

Asymptotic behaviour of the Hodge Laplacian spectrum on graph-like manifolds

Michela Egidi and Olaf Post

Abstract. We consider a family of compact, oriented and connected n -dimensional manifolds X_ε shrinking to a metric graph as $\varepsilon \rightarrow 0$ and describe the asymptotic behaviour of the eigenvalues of the Hodge Laplacian on X_ε . We apply our results to produce manifolds with spectral gaps of arbitrarily large size in the spectrum of the Hodge Laplacian.

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1. Introduction

1.1. Motivation. A graph-like manifold is a family of compact, oriented and connected n -dimensional Riemannian manifolds $\{X_\varepsilon\}_{\varepsilon>0}$ made of building blocks according to the structure of a given metric graph, i.e. a graph where to each edge is associated a length. The manifolds X_ε have the property that they shrink to

the metric graph as $\varepsilon \rightarrow 0$. A graph-like manifold is constructed from *edge neighbourhoods* $X_{\varepsilon,e} = [0, \ell_e] \times \varepsilon Y_e$ and *vertex neighbourhoods* $X_{\varepsilon,v}$ according to the underlying graph. The shrinking parameter ε is, roughly speaking, the radius of the tubular neighbourhood, or in other words, the length scaling factor of the transversal manifold Y_e at an edge e . We assume that all transversal manifolds are compact without boundary. A precise definition is given in Section 4.

Graph-like manifolds have been used in purely mathematical contexts as well as in applications in Physics. One prominent example in spectral geometry is given by Colin de Verdière in [10], where he proved that the first non-zero eigenvalue of a compact manifold of dimension $n \geq 3$ can have arbitrary large multiplicity (using balls as transversal manifolds). In Physics, graph-like manifolds, or more concrete, small neighbourhoods of metric graphs embedded in \mathbb{R}^n are used to model electronic or optic nano-structures. The natural question arising is if the underlying metric graph is a good approximation for the graph-like manifold.

The Laplacian $\Delta_{X_\varepsilon}^0$ on *functions* on graph-like manifolds has been analysed in detail, and the convergence of various objects such as resolvents (in a suitable sense), spectrum etc. is established in many contexts, see again [37, 15] for more details and references.

1.2. Aim of this article and main results. The aim of this article is to consider the eigenvalues of the Laplacian $\Delta_{X_\varepsilon}^\bullet$ acting on *differential forms* on the graph-like manifold and analyse their behaviour as $\varepsilon \rightarrow 0$. To fix the notation in more detail, denote by $\Delta_{X_\varepsilon}^p$ the Laplacian acting on p -forms on X_ε . Any p -form can be decomposed into its exact, co-exact and harmonic component (see (14) for details), and the Laplacian leaves this decomposition invariant. We denote the j -th eigenvalue of the Laplacian acting on exact, resp. co-exact p -forms, on X_ε counted with respect to multiplicity by $\bar{\lambda}_j^p(X_\varepsilon)$, resp. $\bar{\bar{\lambda}}_j^p(X_\varepsilon)$, ($j = 1, 2, \dots$) and call them for short *exact* and *co-exact eigenvalues*. (Throughout this article, we will use the labels $\bar{\bullet}$ and $\bar{\bar{\bullet}}$ for exact and co-exact eigenvalues of Laplacians, respectively.) By Hodge duality and “supersymmetry” (the exterior derivative d , resp. its (formal) adjoint d^* , is an isomorphism between co-exact and exact eigenforms, resp. vice versa, see the proof of Theorem 5.2), we have

$$\bar{\lambda}_j^p(X_\varepsilon) = \bar{\bar{\lambda}}_j^{n-p}(X_\varepsilon) \quad \text{and} \quad \bar{\bar{\lambda}}_j^p(X_\varepsilon) = \bar{\lambda}_j^{p-1}(X_\varepsilon), \quad (j = 1, 2, \dots) \quad (1)$$

so that it suffices to consider only the exact eigenvalues $\bar{\lambda}_j^p(X_\varepsilon)$ for $1 \leq p < n/2 + 1$ or the co-exact eigenvalues $\bar{\bar{\lambda}}_j^p(X_\varepsilon)$ for $0 \leq p < n/2$. In particular, if $n = 2$, then the entire (non-zero) spectrum of the differential form Laplacian is determined by its Laplacian on functions. This case can be considered as “trivial”

in this article, as the convergence for functions has already been established in earlier works (see again [37, 15] and references therein).

On the metric graph, we also have the notion of p -forms, but only co-exact 0-forms and exact 1-forms are non-trivial (apart from the harmonic forms determined by the topology of the graph). We denote the spectrum of the Laplacian on 0-forms (functions) and (exact) 1-forms on the graph X_0 by $\lambda_j^0(X_0)$ and $\lambda_j^1(X_0)$, respectively ($j = 1, 2, \dots$). Note that we have $\lambda_j^1(X_0) = \lambda_j^0(X_0)$ so that we simply write

$$\lambda_j(X_0) := \lambda_j^1(X_0) = \lambda_j^0(X_0) \quad (j = 1, 2, \dots) \tag{2}$$

and speak of *the* eigenvalues of the metric graph. To be consistent with the limit, we also set $\lambda_j(X_\varepsilon) := \bar{\lambda}_j^1(X_\varepsilon) = \bar{\lambda}_j^0(X_\varepsilon)$.

Recall that Y_e denotes the transversal manifold at the edge e and it is a compact and boundaryless manifold. Our main result of this article is now the following:

Theorem 1.1. *Let X_ε be a graph-like Riemannian n -dimensional compact manifold and X_0 its underlying metric graph, then the following is true.*

- (i) *The 0-form eigenvalues, or equivalently, the exact 1-form eigenvalues of X_ε converge to the eigenvalues of X_0 , i.e.,*

$$\lambda_j(X_\varepsilon) = \bar{\lambda}_j^1(X_\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} \lambda_j(X_0) \tag{3a}$$

for all $j = 1, 2, \dots$.

- (ii) *Assume that $n \geq 3$ and $2 \leq p \leq n - 1$, and that all transversal manifolds Y_e have trivial $(p - 1)$ -th cohomology group (i.e., $H^{p-1}(Y_e) = 0$ for all edges e), then we have*

$$\bar{\lambda}_1^p(X_\varepsilon) = \bar{\lambda}_1^{p-1}(X_\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} \infty \tag{3b}$$

for the first eigenvalue of exact p -forms on X_ε .

As a consequence of our eigenvalue ordering, all other eigenvalues $\bar{\lambda}_j^p(X_\varepsilon)$ ($j = 1, 2, \dots$) for $2 \leq p \leq n - 1$ diverge, too. We emphasise that the first part on the eigenvalue convergence is a simple consequence of the convergence for the eigenvalues on 0-forms (functions) already established in earlier works (see again [13, 37] and references therein).

We say that the graph-like manifold X_ε is *transversally trivial* if all transversal manifolds Y_e are Moore spaces, i.e., they have trivial cohomology in the sense that $H^p(Y_e) \neq 0$ only for $p = 0$ and $p = n - 1$ (see Example 4.1 for a construction of such manifolds).

If we assume that X_ε is transversally trivial, we can summarise our result as follows:

- for functions, the convergence result has been established before. Duality gives convergence for n -forms;
- for *exact* 1-forms, we also have convergence due to supersymmetry. Duality gives also convergence for co-exact $(n - 1)$ -forms;
- for *co-exact* 1-forms and hence exact $(n - 1)$ -forms, we have divergence (this case only appears if $n \geq 3$);
- all other p -forms with $2 \leq p \leq n - 2$ diverge (this case only appears if $n \geq 4$).

If the cohomology of Y_e is non-trivial, then the above list is still true, but divergence only happens for higher eigenvalues, i.e., $\bar{\lambda}_j^p(X_\varepsilon) = \bar{\lambda}_j^{p-1}(X_\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} \infty$ if $j \geq N$, where N can be computed using the $(p - 1)$ -th Betti numbers of the transversal manifolds Y_e , see Theorem 5.6 for details. It remains an open question what happens to the first $N - 1$ (non-zero) eigenvalues (see Rem. 5.7).

As a consequence of the above theorem, we obtain the following result (for the notion of Hausdorff convergence, see Section 6.1):

Corollary 1.2. *Assume that the graph-like manifold is transversally trivial (i.e., all transversal manifolds Y_e have trivial cohomology for $p = 1, \dots, n - 2$). Then the spectrum of the differential form Laplacian converges to the spectrum of the metric graph. More precisely, for all $\lambda_0 > 0$ we have that $\sigma(\Delta_{X_\varepsilon}^\bullet) \cap [0, \lambda_0]$ converges in Hausdorff distance to $\sigma(\Delta_{X_0}) \cap [0, \lambda_0]$.*

Furthermore, we asked ourselves about the relation between spectral gaps in the spectrum of the Laplacian acting on 1-forms on X_ε and X_0 , i.e., about intervals (a, b) not belonging to the spectrum. In particular, we have as immediate consequence of the asymptotic description of the spectrum in Theorem 1.1, resp. Corollary 1.2, the following result on spectral gaps (i.e. intervals *disjoint* with the spectrum).

Corollary 1.3. *Assume that the graph-like manifold is transversally trivial and suppose that (a_0, b_0) is a spectral gap for the metric graph X_0 then there exist $a_\varepsilon, b_\varepsilon$ with $a_\varepsilon \rightarrow a_0$ and $b_\varepsilon \rightarrow b_0$ such that $(a_\varepsilon, b_\varepsilon)$ is a spectral gap for the Hodge Laplacian on X_ε on all degrees, i.e., $\sigma(\Delta_{X_\varepsilon}^\bullet) \cap (a_\varepsilon, b_\varepsilon) = \emptyset$.*

In our applications below, $a_\varepsilon = a = 0$ (as 0 is always an eigenvalue of the (entire) Hodge Laplacian on all degrees), hence we can choose $(0, b_\varepsilon)$ as common spectral gap.

