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Hölder continuity of the integrated density of states for quasi-periodic Jacobi operators

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Abstract. We show Hölder continuity for the integrated density of states of a quasi-periodic Jacobi operator with analytic coefficients, in the regime of positive Lyapunov exponent and with a strong Diophantine condition on the frequency. In particular, when the coefficients are trigonometric polynomials we express the Hölder exponent in terms of the degrees of the coefficients.

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1. Introduction

We consider the quasi-periodic Jacobi operators on $l^2(\mathbb{Z})$ defined by

$$(H(x,\omega)\phi)_n = -b(x+(n+1)\omega)\phi_{n+1} - \overline{b(x+n\omega)}\phi_{n-1} + a(x+n\omega)\phi_n,\ n \in \mathbb{Z},$$

where $a\colon \mathbb{T} \to \mathbb{R}$, $b\colon \mathbb{T} \to \mathbb{C}$ ($\mathbb{T} := \mathbb{R}/\mathbb{Z}$) are real analytic functions, b is not identically zero, and ω satisfies a strong Diophantine condition. Specifically, we have

$$\omega \in \mathbb{T}_{c,\alpha} := \Big\{\omega\colon \|n\omega\| \ge \frac{c}{n(\log n)^\alpha},\ n \ge 1\Big\},$$

with some $c \ll 1$ and $\alpha > 1$.

We let $H_N(x,\omega)$ be the restriction of $H(x,\omega)$ to $[0, N-1]$, with Dirichlet boundary conditions. We use $\mathcal{N}(E,\omega)$ and $L(E,\omega)$ to denote the integrated density of states and the Lyapunov exponent for $H(x,\omega)$ (see Section 2 for definitions).

We will be assuming that a and b are trigonometric polynomials of degrees d_a and d_b. Let $d_0 := \max(d_a, d_b)$ and let n_b be the number of zeroes of b on $\mathbb{T}$. Our methods also apply to general a, b. For the meaning of d_0 in this general setting see Remark 5.4. The following is our main result.

Theorem 1.1. *Let $\omega \in \mathbb{T}_{c,\alpha}$ and $I \subset \mathbb{R}$ be an interval such that $L(E,\omega) > \gamma > 0$ for all $E \in I$ and let $p = 1/(n_b + 2d_0)$. Fix $\varepsilon > 0$.*

(1) *There exists $N_0 = N_0(a, b, I, \omega, \gamma, \varepsilon)$ such that for any $N \geq N_0$, $(1/N)^{1/p} \ll \eta \leq 1/N$, and $E \in I$ we have*

$$\int_{\mathbb{T}} |\sigma(H_N(x,\omega)) \cap [E-\eta, E+\eta]| \, dx \leq N\eta^{p-\varepsilon}.$$

(2) *The integrated density of states satisfies*

$$\mathcal{N}(E+\eta,\omega) - \mathcal{N}(E-\eta,\omega) \leq \eta^{p-\varepsilon},$$

for all $E \in I$ and $\eta \leq \eta_0(a, b, I, \omega, \gamma, \varepsilon)$.

Our work generalizes the result of Goldstein and Schlag [5, Theorem 1.1] from the Schrödinger setting ($b = 1$). In the almost Mathieu case ($b = 1$, $a(x) = 2\lambda\cos(2\pi x)$) the Hölder exponent obtained through this approach is $1/2 - \varepsilon$, with arbitrary $\varepsilon > 0$. It is known that the Hölder exponent in this setting cannot be better than $1/2$ (see for example [11, Corollary 20]), so one gets an asymptotically optimal result. In fact, Avila and Jitomirskaya [1] showed that the Hölder exponent is exactly $1/2$ for the almost Mathieu operator with $\lambda \neq -1, 0, 1$ and general analytic potentials with small coupling constant. However, their result covers the positive Lyapunov exponent regime, via Aubry duality, only for the almost Mathieu operator.

The most important particular example of quasi-periodic Jacobi operator is the extended Harper's model:

$$b(x) = \lambda_3 e^{-2\pi i(x+\omega/2)} + \lambda_2 + \lambda_1 e^{2\pi i(x+\omega/2)}, a(x) = 2\lambda\cos(2\pi x).$$

Unlike for the almost Mathieu operator, the positive Lyapunov exponent regime for the extended Harper's model cannot be approached via duality for all the values of the coupling constants (see [8]). Therefore, even for this simple operator our result may cover cases not covered by the methods from [1].

The main difficulty in extending the work of Goldstein and Schlag [5, 6] is dealing with the singularities coming from the zeroes of b. The groundwork for doing this has been laid in [2] and [12], where most of the basic tools needed for this paper have been developed.

The paper is organized as follows. The basic definitions and tools are reviewed in Section 2. The proof of Theorem 1.1 is given in Section 6. The proof relies on the estimate of the number of zeroes for Dirichlet determinants in a small disk, obtained in Section 5. This estimate is obtained through the multiscale method developed in Section 4. Finally, the auxiliary estimates needed for Section 4 are established in Section 3. On a first reading, we recommend to focus on Section 2 and Section 6.

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2. Preliminaries

We begin by recalling the definition of the integrated density of states and some aspects of the transfer matrix formalism for Jacobi operators.

We use $E_j^{(N)}(x,\omega)$ to denote the eigenvalues of $H_N(x,\omega)$ and let

$$\mathcal{N}_N(E,x,\omega) = \frac{1}{N}|\{E_j^{(N)}(x,\omega)\colon E_j^{(N)}(x,\omega) < E\}|.$$

Note that, throughout the paper, given a set $S \subset \mathbb{R}$ we will use $|S|$ to denote its cardinality and $\operatorname{mes}(S)$ to denote its Lebesgue measure. It is known that Kingman's subadditive ergodic theorem implies that there exists $\mathcal{N}(E,\omega)$ such that

$$\mathcal{N}(E,\omega) = \lim_{N\to\infty}\int_{\mathbb{T}} \mathcal{N}_N(E,x,\omega)\,dx \overset{\text{a.s.}}{=} \lim_{N\to\infty} \mathcal{N}_N(E,x,\omega). \tag{2.1}$$

See for example [13, Sec. 5.2]. The quantity $\mathcal{N}(E,\omega)$ is called the integrated density of states.

The methods we are using are complex analytic so we will work with an extension of the operator to a neighbourhood of the real line. We will use the notation

$$\mathbb{H}_y := \{z \in \mathbb{C}\colon |\operatorname{Im} z| < y\}.$$

It is known that a and b admit complex analytic extensions to $\mathbb{H}_{\rho_0}$ with $\rho_0 = \rho_0(a,b)$. It is essential for us that $\det(H_N(\cdot,\omega)-E)$ is a complex analytic function. To achieve this we need to work with the complex analytic extension of $\overline{b}$ instead of $\overline{b}$. More precisely, we let $\tilde{b}(z) = \overline{b(\overline{z})}$ and we have that $H_N(z,\omega)$ is the matrix

$$\begin{bmatrix} a(z) & -b(z+\omega) & 0 & \dots & 0 \\ -\tilde{b}(z+\omega) & a(z+\omega) & -b(z+2\omega) & \dots & 0 \\ \ddots & \ddots & \ddots & \dots & \vdots \\ 0 & \dots & 0 & -\tilde{b}(z+(N-1)\omega) & a(z+(N-1)\omega) \end{bmatrix}. \quad (2.2)$$

The operator is not necessarily self-adjoint off $\mathbb{T}$, but that would have also been the case if we used $\overline{b}$ instead of $\tilde{b}$ (because the values on the diagonal are not necessarily real).

We let M_N be the N-step transfer matrix such that

$$\begin{bmatrix} \phi_N \\ \phi_{N-1} \end{bmatrix} = M_N \begin{bmatrix} \phi_0 \\ \phi_{-1} \end{bmatrix} N \geq 1.$$

for any ϕ satisfying the difference equation $H(z,\omega)\phi = E\phi$. We have that

$$M_N(z,\omega,E) = \prod_{j=N-1}^{0} \left(\frac{1}{b(z+(j+1)\omega)} \begin{bmatrix} a(z+j\omega) - E & -\tilde{b}(z+j\omega) \\ b(z+(j+1)\omega) & 0 \end{bmatrix} \right),$$

for z such that $\prod_{j=1}^{N} b(z+j\omega) \neq 0$. Because $M_N(z)$ is not necessarily analytic we will in fact work with a version that has the singularities removed:

$$M_N^a(z,\omega,E) = \Big(\prod_{j=1}^{N} b(z+j\omega) \Big) M_N(z,\omega,E).$$

Based on the definitions, it is straightforward to check that

$$\log \|M_N(z,\omega,E)\| = -S_N(z+\omega,\omega) + \log \|M_N^a(z)\|, \quad (2.3)$$

where $S_N(z,\omega) = \sum_{k=0}^{N-1} \log |b(z+k\omega)|$. We will also use

$$\tilde{S}_N(z,\omega) = \sum_{k=0}^{N-1} \log |\tilde{b}(z+k\omega)|.$$

Note that $S_N(x,\omega) = \tilde{S}_N(x,\omega)$ for $x \in \mathbb{T}$.

