_k^\kappa}\} \leq \mathrm{e}^{-\frac{\nu_N}{11}L_k^\kappa}, \tag{3.26}$$

where $\nu_N \geq \nu^*$ can be made arbitrarily large for $|g|$ large enough.

Proof. As was noted, the spectrum $\Sigma_{\mathbf{x}}$ is contained in a fixed bounded interval $I \subset \mathbb{R}$. For $k \geq 0$, we have, by $\mathsf{S}(N,k)$, that, with $a = \mathrm{e}^{-\frac{\nu_N}{3}L_k^\kappa} > \mathrm{e}^{-m_N L_k^\delta}$ (recall: $\kappa < \beta/2 < \delta/2$)

$$\mathbb{P}\{\mathbf{F}_{\mathbf{x}}(E) > a\}, \mathbb{P}\{\mathbf{F}_{\mathbf{y}}(E) > a\} \leq \mathrm{e}^{-\nu_N L_k^\kappa}.$$

The LHS of (3.18) can be assessed with the help of Theorem 2.2:

$$\mathbb{P}\{\operatorname{dist}(\Sigma_{\mathbf{x}}, \Sigma_{\mathbf{x}}) \leq \epsilon\} \leq CL_k^A\epsilon^{2/3}.$$

A direct inspection shows that the quantities

$$a_L = \mathrm{e}^{-\frac{\nu_N}{3}L^\kappa},\ b_L = \mathrm{e}^{-\frac{2\nu_N}{3}L^\kappa},\ c_L = \mathrm{e}^{-\frac{\nu_N}{7}L^\kappa},\ q_L = \mathrm{e}^{-\nu_N L^\kappa}, \tag{3.27}$$

satisfy the conditions (3.21)–(3.22), hence Lemma 3.5 applies, with $L = L_k$, and we obtain that, with some $A < +\infty$ and L_0 large enough,

$$\begin{aligned}\mathbb{P}\{\sup_{E\in I} \min[\mathbf{F}_{\mathbf{x}}(E), \mathbf{F}_{\mathbf{y}}(E)] > \mathrm{e}^{-\frac{\nu_N}{3}L_k^\kappa}\} &\leq 2|I|\mathrm{e}^{-\frac{\nu_N}{3}L_k^\kappa} + C'L^A\mathrm{e}^{-\frac{2}{3}\cdot\frac{\nu_N}{7}L_k^\kappa} \\ &\leq \mathrm{e}^{-\frac{\nu_N}{11}L_k^\kappa},\end{aligned}$$

as asserted. □

Lemma 3.7 is an important ingredient of the proof of Theorem 3.2.

3.4. Strong dynamical localization. Now we are going to complete the proof of the assertion **(A)** of Theorem 1.1. The staple here is Lemma 3.8 presenting a more general result[6], under the key assumption (3.30).

Given a *finite* subset $\mathbf{\Lambda} \subset \mathcal{Z}$, we deal with a finite-dimensional random Hamiltonian $\mathbf{H}_{\mathbf{\Lambda}} = \mathbf{H}^{(N)}_{\mathbf{\Lambda}}(\omega)$ in $\ell^2(\mathbf{\Lambda})$:

$$(\mathbf{H}_{\mathbf{\Lambda}} f)(\mathbf{x}) = (-\mathbf{\Delta}_{\mathbf{\Lambda}} f)(\mathbf{x}) + W(\mathbf{x};\omega) f(\mathbf{x}), \quad \mathbf{x} \in \mathbf{\Lambda}. \tag{3.28}$$

Here $(\mathbf{x},\omega) \mapsto W(\mathbf{x},\omega)$ is a bounded real-valued random field on $\mathbf{\Lambda}$. (As in Lemma 3.6, no assumption is made about the distribution of $W(\mathbf{x},\omega)$.) At the same time, we consider Hamiltonians $\mathbf{H}^{(N)}_{\mathbf{B}(\mathbf{u},L)}$ in the balls $\mathbf{B}^{(N)}(\mathbf{u},L) \subseteq \mathbf{\Lambda}$. As in (3.15), let

$$\mathbf{F}_{\mathbf{u}}(E) = C_{\mathcal{Z},N} L^d \max_{\mathbf{z} \in \partial^- \mathbf{B}(\mathbf{u},L)} |\mathbf{G}_{\mathbf{B}(\mathbf{u},L)}(\mathbf{u},\mathbf{z};E)|. \tag{3.29}$$

Like before, denote by $\mathcal{B}_1(\mathbb{R})$ the set of continuous functions $\phi\colon \mathbb{R} \to \mathbb{C}$ with $\|\phi\|_\infty \le 1$.

Lemma 3.8 ([9, Lemma 9]). *Given a positive integer L, assume that the following bound holds true for a pair of balls $\mathbf{B}(\mathbf{x},L), \mathbf{B}(\mathbf{y},L) \subset \mathcal{Z}$ with $\boldsymbol{\rho}(\mathbf{x},\mathbf{y}) \ge L+2$ and some positive functions u, h:*

$$\mathbb{P}\{\textit{there exists } E \in \mathbb{R}\colon \min[\mathbf{F}_{\mathbf{x}}(E), \mathbf{F}_{\mathbf{y}}(E)] > u(L)\} \le h(L). \tag{3.30}$$

Then for any finite connected subset $\mathbf{\Lambda} \supset \mathbf{B}(\mathbf{x},L+1) \cup \mathbf{B}(\mathbf{y},L+1)$ one has

$$\mathbb{E}\Big[\sup_{\phi \in \mathcal{B}_1(\mathbb{R})} |\langle \mathbf{1}_{\mathbf{x}} | \phi(\mathbf{H}_{\mathbf{\Lambda}}) | \mathbf{1}_{\mathbf{y}} \rangle| \Big] \le 4u(L) + h(L). \tag{3.31}$$

Proof. The proof repeats verbatim that of Lemma 9 in [9], except for the quantity $u(L)$ replacing an explicit expression e^{-mL}. ☐

[6] We emphasize that the main idea of the simplified derivation of the strong dynamical localization from the energy-interval MSA bounds is due to Germinet and Klein [23].

For a finite $\mathbf{\Lambda}$, Assertion **(A)** of Theorem 1.1 now follows from

Theorem 3.2. *Given $\nu > 0$, there exist $g_*(\nu)$, $C_*(\nu) \in (0, \infty)$ such that, for $|g| \ge g_*(\nu)$, and $1 \le N \le N^*$, for all $\mathbf{x}, \mathbf{y} \in \mathcal{Z}$ and a finite $\mathbf{\Lambda} \subset \mathcal{Z}$ with $\mathbf{\Lambda} \ni \mathbf{x}, \mathbf{y}$,*

$$\Upsilon_{\mathbf{x},\mathbf{y}} := \mathbb{E}[\sup_{\phi \in \mathcal{B}_1(\mathbb{R})} |\langle \mathbf{1}_{\mathbf{x}} | \phi(\mathbf{H}_{\mathbf{\Lambda}}) | \mathbf{1}_{\mathbf{y}} \rangle|] \le C_*(\nu) \mathrm{e}^{-\nu (\boldsymbol{\rho}_{\mathrm{S}}(\mathbf{x},\mathbf{y}))^{\kappa}}. \tag{3.32}$$

Proof. Without loss of generality, it suffices to prove the assertion for the pairs of points with $\boldsymbol{\rho}_{\mathrm{S}}(\mathbf{x}, \mathbf{y}) > 4NL_0$. Indeed, the EFC correlator is always bounded by 1, so for pairs $\mathbf{x}, \mathbf{y}$ with $\boldsymbol{\rho}_{\mathrm{S}}(\mathbf{x}, \mathbf{y}) \le 4NL_0$ the bound in (3.32) can be attained by taking a sufficiently large constant C_*.

Thus, fix points $\mathbf{x}, \mathbf{y} \in \mathcal{Z}$ with $R := \boldsymbol{\rho}_{\mathrm{S}}(\mathbf{x}, \mathbf{y}) > 4NL_0$. There exists $k \ge 0$ such that $R \in (4NL_k, 4NL_{k+1}]$. Arguing as above, it suffices to consider a finite $\mathbf{\Lambda} \subset \mathcal{Z}$ such that $\mathbf{B}^{(N)}(\mathbf{x}, L_k + 1) \cup \mathbf{B}^{(N)}(\mathbf{y}, L_k + 1) \subset \mathbf{\Lambda}$.

Since $R \le 4NL_{k+1} = 4NYL_k$, we have $L_k \ge R/(4NY)$. Combining Theorem 3.1 and Lemmas 3.7 (cf. (3.26)) and 3.8, we obtain, with $\kappa < \delta$,

$$\Upsilon_{\mathbf{x},\mathbf{y}} \le 4\mathrm{e}^{-\frac{\nu_N}{3(4NY)^{\kappa}} R^{\kappa}} + \mathrm{e}^{-\frac{\nu_N}{11(4NY)^{\kappa}} R^{\kappa}}. \tag{3.33}$$

Furthermore, by the second assertion of Theorem 3.1 (cf. also Lemma 3.2), given an arbitrary $\nu > 0$, we can choose a sufficiently large[7] $|g|$, so that the initial scale estimate $\mathsf{S}(N, 0)$ in (3.4) is fulfilled with $\nu_N \ge 44NY\nu$. Then we obtain

$$\Upsilon_{\mathbf{x},\mathbf{y}} \le 5\mathrm{e}^{-\nu R^{\kappa}} = 5\mathrm{e}^{-\nu (\boldsymbol{\rho}_{\mathrm{S}}(\mathbf{x},\mathbf{y}))^{\kappa}}.$$

This completes the proof of Theorem 3.2. □

The case of an infinite $\mathbf{\Lambda} \subseteq \mathcal{Z}$ requires an additional limiting procedure (making use of Fatou's lemma applied to the EF correlators), developed earlier by Aizenman et. al.; cf., e.g., [1], [2, Appendix A], [3, Sect. 2]. As the argument can be repeated here without any significant change, we omit it from the paper.

4. Exponential decay of eigenfunctions

The aim of this section is to prove the assertion **(B)** of Theorem 1.1. This is achieved along a scheme developed in [16, 17, 18, 19] and modified to include the case of a graph $\mathcal{Z} \in \mathfrak{G}(d, C_d)$ and an infinite-range interaction potential U.

[7] One can also take a slightly smaller $\kappa > 0$ and consider only $R \ge L_k$ with k large enough.

The following table summarises the relations between various parameters, assumed throughout the entire Section 4:

$\tau > \max\left(\zeta^{-1}, 1\right)$	$\mathbb{N} \ni \alpha > 2\tau \quad (\text{hence } \alpha > 2)$
$0 < \beta < \min\left(\frac{1}{2}, \zeta, \frac{7}{8\alpha}\right)$	$\mathbb{N} \ni K + 1 \geq 2\alpha$
$m_N = m^* \, (1 + 3L_0^{-1+\beta})^{N^*-N}$	$m^* \geq 1$
$P(N) = P^* \, (2\alpha)^{N^*-N}$	$P^* > 12N^* d\alpha$

(4.1)

Observe that

$$\text{for all } N = 1, \ldots, N^* \quad P(N) \geq P^* > \max(24Nd, 2Nd\alpha). \tag{4.2}$$

Compared to the scheme used in Section 3, the main distinction is that here we adopt a super-exponential scaling scheme where

$$L_k = \lfloor L_{k-1}^{\alpha} \rfloor \sim (L_0)^{\alpha^k}, \tag{4.3}$$

with the exponent α satisfying the conditions (4.1); it depends upon the value of ζ in condition (U) (cf. eq. (1.8)). (The smaller $\zeta > 0$, the larger is α.) This scheme, going back to Ref. [20], provides weaker (power-law) probabilistic bounds than in Section 3, but allows one to establish exponential decay of the GFs, resulting in *exponential* decay of the EFs, instead of the sub-exponential one, stemming from the analysis of the EFCs in Section 3.