1.3. Related works

Bounds on first non-zero eigenvalues. There has been some work about the spectral gap at the bottom of the spectrum (i.e. estimates of the first non-zero eigenvalue). For example, one can consider the quantity

$$\kappa(L, X, g) := \lambda_1(L)(\text{vol}_n(X, g))^{m/n}, \quad (4)$$

where L is an elliptic operator of order m on the n -dimensional compact Riemannian manifold (X, g) (the powers assure that κ is scale-invariant, i.e., if $\tilde{g} = \tau^2 g$ for some constant $\tau > 0$, then $\tilde{L} = \tau^{-m} L$ and $\text{vol}_n(X, \tilde{g}) = \tau^n (\text{vol}_n(X, g))$, hence $\kappa(\tilde{L}, X, \tilde{g}) = \kappa(L, X, g)$). Berger [2] asked whether

$$\sup_{g \text{ metric on } X} \kappa(\Delta_{(X,g)}, X, g)$$

is finite on a given manifold X . The answer is yes in dimension 2 with constant

$$\kappa(\Delta_{(X,g)}, X, g) \leq 8\pi(\gamma(X) + 1) \quad (5)$$

for a surface of genus $\gamma(X)$ ([40]). Our analysis shows that the bound is optimal in the sense that for any $\delta > 0$ there is a sequence of Riemannian surfaces (X_i, g_i) of genus $\gamma(X_i) \rightarrow \infty$ (graph-like manifolds based on Ramanujan graphs) such that

$$\kappa(\Delta_{(X_i, g_i)}, X_i, g_i) \approx \gamma(X_i)^{1-\delta}, \quad (6)$$

i.e., the bound $\gamma(X_i)$ in (5) is asymptotically optimal (see Corollary 6.4). This result is very much in the spirit of [8], where Colbois and Girouard construct graph-like manifolds X_i based on Ramanujan graphs. They use as operator L the Dirichlet-to-Neumann operator on ∂X_i with metric h_i , and they show that $\kappa(\Lambda_{(\partial X_i, h_i)}, \partial X_i, h_i)/\gamma(X_i)$ is uniformly bounded, in particular, $\kappa(\Lambda_{(\partial X_i, h_i)}, \partial X_i, h_i) \rightarrow \infty$ as $i \rightarrow \infty$.

For higher dimensions, the answer to the finiteness of $\kappa(\Delta_{(X,g)}, X, g)$ is no, as on a compact manifold X of dimension 3 or higher, there are sequences of metrics g_i such that $\kappa(\Delta_{(X,g_i)}, X, g_i) \rightarrow \infty$ as $i \rightarrow \infty$ (see e.g. [7] and references therein).

For L being the Laplacian on p -forms ($2 \leq p \leq n - 2$) with $n \geq 4$, there are also examples of metrics g_i on a given manifold such that $\kappa(L, X, g_i)$ tends to infinity (see [17]). Actually, they proved the result for *exact* p -forms ($2 \leq p \leq n - 1$), allowing also $n = 3$. We rediscover their result (Proposition 6.1). Unfortunately, it seems to be impossible to construct a sequence of metrics such that $\kappa_i^\bullet := \kappa(\Delta_{(X, g_i)}^\bullet, X, g_i)$ tends to infinity for the *entire* Hodge Laplacian $\Delta_{(X, g_i)}^\bullet$ acting on all degrees (including functions), as on any manifold X of

dimension $n \geq 3$ there is a sequence of metrics such that the (rescaled) exact p -form spectrum diverges, while the function spectrum converges (Corollary 6.6).

Conjecture 1.4. *We conjecture that on a compact manifold X of dimension $n \geq 3$, we have a uniform bound on the entire (non-zero) Hodge Laplace spectrum*

$$\sup_{g \text{ metric on } X} \kappa(\Delta_{(X,g)}^\bullet, X, g) < \infty. \quad (7)$$

This conjecture is consistent with the case $n = 2$, as the Hodge Laplacian on a surface is entirely determined by the function spectrum, and hence (7) holds as well.

Colbois and Maerten [9] have shown that on any compact manifold X of dimension $n \geq 2$ there is a sequence of metrics g_i such that $\kappa(\nabla^* \nabla_{(X,g_i)}^p, X, g_i) \rightarrow \infty$, where $\nabla^* \nabla_{(X,g_i)}^p$ denotes the rough (Bochner) Laplacian on p -forms, $1 \leq p \leq n - 1$.

Other related works. There is another research line for “collapsing” manifolds, where one considers families of manifolds with singular limit, but under some curvature bounds. Such a limit usually induces extra structure. We refer to [20] and [25] or [26] for an overview. Let us stress that our graph-like manifolds X_ε always have curvature tending to $\pm\infty$ as $\varepsilon \rightarrow 0$.

Moreover, Jammes showed in [21] (see also the references therein) that on any compact manifold of dimension $n \geq 6$, one can prescribe the volume and any finite part of the spectrum of the Hodge Laplacian acting on p -forms if $1 \leq p < n/2$ and $n \geq 6$ (in the spirit of Colin de Verdière [10, 11], who treated the function case $p = 0$). There are related results on constructing metrics such that the Hodge Laplacian has certain spectral properties also in [19, 18]. In all these “spectral engineering” papers, graph-like manifolds play a prominent role. Chanillo and Trèves showed in [6] a lower bound on the entire Hodge Laplace spectrum in terms of certain “admissible” coverings of the manifold. Indeed, the result they found is not entirely true, as was pointed out by Mantuano (see Remark 4.3 in [28]). The corrected version can be found in [28, Theorem 4.1].

1.4. Organisation of the paper. The paper is organised as follows. In Section 2 we briefly describe discrete and metric graphs and their associated Laplacians on functions and on 1-forms. In Section 3 we review some facts on Laplacians on differential forms as well as some useful facts about their eigenvalues. Section 4 is dedicated to the description of graph-like manifolds and their harmonic forms.

Section 5 contains the proof of our main result Theorem 1.1, namely the convergence of exact 1-forms and the divergence result for higher degree forms. Finally in Section 6 we apply our results to establish the existence of spectral gaps for families of metric graphs and their graph-like manifolds. Moreover, we construct example of (families of) manifolds with (upper or lower) bounds on the first eigenvalue of the Laplacian acting on functions or forms.

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2. Discrete and metric graphs and their Laplacians

The following material sets the notation; for more references and details we refer e.g. to [37, 3] and the references therein. The consideration of functions and forms on discrete and metric graphs can also be found in [36], see also [16].

2.1. Discrete graphs and their Laplacians. Let $G = (V, E, \partial)$ be a finite discrete graph, i.e., $V = V(G)$ and $E = E(G)$ are finite sets (*vertices* and *edges* respectively) and $\partial: E \rightarrow V \times V$ is such that $e \mapsto (\partial_-e, \partial_+e)$ associates to an edge its initial and terminal vertex fixing an orientation for the graph, crucial when working with 1-forms. *We assume (without stating it each time) that all discrete graphs are connected.*

For each vertex $v \in V$ we denote with

$$E_v^\pm = \{e \in E \mid \partial_\pm e = v\}$$

the set of *incoming* and *outgoing edges* at a vertex v and with

$$E_v = E_v^- \cup E_v^+$$

(disjoint union) the set of vertices emanating from v . The *degree of a vertex* is the number of emanating edges, i.e.,

$$\deg v := |E_v|.$$

Note that we allow loops, i.e., $\partial_-e = \partial_+e = v$, and each loop is counted twice in $\deg v$ (as we have taken the disjoint union in $E_v^- \cup E_v^+$). We also allow multiple edges, i.e., edges with the same starting and ending point.

Assume that $G = (V, E, \partial)$ is a discrete graph and $\ell: E \rightarrow (0, \infty)$ a map associating to each edge a number $\ell_e > 0$ (its “length”, as we will interpret it below). Given a function $F: V \rightarrow \mathbb{C}$ on the vertex space of G (or a vector $F \in V^{\mathbb{C}}$, if you prefer), the *discrete Laplacian* (on functions) $\Delta_G = \Delta_G^0$ is defined as

$$(\Delta_G F)(v) = -\frac{1}{\deg v} \sum_{e \in E_v} \frac{1}{\ell_e} (F(v_e) - F(v)),$$

where v_e is the vertex on the opposite site of v on $e \in E_v$. We note that Δ_G can also be defined as $\Delta_G = d_G^* d_G$ where

$$d_G: \ell_2(V, \deg) \longrightarrow \ell_2(E, \ell^{-1}), \quad (d_G F)_e = F(\partial_+ e) - F(\partial_- e),$$

and where $\ell_2(V, \deg) = \mathbb{C}^V$, resp. $\ell_2(E, \ell^{-1}) = \mathbb{C}^E$, carry the norms given by

$$\|F\|_{\ell_2(V, \deg)}^2 = \sum_{v \in V} |F(v)|^2 \deg v \quad \text{and} \quad \|\eta\|_{\ell_2(E, \ell^{-1})}^2 = \sum_{e \in E} |\eta|^2 \frac{1}{\ell_e}$$

and where d_G^* is its adjoint operator with respect to the corresponding inner products. We can equally define a Laplacian on 1-forms by $\Delta_G^1 := d_G d_G^*$, acting on $\ell_2(E, \ell^{-1})$.

2.2. Metric graphs and their Laplacians. Let $G = (V, E, \partial)$ be a discrete graph and $\ell: E \rightarrow (0, \infty)$ a function associating to each edge $e \in E$ a number $\ell_e > 0$ which we will interpret as *length* as follows. We define a *metric graph* associated with the discrete graph G as the quotient

$$X_0 := \bigcup_{e \in E} I_e / \sim,$$

where $I_e := [0, \ell_e]$ and \sim is the relation identifying the end points of the intervals I_e according to the graph: namely, $x \sim y$ if and only if $\psi(x) = \psi(y)$ where $\psi: \bigcup_{e \in E} I_e \rightarrow V$, $0 \in I_e \mapsto \partial_- e$, $\ell_e \in I_e \mapsto \partial_+ e$ and $\psi(x) = x$ for $x \in \bigcup_{e \in E} (0, \ell_e)$.

On a metric graph, we have a natural measure (the Lebesgue measure on each interval) allowing us to define a natural Hilbert space of functions and 1-forms by

$$L^2(X_0) = \bigoplus_{e \in E} L^2(I_e) \quad \text{and} \quad L^2(\Lambda^1(X_0)) = \bigoplus_{e \in E} L^2(\Lambda^1(I_e))$$

with norms given by

$$\|f\|_{L^2(X_0)}^2 := \sum_{e \in E} \|f_e\|_{L^2(I_e)}^2 \quad \text{and} \quad \|\alpha\|_{L^2(\Lambda^1(X_0))}^2 := \sum_{e \in E} \|\alpha_e\|_{L^2(\Lambda^1(I_e))}^2$$

for functions $f: X_0 \rightarrow \mathbb{C}$, $f = (f_e)_{e \in E}$ and 1-forms $\alpha = (\alpha_e)_{e \in E} = (g_e \, ds_e)_{e \in E}$ on X_0 , respectively. Note that functions on I_e can obviously be identified with 1-forms via $g_e \mapsto g_e \, ds_e$; the difference of forms and functions appears only in the domain of the corresponding differential operators below.