We let

$$L_N(y,\omega,E)=\frac{1}{N}\int_{\mathbb{T}}\log\|M_N(x+iy,\omega,E)\|\,dx,$$

$$L(y,\omega,E)=\lim_{N\to\infty}L_N(y,\omega,E)=\inf_{N\geq 1}L_N(y,\omega,E).$$

The limits exist by subadditivity. We also consider the quantities L_N^a and L^a which are defined analogously. Furthermore, let

$$D\,(y)=\int_{\mathbb{T}}\log|b\,(x+iy)|\,dx.$$

When $y=0$ we omit the y argument, so for example we write $L\,(\omega,E)$ instead of $L\,(0,\omega,E)$. From (2.3) it follows that

$$L\,(\omega,E)=-D+L^a\,(\omega,E)\,. \tag{2.4}$$

Given an interval $\Lambda=[a,b]$ we let $H_\Lambda(z,\omega)=H_{b-a+1}(z+a\omega,\omega)$ be the restriction of $H(z,\omega)$ to Λ with Dirichlet boundary conditions and $f_\Lambda^a(z,\omega,E):=\det(H_\Lambda(z,\omega)-E)$. A fundamental property of M_N^a is its relation to the characteristic polynomials of the finite scale restriction of $H(x,\omega)$:

$$\begin{aligned}M_N^a(z)&=\begin{bmatrix}f_N^a(z) & -\tilde{b}(z)f_{N-1}^a(z+\omega)\\ b(z+N\omega)f_{N-1}^a(z) & -\tilde{b}(z)b(z+N\omega)f_{N-2}^a(z+\omega)\end{bmatrix}\\ &=\begin{bmatrix}f_{[0,N-1]}^a(z) & -\tilde{b}(z)f_{[1,N-1]}^a(z)\\ b(z+N\omega)f_{[0,N-2]}^a(z) & -\tilde{b}(z)b(z+N\omega)f_{[1,N-2]}^a(z)\end{bmatrix}.\end{aligned} \tag{2.5}$$

We refer to [13, Chap. 1] for a discussion of such relations.

Next we recall some basic tools that will be used throughout the paper. The main tool is a large deviations estimate for the Dirichlet determinants.

Proposition 2.1. *Let $(\omega,E)\in\mathbb{T}_{c,\alpha}\times\mathbb{C}$ such that $L(y,\omega,E)>\gamma>0$, $y\in(-\rho_0,\rho_0)$. For any $H>0$, $N\geq N_0(a,b,E,\omega,\gamma)$, and $|y|<\rho_0$ we have*

$$\begin{aligned}&\operatorname{mes}\{x\in\mathbb{T}\colon|\log|f_N^a(x+iy,\omega,E)|-NL^a(y,\omega,E)|>H(\log N)^{C_0}\}\\ &\quad\leq C_1\exp(-H),\end{aligned}$$

with $C_0=C_0(\omega)$ and $C_1=C_1(a,b,E,\omega,\gamma)$. Furthermore, the same estimate holds for all the other entries of $M_N^a(x+iy,\omega,E)$.

Corollary 2.2. *Let $(\omega, E) \in \mathbb{T}_{c,\alpha} \times \mathbb{C}$ such that $L(\omega, E) > \gamma > 0$. For any $H > 0$, $N \geq N_0(a, b, E, \omega, \gamma)$, and $|y| \leq 1/N$ we have*

$$\operatorname{mes}\{x \in \mathbb{T}\colon |\log|f_N^a(x + iy, \omega, E)| - NL^a(\omega, E)| > H(\log N)^{C_0}\} \leq C_1 \exp(-H),$$

with $C_0 = C_0(\omega)$ and $C_1 = C_1(a, b, E, \omega, \gamma)$. Furthermore, the same estimate holds for all the other entries of $M_N^a(x + iy, \omega, E)$.

The previous two results are slightly modified versions of [3, Proposition 2.1]. We discuss the modifications in Appendix A. We will only work with Corollary 2.2, but we need Proposition 2.1 to justify the following estimate for the integrability of of the entries of M_N^a.

Corollary 2.3. *Let $(\omega, E) \in \mathbb{T}_{c,\alpha} \times \mathbb{C}$ such that $L(y, \omega, E) > \gamma > 0$, $y \in (-\rho_0, \rho_0)$. There exists a constant $C_0 = C_0(a, b, \omega, E, \gamma)$ such that*

$$\left\| \log|f_N^a(\cdot, \omega, E)| \right\|_{L^p(\mathbb{H}_{\rho_0})} \leq C_0 N p, \quad p \geq 1.$$

The same estimate hold for all the other entries of $M_N^a(\cdot, \omega, E)$.

We will be interested in the number of zeroes of f_N^a in a small disk. The reason for this is the following consequence of the Cartan estimate. See Appendix A for the proof.

Lemma 2.4. *Let $(\omega, E) \in \mathbb{T}_{c,\alpha} \times \mathbb{C}$ be such that $L(\omega, E) > \gamma > 0$. If ζ_j, $j = 1, \ldots, k_0$ are the zeros of f_N^a in $\mathcal{D}(z_0, r_0)$ (counting multiplicities), $|z_0| \ll 1/N$, $r_0 \ll 1/N$, then*

$$\log|f_N^a(z, \omega, E)| > NL^a(\omega, E) - (\log r_0)^2(\log N)^{C_0} + k_0 \min_j \log|z - \zeta_j|,$$

for all $z \in \mathcal{D}(z_0, r_0/2)$, with $C_0 = C_0(a, b, E, \omega, \gamma)$, provided $N \geq N_0(a, b, E, \omega, \gamma, k_0)$. Furthermore, the same estimate holds for all the other entries of $M_N^a(z, \omega, E)$.

The importance of the above result is that it provides an essentially optimal lower bound without any exceptional set. We will also need the following analogous result for b and $\tilde{b}$.

Lemma 2.5. *Let $\omega \in \mathbb{T}_{c,\alpha}$. If ζ_j, $j = 1, \ldots, k_0$ are the zeros of b in $\mathcal{D}(z_0, r_0)$ (counting multiplicities), $|z_0| \ll 1/N$, $r_0 \ll 1/N$, then*

$$\log|b(z)| > D - C_0(\log r_0)^2 + k_0 \min_j \log|z - \zeta_j|, \quad z \in \mathcal{D}(z_0, r_0/2),$$

with $C_0 = C_0(b, \omega)$. Furthermore, the same estimate holds for $\tilde{b}$.

It is possible to count the number of zeros of f_N^a in a small disk via the Jensen formula (see for example [10, Sec. 2.3]). Such a straightforward approach yields the following estimate. We will use the notation

$$\nu_f(z_0, r) = |\{z \in \mathcal{D}(z_0, r) : f(z) = 0\}|.$$

Proposition 2.6 ([2, Theorem 4.13]). *Let $(\omega, E) \in \mathbb{T}_{c,\alpha} \times \mathbb{C}$. There exist constants $C_0 = C_0(a, b, \omega, E, \gamma)$ and $N_0 = N_0(a, b, \omega, E, \gamma)$ such that*

$$\nu_{f_N^a(\cdot,\omega,E)}(x_0, 1/N) \leq (\log N)^{C_0},$$

for any $N \geq N_0$ and $x_0 \in \mathbb{T}$.

The proof of the main result hinges on being able to obtain a constant bound on the zeroes, albeit on an even smaller disk. We will achieve this by using the multiscale counting of zeroes introduced in [6, Sec. 9]. Passing from one scale to the next is done via the Avalanche Principle (see [5, Prop. 3.3]). We will only be using the following particular application of the Avalanche Principle. We refer to [3, Corollary 2.7] for a proof, as the differences between the results are minor.