The property $\mathsf{S}(N, k)$ will be replaced in this section by its counterpart, $\mathsf{S}_{\mathrm{EXP}}(N, k)$, presented in (4.9), adapted to the exponential decay bounds.

The base of induction on N, $\mathsf{S}_{\mathrm{EXP}}(1, k)$, was established in Ref. [12] where one-particle disordered systems on a graph of the class $\mathfrak{G}(d, C_d)$ were studied in the strong disorder regime. Like in Section 3, it also follows from our scaling analysis.

4.1. The analytic step: scaling the GFs. The following definitions are modifications of Definitions 2.1 and 3.1.

Definition 4.1. Given $E \in \mathbb{R}$ and $m^* \geq 1$, an N-particle ball $\mathbf{B} = \mathbf{B}^{(N)}(\mathbf{u}, L)$ is called (E, m_N)-*nonsingular* ((E, m_N)-NS), if for all $\mathbf{y} \in \partial^- \mathbf{B}$,

$$C_{\mathcal{Z},N} L^{Nd} \cdot |\mathbf{G}_{\mathbf{B}}^{(N)}(\mathbf{x}, \mathbf{y}; E)| \leq \mathrm{e}^{-\gamma(m_N, L)L}, \tag{4.4}$$

where

$$\gamma(m_N, L) := m_N(1 + L^{-1/8}). \tag{4.5}$$

Otherwise, $\mathbf{B}$ is called (E, m_N)-*singular* ((E, m_N)-S).

As was said in Section 2.1 (see the paragraph after Definition 2.1), the constant $C_{\mathcal{Z},N}$ is chosen so as to guarantee $\sup_{\mathbf{x}\in\mathcal{Z}^N} \sharp\partial\mathbf{B}^{(N)}(\mathbf{x},L) \le C_{\mathcal{Z},N} L^{Nd}$ for all $L \ge 1$.

Definition 4.2. An N-particle ball $\mathbf{B}^{(N)}(\mathbf{u},L)$, with $N \ge 2$, $\mathbf{u} = (u_1,\dots,u_N)$, is called *weakly interactive* (WI) if

$$\mathrm{diam}(\Pi\mathbf{u}) > 3NL^{\tau}, \tag{4.6}$$

and strongly interactive (SI), otherwise.

The two following statements, Lemma 4.1 and 4.2, are analogous to Lemmas 2.5 and 2.6, and so are their proofs which will be omitted.

Lemma 4.1. *For any* WI *ball* $\mathbf{B}^{(N)}(\mathbf{u},L)$ *there exists a decomposition* $\{1,\dots,N\} = \mathcal{J} \sqcup \mathcal{J}^{\complement}$, *such that,*

$$\mathrm{d}(\Pi_{\mathcal{J}}\mathbf{B}^{(N)}(\mathbf{u},L), \Pi_{\mathcal{J}^{\complement}}\mathbf{B}^{(N)}(\mathbf{u},L)) > L^{\tau}. \tag{4.7}$$

Lemma 4.2. *Let* $\mathbf{B}^{(N)}(\mathbf{x},L)$, $\mathbf{B}^{(N)}(\mathbf{y},L)$ *be a pair of* SI *balls with* $\boldsymbol{\rho}(\mathbf{x},\mathbf{y}) > 8NL^{\tau}$. *Then*

$$\Pi\mathbf{B}^{(N)}(\mathbf{x},L) \cap \Pi\mathbf{B}^{(N)}(\mathbf{y},L) = \varnothing \tag{4.8}$$

and, therefore, the random operators $\mathbf{H}_{\mathbf{B}^{(N)}(\mathbf{x},L)}(\omega)$ *and* $\mathbf{H}_{\mathbf{B}^{(N)}(\mathbf{y},L)}(\omega)$ *are independent.*

Definition 4.3. Given $E \in \mathbb{R}$, $\tau > 0$, $m^* \ge 1$ and integers $k, K \ge 0$, we say that a ball $\mathbf{B}^{(N)}(\mathbf{u},L_{k+1})$ is (E,m_N,K,τ)-good if it contains no collection of $K+1$ (or more) pairwise $8NL_k^{\tau}$-distant (E,m_N)-S balls of radius L_k.

Notice that if $\mathbf{B}^{(N)}(\mathbf{u},L_{k+1})$ is not (E,m_N,K,τ)-good, then it contains

- either at least one (E,m_N)-S WI ball of radius L_k,
- or at least $K+1$ pairwise $8NL_k^{\tau}$-distant, SI, (E,m_N)-S balls of radius L_k.

A pre-requisite for the proof of the following statement is Appendix C. This is a standard result of the MSA with length scales $L_k \sim L_0^{\alpha^k}$, $\alpha > 1$ (cf., e.g., [20]).

Lemma 4.3 (Lemma 3.1). *If* $\mathbf{B}^{(N)}(\mathbf{u},L_{k+1})$ *is* (E,β)-CNR*and* (E,m_N,K,τ)-*good and* L_0 *is large enough, then* $\mathbf{B}^{(N)}(\mathbf{u},L_{k+1})$ *is* (E,m_N)-NS.

Proof. Set $\mathbf{\Lambda} = \mathbf{B}^{(N)}(\mathbf{u},L_{k+1})$, $\mathbf{B} = \mathbf{B}^{(N)}(\mathbf{u},L_{k+1}-1)$ and fix $\mathbf{y} \in \partial^-\mathbf{B}$, so $\mathbf{y} \in \mathbf{\Lambda} \setminus \mathbf{B}$ (the latter is required for the application of Lemma C.2, with $L = L_{k+1}-1$). Let

$$f_{\mathbf{y}}\colon \mathbf{z} \longmapsto |\mathbf{G}_{\mathbf{\Lambda}}^{(N)}(\mathbf{z},\mathbf{y};E)|.$$

By assumption, there is a (possibly empty) collection of balls $\mathbf{B}(\mathbf{u}_j, L_k^{\tau}) \subset \mathbf{\Lambda}$, $1 \le j \le K'$, with $K' \le K$, such that any ball $\mathbf{B}(\mathbf{v}, L_k)$ with $\mathbf{v} \in \mathbf{B} \setminus \bigcup_{j=1}^{K'} \mathbf{B}(\mathbf{u}_j, 8NL_k^{\tau})$ is (E, m_N)-NS. Fix such a collection. As in Definition C.1, item (2), denote by $\mathcal{L}_r(\mathbf{u})$ the spherical layer $\{\mathbf{z} \in \mathcal{Z}: \boldsymbol{\rho}(\mathbf{z}, \mathbf{u}) = r\}$, $r \ge 0$, and set

$$\mathcal{S} := \left\{ \mathbf{x} \in \mathbf{B}_{L_{k+1}-L_k-1}(\mathbf{u}) : \mathcal{L}_{\mathrm{d}(\mathbf{u},\mathbf{x})}(\mathbf{u}) \cap \bigcup_{j=1}^{K'} \mathbf{B}_{8NL_k^{\tau}}(\mathbf{u}_j) \neq \varnothing \right\}$$

(here $\mathcal{S}$ stands for "singular"). Then any ball $\mathbf{B}(\mathbf{v}, L_k) \subset \mathbf{B}$ with $\mathbf{v} \in \mathbf{B} \setminus \mathcal{S}$ is (E, m_N)-NS, and $\mathcal{S}$ is covered by a family of $\le K$ annuli with center $\mathbf{u}$ and total width $\le K(2 \cdot 8NL_k^{\tau} + 1) \le 17NKL_k^{\tau}$, with L_0 large enough. By Lemma C.2 $f_{\mathbf{y}}$ is $(L_k, q, \mathcal{S})$-dominated in $\mathbf{B}$; here (cf. (C.8) with $\delta = 1$)

$$\begin{aligned} -\ln q &= m_N(1 + L_k^{-1/8})L_k - L_{k+1}^{\beta} - \ln(C_{\mathcal{Z}}^{N} L_{k+1}^{Nd}) \\ &\ge m_N L_k + (m_N L_k^{7/8} - 2L_k^{\alpha\beta}) \\ &\ge L_k m_N(1 + \tfrac{1}{2} L_k^{-1/8}), \end{aligned}$$

where the last inequality follows from the assumptions $\alpha\beta < 7/8$ and $m_N \ge m^* \ge 1$ listed in (4.1). By virtue of Lemma C.1 (cf. eq. (C.5), with $L = L_{k+1} - 1$),

$$f_{\mathbf{y}}(\mathbf{u}) \le q^{\frac{(L_{k+1}-1) - 17NKL_k^{\tau} - 1}{1 + L_k}} \mathrm{M}(f_{\mathbf{y}}, \mathbf{B}) \le q^{\frac{L_{k+1} - 18NKL_k^{\tau}}{1 + L_k}} \mathrm{M}(f_{\mathbf{y}}, \mathbf{B}).$$

One can see that, with $\alpha > 2\tau$, $\beta < 1/2$, $m_N \ge 1$,

$$\begin{aligned} -\ln f_{\mathbf{y}}(\mathbf{u}) &\ge -\ln\{(\mathrm{e}^{-m_N(1+\frac{1}{2}L_k^{-1/8})L_k})^{\frac{L_{k+1} - 18NKL_k^{\tau}}{1+L_k}} \mathrm{e}^{L_{k+1}^{\beta}}\} \\ &= m_N(1 + \tfrac{1}{2}L_k^{-1/8})L_k \frac{L_{k+1}(1 - 18NKL_{k+1}^{-1+\frac{\tau}{\alpha}})}{1 + L_k} - L_{k+1}^{\beta} \\ &\ge m_N L_{k+1}\{(1 + \tfrac{1}{4}L_k^{-1/8})(1 - L_{k+1}^{-1/2}) - L_{k+1}^{-1/2}\} \\ &\ge L_{k+1} m_N(1 + 2L_{k+1}^{-1/8}) \\ &\ge \gamma(m_N, L_{k+1})L_{k+1} + \ln(C_{\mathcal{Z},N} L_{k+1}^{Nd}), \end{aligned}$$

assuming L_0 is large enough. This leads to the assertion of Lemma 4.3. □

4.2. Localization in WI balls. The main result of Section 4.2 is Lemma 4.5. We begin with an analog of Lemma 3.3, proved in Appendix B.

Lemma 4.4. *Fix $E \in \mathbb{R}$ and consider a* WI *ball* $\mathbf{B} = \mathbf{B}^{(N)}(\mathbf{u}, L_k)$ *with a canonical factorization* $\mathbf{B} = \mathbf{B}' \times \mathbf{B}''$ *and the respective reduced Hamiltonians* $\mathbf{H}' = \mathbf{H}^{(N')}_{\mathbf{B}'}$ *and* $\mathbf{H}'' = \mathbf{H}^{(N'')}_{\mathbf{B}''}$. *Assume* $\mathbf{B}$ *is* (E, β)-NR. *Suppose in addition that for all* $\lambda'' \in \Sigma(\mathbf{H}'')$ *the* N'*-particle ball* $\mathbf{B}'$ *is* $(E - \lambda'', m_{N'})$-NS *and for all* $\lambda' \in \Sigma(\mathbf{H}')$ *the* N''*-particle ball* $\mathbf{B}''$ *is* $(E - \lambda', m_{N''})$-NS. *Then the ball* $\mathbf{B}$ *is* (E, m_N)-NS.