We define the *exterior derivative* $d = d_{X_0}$ on X_0 as the operator

$$d: \text{dom } d \longrightarrow L^2(\Lambda^1(X_0)), \quad d(f_e)_{e \in E} = (f'_e \, ds_e)_{e \in E}$$

with domain

$$\text{dom } d = H^1(X_0) \cap C(X_0)$$

where $H^1(X_0) = \{f \in L^2(X_0) \mid f' = (f'_e)_{e \in E} \in L^2(X_0)\}$ and where $C(X_0)$ denotes the space of continuous functions on X_0 .

It is not difficult to see that $d = d_{X_0}$ is a *closed* operator with adjoint given by

$$d^*(g_e \, ds_e)_{e \in E} = -(g'_e)_{e \in E}$$

with domain

$$\text{dom } d^* = \left\{ \alpha \in H^1(\Lambda^1(X_0)) \mid \sum_{e \in E} \tilde{\alpha}_e(v) = 0 \right\}$$

with $H^1(\Lambda^1(X_0)) = \{\alpha = (g_e \, ds_e)_{e \in E} \in L^2(\Lambda^1(X_0)) \mid g'_e \in L^2(I_e)\}$ where $\tilde{\alpha}$ is the oriented evaluation of α , i.e.,

$$\tilde{\alpha}_e(v) = \begin{cases} -g_e(0), & v = \partial_- e, \\ g_e(\ell_e), & v = \partial_+ e. \end{cases}$$

The Laplacians acting on functions and 1-forms defined on X_0 are the operators

$$\Delta_{X_0}^0 = d^*d \quad \text{where } \text{dom } \Delta_{X_0}^0 = \{\alpha \in \text{dom } d \mid d\alpha \in \text{dom } d^*\}$$

and

$$\Delta_{X_0}^1 = dd^* \quad \text{where } \text{dom } \Delta_{X_0}^1 = \{\alpha \in \text{dom } d^* \mid d^*\alpha \in \text{dom } d\}.$$

Writing the vertex conditions for the Laplacian on functions explicitly, we obtain the conditions

$$f_e(v) := \begin{cases} f_e(0), & v = \partial_- e, \\ f_e(\ell_e), & v = \partial_+ e, \end{cases} \text{ is independent of } e \in E_v \text{ and } \sum_{e \in E_v} \tilde{f}'_e(v) = 0, \tag{8}$$

called *standard* or *Kirchhoff* vertex conditions. The first condition can be rephrased as *continuity* of f on the metric graph, while the second is a *flux conservation* considering the derivative $df = (f'_e)_e$ as vector field.

Remark 2.1. We remark that $\Delta_{X_0}^0 = d_{X_0}^* d_{X_0}$ and $\Delta_{X_0}^1 = d_{X_0} d_{X_0}^*$ are both non-negative operators and fulfil a “supersymmetry” condition in the sense of [36, Section 1.2]. As a consequence, their non-zero spectrum including multiplicity are the same ([36, Proposition 1.2], see also the proof of Theorem 5.2). This remark applies also to the discrete graph Laplacians $\Delta_G^0 = d_G^* d_G$ and $\Delta_G^1 = d_G d_G^*$ of Section 2.2, as well as to the co-exact and exact Laplacian $\bar{\Delta}_{X_\varepsilon}^{p-1} = d_{X_\varepsilon}^* d_{X_\varepsilon}$ and $\bar{\Delta}_{X_\varepsilon}^p = d_{X_\varepsilon} d_{X_\varepsilon}^*$, respectively, explaining the second relation in (1).

Finally, we remind the reader that whenever the graph is equilateral, i.e., $\ell_e = \ell_0$ for all $e \in E$, the spectra of the discrete Laplacian and the metric Laplacian on 0-forms are related in the following sense. Let

$$\Sigma := \{(j\pi/\ell_0)^2 \mid j = 1, 2, \dots\}$$

be the Dirichlet spectrum of the interval $[0, \ell_0]$, then

$$\lambda \in \sigma(\Delta_{X_0}^0) \iff \varphi(\lambda) := 1 - \cos(\ell_0 \sqrt{\lambda}) \in \sigma(\Delta_G) \tag{9}$$

for all $\lambda \notin \Sigma$ (see e.g. [33, 4] or [37, Section 2.4.1]), and we have the obvious relation also for 1-forms due to Remark 2.1. There is also a relation at the bottom of the spectrum of Δ_G and Δ_{X_0} for *general* (not necessarily equilateral) metric graphs for which we refer to [36, Section 6.1] or [37, Section 2.4.2] for more details.

2.3. Discrete and metric Ramanujan graphs. A discrete graph G is k -regular, if all its vertices have degree k . For ease of notation we assume here that the graph $G = (V, E, \partial)$ is *simple*, and we write $v \sim w$ for adjacent vertices.

Definition 2.2. Let G be a k -regular discrete graph with n vertices and let Δ_G be its (normalised) discrete Laplacian. The graph G is said to be *Ramanujan* if

$$\max\{|1 - \mu| \mid \mu \in \sigma(\Delta_G)\} \leq \frac{2\sqrt{k-1}}{k}.$$

Note that many authors use the eigenvalues of the *adjacency matrix* A_G as the spectrum of a graph. The adjacency matrix is given by $(A_G)_{v,w} = 1$ if $v \sim w$ and $(A_G)_{v,w} = 0$. As $v \sim w$ is equivalent with $w \sim v$, the adjacency matrix is symmetric. For a k -regular graph, we have the relation

$$A_G = k(\text{id} - \Delta_G), \quad \text{or} \quad \Delta_G = \text{id} - \frac{1}{k} A_G \tag{10}$$

with the discrete graph Laplacian (with “length” function $\ell_e = 1$, see Section 2.1).

Note that $\sigma(A_G) \subset [-k, k]$ and $\sigma(\Delta_G) \subset [0, 2]$, and that k , resp. 0 , is always an eigenvalue of A_G , resp. Δ_G , while $-k$, resp. 2 , is an eigenvalue of A_G , resp. Δ_G , if and only if the graph is bipartite (recall that we assume that G is a *finite* graph).

We define the (*maximal*) *spectral gap length* of a discrete graph by

$$\begin{aligned} \gamma(G) &:= \min\{\mu, 2 - \mu \mid \mu \in \sigma(\Delta_G) \setminus \{0, 2\}\} \\ &= 1 - \frac{1}{k} \max\{|\alpha| \mid \alpha \in \sigma(A_G), |\alpha| < k\} \end{aligned} \quad (11)$$

i.e., $\gamma(G)$ is the distance from the non-trivial spectrum 0 (resp. 0 and 2 in the bipartite case) of the Laplacian Δ_G from $\{0, 2\}$ resp. $\{-k, k\}$. Hence, a graph is a Ramanujan graph if its spectral gap length has size at least

$$\gamma(G) \geq 1 - \frac{2\sqrt{k-1}}{k}.$$

It has been shown that the lower bound is optimal, i.e., for any k -regular graph (or even for any graph with maximal degree k) with diameter large enough, the spectral gap length is smaller than $1 - 2\sqrt{k-1}/k + \eta$, where $1/\eta$ is of the same order as the diameter (see [34, Theorem 1] and references therein).

The existence of *infinite* families $\{G^i\}_{i \in \mathbb{N}}$ of k -regular graphs has been shown whenever k is a prime or a power of a prime (see e.g. [27, 30, 32]). Recently, the existence of infinite families of regular bipartite Ramanujan graphs of every degree $k > 2$ has been proved in [29] by showing that any bipartite Ramanujan graph has a 2-lift which is again Ramanujan, bipartite and has twice as many vertices.

Let $\{G^i\}_{i \in \mathbb{N}}$ be a family of Ramanujan graphs such that

$$v_i := |V(G^i)| \longrightarrow \infty \quad (12)$$

and consider the associated family of equilateral metric graphs $\{X_0^i\}_{i \in \mathbb{N}}$ of length ℓ_0 . By (9), the metric graph Laplacians $\Delta_{X_0^i}$ all have a spectral gap

$$(a_0, b_0) = \left(0, \frac{h}{\ell_0^2}\right) \quad \text{with } h = h_k := \arccos^2\left(1 - \frac{2\sqrt{k-1}}{k}\right) > 0 \quad (13)$$

at the bottom of the spectrum.

3. The Hodge Laplacian and their eigenvalues

In this section, we collect some general facts on differential forms, the Hodge Laplacian and its spectrum.

3.1. Differential forms and the Hodge Laplacian. Let (M, g) be a compact, oriented and connected n -dimensional Riemannian manifold. Its Riemannian metric g induces the L^2 -space of p -forms

$$L^2(\Lambda^p(M, g)) = \left\{ \omega: M \rightarrow \mathbb{C} \mid \|\omega\|_{L^2(\Lambda^p(M, g))}^2 = \int_M |\omega|_g^2 \, d\text{vol}_g \, M < \infty \right\}$$

where

$$\|\omega\|_{L^2(\Lambda^p(M, g))}^2 = \langle \omega, \omega \rangle_{L^2(\Lambda^p(M, g))} := \int_M |\omega|_g^2 \, d\text{vol}_g \, M = \int_M \omega \wedge * \omega$$

and where $*$ denotes the Hodge star operator (depending on g). The Laplacian on p -forms on M is formally defined as $\Delta_{(M, g)}^p = \Delta^p = d\delta + d\delta$ where d is the classical exterior derivative and $\delta = (-1)^{np+n+1} * d *$ its formal adjoint with respect to the inner product induced by g . If M has no boundary, then δ is the L^2 -adjoint of d and Δ^p is a non-negative self-adjoint operator with discrete spectrum denoted by $\lambda_j^p(M, g)$ (repeated according to multiplicity).

We allow that M has a boundary ∂M , itself a smooth manifold of dimension $n - 1$. As in the function case, it is possible to impose boundary conditions for functions in the domain of the Hodge Laplacian. To do so, we first decompose a p -form ω in its tangential and normal components on ∂M , i.e., $\omega = \omega_{\text{tan}} + \omega_{\text{norm}}$ where ω_{tan} can be considered as a form on ∂M while $\omega_{\text{norm}} = dr \wedge \omega^\perp$ with ω^\perp being a form on ∂M and r being the distance from ∂M .