Lemma 2.7. *Let $(\omega, E) \in \mathbb{T}_{c,\alpha} \times \mathbb{C}$ such that $L(\omega, E) > \gamma > 0$ and let $A > 1$. Let Λ_j, $j = 1, \ldots, m$ be pairwise disjoint intervals such that their union Λ is also an interval, and $l \leq |\Lambda_j| \leq l^A$. Assume that for some $z \in \mathbb{H}_{(2l^A)^{-1}}$ the large deviations estimate in Proposition 2.1 holds, with some $H \in (0, l(\log l)^{-2C_0})$, for $f_{\Lambda_j}^a(z, \omega, E)$, $j = 1, \ldots, m$ and $f_{\Lambda_j \cup \Lambda_{j+1}}^a(z, \omega, E)$, $j = 1, \ldots, m-1$. Then there exists a constant $l_0(a, b, \omega, E, \gamma, A)$ such that when $l \geq \max(l_0, 2\log m/\gamma)$ we have*

$$\left| \log |f_\Lambda^a(z)| + \sum_{j=2}^{m-1} \log \|A_j(z)\| - \sum_{j=1}^{m-1} \log \|A_{j+1}(z) A_j(z)\| \right| \lesssim m \exp(-\gamma l/2),$$

where $A_j(z) = M_{\Lambda_j}^a(z)$, $j = 2, \ldots, m-1$ and

$$A_1(z) = M_{\Lambda_1}^a(z) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_m(z) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} M_{\Lambda_m}^a(z).$$

Furthermore, we have

$$\left| \log \|M_\Lambda^a(z)\| + \sum_{j=2}^{m-1} \log \|M_{\Lambda_j}^a(z)\| - \sum_{j=1}^{m-1} \log \|M_{\Lambda_{j+1}}^a(z) M_{\Lambda_j}^a(z)\| \right| \\ \lesssim m \exp(-\gamma l/2).$$

It turns out that in conjunction with the Avalanche Principle it is convenient to use the following double integrals introduced in [5, Sec. 5]:

$$J_\varepsilon(u, z_0, r) = \frac{4}{\varepsilon^2} \fint_{\mathcal{D}(z_0,r)} \fint_{\mathcal{D}(z,\varepsilon r)} (u(\zeta) - u(z))\, dA(\zeta) dA(z).$$

Following [5] we refer to this double integral as a Jensen average. In applications u will be subharmonic and in particular of the form $u = \log|f|$. The reason for calling this double integral a Jensen average is that as a consequence of the Jensen formula one gets the following estimate.

Lemma 2.8 ([5, Lemma 5.1]). *Let $f(z)$ be analytic in $\mathcal{D}(z_0, R_0)$. Then for any $r, \varepsilon > 0$ such that $(1+\varepsilon)r < R_0$ we have*

$$\nu_f(z_0, (1-\varepsilon)r) \le J_\varepsilon(\log|f|, z_0, r) \le \nu_f(z_0, (1+\varepsilon)r).$$

Finally, we recall the following uniform upper estimates that are essential to the successful use of the Cartan estimate and the Jensen formula (in conjunction with the deviations estimates).

Proposition 2.9 ([3, Corollary 2.3]). *Let $(\omega_0, E_0) \in \mathbb{T}_{c,\alpha} \times \mathbb{C}$ be such that $L(\omega_0, E_0) > \gamma > 0$. There exist constants $N_0 = N_0(a, b, E_0, \omega_0, \gamma)$, $C_0 = C_0(\omega_0)$, and $C_1 = C_1(a, b, E_0, \omega_0, \gamma)$ such that for $N \ge N_0$ we have*

$$\begin{aligned}\sup\{\log \left\| M_N^a(x+iy, \omega, E)\right\| : x \in \mathbb{T}, |E - E_0|, |\omega - \omega_0| \le N^{-C_1}, |y| \le N^{-1}\}\\ \le NL^a(\omega_0, E_0) + (\log N)^{C_0}.\end{aligned}$$

Lemma 2.10 ([3, Lemma 2.5]). *Let $\omega \in \mathbb{T}_{c,\alpha}$. There exist constants $C_0 = C_0(\omega)$, $C_1 = C_1(b, \omega)$ such that for every $N > 1$ we have*

$$\sup\{S_N(x+iy, \omega) : x \in \mathbb{T},\ |y| \le N^{-1}\} \le ND + C_1(\log N)^{C_0}$$

and

$$\sup\{\tilde{S}_N(x+iy, \omega) : x \in \mathbb{T},\ |y| \le N^{-1}\} \le ND + C_1(\log N)^{C_0}.$$

3. Estimates for Jensen averages

For the purposes of the next section we are interested in the Jensen averages of $\log\|\mathcal{M}_N(z)\|$, where $\mathcal{M}_N(z) = \mathcal{M}_N(z, \omega, E)$ is one of the following matrices:

$$M_N^a(z, \omega, E), \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} M_N^a(z, \omega, E), \quad M_N^a(z, \omega, E) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

It is to be expected that these Jensen averages are related to the number of zeroes of the entries of $\mathcal{M}_N$. In particular we are concerned with the case when the entries have no zeroes and we will show in Proposition 3.6 that in this case the Jensen average is small. A straightforward way of controlling these Jensen averages is by estimating the quotients $\|\mathcal{M}_N(\zeta)\| / \|\mathcal{M}_N(z)\|$, $\zeta \in \mathcal{D}(z, \varepsilon r)$. This will be achieved by using the Taylor formula in Proposition 3.5. The estimate is facilitated by the fact that under the assumption that the entries of $\mathcal{M}_N$ have no zeroes we can take advantage of Harnack's inequality. We recall a version of Harnack's inequality. This is a minor reformulation of [6, Lemma 8.2], that doesn't affect its proof.

Lemma 3.1. *Let $M \gg 1$, $r_0 > 0$, $r_1 = (1 + \log M)^{-2} r_0$, $z_0 \in \mathbb{C}$. If f is an analytic and nonvanishing function on $\mathcal{D}(z_0, r_0)$ such that*

$$\sup_{z \in \mathcal{D}(z_0, r_0)} |f(z)| \le M \quad \text{and} \quad |f(z_0)| \ge M^{-1},$$

then

$$|f(z)| \lesssim |f(z_0)|, \quad z \in \mathcal{D}(z_0, r_1).$$

In what follows we establish the auxiliary results needed for the proof of Proposition 3.5.

Lemma 3.2. *Let $(\omega, E) \in \mathbb{T}_{c,\alpha} \times \mathbb{C}$ be such that $L(\omega, E) > \gamma > 0$. There exists $N_0(a, b, E, \omega, \gamma)$ such that for any $k \ge 0$, $N \ge N_0$, $|z_0| \ll 1/N$, and $0 < r_0 \ll 1/N$ we have that if all the entries of $\mathcal{M}_N(z, \omega, E)$ are either identically zero or have no zeros in $\mathcal{D}(z_0, r_0)$, then*

$$\|\partial_z^k \mathcal{M}_N(z, \omega, E)\| \lesssim k! r_1^{-k} \|\mathcal{M}_N(z_0, \omega, E)\|, \quad z \in \mathcal{D}(z_0, r_1),\ r_1 = r_0^{1+}.$$

Proof. It is convenient for the proof to work with the l^1 matrix norm. Let $f_N(z, \omega, E)$ be any of the not identically zero entries of $\mathcal{M}_N(z, \omega, E)$. By Lemma 2.4 we have

$$\log |f_N(z_0, \omega, E)| \ge N L^a(\omega, E) - (\log r_0)^2 (\log N)^C.$$

At the same time from Proposition 2.9 we know

$$\sup\{\log |f_N(z, \omega, E)| : z \in \mathcal{D}(z_0, r_0)\} \le N L^a(\omega, E) + (\log N)^C.$$

Applying Lemma 3.1 with $f = \exp(N L^a) f_N$, $M = \exp((\log r_0)^2 (\log N)^C)$ we conclude that

$$\|\mathcal{M}_N(z, \omega, E)\| \lesssim \|\mathcal{M}_N(z_0, \omega, E)\|, z \in \mathcal{D}(z_0, r),$$

with

$$r = \frac{r_0}{(1 + (\log r_0)^2 (\log N)^C)^2} \gg r_0^{1+} = r_1,$$

provided N is large enough. From the above and the Cauchy formula we get that for $z \in \mathcal{D}(z_0, r_1)$ we have

$$\begin{aligned} \|\partial_z^k \mathcal{M}_N(z, \omega, E)\| &\lesssim k! r_1^{-k} \sup\{\|\mathcal{M}_N(\zeta, \omega, E)\| : \zeta \in \mathcal{D}(z_0, 2r_1)\} \\ &\lesssim k! r_1^{-k} \|\mathcal{M}_N(z_0, \omega, E)\| . \end{aligned}$$

□

Lemma 3.3. *If B is a 2×2 matrix with top-left entry b, then*

$$\log \left\| \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + zB \right\| = \log|1 + bz| + O(|z|^2) \|B\|^2, \quad \textit{as } z \to 0.$$

For the proof we refer to [5, p. 835]. We note that this result is sensitive to the choice of the norm. For example, with the l^1 norm the error term would be $O(|z|) \|B\|$ (we are using the standard matrix norm induced by the Euclidean norm on $\mathbb{C}^2$).