Consider the following property (replacing $\mathsf{S}(N, k)$; cf. eq. (3.4)).

$\mathsf{S}_{\text{EXP}}(N, k)$ *For all* $E \in \mathbb{R}$, $1 \le n \le N$ *and* $\mathbf{u} \in \mathcal{Z}^n$, *with* $P(n)$ *as in* (4.1),

$$\mathbb{P}\{\, \mathbf{B}^{(n)}(\mathbf{u}, L_k) \text{ is } (E, m_n)\text{-}S \,\} \le L_k^{-P(n)}. \tag{4.9}$$

Lemma 4.5. *Suppose that the property* $\mathsf{S}_{\text{EXP}}(N-1, k)$ *holds for any* $E \in \mathbb{R}$ *and some given* L_0, α, β, τ *and* $m^*, P^* \ge 1$ *and* $P(n)$ *as in* (4.1). *Take* L_0 *large enough. Then for any* $E \in \mathbb{R}$ *and any WI ball* $\mathbf{B}^{(N)}(\mathbf{u}, L_k)$,

$$\mathbb{P}\{\, \mathbf{B}^{(N)}(\mathbf{u}, L_k) \text{ is } (E, m_N)\text{-S} \,\} \le L_{k+1}^{-\frac{3}{2}P(N)}. \tag{4.10}$$

Consequently, if L_0 *is large enough then for any* $E \in \mathbb{R}$ *and any* $\mathbf{u} \in \mathcal{Z}$,

$$\begin{aligned} &\mathbb{P}\{\mathbf{B}^{(N)}(\mathbf{u}, L_{k+1}) \text{ contains a WI } (E, m_N)\text{-S ball of radius } L_k\} \\ &\quad \le C_{\mathcal{Z}}^N L_{k+1}^{Nd} \cdot L_{k+1}^{-\frac{3}{2}P(N)} \\ &\quad \le \frac{1}{4} L_{k+1}^{-\frac{5}{4}P(N)}. \end{aligned} \tag{4.11}$$

Proof. First, we prove the bound (4.10). As in the proof of Lemma 3.4, set $\mathbf{B} = \mathbf{B}^{(N)}(\mathbf{u}, L_k)$ and consider the canonical factorization $\mathbf{B} = \mathbf{B}' \times \mathbf{B}''$, with the reduced Hamiltonians $\mathbf{H}'$ and $\mathbf{H}''$. Given $E \in \mathbb{R}$, introduce the event $\mathcal{S} = \mathcal{S}(E, N)$:

$$\mathcal{S} = \{\omega : \mathbf{B} \text{ is WI and } (E, m_N)\text{-S}\}.$$

We assess its probability with the help of the inequality

$$\mathbb{P}\{\mathcal{S}\} < \mathbb{P}\{\, \mathbf{B} \text{ is not } E\text{-NR}\} + \mathbb{P}\{\, \mathbf{B} \text{ is } E\text{-NRand } (E, m_N)\text{-S} \,\}. \tag{4.12}$$

As earlier, the first term in the RHS of (4.12) can be assessed with the help of Theorem 2.1 and is bounded by $\frac{1}{4} L_{k+1}^{-\frac{3}{2}P(N)}$, so we focus on the second summand. According to Lemma 4.4,

$$\mathbb{P}\{\, \mathbf{B} \text{ is } E\text{-NRand } (E, m_N)\text{-S} \,\} \le \mathbb{P}\{\mathcal{S}'\} + \mathbb{P}\{\mathcal{S}''\}, \tag{4.13}$$

where

$$\mathcal{S}' = \{\text{there exists } \lambda'' \in \Sigma(\mathbf{H}'') \text{ the ball } \mathbf{B}' \text{ is } (E-\lambda'', m_{N'})\text{-S}\},$$

$$\mathcal{S}'' = \{\text{there exists } \lambda' \in \Sigma(\mathbf{H}') \text{ the ball } \mathbf{B}'' \text{ is } (E-\lambda', m_{N''})\text{-S}\}.$$

Further, we have

$$\mathbb{P}\{\mathcal{S}'\} = \mathbb{E}[\mathbb{P}\{\text{there exists } \lambda'' \in \Sigma(\mathbf{H}''){:}\, \mathbf{B}' \text{ is } (E-\lambda'', m_{N'})\text{-S} \,|\, \mathfrak{F}''\}],$$

where the sigma-algebra $\mathfrak{F}''$ is generated by the sample of the random potential V in $\Pi\mathbf{B}''$. Conditioning by $\mathfrak{F}''$ renders $\Sigma(\mathbf{H}'')$ nonrandom. By definition of the canonical decomposition, $\Pi\,\mathbf{B}' \cap \Pi\,\mathbf{B}'' = \varnothing$, and since the random field V is IID, we have

$$\operatorname{ess\,sup}\, \mathbb{P}\{\,\mathbf{B}' \text{ is } (E-\lambda'', m_{N'})\text{-S} \,|\, \mathfrak{F}''\} \le \sup_{E''\in\mathbb{R}} \mathbb{P}\{\,\mathbf{B}' \text{ is } (E'', m_{N'})\text{-S}\},$$

and by the assumed property $\mathsf{S}(N-1,k)$,

$$\mathbb{P}\{\,\mathbf{B}' \text{ is } (E'', m_{N'})\text{-S}\} \le L_k^{-P(N-1)}. \tag{4.14}$$

Therefore,

$$\mathbb{P}\{\mathcal{S}'\} \le \sharp\,\mathbf{B}'' \sup_{E''\in\mathbb{R}} \mathbb{P}\{\mathbf{B}' \text{ is } (E'', m_N)\text{-S}\} \le C_{\mathcal{Z}}^N L_k^{Nd}\, L_k^{-P(N-1)}; \tag{4.15}$$

after the substitution $P(N-1) = 2\alpha P(N)$ (cf. (4.1)), the RHS can be made $\le \frac{1}{4} L_{k+1}^{-\frac{3}{2}P(N)}$, provided $P(N) > 2Nd\alpha$ (and L_0 is large enough). The latter inequality follows from a bound in (4.1).

Summarising this calculation, we obtain

$$\mathbb{P}\{\mathcal{S}'\} \le \frac{1}{4} L_{k+1}^{-\frac{3}{2}P(N)}. \tag{4.16}$$

Similarly,

$$\mathbb{P}\{\mathcal{S}''\} \le \frac{1}{4} L_{k+1}^{-\frac{3}{2}P(N)}. \tag{4.17}$$

Collecting (2.4), (4.12), (4.13), (4.16), and (4.17), the assertion (4.10) follows.

To prove (4.11), notice that the number of WI balls of radius L_k inside $\mathbf{B}^{(N)}(\mathbf{x}, L_{k+1})$ is bounded by the cardinality $\sharp\mathbf{B}^{(N)}(\mathbf{x}, L_{k+1})$, and the probability that a given WI ball is (E, m_N)-S satisfies (4.10). Therefore, the probability in the LHS of (4.11) is upper-bounded, for L_0 large enough, by

$$C_{\mathcal{Z}}^N L_{k+1}^{Nd} L_{k+1}^{-\frac{3}{2}P(N)} = L_{k+1}^{-\frac{5}{4}P(N)} \cdot C_{\mathcal{Z}}^N\, L_{k+1}^{-\frac{1}{4}P(N)+Nd} \le \frac{1}{4} L_{k+1}^{-\frac{5}{4}P(N)}$$

since $P(N) \ge P(N^*) > 4Nd$, by virtue of (4.1) (cf. also (4.2)). □

4.3. The probabilistic scaling step. As in Section 3.2, we introduce the probabilities P_k, Q_{k+1} and S_{k+1}:

$$\mathrm{P}_k = \sup_{\mathbf{u}\in\mathcal{Z}} \mathbb{P}\{\mathbf{B}^{(N)}(\mathbf{u}, L_k) \text{ is } (E, m_N)\text{-S}\},$$

$$\mathrm{Q}_{k+1} = \sup_{\mathbf{u}\in\mathcal{Z}} \mathbb{P}\{\mathbf{B}^{(N)}(\mathbf{u}, L_{k+1}) \text{ is not } (E, \beta)\text{-CNR}\},$$

$$\mathrm{S}_{k+1} = \sup_{\mathbf{u}\in\mathcal{Z}^N} \mathbb{P}\{\mathbf{B}^{N)}(\mathbf{u}, L_{k+1}) \text{ contains a WI } (E, m_N)\text{-S ball } \mathbf{B}^{(N)}(\mathbf{x}, L_k)\}.$$

Theorem 4.1 is an analog of Theorem 3.1.

Theorem 4.1. *Suppose that, for some given α, τ, β, m^*, P^* as in (4.1), the property $\mathsf{S}_{\text{EXP}}(N, 0)$ is satisfied with L_0 large enough. Then $\mathsf{S}_{\text{EXP}}(N, k)$ holds true for all $k \geq 0$ with the same parameters.*

Proof. It suffices to derive $\mathsf{S}_{\text{EXP}}(N, k+1)$ from $\mathsf{S}_{\text{EXP}}(N, k)$, so assume the latter. By virtue of Lemma 4.3, if a ball $\mathbf{B}(\mathbf{u}, L_{k+1})$ is (E, m_N)-S, then

(a) either $\mathbf{B}(\mathbf{u}, L_{k+1})$ is not (E, β)-CNR(with probability $\leq Q_{k+1}$),

(b) or it contains

   (b1) either at least one WI (E, m_N)-NS ball of radius L_k,

   (b2) (b2) or $K' \geq K+1$ pairwise $8NL_k^{\tau}$-distant, SI, (E, m_N)-S balls of radius L_k.

By (4.11), the probability of the event (b1) is bounded by $\mathrm{S}_{k+1} \leq \frac{1}{4} L_{k+1}^{-P(N)}$. Therefore, it remains to assess the probability of the event (b2).

By Lemma 4.2, since the L_k-balls from this singular collection, say, $\mathbf{B}_{L_k}(\mathbf{v}_i)$, $1 \leq i \leq K+1$, are pairwise $8NL_k^{\tau}$-distant and SI, the Hamiltonians $\mathbf{H}_{\mathbf{B}_{L_k}(\mathbf{v}_i)}(\omega)$, $1 \leq i \leq K+1$, are independent. The number of such collections is $\leq C_{\mathcal{Z}}^{(K+1)N} L_{k+1}^{(K+1)Nd}$, thus

$$\mathrm{P}_{k+1} \leq \frac{1}{2} C_{\mathcal{Z}}^{(K+1)N} L_{k+1}^{(K+1)Nd} \mathrm{P}_k^{K+1} + \mathrm{S}_{k+1} + \mathrm{Q}_{k+1}.$$

It follows from Theorem 2.1 that $\mathrm{Q}_{k+1} \leq C\, L_{k+1}^{Nd+1} \mathrm{e}^{-L_{k+1}^{\beta/2}}$ where $\beta > 0$, thus for L_0 large enough, $Q_k \leq \dfrac{1}{4} L_{k+1}^{-P(N)}$ for all $k \geq 0$. Therefore, we can write

$$\mathrm{P}_{k+1} \leq \frac{1}{2} C_{\mathcal{Z}}^{2N} L_{k+1}^{(K+1)Nd} \mathrm{P}_k^{K+1} + \frac{1}{4} L_{k+1}^{-P(N)} + \frac{1}{4} L_{k+1}^{-P(N)}, \tag{4.18}$$

and the RHS can be made $< L_{k+1}^{-P(N)}$, whenever

$$(K+1)P(N) - (K+1)Nd\alpha > \alpha P(N),$$

e.g., with $K+1 \geq 2\alpha$ and $Nd\alpha < \frac{1}{2}P(N)$, provided L_0 is large enough. Again, the conditions $K+1 \geq 2\alpha$ and $P(N) > 2Nd\alpha$ follow from (4.1) and (4.2). ☐

4.4. Conclusion: exponential decay of eigenfunctions. In this section, as before, the condition (V2), as well as the property (RCM) stemming from it (cf. Lemma 2.1), is always assumed, so we do not repeat it in the formulations of Lemmas 4.6 and 4.7.