The Hodge Laplacian with *absolute boundary conditions* is given by those forms ω such that

$$\omega_{\text{norm}} = 0 \quad \text{and} \quad (d\omega)_{\text{norm}} = 0$$

while *relative boundary conditions* require

$$\omega_{\text{tan}} = 0 \quad \text{and} \quad (\delta\omega)_{\text{tan}} = 0.$$

These boundary conditions give rise to two unbounded and self-adjoint operators Δ^{abs} and Δ^{rel} with discrete spectrum, the *Hodge Laplacians* with *absolute* and *relative boundary conditions*, respectively (see e.g. [5] or [31]). Recall that for functions, the absolute correspond to Neumann while the relative correspond to Dirichlet boundary conditions.

Furthermore, since the Hodge star operator exchanges absolute and relative boundary conditions, there is a correspondence between the spectrum of Δ^{abs} and the spectrum of Δ^{rel} , which allows us to study just one of them to cover both cases. *In the sequel, we will only consider **absolute** boundary conditions if the manifold has a boundary, and hence we will mostly suppress the label $(\cdot)^{\text{abs}}$ for ease of notation.*

In an L^2 -framework, we consider d and δ_0 as unbounded operators, defined as the closures \bar{d} and $\bar{\delta}_0$ of d and δ_0 on

$$\text{dom } d = \{\omega \in C^\infty(\Lambda^p(M, g)) \mid d\omega \in L^2(\Lambda^{p+1}(M, g))\},$$

$$\text{dom } \delta_0 = \{\omega \in C^\infty(\Lambda^p(M, g)) \mid \delta\omega \in L^2(\Lambda^{p+1}(M, g)), \omega_{\text{norm}} = 0\},$$

respectively. The Hodge Laplacian with absolute boundary condition is then given by

$$\Delta = \Delta^{\text{abs}} = \bar{d} \bar{\delta}_0 + \bar{\delta}_0 \bar{d}$$

For this operator, Hodge Theory is still valid. In particular, the de Rham theorem holds (see [31, Section 2.1] and references therein): the space of harmonic p -forms (with absolute boundary conditions if the boundary is non-empty) $\mathcal{H}^p(M, g)$, is isomorphic to the p -th de Rham cohomology, $H^p(M)$, and any p -form $\omega \in L^2(\Lambda^p(M, g))$ can be orthogonally decomposed into an *exact* ($d\bar{\omega}$), *co-exact* ($\delta\bar{\bar{\omega}}$) and *harmonic* (ω_0) component, i.e.,

$$\omega = d\bar{\omega} + \delta\bar{\bar{\omega}} + \omega_0 \tag{14}$$

where $\bar{\omega} \in \text{dom } \bar{d}$ is a $(p - 1)$ -form, $\bar{\bar{\omega}} \in \text{dom } \bar{\delta}_0$ is a $(p + 1)$ -form and ω_0 is a harmonic p -form. Moreover, the Hodge Laplacian leaves these spaces invariant and maps p -forms into p -forms. In particular, we can consider the eigenvalues of the Hodge Laplacian acting on exact and co-exact p -forms as $\bar{d} \bar{\delta}_0$ and $\bar{\delta}_0 \bar{d}$, called here *exact* and *co-exact (absolute) p -form eigenvalues*, denoted by

$$\bar{\lambda}_j^p(M, g) \quad \text{and} \quad \bar{\bar{\lambda}}_j^p(M, g),$$

respectively. Let $\bar{E}^p(\lambda) = \ker(d \delta_0 - \lambda)$ and $\bar{\bar{E}}^p(\lambda) = \ker(\delta_0 d - \lambda)$ denote the eigenspaces of exact and co-exact p -forms with eigenvalue λ (as the eigenforms are smooth by elliptic regularity, we can omit the closures). Since d is an isomorphism between $\bar{\bar{E}}^{p-1}(\lambda)$ and $\bar{E}^p(\lambda)$, we have the second equality of (1). For the first equality, we use the Hodge star operator (it interchanges absolute and relative boundary conditions, but in (1) we only consider boundaryless manifolds).

3.2. An estimate from below for exact eigenvalues. We now introduce a simplified but useful version of an estimate from below on the first eigenvalue of the exact p -form Laplacian on a manifold by McGowan ([31, Lemma 2.3]) also used in Gentile and Pagliara in [17, Lemma 1].

Let (M, g) be a n -dimensional compact Riemannian manifold without boundary and let $\{U_i\}_{i=1}^m$ be an open cover of M such that $U_{ij} = U_i \cap U_j$ have a smooth boundary. Moreover, we denote by

$$I_i := \{j \in \{1, \dots, i - 1, i + 1, \dots, m\} \mid U_i \cap U_j \neq \emptyset\}.$$

the index set of *neighbours of U_i* . We say that the cover $\{U_i\}_i$ has *no intersection of degree r* if and only if $U_{i_1} \cap \dots \cap U_{i_r} = \emptyset$ for any r -tuple (i_1, \dots, i_r) with $1 \leq i_1 < i_2 < \dots < i_r \leq m$. We choose a fixed partition of unity $\{\rho_j\}_{j=1}^m$ subordinate to the open cover and we set $\|d\rho\|_\infty := \max_i \sup_{x \in U_i} |d\rho_i(x)|_g$.

Furthermore, we denote by $\bar{\lambda}_1^{p,\text{abs}}(U)$ the first positive eigenvalue on exact p -forms on U satisfying absolute boundary conditions on ∂U . Finally, denote by $H^p(U_{ij})$ the p -th cohomology group of U_{ij} .

Proposition 3.1. *Let M and $\{U_i\}_{i=1}^m$ be as above. Assume that the open cover has no intersection of degree higher than 2 and $H^{p-1}(U_{ij}) = 0$ for all i, j . Then, the first positive eigenvalue of the Laplacian acting on exact p -forms on M satisfies*

$$\bar{\lambda}_1^p(M) \geq \frac{2^{-3}}{\mathfrak{N}_1},$$

where

$$\mathfrak{N}_1 := \sum_{i=1}^m \left(\frac{1}{\bar{\lambda}_1^{p,\text{abs}}(U_i)} + \sum_{j \in I_i} \left(\frac{c_{n,p} \|d\rho\|_\infty^2}{\bar{\lambda}_1^{p-1,\text{abs}}(U_{ij})} + 1 \right) \left(\frac{1}{\bar{\lambda}_1^{p,\text{abs}}(U_i)} + \frac{1}{\bar{\lambda}_1^{p,\text{abs}}(U_j)} \right) \right)$$

and $c_{n,p}$ is a combinatorial constant depending only on p and n .

We remark that these assumptions impose a topological restriction on the manifold as such an open cover does not necessarily exist. Actually, the following general version holds for *higher* exact eigenvalues:

Proposition 3.2. *Let M and $\{U_i\}_i$ be as above and assume that the open cover has no intersection of degree higher than 2. We set $N_1 = \sum_{i,j} \dim H^{p-1}(U_{i,j})$ and $N = N_1 + 1$. Then, the N -th eigenvalue of the Laplacian on exact p -forms on M satisfies*

$$\bar{\lambda}_N^p(M) \geq \frac{1}{\mathfrak{N}_2},$$

where

$$\mathfrak{N}_2 := \sum_{i=1}^m \left(\frac{1}{\bar{\lambda}_1^{p,\text{abs}}(U_i)} + \sum_{j \in I_i} \left(\frac{\|d\rho\|_\infty^p}{\bar{\lambda}_1^{p-1,\text{abs}}(U_{ij})} + 1 \right) \left(\frac{1}{\bar{\lambda}_1^{p,\text{abs}}(U_i)} + \frac{1}{\bar{\lambda}_1^{p,\text{abs}}(U_j)} \right) \right).$$

The proof of this proposition uses the same argument of the proof of McGowan’s lemma. The generalisation to p -forms is trivial since we have particular assumptions on the cover, i.e., no intersections of degree higher than 2 (see the remark after the proof of his “technical lemma” [31, Lemma 2.3]).

3.3. A variational characterisation of exact eigenvalues. We will make use of the following characterisation of eigenvalues of the Hodge Laplacian acting on exact p -forms by Dodziuk [12, Proposition 3.1] whose proof can be found in [31, Proposition 2.1]. Its advantage is that it does not make use of the adjoint δ of the exterior derivative, and hence the metric g does not enter in a complicated way.

Proposition 3.3. *Let M be a compact Riemannian manifold, then the spectrum of the Laplacian $0 < \bar{\lambda}_1^p \leq \bar{\lambda}_2^p \leq \dots$ on exact p -forms on M satisfying absolute boundary conditions can be computed by*

$$\bar{\lambda}_j^p(M) = \inf_{V_j} \sup \left\{ \frac{\langle \eta, \eta \rangle_{L^2(\Lambda^p(M))}}{\langle \theta, \theta \rangle_{L^2(\Lambda^{p-1}(M))}} \mid \eta \in V_j \setminus \{0\} \text{ such that } \eta = d\theta \right\},$$

where V_j ranges over all j -dimensional subspaces of smooth exact p -forms and θ is a smooth $(p - 1)$ -form.

The advantage of this characterisation is that the metric only enters via the L^2 -norm, and no derivatives of the metric or its coefficients are needed.

As a consequence we have (see [12, Proposition 3.3] or [31, Lem. 2.2]):

Proposition 3.4. *Assume that g and \tilde{g} are two Riemannian metrics on M such that $c_-^2 g \leq \tilde{g} \leq c_+^2 g$ for some constants $0 < c_- \leq c_+ < \infty$, i.e.,*

$$c_-^2 g_x(\xi, \xi) \leq \tilde{g}_x(\xi, \xi) \leq c_+^2 g_x(\xi, \xi) \quad \text{for all } \xi \in T_x^* M \text{ and } x \in M,$$

then the eigenvalues of exact p -forms with absolute boundary conditions fulfil

$$\frac{1}{c_-^2} \left(\frac{c_-}{c_+} \right)^{n+2p} \bar{\lambda}_j^p(M, g) \leq \bar{\lambda}_j^p(M, \tilde{g}) \leq \frac{1}{c_+^2} \left(\frac{c_+}{c_-} \right)^{n+2p} \bar{\lambda}_j^p(M, g)$$

for all $j = 1, 2, \dots$

As a consequence, the eigenvalues $\bar{\lambda}_j^p(M, g)$ depend continuously on g in the sup-norm defined, e.g., in [37, Section 5.2]. In particular, this proposition allows us to consider also perturbation of graph-like manifolds as defined in the next section. For a discussion of possible cases we refer to [37, Sections 5.2–5.6]). As an example, we could consider tubular neighbourhoods of graphs embedded in \mathbb{R}^n .