Lemma 3.4. *Let $(\omega, E) \in \mathbb{T}_{c,\alpha} \times \mathbb{C}$ be such that $L(\omega, E) > \gamma > 0$. There exists $N_0(a, b, E, \omega, \gamma)$ such that for $N \geq N_0$, $|z_0| \ll 1/N$, and $\exp(-N^{1/2-}) \lesssim r_0 \ll 1/N$ we have that if all the entries of $\mathcal{M}_N(z, \omega, E)$ are either identically zero or have no zeros in $\mathcal{D}(z_0, r_0)$, then*

$$\frac{|\det \mathcal{M}_N(z_0, \omega, E)|}{\|\mathcal{M}_N(z_0, \omega, E)\|^2} \leq \exp(-NL(\omega, E)).$$

Proof. We are only concerned with the case $\mathcal{M}_N = M_N^a$ because the other cases are trivial. Since we have

$$\det M_N^a(z_0, \omega, E) = \exp(\tilde{S}_N(z_0, \omega) + S_N(z_0 + \omega, \omega)),$$

it follows from Lemma 2.10 that

$$|\det M_N^a(z_0, \omega, E)| \leq \exp(2ND + (\log N)^C).$$

On the other hand, Lemma 2.4 yields that

$$\begin{aligned} \|M_N^a(z_0, \omega, E)\|^2 &\geq \exp(2NL^a(\omega, E) - (\log r_0)^2 (\log N)^C) \\ &\geq \exp(2NL^a(\omega, E) - N^{1-}). \end{aligned}$$

The conclusion follows by recalling that we have (2.4). □

Proposition 3.5. *Let $(\omega, E) \in \mathbb{T}_{c,\alpha} \times \mathbb{C}$ be such that $L(\omega, E) > \gamma > 0$. There exists $N_0(a, b, E, \omega, \gamma)$ such that for $N \geq N_0$, $|z_0| \ll 1/N$, and* $\exp(-N^{1/2-}) \lesssim r_0 \ll 1/N$ *we have that if all the entries of $\mathcal{M}_N(z, \omega, E)$ are either identically zero or have no zeros in $\mathcal{D}(z_0, r_0)$, then for $z \in \mathcal{D}(z_0, r_1^{1+})$, $r_1 = r_0^{1+}$, we have*

$$\log \frac{\|\mathcal{M}_N(z, \omega, E)\|}{\|\mathcal{M}_N(z_0, \omega, E)\|} = \log|1 + b(z - z_0)| + O(|z - z_0|^2) r_1^{-2} + O(1) \exp(-NL(\omega, E)),$$

with $b = b(z_0)$ and $|b| \lesssim r_1^{-1}$.

Proof. Let

$$\mathcal{M}_N(z_0) = U \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix} V$$

be the singular value decomposition of $\mathcal{M}_N(z_0)$. So, U and V are unitary and the singular values are

$$\mu_1 = \|\mathcal{M}_N(z_0)\| \text{ and } \mu_2 = \frac{|\det \mathcal{M}_N(z_0)|}{\|\mathcal{M}_N(z_0)\|}.$$

Using Taylor's theorem, Lemma 3.2, and Lemma 3.4 we get that for $z \in \mathcal{D}(z_0, r_1)$ we have

$$\begin{aligned} \frac{\|\mathcal{M}_N(z)\|}{\|\mathcal{M}_N(z_0)\|} &= \left\| \frac{1}{\mu_1} U^{-1} \mathcal{M}_N(z) V^{-1} \right\| \\ &= \left\| \begin{bmatrix} 1 & 0 \\ 0 & \mu_2/\mu_1 \end{bmatrix} + (z - z_0) B \right\| + O(|z - z_0|^2) r_1^{-2} \\ &= \left\| \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (z - z_0) B \right\| + O(|z - z_0|^2) r_1^{-2} + O(1) \exp(-NL), \end{aligned}$$

with

$$\|B\| \lesssim r_1^{-1}.$$

It follows that for $z \in \mathcal{D}(z_0, r_1^+)$ we have

$$\log \frac{\|\mathcal{M}_N(z)\|}{\|\mathcal{M}_N(z_0)\|} - \log \left\| \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (z - z_0) B \right\| = O(|z - z_0|^2) r_1^{-2} + O(1) \exp(-NL).$$

The conclusion now holds due to Lemma 3.3. □

Proposition 3.6. *Let $(\omega, E) \in \mathbb{T}_{c,\alpha} \times \mathbb{C}$ be such that $L(\omega, E) > \gamma > 0$. There exists $N_0(a, b, E, \omega, \gamma)$ such that for $N \geq N_0$, $|z_0| \ll 1/N$, and* $\exp(-N^{1/2-}) \lesssim$ *$r_0 \ll 1/N$ we have that if all the entries of $\mathcal{M}_N(z, \omega, E)$ are either identically zero or have no zeros in $\mathcal{D}(z_0, r_0)$, then for $x_0 \in \mathbb{T}$, $\varepsilon \in (0, 1)$, and $r \leq r_1^{1+}/2$, $r_1 = r_0^{1+}$, we have*

$$J_\varepsilon(\log \|\mathcal{M}_N(\cdot, \omega, E)\|, x_0, r) = O(r^2) r_1^{-2} + O(1)\varepsilon^{-2} \exp(-NL(\omega, E)).$$

Proof. The result follows from Proposition 3.5, the fact that

$$\fint_{\mathcal{D}(z,\varepsilon r)} \log|1 + b(\zeta - z)|\, dA(\zeta) = 0$$

(due to the mean value property for harmonic functions; it is essential that we have $|b| \lesssim r_1^{-1}$ and $|\zeta - z| \lesssim r_1^{1+}$) and

$$\fint_{\mathcal{D}(z,\varepsilon r)} |\zeta - z|^2\, dA(\zeta) = \frac{\varepsilon^2 r^2}{2}. \qquad \square$$

We will also need an estimate for the case when we don't have further information on the entries of $\mathcal{M}_N$. For this we use the following result on the Jensen averages of subharmonic functions.

Lemma 3.7 ([5, Lemma 5.4]). *Let*

$$u(z) = \int \log|z - \zeta| \mu(d\zeta) + h(z),\ z \in \Omega,$$

where h is harmonic and μ is a non-negative measure on some domain Ω. Then

$$\mu(\mathcal{D}(z_0, (1-\varepsilon)r)) \leq J_\varepsilon(u, z_0, r) \leq \mu(\mathcal{D}(z_0, (1+\varepsilon)r)),$$

for any z_0, ε, r such that $\mathcal{D}(z_0, (1+\varepsilon)r) \subset \Omega$.

Proposition 3.8. *If $\mathcal{M}_N(z)$ is analytic on a neighbourhood of the closure of $\mathbb{H}_{\rho_0}$, then there exists $C_0(a, b, E, \rho_0)$ such that*

$$0 \leq J_\varepsilon(\log \|\mathcal{M}_N(\cdot, \omega, E)\|, z_0, r) \leq C_0 N,$$

for any z_0, ε, r such that $\mathcal{D}(z_0, (1+\varepsilon)r) \subset \mathbb{H}_{\rho_0}$.

Proof. Since $\log\|\mathcal{M}_N(z)\|$ is subharmonic it admits a Riesz representation:

$$\log\|\mathcal{M}_N(z)\| = \int \log|\zeta - z|\,\mu_N(d\zeta) + h_N(z),$$

where μ_N is a positive measure and h_N is harmonic. It is known that

$$\mu_N(\mathbb{H}_{\rho_0}) \le CN.$$

For a proof we refer to [2, Lemma 3.4]. Now the conclusion follows from Lemma 3.7. □

4. Multiscale counting of zeroes

Given an interval Λ together with a partition into intervals $\{\Lambda_j\}$, $j = 1,\dots,m$ (ordered from leftmost to rightmost) it's easy to see that

$$M_\Lambda^a = \prod_{j=m}^{1} M_{\Lambda_j}^a.$$

Such a factorization doesn't hold for f_Λ^a, but an approximation of this relation is available by using the Avalanche Principle. This allows one to relate the number of zeroes of f_Λ to that of f_{Λ_j}, $j = 1,\dots,m$. This is achieved by using Jensen averages and it is therefore crucial to control the Jensen averages of the extraneous terms that result from the application of the Avalanche Principle. For this it is natural to introduce the following notion.