Recall that under the assumption (V2), the spectrum of the Hamiltonian $\mathbf{H}(\omega)$ is a.s. bounded by a value $O(|g|, N, d)$, and so are the spectra of its restrictions to arbitrary finite balls, hence we can restrict our analysis to a compact energy interval $I_g^* = I_g^*(N,d) \subset \mathbb{R}$ of length $|I_g^*|$. Below we assume that such an interval is fixed.

An analog of Lemma 3.5 is the following

Lemma 4.6. *Suppose we are given two $4NL$-distant balls $\mathbf{B}_L(\mathbf{x})$, $\mathbf{B}_L(\mathbf{y})$, and numbers $a_L, q_L > 0$ such that*

$$\sup_{E\in\mathbb{R}} \max[\mathbb{P}\{\mathbf{F_x}(E) > a_L\}, \mathbb{P}\{\mathbf{F_y}(E) > a_L\}] \leq q_L,$$

with $\mathbf{F_x}, \mathbf{F_y}$ defined as in (3.15). Then for any $b > 0$, one has

$$\begin{aligned} &\mathbb{P}\{\textit{there exists } E \in I_g^*\colon \min(\mathbf{F_x}(E), \mathbf{F_y}(E)) \geq a_L\} \\ &\quad \leq 2|I_g^*| b^{-1} q_L + C''' L^{4Nd} b^{2/3}. \end{aligned} \tag{4.19}$$

The reason why we need a separate bound (4.19) is that the derivation of the variable-energy estimates based on Lemma 3.5 gives rise to exponential decay of eigenfunctions only if the probabilistic bounds obtained by the fixed-energy analysis in the balls of size L are also exponential in L (which is never the case in the MSA); this can be seen in the condition (3.16).

In the proof given below, we will use the following auxiliary result, which, unlike Lemma 3.5, is better adapted to the proof of *exponential* decay of the EFs in a situation where the variable-energy MSA bounds (on the GFs) at the scale L decay slower than exponentially in L.

Lemma 4.7 ([12, Theorem 4]). *Let be given a ball* $\mathbf{B}_L(\mathbf{x})$*, a bounded interval* $I \subset \mathbb{R}$ *and numbers* $a_L, q_L > 0$ *such that*

$$\sup_{E \in I} \mathbb{P}\{\mathbf{F}_{\mathbf{x}}(E) > a_L\} \leq q_L. \tag{4.20}$$

Set $K = \sharp\, \mathbf{B}_L(\mathbf{x})$*. Then the following properties* (A), (B) *hold true.*

(A) *For any* $b > q_L$ *there exists an event* $\mathcal{S}_b$ *with* $\mathbb{P}\{\mathcal{S}_b\} \leq b^{-1}|I|\, q_L$ *and such that for any* $\omega \notin \mathcal{S}_b$*, the set of energies*

$$\mathcal{E}_{\mathbf{x}}(a_L) = \mathcal{E}_{\mathbf{x}}(a_L; \omega) := \{\mathbf{F}_{\mathbf{x}}(E) \geq a_L\}$$

is covered by $K' < 3K^2$ *intervals* $J_{\mathbf{x},i} = [E^-_{\mathbf{x},i}, E^+_{\mathbf{x},i}]$*, of total length* $\sum_i |J_{\mathbf{x},i}| \leq b$*.*

(B) *Consider the parametric operator family* $\mathbf{A}(t) = \mathbf{H}_{\mathbf{\Lambda}} + t\mathbf{1}$*,* $t \in \mathbb{R}$*. The endpoints* $E^{\pm}_{\mathbf{x},i}(t)$ *for the operators* $\mathbf{A}(t)$ *(replacing* $\mathbf{H}_{\mathbf{B}_L(\mathbf{x})}$*) have the form*

$$E^{\pm}_{\mathbf{x},i}(t) = E^{\pm}_{\mathbf{x},i} + t, \quad t \in \mathbb{R}.$$

Proof of Lemma 4.7. (A) Set for brevity $\mathbf{B} = \mathbf{B}_L(\mathbf{x})$. We have that

$$\mathbf{F}_{\mathbf{x}} = \max_{\mathbf{y} \in \partial^- \mathbf{B}} |\mathbf{F}_{\mathbf{x},\mathbf{y}}|, \ \text{ where } \mathbf{F}_{\mathbf{x},\mathbf{y}} := \mathbf{G}_{\mathbf{\Lambda}}(\mathbf{x}, \mathbf{y}; E).$$

Fix $\mathbf{y}$; the derivative of the rational function

$$\mathbf{F}_{\mathbf{x},\mathbf{y}} \colon E \mapsto \sum_{k=1}^{K} \frac{c_k}{E_k - E} = \sum_{k=1}^{K} \frac{\langle \mathbf{1_x} | \psi_k \rangle \, \langle \psi_k | \mathbf{1_y} \rangle}{E_k - E} \tag{4.21}$$

is a ratio of two real polynomials (we choose the EFs real):

$$\frac{\mathrm{d}}{\mathrm{d}E} \mathbf{F}_{\mathbf{x},\mathbf{y}}(E) = -\sum_k c_k (E_k - E)^{-2} =: \mathcal{P}(E)/\mathcal{Q}(E),$$

with $\deg \mathcal{P} \leq 2K - 2$. Hence, it has $\leq 2K - 2$ zeros and $\leq K$ poles, so $\mathbf{F}_{\mathbf{x},\mathbf{y}}$ has $< 3K$ intervals of monotonicity. Then the total number of monotonicity intervals for all functions $\mathbf{F}_{\mathbf{x},\mathbf{y}}$ is upper-bounded by $(\sharp\, \partial^- \mathbf{B}_L(\mathbf{u})) \cdot 3K \leq 3K^2$. Admitting the value $+\infty$ for the functions $|\mathbf{F}_{\mathbf{x},\mathbf{y}}|$, we can write

$$\bigcup_{\mathbf{y}} \{E \colon |\mathbf{F}_{\mathbf{x},\mathbf{y}}(E)| \geq a\} \subset \bigcup_{i=1}^{3K^2} J_{\mathbf{x},i}, \quad J_{\mathbf{x},i} = [E^-_{\mathbf{x},i}, E^+_{\mathbf{x},i}] \subset I.$$

Introduce the event $\mathcal{S}_{b,\mathbf{x}} = \{\omega : \mathrm{mes}\{E \in I : \mathbf{F}_{\mathbf{x}}(E) \ge a\} \ge b\}$. By the Chebychev inequality and the Fubini theorem, we have

$$\begin{aligned} \mathbb{P}\{\mathcal{S}_{b,\mathbf{x}}\} &\le b^{-1}\mathbb{E}[\int_I \mathbf{1}_{\{\mathbf{F}_{\mathbf{x}}(E)\ge a\}}\, dE] \\ &= b^{-1}\int_I \mathbb{E}[\mathbf{1}_{\{\mathbf{F}_{\mathbf{x}}(E)\ge a\}}]\, dE \\ &= b^{-1}\int_I \mathbb{P}\{\mathbf{F}_{\mathbf{x}}(E) \ge a\}\, dE \\ &\le b^{-1}|I|q_L. \end{aligned} \tag{4.22}$$

So, for all $\omega \notin \mathcal{S}_{b,\mathbf{x}}$, $\sum_i |J_{\mathbf{x},i}| \le \mathrm{mes}\{E \in I : \mathbf{F}_{\mathbf{x}} \ge a\} \le b$. This yields (A).

(B) The operators $\mathbf{A}(t)$ share common eigenvectors; the latter determine the coefficients c_k in (4.21), so we can choose the eigenfunctions $\psi_k(t)$ constant in t and obtain $c_k(t) \equiv c_k(0)$. The eigenvalues of $\mathbf{A}(t)$ have the form $E_{\mathbf{x},i}(t) = E_{\mathbf{x},i} + t$. Thus $\mathbf{F}_{\mathbf{x},\mathbf{y}}(E;t) = \mathbf{F}_{\mathbf{x},\mathbf{y}}(E-t;0)$, and $J_{\mathbf{x},i}(t) = [E^-_{\mathbf{x},i} + t, E^+_{\mathbf{x},i} + t]$. ☐

Proposition 4.7 operates with an arbitrary interval $I \subset \mathbb{R}$; we proved the fixed-energy bounds on the GFs for all $E \in \mathbb{R}$, so we are entitled to apply below the assertion of Proposiiton 4.7 with $I = I^*_g$ containing the spectrum of $\mathbf{H}(\omega)$.

Proof of Lemma 4.6. Fix $b > 0$ and let $\mathcal{S}_{b,\mathbf{z}} = \{\omega : \mathrm{mes}\{E : \mathbf{F}_{\mathbf{z}}(E) \ge a\} \ge b\}$ for $\mathbf{z} \in \{\mathbf{x}, \mathbf{y}\}$, $\mathcal{S}_b = \mathcal{S}_{b,\mathbf{x}} \cup \mathcal{S}_{b,\mathbf{y}}$. Let $\mathcal{S}$ be the event figuring in the LHS of (4.19). Using the bounds of the form (4.22) on $\mathbb{P}\{\mathcal{S}_{b,\mathbf{x}}\}$ and $\mathbb{P}\{\mathcal{S}_{b,\mathbf{y}}\}$, we have

$$\mathbb{P}\{\mathcal{S}\} \le \mathbb{P}\{\mathcal{S}_b\} + \mathbb{P}\{\mathcal{S} \cap \mathcal{S}^{\complement}_b\} \le 2|I^*_g|b^{-1}q_L + \mathbb{P}\{\mathcal{S} \cap \mathcal{S}^{\complement}_b\}.$$

It remains to asses $\mathbb{P}\{\mathcal{S} \cap \mathcal{S}^{\complement}_b\}$.

By Lemma 2.3, the $4NL$-distant balls $\mathbf{B}_L(\mathbf{x})$, $\mathbf{B}_L(\mathbf{y})$ are weakly $\mathcal{Q}$-separated for some $\mathcal{Q} \subset \mathcal{Z}$. As in the definition of weak separation, we associate with these balls the integers ("occupation numbers") $n_1 > n_2 \ge 0$. Consider the random variables $\xi = \xi_{\mathcal{Q}} = \langle V(\cdot;\omega)\rangle_{\mathcal{Q}}$, $\eta_z(\omega) = V(z;\omega) - \xi(\omega)$, $z \in \mathcal{Q}$, and let $\mathfrak{F}_{\mathcal{Q}}$ be the sigma-algebra generated by $\{\eta_z, z \in \mathcal{Q}; V(u;\cdot), u \notin \mathcal{Q}\}$. Introduce the continuity modulus $\mathfrak{s}_{\mathcal{Q}}(\cdot|\mathfrak{F}_{\mathcal{Q}})$ of the conditional probability distribution function $F_\xi(t|\mathfrak{F}_{\mathcal{Q}}) = \mathbb{P}\{\xi \le t|\mathfrak{F}_{\mathcal{Q}}\}$; it satisfies the condition (RCM) with $\theta' = \theta'' = 2/3$, $C' = 1$, $A' = 0$, $A'' = 2$. The representation $V(z;\omega) = \xi(\omega) + \eta_z(\omega)$ for $z \in \mathcal{Q}$ implies $\mathbf{H}_{\mathbf{B}(\mathbf{x},L)} = n_1\xi(\omega) + \mathbf{A}_1(\omega)$, $\mathbf{H}_{\mathbf{B}(\mathbf{y},L)} = n_2\xi(\omega) + \mathbf{A}_2(\omega)$, where $\mathbf{A}_1$, $\mathbf{A}_2$ are $\mathfrak{F}_{\mathcal{Q}}$-measurable.