3.4. Scaling behaviour. We say that a Riemannian manifold M_ε with a metric g_ε is ε -homothetic, if $(M_\varepsilon, g_\varepsilon)$ is conformally equivalent with a Riemannian manifold (M, g) with (constant) conformal factor ε^2 , i.e., $g_\varepsilon = \varepsilon^2 g$. For short, we write $M_\varepsilon = \varepsilon M$. It is often convenient to think of the Riemannian manifold M_ε as the (ε -independent) manifold M with metric $g_\varepsilon = \varepsilon^2 g$.

Obviously, the scaling with a constant factor leads to the following simple result for p -forms on a Riemannian manifold.

Lemma 3.5. *Let ω be a p -form on a n -dimensional Riemannian manifold M with metric g , and let εM be the Riemannian manifold $(M, \varepsilon^2 g)$, then we have*

$$\|\omega\|_{L^2(\Lambda^p(\varepsilon M))}^2 = \varepsilon^{n-2p} \|\omega\|_{L^2(\Lambda^p(M))}^2 \tag{15a}$$

and

$$\bar{\lambda}_j^p(\varepsilon M) = \varepsilon^{-2} \bar{\lambda}_1^p(M) \tag{15b}$$

Proof. The first assertion follows from the fact that we have $|w|_{\varepsilon^2 g}^2 = \varepsilon^{-2p} |w|_g^2$ and $\text{dvol}_{\varepsilon^2 g} M = \varepsilon^n \text{dvol}_g M$ pointwise. The second follows from the variational characterisation of the j -th eigenvalue of Proposition 3.3, as we have the scaling behaviour

$$\frac{\|\eta\|_{L^2(\Lambda^p(\varepsilon M))}^2}{\|\theta\|_{L^2(\Lambda^{p-1}(\varepsilon M))}^2} = \frac{\varepsilon^{n-2p} \|\eta\|_{L^2(\Lambda^p(M))}^2}{\varepsilon^{n-2(p-1)} \|\theta\|_{L^2(\Lambda^{p-1}(M))}^2} = \varepsilon^{-2} \frac{\|\eta\|_{L^2(\Lambda^p(M))}^2}{\|\theta\|_{L^2(\Lambda^{p-1}(M))}^2}.$$

Note that the condition $\eta = d\theta$ is independent of the metric. (This is the advantage of the characterisation of Proposition 3.3). □

4. Graph-like manifolds and their harmonic forms

4.1. Graph-like manifolds. A graph-like manifold associated with a metric graph X_0 is a family of oriented and connected n -dimensional Riemannian manifolds $(X_\varepsilon)_{0 < \varepsilon \leq \varepsilon_0}$ (ε_0 small enough) shrinking to X_0 as $\varepsilon \rightarrow 0$ in the following sense. We assume that X_ε decomposes as

$$X_\varepsilon = \bigcup_{e \in E} X_{\varepsilon,e} \cup \bigcup_{v \in V} X_{\varepsilon,v}, \tag{16}$$

where $X_{\varepsilon,v}$ and $X_{\varepsilon,e}$ are called *edge* and *vertex neighbourhood*, respectively. More precisely, we assume that $X_{\varepsilon,v}$ and $X_{\varepsilon,e}$ are closed subsets of X_ε such that

$$X_{\varepsilon,v} \cap X_{\varepsilon,e} = \begin{cases} Y_{\varepsilon,e} & e \in E_v, \\ \emptyset & e \notin E_v, \end{cases}$$

where $Y_{\varepsilon,e}$ is a boundaryless smooth connected Riemannian manifold of dimension $n - 1$. Furthermore, we assume that $X_{\varepsilon,v}$ is ε -homothetic to a fixed connected Riemannian manifold X_v (with metric g_v) as well as $Y_{\varepsilon,e}$ is ε -homothetic to a fixed Riemannian manifold Y_e (with metric h_e), i.e., $X_{\varepsilon,v} = \varepsilon X_v$ and $Y_{\varepsilon,e} = \varepsilon Y_e$. In particular, X_v is a manifold with boundary, whose boundary is $\bigcup_{e \in E_v} Y_e$. Moreover, we assume that $X_{\varepsilon,e}$ is isometric to the product $I_e \times \varepsilon Y_e$. If g_ε denotes the metric of X_ε and $g_{\varepsilon,e}$, resp. $g_{\varepsilon,v}$, the restriction of g_ε to the edge, resp. vertex neighbourhood, then we have

$$g_{\varepsilon,e} = ds^2 + \varepsilon^2 h_e \quad \text{and} \quad g_{\varepsilon,v} = \varepsilon^2 g_v \quad (17)$$

(after some obvious identifications). We often refer to a single manifold X_ε as graph-like manifold instead of the family $(X_\varepsilon)_\varepsilon$ as in the definition above.

Assume for simplicity that $\text{vol}_{n-1} Y_e = 1$ for all $e \in E$ (the general case would lead to the weighted vertex condition $\sum_{e \in E_v} (\text{vol}_{n-1} Y_e) \widehat{f}'_e(v) = 0$ instead of (8) for the metric graph Laplacian, see [37, 14] for details).

We call a graph-like manifold $(X_\varepsilon)_\varepsilon$ *transversally trivial* if all transversal manifolds are Moore spaces, i.e., if $H^p(Y_e) = 0$ for all $1 \leq p \leq n - 2$ and all $e \in E$. Note that a member of a transversally trivial graph-like manifold X_ε is not necessarily homotopy-equivalent to the metric graph X_0 , as the vertex neighbourhoods need not to be contractible.

Example 4.1. Let us construct a typical example of a transversally trivial graph-like manifold. Let $n \geq 2$. For each vertex v fix a manifold \widehat{X}_v . Remove $\text{deg } v$ open balls from \widehat{X}_v hence the resulting manifold X_v has a boundary consisting of $\text{deg } v$ many components each diffeomorphic to an $(n - 1)$ -sphere S^{n-1} . For $e \in E_v$ let $Y_e = S^{n-1}$ with a metric such that its volume is 1. As (unscaled) edge neighbourhood, we choose $X_{1,e} := [0, \ell_e] \times Y_e$ with the product metric. Then we can construct a graph-like (topological) manifold X_1 with a canonical decomposition as in (16) (for $\varepsilon = 1$) by identifying the e -th boundary component of X_v with the corresponding end of the edge neighbourhood $X_{1,e}$. By a small local change we can assume that the resulting manifold X_1 is smooth; the corresponding family of graph-like manifolds $(X_\varepsilon)_{\varepsilon > 0}$ is now given as above by choosing the metric accordingly.

Remark 4.2. Let X be a compact manifold without boundary. The aim of the remark is to show that X can be turned into a graph-like manifold with underlying metric graph being a finite tree graph: think of “growing” the tree out of the

original manifold. More formally, construct a graph-like manifold according to a tree graph and leave one cylinder of a leaf (a vertex of degree 1) “uncapped”; glue the original manifold X with one disc removed together with the free cylinder. Obviously, the resulting manifold is homeomorphic to the original manifold X .

We could also modify X such that it becomes a graph-like manifold with respect to topologically more complicated graphs.

We can now define on X_ε the corresponding Hilbert spaces of p -forms as in Section 3.1. Since X_ε has no boundary, the formal adjoint δ of d is also its Hilbert space adjoint. Moreover, the Hodge Laplacian on p -forms on X_ε is given by $\Delta_{X_\varepsilon}^p = dd^* + d^*d$ where $d = d_{X_\varepsilon}$ and $d^* = d_{X_\varepsilon}^*$ are the classical exterior derivative and co-derivative on a manifold (as unbounded operators in the L^2 -spaces).

4.2. Harmonic forms. For completeness we finally turn to the dimension of the class of harmonic 1-forms.

For the graph, this dimension is given by its first Betti number, i.e., $b_1(X_0) = |E| - |V| + 1$, while for the manifold X_ε it is given by the dimension of its first cohomology group $H^1(X_\varepsilon)$. Since X_ε arises from the graph X_0 , the dimension of $H^1(X_\varepsilon)$ is the sum of $b_1(X_0)$ and the dimension of a subset of the first cohomology group of $\bigcup_{v \in V} X_{\varepsilon,v}$, meaning that the graph-like manifold inherits part of the topology of the underlying metric graph.

In particular, if Y_e has trivial p -th cohomology group for $1 \leq p \leq n - 2$, i.e., $H^p(Y_e) = 0$ for all $e \in E$, then the cohomology groups of X_ε can be computed explicitly. Using Mayer-Vietoris sequence, the natural splitting (16) and Poincaré duality, we obtain

$$H^k(X_\varepsilon) = \begin{cases} \mathbb{R} & k \in \{0, n\}, \\ \bigoplus_{v \in V} H^1(X_v) \oplus H^1(X_0) & k \in \{1, n - 1\}, \\ \bigoplus_{v \in V} H^k(X_v) & k \in \{2, \dots, n - 2\}. \end{cases}$$

For the general case, i.e., when some or all of the Y_e have non-trivial p -th cohomology groups for $1 \leq p \leq n - 2$, we do not have a general formula. However, again using Mayer-Vietoris sequence, it is possible to compute the cohomology groups explicitly for concrete examples of edge and vertex neighbourhoods.

5. Proof of the main theorem

Let us now prove the main result of this article. The convergence result of our main theorem, i.e., (3a) of Theorem 1.1, is more or less “trivial” in the sense that it follows from previous convergence results for functions by a simple supersymmetry argument.

The divergence (3b) of Theorem 1.1 is new and proven in Subsection 5.3. As preparation, we need some estimates of exact p -forms with absolute boundary conditions on the building blocks of our graph-like manifolds provided in Subsection 5.2.

5.1. Convergence for exact 1-forms. Let X_ε be a compact graph-like manifold as constructed in Section 4.1 associated with a metric graph X_e . We have already noticed in (1)–(2) and in Remark 2.1, that the co-exact 1-form eigenvalues equal the (exact) 0-form eigenvalues, i.e., the eigenvalues of the Laplacian on functions on X_0 and X_ε . For the functions, we have the following result, first proven in the manifold case in [13] (based on the results [23, 39]). For a detailed overview and detailed proofs of the result, we refer to [37].