Definition 4.1. *We say that $s \in \mathbb{Z}$ is **adjusted** to $(\mathcal{D}(z_0,r_0),\omega,E)$ at scale l if for all $l \le k \le 100l$ and $|m| \le 100$ all the entries of $M_l^a(\cdot + (s+m)\omega,\omega,E)$ have no zeros in $\mathcal{D}(z_0,r_0)$.*

Note that if s is adjusted then by the results of the previous section we have good control on the Jensen averages of $\log\|M_{\Lambda'}^a\|$, where Λ' can be any interval of size $l \le |\Lambda'| \le 100l$ that is "sufficiently close" to s. The notion of being adjusted is useful because we can find many adjusted integers.

Lemma 4.2. *Let $(\omega,E) \in \mathbb{T}_{c,\alpha} \times \mathbb{C}$, $x_0 \in \mathbb{T}$ and $n_0 \in \mathbb{Z}$. Given $l \gg 1$ and $r_0 = \exp(-(\log l)^A)$, $A > 1$, there exists $n_0' \in [n_0 - l^6, n_0 + l^6]$ such that n_0' is adjusted to $(\mathcal{D}(x_0,r_0),\omega,E)$ at scale l.*

For the proof we refer to [6, Lemma 9.7].

We can now prove the result on multiscale counting of zeroes.

Proposition 4.3. *Let* $(\omega, E) \in \mathbb{T}_{c,\alpha} \times \mathbb{C}$ *such that* $L(\omega, E) > \gamma > 0$ *and let* $A > 1$. *Let* Λ_j, $j = 1, \ldots, m$ *be pairwise disjoint intervals such that their union* Λ *is also an interval, and* $l \ll |\Lambda_j| \leq l^A$. *There exists* $l_0 = l_0(a, b, \omega, E, \gamma, A)$ *such that if* $l \geq \max(l_0, (\log m)^{1+})$ *and all but* k *of the intervals* Λ_j *have the endpoints adjusted to* $(\mathcal{D}(x_0, r_0), \omega, E)$ *at scale* l, $x_0 \in \mathbb{T}$, $\exp(-l^{1/2-}) \lesssim r_0 \ll 1/l$, *then*

$$\begin{aligned} J_\varepsilon(\log|f_\Lambda^a|, x_0, r) - \sum_{j=1}^m J_\varepsilon(\log|f_{\Lambda_j}^a|, x_0, r) \\ = O(1)\varepsilon^{-4} r^{-2} \exp(-l^{1-}) + (m-k) O(r^2) r_1^{-2} + k O(1) C_0 l, \end{aligned}$$

with $C_0 = C_0(a, b, \omega, E, \gamma)$, *and for any* $\varepsilon \in (0, 1)$, $r \leq r_1^{1+}/2$, $r_1 = r_0^{1+}$.

Proof. The proof is essentially the same as for [6, Prop. 9.3]. We partition each Λ_j into five intervals $\Lambda_j^{(i)}$, $i = 1, \ldots, 5$ such that $|\Lambda_j^{(i)}| = l$ for $i \neq 3$. Applying the Avalanche Principle expansion to $\log|f_\Lambda^a|$, $\log|f_{\Lambda_j}^a|$ (i.e. using Lemma 2.7 and Proposition 2.1) we get

$$\log|f_\Lambda^a(z)| - \sum_{j=1}^m \log|f_{\Lambda_j}^a(z)| = \sum \pm \log\|A_{\Lambda'}(z)\| + O(1)\exp(-cl),$$

for $z \in \mathcal{D}(z_0, r_0) \setminus \mathcal{B}$, $\operatorname{mes}(\mathcal{B}) \leq \exp(-l^{1-})$, with $A_{\Lambda'}(z)$ of the form

$$M_{\Lambda'}^a(z), \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} M_{\Lambda'}^a(z), \text{ or } M_{\Lambda'}^a(z) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

where Λ' is an interval of length l or $2l$ containing an endpoint of one the intervals Λ_j. By using Corollary 2.3 it follows that

$$\begin{aligned} J_\varepsilon(\log|f_\Lambda^a|, z_0, r) - \sum_{j=1}^m J_\varepsilon(\log|f_{\Lambda_j}^a|, z_0, r) \\ = \sum \pm J_\varepsilon(\log\|A_{\Lambda'}\|, z_0, r) + O(1)\varepsilon^{-4} r^{-2} \exp(-l^{1-}). \end{aligned} \tag{4.1}$$

Indeed we have

$$\begin{aligned} J_\varepsilon(\log|f_\Lambda^a|, z_0, r) = \frac{4}{\pi^2 \varepsilon^4 r^4} \int_{\mathcal{D}(x_0, r)} \int_{\mathcal{D}(z, \varepsilon r)} \log|f_\Lambda^a(\zeta)|\, dA(\zeta)\, dA(z) \\ - \frac{4}{\pi \varepsilon^2 r^2} \int_{\mathcal{D}(x_0, r)} \log|f_\Lambda^a(z)|\, dA(z), \end{aligned}$$

$$\begin{aligned}\frac{4}{\pi^2\varepsilon^4 r^4}\int_{\mathcal{D}(x_0,r)}\int_{\mathcal{D}(z,\varepsilon r)\cap\mathcal{B}} &|\log|f_\Lambda^a(\zeta)||\,dA(\zeta)\,dA(z)\\ &\lesssim \frac{1}{\varepsilon^4 r^4}\int_{\mathcal{D}(x_0,r)} Cml^A\sqrt{|\mathcal{B}|}\,dA(z)\\ &\lesssim \frac{1}{\varepsilon^4 r^2}\exp(-l^{1-}),\end{aligned}$$

and

$$\begin{aligned}\frac{4}{\pi\varepsilon^2 r^2}\int_{\mathcal{D}(x_0,r)\cap\mathcal{B}}|\log|f_\Lambda^a(z)||\,dA(z) &\lesssim \frac{1}{\varepsilon^2 r^2}Cml^A\sqrt{|\mathcal{B}|}\\ &\lesssim \frac{1}{\varepsilon^2 r^2}\exp(-l^{1-}).\end{aligned}$$

Note that we used the assumption that $l \geq (\log m)^{1+}$. The other terms are dealt with in the same way.

The conclusion follows immediately by applying either Proposition 3.6 or Proposition 3.8 to the averages on the right-hand side of (4.1). □

5. Count of zeroes in a small disk

We will show in Proposition 5.3 that if Λ has adjusted endpoints then we can use Proposition 4.3 to obtain a bound on the number of zeroes of f_Λ^a. The idea is simply that the zeroes on Λ can be shifted around resulting in more zeroes at a larger scale. The assumption that a, b are trigonometric polynomials comes into play via the fact that in this case $f_N^a(\cdot,\omega,E)$ is a rational function of degree at most $2d_0N$. This is easily seen from (2.2).

We will be using the following known results on the equidistribution of the orbit of an irrational shift.

Lemma 5.1. *Let $\omega \in \mathbb{T}_{c,\alpha}$ and $N > 1$. There exists a constant $C_0(\omega)$ such that for any interval $I \subset \mathbb{T}$ we have*

$$|\{m \in [0, N-1] : m\omega \in I\}| = N|I| + O(1)C_0(\log N)^{\alpha+2}.$$

This lemma is a consequence of the Erdös-Turán theorem on the discrepancy of a sequence of real numbers, and of the Diophantine condition imposed on ω. See [9, Lemma 2.3.2-3] for the resulting estimates for irrational shifts that yield the above lemma as a particular case.

Corollary 5.2. *Let $\omega \in \mathbb{T}_{c,\alpha}$ and $N > 1$. There exists $C_0(\omega)$ such that the distance between any two consecutive points of the set $\{m\omega : m \in [0, N-1]\} \subset \mathbb{T}$ is between $cN^{-1}(\log N)^{-\alpha}$ and $C_0 N^{-1}(\log N)^{\alpha+2}$.*

This is an immediate consequence of the previous lemma and the of the Diophantine condition.

Proposition 5.3. *Let $(\omega, E) \in \mathbb{T}_{c,\alpha} \times \mathbb{C}$ such that $L(\omega, E) > \gamma > 0$ and let $A > 1$. If the endpoints of Λ are adjusted to $(\mathcal{D}(x_0, r_0), \omega, E)$ at scale l, $\exp(-l^{1/2-}) \lesssim r_0 \ll 1/l$, $|\Lambda|^{1/A} \le l \ll |\Lambda|$, then $f_\Lambda^a(\cdot, \omega, E)$ has at most $2d_0$ zeroes in $\mathcal{D}(x_0, r_0 \exp(-(\log l)^{C_0}))$, for some $C_0 = C_0(a, b, \omega, E, A)$, provided $l \ge l_0(a, b, \omega, E, A)$.*

Proof. Let $N \simeq \exp(l^{1-})$. The idea of the proof is that Proposition 4.3 implies that if f_Λ^a has too many zeroes then f_N^a has too many zeroes.