For any $\omega \notin \mathcal{S}_b$, the energies E where $\mathbf{F}_{\mathbf{x}}(E) \geq a$ are covered by a union of intervals $J_{\mathbf{x},i}$ with $|J_{\mathbf{x},i}| =: \epsilon_{\mathbf{x},i} \leq \sum_j \epsilon_{\mathbf{x},j} \leq 2b$. By assertion (B) of Lemma 4.7, we have

$$J_{\mathbf{x},i}(\omega) = [E^-_{\mathbf{x},i} + n_1\xi(\omega), E^+_{\mathbf{x},i} + n_1\xi(\omega)],$$

where $E^{\pm}_{\mathbf{x},i}$ are $\mathfrak{F}_{\mathcal{Q}}$-measurable.

Similarly, introduce the intervals $J_{\mathbf{y},j}$ with $|J_{\mathbf{y},j}| =: \epsilon_{\mathbf{y},j}$, $\sum_j \epsilon_{\mathbf{y},j} \leq 2b$, and

$$J_{\mathbf{y},j}(\omega) = [E^-_{\mathbf{y},j} + n_2\xi(\omega), E^+_{\mathbf{y},j} + n_2\xi(\omega)], \quad n_2 < n_1.$$

It is readily seen that

$$\begin{aligned}\{\omega\colon J_{\mathbf{x},i} \cap J_{\mathbf{y},j} \neq \varnothing\} \cap \mathcal{S}_b^{\complement} &\subset \{\omega\colon |E_{\mathbf{x},i} - E_{\mathbf{y},j}| \leq \epsilon_{\mathbf{x},i} + \epsilon_{\mathbf{y},j}\} \cap \mathcal{S}_b^{\complement} \\ &\subset \{\omega\colon |(n_1 - n_2)\xi - \mu_{i,j}(\omega)| \leq 4b\},\end{aligned}$$

with some $\mathfrak{F}_{\mathcal{Q}}$-measurable $\mu_{i,j}$. Let $s = 4b$ and recall that $n_1 - n_2 \geq 1$. For any given pair of indices (i, j),

$$\mathbb{P}\{|(n_1 - n_2)\xi - \mu_{i,j}| \leq s\} \leq \mathbb{E}[\mathbb{P}\{|(n_1 - n_2)\xi - \mu_{i,j}| \leq s \,|\, \mathfrak{F}_{\mathcal{Q}}\}] \tag{4.23}$$

For any ω such that $\mathfrak{s}_Q(s\,|\mathfrak{F}_{\mathcal{Q}}) \leq C'(\sharp Q)^{A'} s^{\vartheta'} \equiv s^{2/3}$, the internal conditional probability in the RHS of (4.23) is obviously bounded by the latter value, $s^{2/3}$, for each pair (i, j), while the probability of the complementary event (which is *the same* for all pairs (i, j)) is bounded by $C''(\sharp Q)^{2'} s^{2/3}$. Taking the sum over all pairs (i, j) and using (1.1) to bound the cardinality of the set $\mathcal{Q}$ of diameter $\leq 2NL$, we obtain, with $s = 4b$,

$$\begin{aligned}\mathbb{P}\{\mathcal{S} \cap \mathcal{S}_b^{\complement}\} &\leq s^{2/3} + \sharp\mathbf{B}(\mathbf{x}, L)\, \sharp\mathbf{B}(\mathbf{y}, L) C''(\sharp\mathcal{Q})^2 s^{2/3} \\ &\leq C''' L^{4Nd} b^{2/3},\end{aligned}$$

yielding the asserted bound. □

Setting $L = L_k$, $k \geq 0$, and

$$a_{L_k} = \mathrm{e}^{-m_N L_k}, \quad q_{L_k} = L_k^{-P(N)}, \quad b = b_{L_k} = L_k^{-P(N)/2}, \tag{4.24}$$

we come to the following result, marking the end of the proof of our main theorem. The assertion on the p.p. spectrum becomes trivial for finite graphs $\mathcal{Z}$, but the bound (4.25) is useful in this case, too.

Corollary 4.8. *For $k \geq 0$ and any pair of $4NL_k$-distant balls $\mathbf{B}_{L_k}(\mathbf{x})$, $\mathbf{B}_{L_k}(\mathbf{y})$ the following bound holds true*:

$$\begin{aligned}\mathbb{P}\{\text{there exists } E \in \mathbb{R}\colon \mathbf{B}_{L_k}(\mathbf{x}) \text{ and } \mathbf{B}_{L_k}(\mathbf{y}) \text{ are } (E, m_N)\text{-S}\} &\leq CL_k^{-\frac{1}{6}P(N)} \\ &\leq CL_k^{-4Nd}.\end{aligned} \tag{4.25}$$

Consequently, for $|g|$ large enough, with probability one, the operator $\mathbf{H}^{(N)}(\omega)$ has pure point spectrum, and all its eigenfunctions obey (1.10).

Proof. The first assertion follows from Lemma 4.6, with $a_{L_k}, q_{L_k}, b = b_{L_k}$ given in (4.24):

$$\begin{aligned}\mathbb{P}\{\mathcal{S}\} &\leq \frac{2|I_g^*|q_L}{b_{L_k}} + C'''L^{4Nd} b_{L_k}^{2/3} \\ &\leq C_1 L_k^{-P(N)+\frac{1}{2}P(N)} + C_2 L_k^{4Nd} L_k^{-\frac{2}{3}\cdot\frac{1}{2}P(N)} \\ &\leq C_3 L_k^{-\frac{1}{3}P(N)+4Nd} \\ &\leq C_3 L_k^{-\frac{1}{6}P(N)} \\ &\leq C_3 L_k^{-4Nd};\end{aligned}$$

in the last line we used that $P(N) \geq P* \geq 24Nd$ (cf. (4.1)).

The second assertion is a well-known result going back to [20]. In fact, the proof of Lemma 3.1 from [20] can be adapted to the pairs of balls $\mathbf{B}_{L_k}(\mathbf{x}), \mathbf{B}_{L_k}(\mathbf{y}) \subset \mathcal{Z}^N$ at distance $\geq CL_k$, with $C \in (0, +\infty)$. The key point is that structure of the random potential (single- or multi-particle) becomes irrelevant to the proof of [20, Lemma 3.1], once the "double singularity" bound of the form (4.25) is proved, with $P(N)$ large enough. ☐

Appendices

A. Proof of Lemma 3.3

Step 1. Approximate decoupling. Consider a WI ball $\mathbf{B} = \mathbf{B}(\mathbf{u}, L_k)$ with the canonical factorization $\mathbf{B} = \mathbf{B}' \times \mathbf{B}''$, where $\mathbf{B}' = \mathbf{B}^{(N')}(\mathbf{u}', L_k)$ and $\mathbf{B}'' = \mathbf{B}^{(N'')}(\mathbf{u}'', L_k)$, with $\mathbf{u} = (\mathbf{u}', \mathbf{u}'')$, $\mathbf{u}' = \mathbf{u}_{\mathcal{J}}$, $\mathbf{u}'' = \mathbf{u}_{\mathcal{J}^{\complement}}$, $\mathcal{J} \subset \{1, \ldots, N\}$, cf. (2.20)–(2.21). We have the representation (2.22)

$$\mathbf{H}_{\mathbf{B}}^{(N)} = \mathbf{H}_{\mathbf{B}}^{\mathrm{ni}} + \mathbf{U}_{\mathbf{B}',\mathbf{B}''}, \tag{A.1a}$$

$$\mathbf{H}_{\mathbf{B}}^{\mathrm{ni}} = \mathbf{H}_{\mathbf{B}'}^{(N')} \otimes \mathbf{I}^{(N'')} + \mathbf{I}^{(N')} \otimes \mathbf{H}_{\mathbf{B}''}^{(N'')}. \tag{A.1b}$$

The interaction energy $\mathbf{U}_{\mathbf{B}',\mathbf{B}''}$ has the following form:

$$\mathbf{U}_{\mathbf{B}',\mathbf{B}''}(\mathbf{x}) = \sum_{i\in\mathcal{J},\, j\in\mathcal{J}^{\complement}} U(\mathrm{d}(x_i, x_j)). \tag{A.2}$$

By Lemma 2.5, for any configuration $\mathbf{x} \in \mathbf{B}$ the graph distance between the projected sub-configurations (in $\mathcal{Z}$) satisfies $\mathrm{d}(\Pi_{\mathcal{J}}\mathbf{x}, \Pi_{\mathcal{J}^{\complement}}\mathbf{x}) > L_k$, thus by assumption (U), the norm of the multiplication operator $\mathbf{U}_{\mathbf{B}',\mathbf{B}''}$ obeys

$$\|\mathbf{U}_{\mathbf{B}',\mathbf{B}''}\| \le C\, N'\, N''\, \mathrm{e}^{-L_k^{\zeta}} \le CN^2\mathrm{e}^{-L_k^{\zeta}}, \tag{A.3}$$

with $C = C_U$ as in (1.8).

The eigenvalues of $\mathbf{H}^{\mathrm{ni}}_{\mathbf{B}}$ are the sums $E_{a,b} = \lambda_a + \mu_b$, where λ_a form the spectrum $\Sigma' := \Sigma(\mathbf{H}^{(N')}_{\mathbf{B}'})$ and μ_b the spectrum $\Sigma'' := \Sigma(\mathbf{H}^{(N'')}_{\mathbf{B}''})$. The EFs of $\mathbf{H}^{\mathrm{ni}}_{\mathbf{B}}$ can be chosen in the form $\boldsymbol{\phi}_a \otimes \boldsymbol{\psi}_b$ where $\{\boldsymbol{\phi}_a\}$ are EFs of $\mathbf{H}^{(N')}_{\mathbf{B}'}$ and $\{\boldsymbol{\psi}_b\}$ of $\mathbf{H}^{(N'')}_{\mathbf{B}''}$.