Denote by $\lambda_j(X_\varepsilon)$ and $\lambda_j(X_0)$ the eigenvalues (in increasing order, repeated according to their multiplicity) of the Laplacian acting on functions on the manifold and the metric graph (see (8) for the metric graph Laplacian).

Proposition 5.1 ([13, 37]). *Let X_ε be a compact graph-like manifold associated with a metric graph X_e . Then we have*

$$|\lambda_j(X_\varepsilon) - \lambda_j(X_0)| = \mathcal{O}(\varepsilon^{1/2}/\ell_0) \quad \text{for all } j = 1, 2, \dots$$

where $\ell_0 = \min_e\{\ell_e, 1\} > 0$ denotes the minimal edge length. Moreover, the error depends only on j , and the building blocks X_v, Y_e of the graph-like manifold.

We will need the precise dependency on the edge length and other parameters in Section 6 when considering *families* of metric graphs and graph-like manifolds. The exact statement on the error term follows from a combination of Theorems 6.4.1, 7.1.2 and 4.6.4 of [37].

Denote by $\bar{\lambda}_j^1(X_\varepsilon)$ and $\bar{\lambda}_j^1(X_0)$ the j -th eigenvalues of the exact 1-form Laplacian on X_ε and X_0 , respectively. The above-mentioned convergence for the eigenvalues for functions immediately gives the convergence for exact 1-forms, using a simple supersymmetry argument as in [36, Section 1.2]. For the convenience of the reader, we give a simple proof here.

Theorem 5.2. *Let X_ε be a graph-like manifold with underlying metric graph X_0 . Denote by $\bar{\lambda}_j^1(X_\varepsilon)$ and $\bar{\lambda}_j^1(X_0)$ the j -th exact 1-form eigenvalue on X_ε and X_0 , respectively. Then*

$$\bar{\lambda}_j^1(X_\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} \bar{\lambda}_j^1(X_0) \quad \text{for all } j = 1, 2, \dots$$

Proof. We will just show that the eigenspaces for non-zero eigenvalues of $\Delta_{X_\varepsilon}^1 = \Delta^1 = dd^*$ and $\Delta_{X_0}^0 = \Delta^0 = d^*d$ are isomorphic (the argument works for $\varepsilon > 0$ and $\varepsilon = 0$ as well). The convergence result then follows immediately from Proposition 5.1.

As isomorphism, we choose

$$d: \ker(\Delta^0 - \lambda) \longrightarrow \ker(\Delta^1 - \lambda)$$

for $\lambda \neq 0$. First, note that if $f \in \ker(\Delta^0 - \lambda)$, then

$$\Delta^1 df = dd^*df = d\Delta^0 f = \lambda f,$$

i.e., $df \in \ker(\Delta^1 - \lambda)$, hence the above map is properly defined. The map d as above is injective. If $df = 0$ for $f \in \ker(\Delta^0 - \lambda)$ then $\lambda f = \Delta^0 f = d^*df = 0$. As $\lambda \neq 0$ we have $f = 0$. For the surjectivity, let $\alpha \in \ker(\Delta^1 - \lambda)$. Set $f := \lambda^{-1}d^*\alpha$ (we use again that $\lambda \neq 0$). Then

$$df = d(\lambda^{-1}d^*\alpha) = \lambda^{-1}\Delta^1\alpha = \alpha,$$

i.e., d as above is surjective. In particular, we have shown that the spectrum of Δ^0 and Δ^1 away from 0 is the same, including multiplicity. \square

5.2. Eigenvalue asymptotics on the building blocks. We will now provide some eigenvalue asymptotics for eigenvalues of exact p -forms with absolute boundary conditions on the building blocks of our graph-like manifold. These asymptotics are needed in order to make use of the eigenvalue estimate from below of Proposition 3.1.

A vertex neighbourhood $X_{\varepsilon,v}$ is by definition ε -homothetic, i.e., $X_{\varepsilon,v} = \varepsilon X_v$. As a result of Lemma 3.5 we have:

Corollary 5.3. *Let $X_{\varepsilon,v}$ be a vertex neighbourhood of a graph-like manifold X_ε . Then, the smallest positive eigenvalue of the Laplacian acting on exact p -forms on $X_{\varepsilon,v}$ with absolute boundary conditions satisfies*

$$\bar{\lambda}_1^p(X_{\varepsilon,v}) = \varepsilon^{-2}\bar{\lambda}_1^p(X_v). \tag{18}$$

For the edge neighbourhood, we have to work a little bit more. Note that we cannot make use of the product structure of the underlying space as the product does not respect exact and co-exact forms.

Proposition 5.4. *Let $X_{\varepsilon,e}$ be an edge neighbourhood of a n -dimensional graph-like manifold X_ε . Then, the smallest eigenvalue of the Laplacian acting on exact p -forms ($2 \leq p \leq n - 1$) with absolute boundary conditions satisfies*

$$\bar{\lambda}_1^p(X_{\varepsilon,e}) = \varepsilon^{-2}c_p(\varepsilon), \tag{19}$$

where $c_p(\varepsilon) \rightarrow \bar{\lambda}_1^p(Y_e) > 0$ as $\varepsilon \rightarrow 0$, and where $\bar{\lambda}_1^p(Y_e)$ denotes the first eigenvalue of the Laplacian acting on exact p -forms on Y_e .

Proof. By Proposition 3.3 we have to analyse the quotient $\|\eta\|^2/\|\theta\|^2$ for an exact p -form η and a $(p - 1)$ -form θ such that $\eta = d\theta$. Recall that $X_{\varepsilon,e} = I_e \times \varepsilon Y_e$ (i.e., $I_e \times Y_e$ with metric $g_{\varepsilon,e} = ds^2 + \varepsilon^2 h_e$). Then, the $(p - 1)$ -form θ on $X_{\varepsilon,e}$ can be written as

$$\theta = \theta_1 \wedge ds + \theta_2 \tag{20}$$

where θ_1 , resp. θ_2 , is a $(p - 2)$ -form, resp. $(p - 1)$ -form, on Y_e . Using the scaling behaviour of the metric in a similar way as in Lemma 3.5, we have

$$\begin{aligned} \|\theta\|_{L^2(\Lambda^{p-1}(X_{\varepsilon,e}))}^2 &= \int_{X_{\varepsilon,e}} |\theta|_{g_{\varepsilon,e}}^2 \, \text{dvol } X_{\varepsilon,e} \\ &= \int_{I_e} \int_{Y_e} (\varepsilon^{-2(p-2)}|\theta_1|_{h_e}^2 + \varepsilon^{-2(p-1)}|\theta_2|_{h_e}^2) \varepsilon^{n-1} \, ds \, \text{dvol } Y_e \\ &= \varepsilon^{n-2p+1} \int_{I_e} \int_{Y_e} (\varepsilon^2|\theta_1|_{h_e}^2 + |\theta_2|_{h_e}^2) \, ds \, \text{dvol } Y_e, \end{aligned} \tag{21}$$

where the ε -factors appears due to the scaled metric $\varepsilon^2 h_e$. The decomposition of $d\theta$ according to (20) is given by

$$d\theta = (d_{Y_e}\theta_1 + \partial_s\theta_2) \wedge ds + d_{Y_e}\theta_2, \tag{22}$$

hence

$$\begin{aligned} \|d\theta\|_{L^2(\Lambda^p(X_{\varepsilon,e}))}^2 &= \int_{X_{\varepsilon,e}} |d\theta|_{g_{\varepsilon,e}}^2 \, \text{dvol } X_{\varepsilon,e} \\ &= \int_{I_e} \int_{Y_e} (\varepsilon^{-2(p-1)}|d_{Y_e}\theta_1 + \partial_s\theta_2|_{h_e}^2 + \varepsilon^{-2p}|d_{Y_e}\theta_2|_{h_e}^2) \varepsilon^{n-1} \, ds \, \text{dvol } Y_e \\ &= \varepsilon^{n-2p-1} \int_{I_e} \int_{Y_e} (\varepsilon^2|d_{Y_e}\theta_1 + \partial_s\theta_2|_{h_e}^2 + |d_{Y_e}\theta_2|_{h_e}^2) \, ds \, \text{dvol } Y_e. \end{aligned} \tag{23}$$

In particular, if we substitute (21) and (23) into the quotient $\|\eta\|^2/\|\theta\|^2$ we conclude

$$\frac{\|d\theta\|_{L^2(\Lambda^p(X_{\varepsilon,e}))}^2}{\|\theta\|_{L^2(\Lambda^{p-1}(X_{\varepsilon,e}))}^2} = \varepsilon^{-2} \frac{\int_{I_e} \int_{Y_e} (\varepsilon^2 |\theta_1 + \partial_s \theta_2|_{h_e}^2 + |d_{Y_e} \theta_2|_{h_e}^2) \, ds \, \text{dvol } Y_e}{\int_{I_e} \int_{Y_e} (\varepsilon^2 |\theta_1|_{h_e}^2 + |\theta_2|_{h_e}^2) \, ds \, \text{dvol } Y_e}.$$

In particular, together with Proposition 3.3 this yields

$$\bar{\lambda}_1^p(X_{\varepsilon,e}) = \varepsilon^{-2} c_p(\varepsilon)$$

with

$$c_p(\varepsilon) = \sup \left\{ \frac{\int_{I_e} \int_{Y_e} (\varepsilon^2 |\theta_1 + \partial_s \theta_2|_{h_e}^2 + |d_{Y_e} \theta_2|_{h_e}^2) \, ds \, \text{dvol } Y_e}{\int_{I_e} \int_{Y_e} (\varepsilon^2 |\theta_1|_{h_e}^2 + |\theta_2|_{h_e}^2) \, ds \, \text{dvol } Y_e} \mid \begin{array}{l} \theta = \theta_1 \wedge ds + \theta_2 \neq 0, \\ \theta_1 \text{ (} p-2\text{)-form,} \\ \theta_2 \text{ (} p-1\text{)-form} \end{array} \right\}.$$

In the limit $\varepsilon \rightarrow 0$, this constant tends to a number $c_p(0)$ given by

$$c_p(0) = \sup \left\{ \frac{\int_{I_e} \int_{Y_e} |d_{Y_e} \theta_2|_{h_e}^2 \, ds \, \text{dvol } Y_e}{\int_{I_e} \int_{Y_e} |\theta_2|_{h_e}^2 \, ds \, \text{dvol } Y_e} \mid \theta_2 \neq 0 \text{ (} p-1\text{)-form} \right\}.$$

This constant is the min-max characterisation of the first eigenvalue of the operator $\text{id} \otimes \bar{\Delta}_{Y_e}^p$ acting on $L^2(I_e) \otimes L^2(\Lambda^p(Y_e))$, whose spectrum agrees with the one of $\bar{\Delta}_{Y_e}^p$ (see e.g. [38, Theorem XIII.34]). Hence, we have $c_p(0) = \bar{\lambda}_1^p(Y_e)$. \square

5.3. Divergence for co-exact forms. We will assume for the rest of this section that $n \geq 3$. If $\dim X_e = 2$, then the spectrum of exact and co-exact 1-forms coincide by duality. Hence, the spectrum of the Hodge Laplacian is entirely determined by the spectrum on functions, and hence its behaviour is covered by the results of Subsection 5.1.