Proposition 2.6 guarantees that there exists $n \in [1, (\log l)^C]$ such that f_Λ^a has no zeroes in $\mathcal{D}(x_0, \rho_0) \setminus \mathcal{D}(x_0, \rho_1)$, $\rho_0 = r_0^{2+} \exp(-n \log l)$, $\rho_1 = \rho_0 / l$. Let $\Lambda_m = m + \Lambda$, $x_m = x_0 + m\omega$, and

$$S = \{m \in [0, N-1] \colon \Lambda_m \subset [l, N-l-1] \text{ and } x_m \in \mathcal{D}(x_0, (1-2\varepsilon)\rho_0)\},$$

with $\varepsilon = \varepsilon(d_0) \ll 1$ to be chosen later. Note that Lemma 5.1 gives us that

$$|S| = 2N(1-2\varepsilon)\rho_0 + O(1)C(\log N)^{\alpha+2} \tag{5.1}$$

and due to the Diophantine condition we have that if $m_1, m_2 \in S$, $m_1 \ne m_2$ then

$$\operatorname{dist}(\Lambda_1, \Lambda_2) \gg l.$$

If $x_m \in \mathcal{D}(x_0, (1-2\varepsilon)\rho_0)$ then $\mathcal{D}(x_0, r_0/2) \subset \mathcal{D}(x_m, r_0)$, because $\rho_0 \ll r_0$. Since we have that the endpoints of Λ_m are adjusted to $(\mathcal{D}(x_m, r_0), \omega, E)$ at scale l it follows that they are also adjusted to $(\mathcal{D}(x_0, r_0/2), \omega, E)$ at scale l, provided $m \in S$. It is now easy to see that we can find a partition of $[0, N-1]$ containing the intervals Λ_m, $m \in S$, that satisfies the requirements of Proposition 4.3 and such that 0 and $N-1$ are the only unadjusted endpoints (we are using Lemma 4.2; to make sure that we can apply the lemma, we can replace r_0 by $r_0 \exp(-(\log l)^C)$, as this won't affect the final result). It then follows that

$$\begin{aligned} &\frac{1}{N} J_\varepsilon(\log |f_N^a|, x_0, \rho_0) \\ &\quad \ge \frac{1}{N} \sum_{m \in S} J_\varepsilon(\log |f_{\Lambda_m}^a|, x_0, \rho_0) - C(\exp(l^{1-}) + \rho_0^2 (r_0^{1+})^{-2}). \end{aligned} \tag{5.2}$$

We used the fact that the Jensen averages of subharmonic functions are non-negative (due to the sub-mean-value property of subharmonic functions). Let $Z = \nu_{f_\Lambda^a}(x_0, \rho_0)$. We obviously have that

$$Z = \nu_{f_{\Lambda_m}^a}(x_m, \rho_0) = \nu_{f_{\Lambda_m}^a}(x_m, \rho_1),$$

for any m. If $m \in S$ then $\mathcal{D}(x_m, \rho_1) \subset \mathcal{D}(x_0, (1-\varepsilon)\rho_0)$ and therefore

$$\nu_{f_{\Lambda_m}^a}(x_0, (1-\varepsilon)\rho_0) \geq Z.$$

This, together with (5.1), (5.2), and (2.8) imply that

$$\frac{1}{N}\nu_{f_N^a}(x_0, (1+\varepsilon)\rho_0) \geq 2(1-2\varepsilon)\rho_0 Z - C(\exp(l^{1-}) + \rho_0^2(r_0^{1+})^{-2}).$$

We can repeat the above reasoning with Λ_m instead of Λ, x_m instead of x_0, and the same r_0, ρ_0, ρ_1 to get

$$\frac{1}{N}\nu_{f_N^a}(x_m, (1+\varepsilon)\rho_0) \geq 2(1-2\varepsilon)\rho_0 Z - C(\exp(l^{1-}) + \rho_0^2(r_0^{1+})^{-2}).$$

We can find at least $[2\rho_0(1+2\varepsilon)]^{-1}$ pairwise disjoint disks $\mathcal{D}(x_m, (1+\varepsilon)\rho_0)$ (we are using Corollary 5.2 and $(\log N)^{\alpha+2}/N \ll \rho_0$), so it follows that

$$2d_0 \geq \frac{1}{2\rho_0(1+2\varepsilon)}(2(1-2\varepsilon)\rho_0 Z - C(\exp(l^{1-}) + \rho_0^2(r_0^{1+})^{-2})).$$

For $\varepsilon = \varepsilon(d_0)$ small enough and l large enough, the above inequality implies that $2d_0 + 1 > Z$. So we can conclude that $Z \leq 2d_0$. □

Remark 5.4. For general a, b it follows from the Jensen formula (together with the large deviations estimate and the uniform upper bound) that the number of zeroes of $f_N^a(\cdot, \omega, E)$ in a strip around $\mathbb{T}$ is bounded by $C_0 N$, with $C_0 = C_0(a, b, \omega, E, \gamma)$. It is clear from the proof that in this case the previous lemma holds with $d_0 = C_0/2$.

6. Proof of the main result

One can get information on the regularity of the integrated density of states from finite scale estimates via the following standard result.

Lemma 6.1. *For any $N, m \geq 1$, $\omega \in \mathbb{T}$, and any interval $I \subset \mathbb{R}$ we have*

$$\frac{1}{mN}\int_{\mathbb{T}} |\sigma(H_{mN}(x,\omega)) \cap I|\, dx \leq \frac{1}{N}\int_{\mathbb{T}} |\sigma(H_N(x,\omega)) \cap I|\, dx + \frac{4}{N}.$$

Proof. We have that

$$H_{mN}(x) = \bigoplus_{k=0}^{m-1} H_N(x + kN\omega) + R,$$

with $\operatorname{rank} R \leq 2m$. It follows from Weyl's interlacing inequalities (see [7, Theorem 4.3.6]) that

$$|\sigma(H_{mN}(x)) \cap I| \leq \sum_{k=0}^{m-1} |\sigma(H_N(x + kN\omega)) \cap I| + 4m.$$

The conclusion follows immediately. □

Let $\Lambda = [\alpha, \beta]$. The following estimate is well-known from the proof of the Wegner estimate for the Anderson model:

$$\begin{aligned} |\sigma(H_\Lambda) \cap [E - \eta, E + \eta]| &\leq 2\eta \sum_{j=\alpha}^{\beta} \frac{\eta}{(E_j^\Lambda - E)^2 + \eta^2} \\ &= 2\eta \operatorname{Im} \operatorname{Tr}(H_\Lambda - E - i\eta)^{-1} \\ &\leq 2\eta \sum_{k=\alpha}^{\beta} |\langle \delta_k, (H_\Lambda - E - i\eta)^{-1} \delta_k \rangle|. \end{aligned}$$

We are left now with finding a bound on the diagonal entries of Green's function. For the Anderson model this is straightforward using Schur's complement and the independence of the single-site potentials (assuming the common distribution has bounded density). In the quasi-periodic setting such a simple approach fails due to the correlations between the single-site potentials. Instead, we will use the fact that due to Cramer's formula we have

$$\begin{aligned} &|\langle \delta_k, (H_\Lambda(x, \omega) - E - i\eta)^{-1} \delta_k \rangle| \\ &\qquad = \frac{|f^a_{[\alpha,k-1]}(x, \omega, E + i\eta)|\,|f^a_{[k+1,\beta]}(x, \omega, E + i\eta)|}{|f^a_{[\alpha,\beta]}(x, \omega, E + i\eta)|}. \end{aligned}$$

We can immediately write an estimate by using the uniform upper bound for the terms on top and the large deviations theorem for the bottom. This estimate is not of the right order of magnitude, but it can be improved by using the Avalanche Principle. The idea is simply that if we write the Avalanche Principle expansion for the determinants, after cancellations, we would be left with a similar quantity but at a much smaller scale. There are two issues with this approach. First, working with the determinants results in some extra terms that won't cancel out (namely the A_1, A_m terms in Lemma 2.7). Second, $[\alpha, k-1]$ and $[k+1, \beta]$ don't partition $[\alpha, \beta]$ so we'd be left with some extra terms that we don't want. These issues are addressed by the following lemma. We will use the notation

$$\mathcal{W}_{N,k}(x, \omega, E) = \frac{\|M^a_{[0,k-1]}(x, \omega, E)\| \, \|M^a_{[k,N-1]}(x, \omega, E)\|}{\|M^a_{[0,N-1]}(x, \omega, E)\|}.$$

Lemma 6.2. *Let* $(\omega, E) \in \mathbb{T}_{c,\alpha} \times \mathbb{R}$, $x \in \mathbb{T}$, $\eta > 0$, $\mathcal{K} \subset [0, N-1]$, $N \geq 1$. *Then we have*

$$\begin{aligned} &|\sigma(H_N(x, \omega)) \cap [E-\eta, E+\eta]| \\ &\quad \leq 4\eta \sum_{k \notin \mathcal{K}} \frac{1}{|\tilde{b}(x+k\omega)|} \mathcal{W}_{N,k}(x, \omega, E+i\eta) + 2|\mathcal{K}| + 10. \end{aligned}$$

Proof. We assume that the entry of $M^a_{[0,N-1]}(x)$ with the largest absolute value is

$$-\tilde{b}(x) b(x+N\omega) f^a_{[1,N-2]}(x).$$

The case when the largest entry is one of the other entries can be treated analogously to this one. We singled out this case because it captures all the needed ideas.