Step 2. Nonresonance properties. By the min-max principle, the assumed (E,β)-NR property of $\mathbf{B}$ (with regard to $\mathbf{G}^{(N)}_{\mathbf{B}}(E) = (\mathbf{H}^{(N)}_{\mathbf{B}} - E\mathbf{I})^{-1}$) implies a slightly weaker property for $\mathbf{G}^{\mathrm{ni}}(E) = (\mathbf{H}^{\mathrm{ni}}_{\mathbf{B}} - E\mathbf{I})^{-1}$:

$$\begin{aligned} \mathrm{dist}(\Sigma(\mathbf{H}^{\mathrm{ni}}_{\mathbf{B}}), E) &\ge \mathrm{dist}(\Sigma(\mathbf{H}^{(N)}_{\mathbf{B}}), E) - \|\mathbf{U}_{\mathbf{B}',\mathbf{B}''}\| \\ &\ge 2\mathrm{e}^{-L_k^{\beta}} - C\mathrm{e}^{-L_k^{\zeta}} \\ &\ge \mathrm{e}^{-L_k^{\beta}}, \end{aligned} \tag{A.4}$$

provided that $\beta < \zeta$ (which is one of conditions (3.2)) and L_0 is large enough. In terms of the resolvents $\mathbf{G}^{(N)}_{\mathbf{B}}(E)$ and $\mathbf{G}^{\mathrm{ni}}_{\mathbf{B}}(E)$ we then have

$$\|\mathbf{G}^{(N)}_{\mathbf{B}}(E)\| \le \frac{1}{2}\mathrm{e}^{L_k^{\beta}} < \mathrm{e}^{L_k^{\beta}}, \quad \|\mathbf{G}^{\mathrm{ni}}_{\mathbf{B}}(E)\| \le \mathrm{e}^{L_k^{\beta}}. \tag{A.5}$$

Step 3. Analytic perturbation estimates. We begin with the following identities for the GF $\mathbf{G}^{\mathrm{ni}}_{\mathbf{B}}(\mathbf{u},\mathbf{y};E)$:

$$\mathbf{G}^{\mathrm{ni}}_{\mathbf{B}}(\mathbf{u},\mathbf{y};E) = \sum_{\lambda_a\in\Sigma'} \boldsymbol{\phi}_a(\mathbf{u}')\boldsymbol{\phi}_a(\mathbf{y}')\, \mathbf{G}^{(N'')}_{\mathbf{B}''}(\mathbf{u}'',\mathbf{y}''; E-\lambda_a) \tag{A.6}$$

$$= \sum_{\mu_b\in\Sigma''} \boldsymbol{\psi}_b(\mathbf{u}'')\boldsymbol{\psi}_b(\mathbf{y}'')\, \mathbf{G}^{(N')}_{\mathbf{B}'}(\mathbf{u}',\mathbf{y}'; E-\mu_b). \tag{A.7}$$

By assumptions of the lemma,

- for all $\mu_b \in \Sigma''$, the ball $\mathbf{B}'$ is $(E - \mu_b, \delta, m_{N'})$-NS, (A.8a)
- for all $\lambda_a \in \Sigma'$, the ball $\mathbf{B}''$ is $(E - \lambda_a, \delta, m_{N''})$-NS. (A.8b)

For any $\mathbf{y} \in \partial^- \mathbf{B}$, either $\boldsymbol{\rho}^{(N')}(\mathbf{u}', \mathbf{y}') = L_k$ or $\boldsymbol{\rho}^{(N'')}(\mathbf{u}'', \mathbf{y}'') = L_k$. In the first case we infer from (A.7), combined with $(E - \mu_b, \delta, m_{N'})$-NS property of ball $\mathbf{B}'$, that

$$|\mathbf{G}^{\mathrm{ni}}_{\mathbf{B}}(\mathbf{u}, \mathbf{y}; E)| \le \sharp\, \mathbf{B}''\, \mathrm{e}^{-m_{N'} L_k^{\delta} + 2L_k^{\beta}}. \tag{A.9}$$

Similarly, in the second case we obtain with the help of (A.6) that

$$|\mathbf{G}^{\mathrm{ni}}_{\mathbf{B}}(\mathbf{u}, \mathbf{y}; E)| \le \sharp\, \mathbf{B}'\, \mathrm{e}^{-m_{N''} L_k^{\delta} + 2L_k^{\beta}}. \tag{A.10}$$

By (3.1), $m_{N'}, m_{N''} \ge m_{N-1} = m_N(1 + 3L_0^{-\delta+\beta})$; in either case, the LHS is bounded by

$$C_{\mathbb{Z}}^{N} L_k^{Nd}\, \mathrm{e}^{-m_{N-1} L_k^{\delta} + 2L_k^{\beta}} \le \mathrm{e}^{-m_N L_k^{\delta} - L_k^{\beta}} \le \tfrac{1}{2} \mathrm{e}^{-m_N L_k^{\delta}}, \tag{A.11}$$

provided that L_0 is large enough.

To assess $\mathbf{G}^{(N)}_{\mathbf{B}}(\mathbf{u}, \mathbf{y}; E)$, we use the second resolvent equation and write

$$\begin{aligned}
\|\mathbf{G}^{(N)}_{\mathbf{B}}(E) - \mathbf{G}^{\mathrm{ni}}_{\mathbf{B}}(E)\| &\le \|\mathbf{G}^{\mathrm{ni}}_{\mathbf{B}}(E)\|\, \|\mathbf{U}_{\mathbf{B}',\mathbf{B}''}\|\, \|\mathbf{G}^{(N)}_{\mathbf{B}}(E)\| \\
&\le C \mathrm{e}^{2L_k^{\beta} - L_k^{\zeta}} \\
&\le \mathrm{e}^{-\frac{1}{2} L_k^{\zeta}} \\
&\le \frac{1}{2} \mathrm{e}^{-m_N L_k^{\delta}},
\end{aligned} \tag{A.12}$$

provided that $\beta < \delta < \zeta$ and L_0 large enough.

Collecting (A.6), (A.7), (A.11), and (A.12), we get

$$\max_{\mathbf{y} \in \partial^- \mathbf{B}} |\mathbf{G}^{(N)}_{\mathbf{B}}(\mathbf{u}, \mathbf{y}; E)| \le \frac{1}{2} \mathrm{e}^{-m_N L_k^{\delta}} + \frac{1}{2} \mathrm{e}^{-m_N L_k^{\delta}} = \mathrm{e}^{-m_N L_k^{\delta}}. \tag{A.13}$$

Thus the ball $\mathbf{B}$ is (E, δ, m_N)-NS.

B. Proof of Lemma 4.4

Step 1. Approximate decoupling. We start as in the proof of Lemma 3.3, but have to achieve an exponential bound upon the GFs. The definitions of the operators $\mathbf{H}^{\mathrm{ni}}_{\mathbf{B}}$

and $\mathbf{U}_{\mathbf{B}',\mathbf{B}''}$ (see (A.1) (A.2)) remain in force. The bound on the interaction (A.3) is to be modified: since $\boldsymbol{\rho}\,(\Pi_{\mathcal{J}}\mathbf{B}, \Pi_{\mathcal{J}^c}\mathbf{B}) \geq L_k^{\tau}$ and $\tau\zeta > 1$ (cf. (4.1)), we have

$$\|\mathbf{U}_{\mathbf{B}',\mathbf{B}''}\| \leq C_U N^2 \mathrm{e}^{-L_k^{\tau\zeta}} \leq \mathrm{e}^{-\tilde{m}L_k} \tag{B.1}$$

where $\tilde{m} > 0$ can be chosen arbitrarily large, provided L_0 is large enough. Specifically, we require that $\tilde{m} \geq 2m_1$, hence $\tilde{m} \geq 2m_N$ for $1 \leq N \leq N^*$, cf. (4.1).

Step 2. Nonresonance properties. A direct analog of (A.4) is

$$\mathrm{dist}[\Sigma(\mathbf{H}^{\mathrm{ni}}_{\mathbf{B}}), E] \geq 2\mathrm{e}^{-L_k^{\beta}} - \mathrm{e}^{-\tilde{m}L_k} \geq \mathrm{e}^{-L_k^{\beta}}; \tag{B.2}$$

it implies, as before, that

$$\|\mathbf{G}^{(N)}_{\mathbf{B}}(E)\| \leq \frac{1}{2}\mathrm{e}^{L_k^{\beta}} < \mathrm{e}^{L_k^{\beta}}, \quad \|\mathbf{G}^{\mathrm{ni}}_{\mathbf{B}}(E)\| \leq \mathrm{e}^{L_k^{\beta}}. \tag{B.3}$$

Step 3. Analytic perturbation estimates. We can use again the general identities (A.6)–(A.7) and the assumed properties (A.8) (this time, with $\delta = 1$). The estimates (A.9)–(A.9) are to be modified as follows.

Given $\mathbf{y} \in \partial^{-}\mathbf{B}$, we have two possibilities.

(i) $\boldsymbol{\rho}^{(N')}(\mathbf{u}', \mathbf{y}') = L_k$, in which case we deduce from (A.7), combined with $(E - \mu_b, m_{N'})$-NS property of the ball $\mathbf{B}'$ (for all EVs μ_b), that

$$|\mathbf{G}^{\mathrm{ni}}(\mathbf{u}, \mathbf{y}; E)| \leq \sharp\,\mathbf{B}''\, \mathrm{e}^{-m_{N'}L_k + 2L_k^{\beta}}. \tag{B.4}$$

(ii) $\boldsymbol{\rho}^{(N'')}(\mathbf{u}'', \mathbf{y}'') = L_k$. Then we obtain a similar bound, using the assumed exponential bounds on the GF in the ball $\mathbf{B}''$:

$$|\mathbf{G}^{\mathrm{ni}}_{\mathbf{B}}(\mathbf{u}, \mathbf{y}; E)| \leq \sharp\,\mathbf{B}'\, \mathrm{e}^{-m_{N''}L_k + 2L_k^{\beta}}. \tag{B.5}$$

In either case, $m_{N'}, m_{N''} \geq m_{N-1} = m_N(1 + 3L_0^{-1+\beta})$, so that we have

$$m_{N-1}L_k = m_N(1 + 3L_0^{-1+\beta})L_k \geq m_N L_k + 3L_k^{\beta},$$

thus for L_0 large enough, we obtain from (A.6)–(A.7)

$$\begin{aligned} |\mathbf{G}^{\mathrm{ni}}_{\mathbf{B}}(\mathbf{u}, \mathbf{y}; E)| &\leq C_{\mathcal{Z}}^{N} L_k^{Nd} \mathrm{e}^{-m_{N-1}L_k + 2L_k^{\beta}} \\ &\leq \mathrm{e}^{-m_N L_k - 3L_k^{\beta} + 2L_k^{\beta}} \\ &\leq \mathrm{e}^{-m_N L_k - L_k^{\beta}}. \end{aligned} \tag{B.6}$$

Next, by virtue of the second resolvent identity and (B.1), we have

$$\begin{aligned}\|\mathbf{G}^{(N)}_{\mathbf{B}}(E) - \mathbf{G}^{\rm ni}_{\mathbf{B}}(E)\| &\le \|\mathbf{G}^{\rm ni}_{\mathbf{B}}(E)\| \, \|\mathbf{U}_{\mathbf{B}',\mathbf{B}''}\| \, \|\mathbf{G}^{(N)}_{\mathbf{B}}(E)\| \\ &\le \mathrm{e}^{-\frac{3}{2} m_N L_k},\end{aligned} \tag{B.7}$$

since $\tilde{m} \ge 2m_N$, $L_0 > 1$, $\beta < 1/2$. Collecting the bounds (B.6)–(B.7), we obtain

$$\begin{aligned}\max_{\mathbf{y}\in\partial^-\mathbf{B}} |\mathbf{G}^{(N)}_{\mathbf{B}}(\mathbf{u},\mathbf{y};E)| &\le \mathrm{e}^{-m_N L_k - L_k^\beta} + \mathrm{e}^{-\frac{3}{2} m_N L_k} \\ &\le (C_{\mathcal{Z},N} L_k^{Nd})^{-1} \mathrm{e}^{-m_N L_k},\end{aligned}$$

for L_0 large enough. Therefore, the ball $\mathbf{B}$ is (E, m_N)-NS.

C. Dominated decay of functions on $\mathcal{Z}$

In this section we establish Lemmas C.1 and C.2 applicable to finite connected subgraphs of arbitrary locally finite, connected graphs, including $\mathcal{Z} = \mathcal{Z}^N$, $N \ge 2$. These lemmas are used in the proofs of Lemmas 3.1 and 4.3. The argument here is nothing more than a variant of the one used in the proof of [20, Lemma 4.2] and in a number of subsequent papers; in the case where $\mathcal{Z} = \mathbb{Z}^d$, it was presented in [19, Sect. 2.6].