We now come to the proof of the divergence of our main theorem, namely to (3b) of Theorem 1.1. We will make use of Proposition 3.1 for $2 \leq p \leq n - 1$ assuming that $H^{p-1}(Y_e) = 0$ for all $e \in E$. Then, we will briefly explain how

the same argument works for $p = 2$ and non-trivial cohomology $H^1(Y_e) \neq 0$ for some $e \in E$ using Proposition 3.2.

Let $H^1(Y_e) = 0$ for all $e \in E$. Let

$$\mathcal{U}_\varepsilon = \{U_{\varepsilon,v}\}_{v \in V} \cup \{X_{\varepsilon,e}\}_{e \in E}$$

be an open cover of X_ε , where $U_{\varepsilon,v}$ is the open ε -neighbourhood of $X_{\varepsilon,v}$ in X_ε , or in other words, a slightly enlarged vertex neighbourhood $X_{\varepsilon,v}$ to ensure that \mathcal{U}_ε is an open cover.

It is easily seen that \mathcal{U}_ε has intersection up to degree 2 only (no three different sets of \mathcal{U}_ε have non-trivial intersection). The intersections of degree 2 are given by $X_{\varepsilon,v,e} = U_{\varepsilon,v} \cap X_{\varepsilon,e}$ which is empty ($e \notin E_v$) or otherwise isometric to the product $(0, \varepsilon) \times Y_{\varepsilon,e}$, hence ε -homothetic with the product $(0, 1) \times Y_e$ (recall that we enlarged $X_{\varepsilon,v}$ by an ε -neighbourhood). Moreover, $X_{\varepsilon,v,e}$ is homeomorphic to $(0, 1) \times Y_e$, and hence homotopy-equivalent with Y_e . In particular, $H^{p-1}(X_{\varepsilon,v,e}) = H^{p-1}(Y_e)$.

Recall that $\bar{\lambda}_j^p(X_\varepsilon)$ denotes the j -th exact p -form eigenvalue on X_ε , which equals the j -th co-exact $(p - 1)$ -eigenvalue $\bar{\bar{\lambda}}_j^{p-1}(X_\varepsilon)$. We assume $n \geq 3$, as in dimension 2 the Hodge Laplace spectrum is entirely determined by the scalar case. Denote by $H^p(Y_e)$ the p -th cohomology group of the transversal manifold Y_e of the edge neighbourhood $X_{\varepsilon,e}$.

Theorem 5.5. *Let X_ε be a graph-like manifold of dimension $n \geq 3$ with underlying metric graph X_0 . Assume that $2 \leq p \leq n - 1$ and that the $(p - 1)$ -th cohomology group of the transversal manifold Y_e vanishes for all $e \in E$, i.e., $H^{p-1}(Y_e) = 0$. Then, the first eigenvalue of the Laplacian acting on exact p -forms on X_ε satisfies*

$$\bar{\lambda}_1^p(X_\varepsilon) \geq \tau_p \varepsilon^{-2},$$

where $\tau_p > 0$ is a constant depending only on the building blocks X_v and Y_e of the graph-like manifold, the minimal length $\ell_0 = \min_{e \in E} \{\ell_e, 1\}$ and p . In particular, all eigenvalues $\bar{\lambda}_j^p(X_\varepsilon)$ of exact p -forms and all eigenvalues $\bar{\bar{\lambda}}_j^{p-1}(X_\varepsilon)$ of co-exact $(p - 1)$ -forms tend to ∞ as $\varepsilon \rightarrow 0$.

Proof. We will apply Proposition 3.1 to the manifold X_ε and the cover \mathcal{U}_ε (having no intersection of degree higher than 2). The assumptions on the cohomology are fulfilled as the $(p - 1)$ -th cohomology of the intersections of degree 2 of the cover vanishes (as we have already stated above). We first look at the denominator of the right hand side of the estimate in Proposition 3.1 and obtain in our situation

here

$$\begin{aligned}
 & \sum_{v \in V} \left(\frac{1}{\bar{\lambda}_1^p(X_{\varepsilon,v})} + \sum_{e \in E_v} \left(\frac{c_{n,p} \|d\rho_\varepsilon\|_\infty^2}{\bar{\lambda}_1^{p-1}(X_{\varepsilon,v,e})} + 1 \right) \left(\frac{1}{\bar{\lambda}_1^p(X_{\varepsilon,v})} + \frac{1}{\bar{\lambda}_1^p(X_{\varepsilon,e})} \right) \right) \\
 & + \sum_{e \in E} \left(\frac{1}{\bar{\lambda}_1^p(X_{\varepsilon,e})} + \sum_{v=\partial_\pm e} \left(\frac{c_{n,p} \|d\rho_\varepsilon\|_\infty^2}{\bar{\lambda}_1^{p-1}(X_{\varepsilon,v,e})} + 1 \right) \left(\frac{1}{\bar{\lambda}_1^p(X_{\varepsilon,v})} + \frac{1}{\bar{\lambda}_1^p(X_{\varepsilon,e})} \right) \right) \\
 & = \sum_{v \in V} \left(\frac{1}{\bar{\lambda}_1^p(X_{\varepsilon,v})} + \frac{\deg v}{\bar{\lambda}_1^p(X_{\varepsilon,e})} \right. \\
 & \quad \left. + 2 \sum_{e \in E_v} \left(\frac{c_{n,p} \|d\rho_\varepsilon\|_\infty^2}{\bar{\lambda}_1^{p-1}(X_{\varepsilon,v,e})} + 1 \right) \left(\frac{1}{\bar{\lambda}_1^p(X_{\varepsilon,v})} + \frac{1}{\bar{\lambda}_1^p(X_{\varepsilon,e})} \right) \right) \\
 & = \varepsilon^2 \sum_{v \in V} \left(\frac{1}{\bar{\lambda}_1^p(X_v)} + \frac{\deg v}{c_p(\varepsilon)} \right. \\
 & \quad \left. + 2 \sum_{e \in E_v} \left(\frac{c_{n,p} \varepsilon^2 \|d\rho_\varepsilon\|_\infty^2}{\bar{\lambda}_1^{p-1}(X_{v,e})} + 1 \right) \left(\frac{1}{\bar{\lambda}_1^p(X_v)} + \frac{1}{c_p(\varepsilon)} \right) \right) \\
 & =: \varepsilon^2 C_p(\varepsilon).
 \end{aligned}$$

Note that the cover \mathcal{U}_ε is labelled by $v \in V$ and $e \in E$, and we have rewritten the sum over the edges as a sum over the vertices (leading to the extra term with $\deg v$ and the factor 2) for the second equality. For the third equality, we have used the scaling behaviour of the eigenvalues in equations (18) and (19), and a similar one for the ε -homothetic overlap manifold $X_{\varepsilon,v,e}$.

Let us now analyse the constant $C_p(\varepsilon)$ as $\varepsilon \rightarrow 0$. First, we have seen in Proposition 5.4 that $c_p(\varepsilon) \rightarrow \bar{\lambda}_1^p(Y_e) > 0$. Moreover, the norm of the derivative of the partition of unit norm depends on ε as these functions have to change from 0 to 1 on a length scale of order ε on the vertex neighbourhoods and on a length scale of order ℓ_0 on the edge neighbourhood, hence the derivative is of order $\varepsilon^{-1} + \ell_0^{-1}$ and $\varepsilon^2 \|d\rho_\varepsilon\|_\infty^2 = \mathcal{O}(1) + \mathcal{O}((\varepsilon/\ell_0)^2)$ (we will need the dependency on ℓ_0 for Section 6 when we allow ℓ_0 also to depend on e). In particular, $C_p(\varepsilon) \rightarrow C_p(0)$ as $\varepsilon \rightarrow 0$ provided ε/ℓ_0 remains bounded, where $C_p(0)$ depends only on some data of the building blocks.

Proposition 3.1 now gives

$$\bar{\lambda}_1^p(X_\varepsilon) \geq \frac{2^{-3}}{\varepsilon^2 C_p(\varepsilon)}$$

which proves our assertion. □

Removing the assumption of vanishing cohomology groups we have the following theorem whose proof follows the line of the previous one where we use Proposition 3.2 for the estimate of a higher eigenvalue for exact p -forms on X_ε .

Theorem 5.6. *Let X_ε be a graph-like manifold of dimension $n \geq 3$ with underlying metric graph X_0 . Then the N -th eigenvalue of the Laplacian acting on exact p -forms on X_ε satisfies*

$$\bar{\lambda}_N^p(X_\varepsilon) \geq \tilde{\tau}_p \varepsilon^{-2},$$

where $\tilde{\tau}_p > 0$ is as before and where

$$N = 1 + \sum_{v \in V} \sum_{e \in E_v} \dim H^{p-1}(Y_e) = 1 + 2 \sum_{e \in E} \dim H^{p-1}(Y_e).$$

Remark 5.7. We point out that the first $N - 1$ eigenvalues of the Theorem above are strictly positive since we consider the spectrum away from zero. However, it is an open question how the eigenvalues behave asymptotically as $\varepsilon \rightarrow 0$.

6. Examples

Let us discuss some consequences of our asymptotic description of the Hodge Laplacian spectrum.