From our assumption we get that

$$\|M^a_{[0,N-1]}(x)\| \leq 2|\tilde{b}(x) b(x+N\omega) f^a_{[1,N-2]}(x)|.$$

To take advantage of this relation we need to work with $H_{[1,N-2]}$ instead of $H_{[0,N-1]}$. This is not a problem because we have

$$H_{[0,N-1]} = H_{\{0\}} \oplus H_{[1,N-2]} \oplus H_{\{N-1\}} + R,$$

with rank $R \leq 4$, and then Weyl's interlacing inequalities (see [7, Theorem 4.3.6]) imply

$$\begin{aligned} &|\sigma(H_N(x, \omega)) \cap [E-\eta, E+\eta]| \\ &\quad \leq |\sigma(H_{[1,N-2]}(x, \omega)) \cap [E-\eta, E+\eta]| + 2 \operatorname{rank} R + 2 \\ &\quad \leq |\sigma(H_{[1,N-2]}(x, \omega)) \cap [E-\eta, E+\eta]| + 10. \end{aligned}$$

We know that

$$\begin{aligned} &|\sigma(H_{[1,N-2]}(x, \omega)) \cap [E-\eta, E+\eta]| \\ &\quad \leq 2\eta \sum_{k=1}^{N-2} |\langle \delta_k, (H_{[1,N-2]}(x, \omega) - E - i\eta)^{-1} \delta_k \rangle|. \end{aligned}$$

We have

$$
\begin{aligned}
&|\langle\delta_k, (H_{[1,N-2]}(x,\omega) - E - i\eta)^{-1}\delta_k\rangle| \\
&\quad = \frac{\|f^a_{[1,k-1]}(x)\|\,\|f^a_{[k+1,N-2]}(x)\|}{\|f^a_{[1,N-1]}(x)\|} \\
&\quad \le \frac{\|M^a_{[0,k-1]}(x)\|}{|\tilde{b}(x)|}\,\frac{\|M^a_{[k,N-1]}(x)\|}{|\tilde{b}(x+k\omega)||b(x+N\omega)|}\,\frac{2|\tilde{b}(x)||b(x+N\omega)|}{\|M^a_{[0,N-1]}(x)\|} \\
&\quad = \frac{2}{|\tilde{b}(x+k\omega)|}\mathcal{W}_{N,k}(x).
\end{aligned}
$$

At the same time we have

$$
|\langle\delta_k, (H_{[1,N-2]}(x,\omega) - E - i\eta)^{-1}\delta_k\rangle| \le \|(H_{[1,N-2]}(x,\omega) - E - i\eta)^{-1}\| \le \frac{1}{\eta},
$$

so we get

$$
|\sigma(H_{[1,N-2]}(x,\omega)) \cap [E-\eta, E+\eta]| \le 4\eta \sum_{k\notin\mathcal{K}} \frac{1}{|\tilde{b}(x+k\omega)|}\mathcal{W}_{N,k}(x) + 2|\mathcal{K}|,
$$

and the conclusion follows immediately. □

We will now see how to estimate $\mathcal{W}_{N,k}$ by using the Avalanche Principle. Given an interval $\Lambda = [\alpha, \beta]$ such that $0 \in \Lambda$ we will use the notation

$$
\mathcal{W}_\Lambda(x,\omega,E) = \frac{\|M^a_{[\alpha,0]}(x,\omega,E)\|\,\|M^a_{[1,\beta]}(x,\omega,E)\|}{\|M^a_{[\alpha,\beta]}(x,\omega,E)\|}.
$$

Lemma 6.3. *Let $(\omega, E) \in \mathbb{T}_{c,\alpha} \times \mathbb{C}$ such that $L(\omega, E) > \gamma > 0$ and. There exists a constant $N_0 = N_0(a, b, \omega, E, \gamma)$ such that if $N \ge N_0$ and Λ is an interval such that $\Lambda \supset [-|\Lambda|/4, |\Lambda|/4]$, $(\log N)^{1+} \le |\Lambda| \ll N$, then*

$$
\log|\mathcal{W}_{N,k}(x,\omega,E)| = \log|\mathcal{W}_\Lambda(x+(k-1)\omega,\omega,E)| + O(1)\exp(-|\Lambda|^{1-}),
$$

for $k \in [2|\Lambda|, N-2|\Lambda|]$ and $x \in \mathbb{T} \setminus \mathcal{B}_{N,\Lambda}(\omega,E)$, with $|\mathcal{B}_{N,\Lambda}| \le \exp(-|\Lambda|^{1-})$.

Proof. Fix $k \in [2|\Lambda|, N-2|\Lambda|]$. We can partition $[0, N-1]$ into intervals of size proportional to $|\Lambda|$ (between, say, $1/4|\Lambda|$ and $4|\Lambda|$) one of which is $(k-1)+\Lambda$. Partitioning $(k-1)+\Lambda$ as

$$
[\alpha+(k-1), k-1] \cup [k, \beta+(k-1)],
$$

we also induce partitions on $[0, k-1]$ and $[k, N-1]$. The conclusion follows by applying the Avalanche Principle expansion (i.e. using Lemma 2.7 and Proposition 2.1) to all three factors in the expression of $\mathcal{W}_{N,k}(x,\omega,E)$. □

We note that for $x \in \mathbb{T} \setminus \mathcal{B}_{N,\Lambda}$, with $\mathcal{B}_{N,\Lambda}$ as in the previous lemma, we have

$$\log \|M^a_{(k-1)+\Lambda}(x)\| \geq \log |f^a_{(k-1)+\Lambda}(x)| \geq |\Lambda| L^a - |\Lambda|^{1-}.$$

This, together with the uniform upper bound from Proposition 2.9, imply that

$$|\mathcal{W}_\Lambda(x + (k-1)\omega)| \leq \exp(|\Lambda|^{1-}).$$

Such an estimate is not good enough. It will be clear that we need $(\log |\Lambda|)^C$ instead of $|\Lambda|^{1-}$. While it is certainly possible to apply the large deviations estimate with a deviation of size $(\log |\Lambda|)^C$, the resulting exceptional set would be too large for the Avalanche Principle and also for bounding the integral of $|\sigma(H_N) \cap [E-\eta, E+\eta]|$ over it. This difficulty will be overcome by using Lemma 2.4.

We will also use the following standard estimate.

Lemma 6.4. *Let $\omega \in \mathbb{T}_{c,\alpha}$ and $p > 1$. There exists a constant $C_0(\omega, p)$ such that for any $N > 1$ and $\rho \gg 1/N$ we have*

$$\sum_{k \in S} \|k\omega\|^{-p} \leq C_0 N (\log N)^\alpha \rho^{1-p},$$

where

$$S = \{k \in [0, N-1] \colon \|k\omega\| \geq \rho\}.$$

Proof. Let $x_1 \leq \ldots \leq x_n$ be the elements of the set $\{k\omega (\mod 1) : k \in S\}$ and $x_0 = x_1 - 1/N$. Note that we have $x_0 \geq \rho/2$. Also, due to the Diophantine restriction on ω we have $x_{i+1} - x_i \geq C N^{-1} (\log N)^{-\alpha}$. We can now conclude that

$$\begin{aligned}\sum_{k \in S} \|k\omega\|^{-p} = \sum_{i=1}^n x_i^{-p} &\leq \sum_{i=1}^n \frac{1}{x_i - x_{i-1}} \int_{x_{i-1}}^{x_i} t^{-p}\, dt \\ &\leq CN(\log N)^\alpha \int_{\rho/2}^1 t^{-p}\, dt \leq C' N (\log N)^\alpha \rho^{1-p}. \qquad \square\end{aligned}$$

Proof of Theorem 1.1. We just have to prove the first part of the theorem. The second part follows from the first, Lemma 6.1, and (2.1).