Definition C.1 ([19, Definition 2.6.1]). Let be given a finite connected subgraph $\mathbf{\Lambda} \subset \mathcal{Z}$, a non-negative function $f : \mathbf{\Lambda} \to [0, \infty)$, a number $q \in (0, 1)$, two integers $L \ge \ell \ge 1$, and a ball $\mathbf{B}(\mathbf{u}, L) \subsetneq \mathbf{\Lambda}$.

(1) A point $\mathbf{x} \in \mathbf{B}(\mathbf{u}, L - \ell)$ is called (ℓ, q)-*regular* for the function f, if

$$f(\mathbf{x}) \le q\, \mathrm{M}(f, \mathbf{B}(\mathbf{x}, \ell + 1)). \tag{C.1}$$

Here and below, we set $\mathrm{M}(f, \mathcal{W}) := \sup\,[f(\mathbf{y}) : \mathbf{y} \in \mathcal{W}]$, $\quad \mathcal{W} \subseteq \mathbf{\Lambda}$. The set of all (ℓ, q)-regular points $\mathbf{x} \in \mathbf{B}(\mathbf{u}, L)$ for f is denoted by $\mathcal{R}_f(\mathbf{u})$.

(2) A spherical layer $\mathcal{L}_r(\mathbf{u}) = \{\mathbf{y} : \mathrm{d}(\mathbf{u}, \mathbf{y}) = r\}$ is called *regular* if $\mathcal{L}_r(\mathbf{u}) \subset \mathcal{R}_f(\mathbf{u})$.

(3) For $\mathbf{x} \in \mathbf{B}(\mathbf{u}, L - \ell)$, set

$$\bar{r}(\mathbf{x}) := \begin{cases} \min[r \ge \mathrm{d}(\mathbf{u}, \mathbf{x}) : \mathcal{L}_r(\mathbf{u}) \subset \mathcal{R}_f(\mathbf{u})], \\ \qquad\qquad \text{if a regular layer } \mathcal{L}_r(\mathbf{u}) \text{ exists, with } r \ge \mathrm{d}(\mathbf{u}, \mathbf{x}), \\ +\infty, \quad \text{if no such layer } \mathcal{L}_r(\mathbf{u}) \text{ exists,} \end{cases} \tag{C.2}$$

and

$$\mathrm{R}_f(\mathbf{x}) = \begin{cases} \bar{r}(\mathbf{x}) + \ell, & \bar{\mathbf{x}} < +\infty, \\ +\infty, & \text{otherwise}. \end{cases} \tag{C.3}$$

(4) Given a set $\mathcal{S} \subseteq \mathbf{\Lambda}$, the function f is called $(\ell, q, \mathcal{S})$-*dominated* in $\mathbf{B}(L, \mathbf{u})$ if $\mathbf{B}(L, \mathbf{u}) \setminus \mathcal{S} \subset \mathcal{R}_f(\mathbf{u})$, and for any $\mathbf{x} \in \mathbf{B}(\mathbf{u}, L - \ell)$ with $\mathrm{R}_f(\mathbf{x}) < +\infty$, one has

$$f(\mathbf{x}) \leq q\mathrm{M}(f, \mathbf{B}(\mathbf{u}, \mathrm{R}_f(\mathbf{x}))). \tag{C.4}$$

Lemma C.1 ([19, Theorem 2.6.1]). *Let a function $f: \mathbf{\Lambda} \to \mathbb{R}_+$ be $(\ell, q, \mathcal{S})$-dominated in an N-particle ball $\mathbf{B}(\mathbf{u}, L)$, where $L \geq \ell \geq 0$. Assume that the set $\mathcal{S}$ is covered by a union $\mathfrak{U}$ of concentric annuli $\mathbf{B}(\mathbf{u}, b_j) \setminus \mathbf{B}(\mathbf{u}, a_j - 1)$, with total width $\mathrm{w}(\mathfrak{U}) := \sum_j (b_j - a_j + 1) \leq L - \ell$. Then*

$$f(\mathbf{u}) \leq q^{\frac{L-\ell-\mathrm{w}(\mathfrak{U})}{\ell+1}} \mathrm{M}(f, \mathbf{B}(\mathbf{u}, L+1)). \tag{C.5}$$

The proof of Lemma C.1 repeats almost *verbatim* that of Theorem 2.6.1 in [19], and we omit it from the paper. The following result is a minor modification of [19, Theorem 2.6.2], adapted to the sub-exponential decay bounds. It explains the relevance of Lemma C.1 in the context of the MSA.

Lemma C.2. *Fix $0 < \beta, \delta \leq 1$, $m > 0$ and $E \in \mathbb{R}$. Suppose that, for some integer $L \geq 1$ and $\mathbf{u} \in \mathbb{Z}^N$, the N-particle ball $\mathbf{B}(\mathbf{u}, L)$ is (E, β)-*CNR*. Take a finite connected subgraph $\mathbf{\Lambda} \subset \mathbb{Z}^N$ such that $\mathbf{\Lambda} \supset \mathbf{B}(\mathbf{u}, L)$ and $\mathbf{y} \in \mathbf{\Lambda} \setminus \mathbf{B}(\mathbf{u}, L)$, and consider the function $f = f_{\mathbf{y}}: \mathbf{x} \in \mathbf{\Lambda} \to \mathbb{R}_+$ given by*

$$f: \mathbf{x} \longmapsto |\mathbf{G}_{\mathbf{\Lambda}}^{(N)}(\mathbf{x}, \mathbf{y}; E)|. \tag{C.6}$$

*Given $\ell = 0, \ldots, L - 1$, let $\mathcal{S} = \mathcal{S}(E) \subset \mathbf{B}(\mathbf{u}, L - \ell - 1)$ be a (possibly empty) subset such that any ball $\mathbf{B}(\mathbf{x}, \ell)$ with $\mathbf{x} \in \mathbf{B}(\mathbf{u}, L - \ell - 1) \setminus \mathcal{S}$ is (E, δ, m)-*NS*. If*

$$m\ell^\delta > 2L^\beta > L^\beta + \ln(C_{\mathbb{Z}} L^{Nd}), \tag{C.7}$$

then for all $\mathbf{y} \in \partial^- \mathbf{B}(\mathbf{u}, L)$, the function f is $(\ell, q, \mathcal{S})$-dominated in $\mathbf{B}(\mathbf{u}, L)$, with

$$q = \mathrm{e}^{-m'\ell^\delta}, \quad \text{where } m' := m - 2\ell^{-\delta} L^\beta. \tag{C.8}$$

Proof. First, note that for any $\mathbf{x} \in \mathbf{B}(\mathbf{u}, L - \ell) \setminus \mathcal{S}$ we have

$$f(\mathbf{x}) \leq \mathrm{e}^{-m\ell^\delta} \mathrm{M}(f, \mathbf{B}(\mathbf{x}, \ell)) \leq q\mathrm{M}(f, \mathbf{B}(\mathbf{x}, \ell)),$$

since ball $\mathbf{B}(\mathbf{x}, \ell)$ must be (E, δ, m)-NS, by definition of the set $\mathcal{S}$.

Further, define the functions $\mathbf{x} \mapsto \bar{r}(\mathbf{x})$, $\mathbf{x} \mapsto \mathrm{R}_f(\mathbf{x})$ in the same way as in (C.2)–(C.3). Suppose that $\mathbf{x} \in \mathcal{S}$ and $\mathrm{R}_f(\mathbf{x}) < \infty$, i.e., each point $\mathbf{y} \in \mathcal{L}_{\bar{r}(\mathbf{x})}(\mathbf{u})$ is regular. Applying the GRI (2.1) to the ball $\mathbf{B}(\mathbf{u}, \bar{r}(\mathbf{x}) - 1)$, we get, by the assumed (E, β)-CNRproperty of the ball $\mathbf{B}(\mathbf{u}, L)$,

$$
\begin{aligned}
f(\mathbf{x}) &\le C_{\mathcal{Z}} (\bar{r})^{Nd} \| \mathbf{G}_{\mathbf{B}(\mathbf{u}, \bar{r}(\mathbf{x})-1)}(E) \| \max_{\mathbf{z} \in \mathcal{L}_{\bar{r}}(\mathbf{u})} |\mathbf{G}_{\mathbf{\Lambda}}(\mathbf{z}, \mathbf{y}; E)| \\
&\le C_{\mathcal{Z}} L^{Nd} \mathrm{e}^{L^\beta} \mathrm{M}(f, \mathcal{L}_{\bar{r}}(\mathbf{u})).
\end{aligned}
$$

Next, applying the GRI to each ball $\mathbf{B}(\mathbf{z}, \ell)$ with $\mathbf{z} \in \mathcal{L}_{\bar{r}}(\mathbf{u})$, we obtain

$$
f(\mathbf{x}) \le C_{\mathcal{Z}} L^{Nd} \mathrm{e}^{L^\beta} \mathrm{e}^{-m\ell^\delta} \mathrm{M}(f, \mathbf{B}(\mathbf{u}, \bar{r} + \ell)) \le \mathrm{e}^{-m'\ell^\delta} \mathrm{M}(f, \mathbf{B}(\mathbf{u}, \mathrm{R}_f(\mathbf{x}))),
$$

with m' given by (C.8), provided that the condition (C.7) is fulfilled. Thus f is indeed $(\ell, q, \mathcal{S})$-dominated in $\mathbf{B}(\mathbf{u}, L)$, with q given by (C.8). □

Remark C.1. Lemma C.2 is used in the proof of Lemmas 3.1 and 4.3. In the scaling procedure with $L_{k+1} = Y L_k$, the condition upon the key exponents in (C.7) becomes (with $\ell = L_k$, $L = Y L_k$) $\beta < \delta$, which is assumed in (3.2). In the case where (as in Section 4) $L_{k+1} \sim L_k^\alpha$ and $\delta = 1$, it is required that $\beta < \delta/\alpha = 1/\alpha$, which follows from the assumption $\beta < 7/(8\alpha)$ specified in the table (4.1).

D. Proof of Lemma 2.1

1. First, we establish the property (RCM) for the uniform marginal distribution $\mathrm{Unif}([0, \ell])$, $\ell > 0$, which is the prototypical example. It will be extended to the case of smooth positive density by simple approximation arguments.

Let be given an integer $n > 1$ and IID random variables (r.v.) $X_1, \ldots, X_n$ uniformly distributed in $[0, \ell]$. For brevity, below we use notation $\mathbf{X}$ for the real vector $(X_1, \ldots, X_n)$, uniformly distributed in the cube $\mathbf{C}_\ell = \mathbf{C}_\ell^{(n)} = [0, \ell]^n$. Recall: n corresponds in (RCM) to the cardinality $|\mathcal{Q}|$ of a finite set $\mathcal{Q}$.

Next, make an orthogonal change of variables in $\mathbb{R}^n$, taking as the first coordinate $\tilde{\xi} := \sqrt{n}\, \xi$ (the factor $\sqrt{n}$ is required for the Euclidean normalization) and choosing in some way (irrelevant for further considerations) complementary coordinates $Y_1, \ldots, Y_{n-1}$; denote $\mathbf{Y} = (Y_1, \ldots, Y_{n-1})$. When $\mathbf{X}$ varies in the cube $\mathbf{C}_\ell$, the range $\mathcal{Y}$ of $\mathbf{Y}$ is a polyhedron (obtained by projecting $\mathbf{C}_\ell$ onto the subspace orthogonal to $(1, \ldots, 1)$). The Euclidean space $\mathbb{R}^n$ is stratified into affine lines $\mathcal{L}(\mathbf{y})$ parallel to $(1, \ldots, 1)$ labeled by their projections $\mathbf{y}$ onto the latter hyperplane.

We set $J_{\mathbf{y}} := \mathcal{L}(\mathbf{y}) \cap \mathbf{C}_\ell$. The uniform probability distribution on $\mathbf{C}_\ell$ induces marginal distributions for the r.v. ξ (resp., $\tilde{\xi}$) and $\mathbf{Y}$. The distribution of $\tilde{\xi}$ conditional on $\mathbf{Y} = \mathbf{y}$ is a uniform distribution on the interval $J_{\mathbf{y}}$, of length $\mathfrak{l}(\mathbf{y}) := |J_{\mathbf{y}}|$, with constant density $|J_{\mathbf{y}}|^{-1}$ – except, of course, for a finite number of projection points $\mathbf{y}$ where $|J_{\mathbf{y}}| = 0$; these points can be safely ignored.

Due to the Euclidean normalization $\tilde{\xi} = \sqrt{n}\,\xi$ varies in some interval of length $s\sqrt{n}$ when ξ varies in an interval of length s. For the original form of (RCM) (cf. (2.9)), we have to assess the conditional measure of arbitrary sub-intervals $\mathcal{X}(\mathbf{y}) \subset J_{\mathbf{y}}$ of length $s\sqrt{n}$; for the modified form suitable for the proof of Lemma 2.4, we fix first a measurable family of sub-intervals $\mathcal{X}(\mathbf{y}) = \mathcal{X}_s(\mathbf{y}) = [\zeta(\mathbf{y}), \zeta(\mathbf{y}) + s\sqrt{n}]$ with $\mathfrak{F}_\eta$-measurable ζ; they play the role of $I(s;\omega)$) figuring in (2.10). The rest of the argument is essentially analytic and applies to both settings; for definiteness, we will assess the probability of a set $\mathcal{A}_s = \{\xi(\omega) \in I(s;\omega)\} \equiv \{\xi(\mathbf{x}) \in \mathcal{X}(\mathbf{y})\}$, with $|\mathcal{X}(\mathbf{y})| = s\sqrt{n}$, $s > 0$.

Using the Fubini theorem for integration in variables $(\tilde{\xi}, \mathbf{Y})$, we obtain

$$\mathbb{P}\{\mathcal{A}_s\} = \int_{\mathcal{Y}} d\mathbf{y}\, p_{\mathbf{Y}}(\mathbf{y})\, \mathfrak{l}^{-1}(\mathbf{y})\, |\mathcal{X}(\mathbf{y})| = \mathbb{E}^{\mathcal{Y}}[\mathfrak{l}^{-1}(\mathbf{y})\, |\mathcal{X}(\mathbf{y})|]. \tag{D.1}$$

Further, since $|\mathcal{X}(\mathbf{y})| \le \mathfrak{l}(\mathbf{y})$, we have for any $\delta > 0$

$$\frac{|\mathcal{X}(\mathbf{y})|}{\mathfrak{l}(\mathbf{y})} \le \frac{|\mathcal{X}(\mathbf{y})|}{\mathfrak{l}(\mathbf{y})} \mathbf{1}_{\mathfrak{l} \ge \delta\sqrt{n}} + \mathbf{1}_{\mathfrak{l} < \delta\sqrt{n}} \le \frac{s\sqrt{n}}{\delta\sqrt{n}} \mathbf{1}_{\mathfrak{l} \ge \delta\sqrt{n}} + \mathbf{1}_{\mathfrak{l} < \delta\sqrt{n}},$$

hence, denoting $\mathcal{S}_\delta := \{\mathfrak{l} < \delta\sqrt{n}\}$,

$$\mathbb{P}\{\mathcal{A}_s\} \le s\delta^{-1} + \mathbb{E}[\mathbf{1}_{\mathfrak{l} < \delta\sqrt{n}}] = s\delta^{-1} + \mathbb{P}\{\mathcal{S}_\delta\}. \tag{D.2}$$

It remains to assess the probability $\mathbb{P}\{\mathcal{S}_\delta\}$ (*with a specific choice $\delta = s^{1/3}$ made below, this is the analog of the exceptional set $\widetilde{\mathcal{S}}_s$ figuring in* (2.10)).

Let $\underline{X} = \min\{X_1, \dots, X_n\}$, $\overline{X} = \max\{X_1, \dots, X_n\}$. It is straightforward that $|\mathcal{X}(\mathbf{Y})| = \sqrt{n}(\underline{X} + (1 - \overline{X}))$, thus $\mathcal{S}_\delta$ is equivalently determined by the condition $\underline{X} + (\ell - \overline{X}) < \delta\sqrt{n}/\sqrt{n} = \delta$, and we have

$$\mathcal{S}_\delta \subset \bigcup_{i,j} A_{ij}(\delta), \quad A_{ij}(\delta) = \{\mathbf{X}: 0 \le X_i + (\ell - X_j) \le \delta\}.$$

For any $\delta < \ell/2$, $A_{ii}(\delta) = \varnothing$, while for $i \ne j$, by independence of (X_i), we have a crude estimate $\mathbb{P}\{A_{ij}(\delta)\} \le (\delta/\ell)^2$ (a more accurate calculation improves it by the factor $1/2$, irrelevant for our purposes), hence, counting the pairs (i, j), we get

$$\mathbb{P}\{\mathcal{S}_\delta\} \le n(n-1)\,\frac{\delta^2}{\ell^2} < \frac{n^2\delta^2}{\ell^2}. \tag{D.3}$$

Collecting (D.2)–(D.3) and setting $\delta = s^{1/3}$, we come to the asserted bound, with $\vartheta' = \vartheta'' = 2/3$, in the case of a uniform marginal distribution $\mathrm{Unif}([0,\ell])$, $\ell = c$.

2. The next step is mainly a preparation for the final one, but it can also be used alone to extend the above result to a larger class of probability distributions.

Consider a probability distribution supported by a finite of infinite number of intervals $\{J_k, k \in \mathcal{K}\}$, $\mathcal{K} \in \mathbb{Z}$, with probability density constant on each interval J_k: $p(x) = \sum_{k\in\mathcal{K}} c_k \mathbf{1}_{J_k}$. Given an integer $n > 1$, the sample space of n IID r.v. X_i with density p is a union of parallelepipeds

$$\Omega \cong \bigcup_{\mathbf{k}\in\mathbf{K}=\mathcal{K}^n} \mathbf{J}_\mathbf{k}, \quad \mathbf{J}_\mathbf{k} = \bigtimes_{i=1}^{n} J_{k_i}, \quad \mathbf{k} = (k_1, \dots, k_n),$$

and the restriction of the joint density of the r.v. $X_1, \dots, X_n$ on each parallelepiped $\mathbf{J}_\mathbf{k}$ is constant. Obviously, for any measurable subset $\mathcal{A} \subset \Omega$,

$$\mathbb{P}\{\mathcal{A}\} = \mathbb{E}[\mathbb{P}\{\mathcal{A} \,|\, \mathfrak{F}_\mathbf{K}\}] \le \sup_{\mathbf{k}\in\mathbf{K}} \mathbb{P}\{\mathcal{A} \,|\, \mathbf{J}_\mathbf{k}\}.$$

Since the conditional distribution on $\mathbf{J}_\mathbf{k}$ has constant density, one can easily adapt the results of the first step to the product of intervals of different lengths. Below we consider a probability measure with support $[0,c]$, as in (V1), and partition $[0,c]$ into n sub-intervals of identical length, $J_k = [kc/n, (k+1)c/n]$, $k \in \mathcal{K} = \{0, \dots, n-1\}$. In terms of the estimates obtained at the step 1, one has the length $\ell = cn^{-1}$, resulting in factors polynomially bounded in n, thus contributing only to the pre-factor $C|\Omega|^A$ in (RCM), without changing the s-dependence of the regularity bound (2.9).

3. Whenever the density p is non-constant, hence takes at least two values $0 < \rho_1 < \rho_2$, we cannot reduce the analysis to that performed for the uniform distribution, by formally using the fact that the product measure with density $p^{\otimes n}$ is absolutely continuous with respect to the normalized Lebesgue measure. Indeed, $p^{\otimes n}$ takes at least two values with ratio $(\rho_2/\rho_1)^n$ exponentially large in n, and this would ruin all applications to the EVC estimation we are aiming at. However, dividing the interval $[0,c]$ into sub-intervals $J_k = [kc/n, (k+1)c/n]$, we infer from the uniform boundedness of the logarithmic derivative of p (stemming directly from (V1)) that the restriction of the joint density $p^{\otimes n}$ to any sub-cube $\mathbf{J}_\mathbf{k}$ of $[0,c]^n$ of side length $O(n^{-1})$ admits the representation

$$p^{\otimes n}(\mathbf{x}) = \widetilde{C} \prod_{j=1}^{n} (1 + O(n^{-1})) = \widetilde{C} \cdot \mathrm{e}^{\alpha(\mathbf{x})}, \tag{D.4}$$

where $|\alpha(\mathbf{x})| \le c_1 < +\infty$ and $0 < \widetilde{C} < +\infty$. Indeed, let $\mathbf{J_k} = \times_{i=1}^{n}[a_i, a_i + cn^{-1}]$, $\mathbf{a} = (a_1, \dots, a_n]$, then for any point $\mathbf{x}, \mathbf{y}$ in a cube of side length c/n

$$\alpha(\mathbf{x}) := \sum_{i=1}^{n} |\ln p(x_i) - \ln p(a_i)| \le nC' \max_i |x_i - a_i| \le C'cn/n = C'',$$

where C' is determined by the upper bound on $p'(x)/p(x)$ stemming from (V1). In fact, all we need from (V1) is uniform boundedness of the logarithmic derivative $p'(x)/p(x)$ on the segment supporting the marginal measure (except for the negligible endpoints of the segment). Hence we get (D.4) with $\widetilde{C} := p(a_1)\cdots p(a_n)$. The factor $\widetilde{C}$ is eliminated by normalization of the conditional distribution on the sub-cube $\mathbf{J_k}$:

$$p_{\mathbf{k}}(\mathbf{x}) = \frac{p^{\otimes n}(\mathbf{x})}{\int_{\mathbf{J_k}} p^{\otimes n}(\mathbf{y})\, d\mathbf{y}} = |\mathbf{J_k}|^{-1} \frac{\mathrm{e}^{a(\mathbf{x})}}{|\mathbf{J_k}|^{-1} \int_{\mathbf{J_k}} \mathrm{e}^{a(\mathbf{y})}\, d\mathbf{y}} = O(1)\, |\mathbf{J_k}|^{-1},$$

thus the conditional measure on the sub-cube $\mathbf{J_k}$ has bounded Radon-Nikodym derivative with respect to the normalized Lebesgue measure on $\mathbf{J_k}$. Hence for any measurable subset $\mathcal{A} \subset \mathbf{J_k}$, one has

$$\int_{\mathbf{J_k}} \mathbf{1}_{\mathcal{A}}(\mathbf{x})\, p_{\mathbf{k}}(\mathbf{x})\, d\mathbf{x} \le \mathrm{Const}\, |\mathbf{J_k}|^{-1} \int_{\mathbf{J_k}} \mathbf{1}_{\mathcal{A}}(\mathbf{x})\, d\mathbf{x},$$

where the RHS can be assessed with the help of the bounds from step **1**. Now the asserted general bound follows by combining the results of steps **1** and **2**.

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