Let $l = (\log N)^2$, $r_0 = \exp(-(\log l)^2)$, and $r_1 = r_0 \exp(-(\log l)^{C_0})$, with C_0 as in Proposition 5.3, with the given r_0 and $A = 10$. Let $\{x_j\}$ be a minimal set of points such that the disks $\mathcal{D}(x_j, r_1/2)$ cover $\mathbb{T}$. By Lemma 4.2 we can find intervals $\Lambda_j = [\alpha_j, \beta_j]$, $\alpha_j \simeq -l^7$, $\beta_j \simeq l^7$, such that α_j, β_j are adjusted to $(\mathcal{D}(x_j, r_0), \omega, E)$ at scale l. It follows from Proposition 5.3 that $f^a_{\Lambda_j}(\cdot, \omega, E)$ has at most $2d_0$ zeroes in $\mathcal{D}(x_j, r_1)$. Furthermore, for N large enough, $\tilde{b}$ has at most n_b zeroes in $\mathcal{D}(x_j + \omega, r_1)$. Therefore, from Lemma 2.4, Proposition 2.9, and Lemma 2.5 we get

$$\begin{aligned}&\Big|\frac{1}{\tilde{b}(x+\omega)}\mathcal{W}_{\Lambda_j}(x)\Big|\\&\quad\le \exp((\log l)^C)|x-\zeta_j|^{-2d_0}|x+\omega-\zeta_j'|^{-n_b}\\&\quad\le \exp((\log l)^C)\max(|x-\zeta_j|^{-(2d_0+n_b)}, |x+\omega-\zeta_j'|^{-(2d_0+n_b)})\end{aligned}$$

for all $x \in \mathcal{D}(x_j, r_1/2)$.

Let $\mathcal{B} = \cup\mathcal{B}_{N,\Lambda_j}$, with $\mathcal{B}_{N,\Lambda_j}$ as in Lemma 6.3 and $\mathcal{K}$ be the set of integers k that are not in $[l^8, N-l^8]$ (i.e., to which we cannot apply Lemma 6.3), such that $x+(k-1)\omega$ is at distance less than ρ_0 from the zeroes of $f^a_{\Lambda_j}$ in $\mathcal{D}(x_j, r_1)$, or such that $x + k\omega$ is at distance at least ρ_0 from the zeroes of $\tilde{b}$, with $\rho_0 \gg 1/N$ to be chosen later. We have that $|\mathcal{B}| \le \exp(-(\log N)^{14-})$ and

$$|\mathcal{K}| \lesssim d_0 N\exp((\log l)^C)\rho_0 + n_b N\rho_0 + (\log N)^C.$$

Applying Lemma 6.2 and Lemma 6.3 we get that for $x \in \mathbb{T}\setminus\mathcal{B}$ be have

$$\begin{aligned}&|\sigma(H_N(x,\omega))\cap[E-\eta, E+\eta]|\\&\quad\lesssim \eta\sum_{k\notin\mathcal{K}}\frac{1}{|\tilde{b}(x+k\omega)|}\mathcal{W}_{N,k}(x,\omega,E+i\eta) + |\mathcal{K}| \qquad (6.1)\\&\quad\lesssim N\eta\exp((\log l)^C)\rho_0^{1-(2d_0+n_b)} + |\mathcal{K}|.\end{aligned}$$

We obtained the $\rho_0^{1-(2d_0+n_b)}$ factor instead of a $\rho_0^{-(2d_0+n_b)}$ factor by using Lemma 6.4 (this is the reason for needing $\rho_0 \gg 1/N$). At this point we are essentially looking for a choice of ρ_0 such that

$$\eta\rho_0^{1-(2d_0+n_b)} + \rho_0 \lesssim \eta^p,$$

with p as large as possible. An elementary analysis yields that the largest possible Hölder exponent is $p = 1/(2d_0+n_b)$ and it is attained when $\rho_0 = \eta^p$. Now we get that for any $(1/N)^{1/p} \ll \eta \le 1/N$ (in fact, for the upper bound all we need is that $\eta^{0+}\exp((\log l)^C) \le 1$) we have

$$|\sigma(H_N(x,\omega))\cap[E-\eta, E+\eta]| \le N\eta^{p-},$$

for any $x \in \mathbb{T}\backslash\mathcal{B}$. Note that for (6.1) to hold we need to ensure that $L(E+i\eta,\omega) \gtrsim \gamma$. This is true for N large enough, by continuity of the Lyapunov exponent (see [8]; in fact, the continuity in the imaginary direction also follows immediately from the Thouless formula, see [13, Theorem 5.15]). Since for any $x \in \mathbb{T}$ we have

$$|\sigma(H_N(x,\omega)) \cap [E-\eta, E+\eta]| \leq N$$

and $|\mathcal{B}| \leq \exp(-(\log N)^{14-})$ it follows that

$$\int_{\mathbb{T}} |\sigma(H_N(x,\omega)) \cap [E-\eta, E+\eta]|\, dx \lesssim N\eta^{p-}.$$

Finally, let us note that to obtain the first part by using Lemma 6.1 one needs that $\eta^{p-} \gtrsim N^{-1}$, which is not a problem. □

Appendix A. Discussion of some results from Section 2

First we discuss Proposition 2.1 and Corollary 2.2. Proposition 2.1 for the determinants is just [3, Proposition 2.1] stated for general y instead of just $y = 0$. This is fine because the large deviations estimate depends only on the positivity of the Lyapunov exponent. In particular, the fact that the operator is Hermitian for $y = 0$ is not used. The statement for the other entries follows from the estimate for f_N^a. It is clear from (2.5) that one needs to control the deviations of b and $\tilde{b}$. This is easily achieved by applying the large deviations estimate for subharmonic functions [4, Theorem 3.8]. To get Corollary 2.2 we simply use the fact that

$$|NL^a(y,\omega,E) - NL^a(\omega,E)| \leq C(N|y| + (\log N)^2).$$

This follows from the estimates

$$0 \leq L_N^a(y,\omega,E) - L^a(y,\omega,E) < C\frac{(\log N)^2}{N}$$

and

$$|L_N^a(y,\omega,E) - L_N^a(\omega,E)| \leq C|y|$$

which were established in [2, Lemma 3.9, Corollary 3.13].

Next we prove Lemma 2.4. We will use the following formulation of Cartan's estimate (cf. [10, Theorem 11.4] and [6, Lemma 2.4]).

Lemma A.1. *Let ϕ be an analytic function on $\mathcal{D}(z_0, r_0)$, $z_0 \in \mathbb{C}$ and let m, M be such that*

$$\sup_{\mathcal{D}(z_0,r_0)} \log|\phi(z)| \le M, \quad m \le \log|\phi(z_0)|.$$

Given $H \gg 1$, there exists

$$\mathcal{B} = \bigcup_{j=1}^{K} \mathcal{D}(z_j, r_j), \quad K \lesssim H(M-m), \sum_{j=1}^{K} r_j \le r_0 \exp(-H),$$

such that

$$\log|\phi(z)| - M \gtrsim H(M-m),$$

for $z \in \mathcal{D}(z_0, r_0/6) \setminus \mathcal{B}$.

Proof. (of Lemma 2.4) From Corollary 2.2 with $H = -C \log r_0$, $C \gg 1$ we know that there exists z_1, $|z_1 - z_0| \ll r_0$ such that

$$\log|f_N^a(z_1)| > NL^a + (\log r_0)(\log N)^C.$$

We can now apply Cartan's estimate on $\mathcal{D}(z_1, 100r_0)$, with

$$H = -C \log r_0, \quad M = NL^a + (\log N)^C, \quad m = NL^a + (\log r_0)(\log N)^C,$$

to get that

$$\log|f_N^a(z)| > NL^a - (\log r_0)^2(\log N)^C,$$

for $z \in \mathcal{D}(z_0, r_0) \setminus \mathcal{B}$, with $\mathcal{B}$ as in Lemma A.1. We can guarantee that there exists $r \in (r_0/2, r_0)$ such that $\partial\mathcal{D}(z_0, r) \subset \mathcal{D}(z_0, r_0) \setminus \mathcal{B}$ and

$$\min_j \operatorname{dist}(\zeta_j, \partial\mathcal{D}(z_0, r)) \gtrsim \frac{r_0}{k_0 + 1}.$$

The minimum principle now implies that

$$\begin{aligned}\log\left|\frac{f_N^a(z)}{\prod(z-\zeta_j)}\right| &> NL^a - (\log r_0)^2(\log N)^C + k_0 \log c\frac{r_0}{k_0+1} \\ &> NL^a - 2(\log r_0)^2(\log N)^C,\end{aligned}$$

for $z \in \mathcal{D}(z_0, r)$. The conclusion follows immediately. □

Finally, we note that Lemma 2.5 follows analogously by using the large deviations estimate for subharmonic functions [4, Theorem 3.8].

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