6.1. Hausdorff convergence of the spectrum and spectral gaps. Let us first come to Corollary 1.2, the Hausdorff convergence of the entire Hodge Laplace spectrum. Let $A, B \subset \mathbb{R}$ be two compact sets. The *Hausdorff distance* of A and B is defined as

$$d(A, B) := \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}, \quad \text{where } d(a, B) := \inf_{b \in B} |a - b|. \quad (24)$$

A sequence $(A_n)_n$ of compact sets $A_n \subset \mathbb{R}$ converges in Hausdorff distance to A_0 if and only if $d(A_n, A) \rightarrow 0$ as $n \rightarrow \infty$. In particular, $d(A_n, A) \rightarrow 0$ if and only if for all $\lambda_0 \in A_0$ there exists $\lambda_\varepsilon \in A_\varepsilon$ such that $|\lambda_0 - \lambda_\varepsilon| \rightarrow 0$ and for all $x \in \mathbb{R} \setminus A_0$ there exists $\eta > 0$ such that $[x - \eta, x + \eta] \cap A_\varepsilon = \emptyset$ for ε sufficiently small (see e.g. [37, Proposition A.1.6]).

Corollary 1.2 about spectral convergence is now an immediate consequence of Theorem 1.1, as in a compact interval $[0, \lambda_0]$, eventually all divergent eigenvalues from higher forms leave this interval, and the remaining ones converge.

A spectral gap of an operator $\Delta \geq 0$ is a non-empty interval (a, b) such that

$$\sigma(\Delta) \cap (a, b) = \emptyset.$$

Corollary 1.3 on spectral gap is again an immediate consequence of the Hausdorff convergence of Corollary 1.2 under the assumption that the manifold is transversally trivial (i.e., all transversal manifolds Y_e have trivial p -th cohomology for all $1 \leq p \leq n - 2$).

Examples of manifolds with spectral gaps can be generated in different ways. In [35, 24] we constructed (non-compact) abelian covering manifolds having an arbitrary large number of gaps in their essential spectrum of the scalar Laplacian, and in [1], we extended the analysis to the Hodge Laplacian on certain manifolds.

One can construct metric graphs with spectral gaps (and hence graph-like manifolds with spectral gaps) with a technique called *graph decoration* that works as follows. We consider a finite metric graph X_0 and a second finite metric graph \tilde{X}_0 . For each $v \in V(X_0)$, let $\tilde{X}_0 \times \{v\}$ be a copy of a finite metric graph \tilde{X}_0 . Fix a vertex \tilde{v} of \tilde{X}_0 . Then the graph decoration of X_0 with the graph \tilde{X}_0 is the graph obtained from X_0 by identifying the vertex \tilde{v} of $\tilde{X}_0 \times \{v\}$ with v . This decoration opens up a gap in the spectrum of the Laplacian on function on X_0 as described in [22] and therefore in its 1-form Laplacian. Consequently, the associated graph-like manifold has a spectral gap in its 1-form Laplacian (and no spectrum away from 0 for higher forms due to the divergence).

More examples of *families* of graphs and their graph-like manifolds with spectral gaps are given in Section 6.3.

6.2. Manifolds with special spectral properties. Let $(X_\varepsilon)_{\varepsilon>0}$ be a graph-like manifold constructed from a metric graph X_0 with underlying graph (V, E, ∂) . We assume that the graph-like manifold is transversally trivial, i.e., all transversal manifolds Y_e have trivial homology $H^p(Y_e) = 0$ for all $1 \leq p \leq n - 2$. An example of a construction of such graph-like manifolds is given in Example 4.1.

For simplicity, we assume that X_0 is equilateral, i.e., all edge lengths are given by a number $\ell > 0$. (One can easily extend the results to the case when $c_-\ell \leq \ell_e \leq c_+\ell$ for all $e \in E$ and some constants $c_\pm > 0$.)

We write

$$a_\varepsilon \lesssim b_\varepsilon, \quad a_\varepsilon \gtrsim b_\varepsilon, \quad a_\varepsilon \simeq b_\varepsilon \tag{25}$$

if

$$a_\varepsilon \leq \text{const}_+ b_\varepsilon, \quad a_\varepsilon \geq \text{const}_- b_\varepsilon, \quad \text{const}_- a_\varepsilon \leq b_\varepsilon \leq \text{const}_+ a_\varepsilon \tag{25'}$$

for all $\varepsilon > 0$ small enough and constants const_\pm independent of ε .

Let us first summarise the asymptotic spectral behaviour of a graph-like manifold X_ε and its dependence on the parameters ε , ℓ , $|V|$, and $|E|$. In particular, we have for 0-forms, exact p -forms and co-exact $(p - 1)$ -forms and the volume:

$$|\lambda_j^0(X_\varepsilon) - \lambda_j^0(X_0)| \lesssim \frac{\varepsilon^{1/2}}{\ell_0} \quad (\ell_0 = \min\{\ell, 1\}), \tag{26}$$

$$\bar{\lambda}_1^p(X_\varepsilon) = \bar{\lambda}_1^{p-1}(X_\varepsilon) \gtrsim \frac{1}{\varepsilon^2|E|(1 + \varepsilon^2/\ell^2)}, \tag{27}$$

$$\text{vol } X_\varepsilon \approx \varepsilon^n|V| + \varepsilon^{n-1}\ell|E|, \tag{28}$$

where the constants in \lesssim etc. depend only on the building blocks X_v and Y_e of the (unscaled, i.e. $\varepsilon = 1$) graph-like manifold. Equation (27) follows from analysing the lower bound constant τ_p in Theorem 5.5 (or Theorem 5.6). We see that the constant $C_p(\varepsilon)$ in its proof is bounded from above by

$$C_p(\varepsilon) \lesssim (|V| + |E|(1 + \varepsilon^2/\ell^2)) \lesssim |E|(1 + \varepsilon^2/\ell^2)$$

where again the constants in \lesssim depend only on the building blocks and where we used $|V| \leq \sum_{v \in V} \deg v = 2|E|$ for any graph G (assuming that there are no isolated vertices, i.e., vertices of degree 0).

Let us now assume that $\ell = \ell_\varepsilon = \varepsilon^\gamma$ depends on ε for some $\gamma \in \mathbb{R}$ (negative γ 's are not excluded). In particular, X_0 now also depends on ε , and we write $\varepsilon^\gamma X_0$ for metric graph with all edge lengths multiplies by ε^γ .

- – For the closeness in (26) to hold we need $\gamma < 1/2$, as the error term is of order $\varepsilon^{1/2} / \min\{\varepsilon^\gamma, 1\} = \varepsilon^{1/2 - \max\{\gamma, 0\}}$.
- For the metric graph eigenvalue, we have $\lambda_j(\varepsilon^\gamma X_0) = \varepsilon^{-2\gamma} \lambda_j(X_0)$.
- For the metric graph eigenvalue (of order $\varepsilon^{-2\gamma}$) to be dominant with respect to the error (of order $\varepsilon^{1/2 - \max\{\gamma, 0\}}$), we need $\gamma > -1/4$. Hence

$$\lambda_j^0(X_\varepsilon) \begin{cases} \approx \varepsilon^{-2\gamma}, & -1/4 < \gamma (< 1/2), \\ \lesssim \varepsilon^{1/2}, & \gamma \leq -1/4. \end{cases} \tag{26'}$$

- For the divergence in (27) to hold we need $\gamma < 2$. In particular, we have

$$\bar{\lambda}_1^p(X_\varepsilon) \gtrsim \begin{cases} \varepsilon^{-2}, & \gamma \leq 1, \\ \varepsilon^{-4+2\gamma}, & 1 \leq \gamma (< 2). \end{cases} \tag{27'}$$

- For the volume, we have

$$\text{vol } X_\varepsilon \approx \varepsilon^n|V| + \varepsilon^{n-1+\gamma}|E| \approx \begin{cases} \varepsilon^{n-1+\gamma}|E|, & \gamma \leq 1, \\ \varepsilon^n|V|, & \gamma \geq 1. \end{cases} \tag{28'}$$

Constant volume and arbitrarily large form eigenvalues. The following example gives another answer to a question of Berger [2], answered already in [17] (see also the references therein for further contributions). Their Theorem 1 says that for any closed manifold X of dimension $n \geq 4$ there exists a metric of volume 1 such that $\lambda_1^p(X)$ (the non-harmonic spectrum) is arbitrarily large. Note that their construction corresponds to a simple graph with one edge and two vertices. We have the following result.

Proposition 6.1. *On any transversally trivial graph-like manifold of dimension $n \geq 3$ there exists a family of metrics \tilde{g}_ε of volume 1 such that for the first eigenvalue on exact p -forms we have*

$$\bar{\lambda}_1^p(X, \tilde{g}_\varepsilon) \rightarrow \infty, \quad \text{as } \varepsilon \rightarrow 0,$$

for $2 \leq p \leq n - 1$. Moreover, the function ($p = 0$) and exact 1-form spectrum converges to 0, i.e.,

$$\lambda_1^0(X, \tilde{g}_\varepsilon) = \bar{\lambda}_1^1(X, \tilde{g}_\varepsilon) \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0.$$

Proof. Let g_ε be the metric of the graph-like manifold as constructed in Section 4.1. For any $\gamma < 1$, we have

$$\bar{\lambda}_1^p(X, g_\varepsilon)(\text{vol}(X, g_\varepsilon))^{2/n} \gtrsim \varepsilon^{-2} \varepsilon^{2(n-1+\gamma)/n} = \varepsilon^{-2(1-\gamma)/n} \rightarrow \infty, \quad \text{as } \varepsilon \rightarrow 0$$

by (27') and (28'). Set now $\tilde{g}_\varepsilon := \text{vol}(X, g_\varepsilon)^{-2/n} g_\varepsilon$, then $\text{vol}(X, \tilde{g}_\varepsilon) = 1$ and

$$\bar{\lambda}_1^p(X, \tilde{g}_\varepsilon) = \text{vol}(X, g_\varepsilon)^{2/n} \bar{\lambda}_1^p(X, g_\varepsilon) \gtrsim \varepsilon^{-2(1-\gamma)/n} \rightarrow \infty, \quad \text{as } \varepsilon \rightarrow 0.$$

If $-1/4 < \gamma < 1/2$, then the 0-form (and exact 1-form) eigenvalues of the metric graph and the manifold are close and $\lambda_j^0(X, g_\varepsilon) \approx \varepsilon^{-2\gamma}$, hence

$$\lambda_j^0(X, \tilde{g}_\varepsilon) = \text{vol}(X, g_\varepsilon)^{2/n} \lambda_j^0(X, g_\varepsilon) \approx \varepsilon^{2(n-1+\gamma)/n} \varepsilon^{-2\gamma} = \varepsilon^{2(n-1)(1-\gamma)/n} \rightarrow 0. \quad \square$$

The transversal length scale (the one of the transversal manifolds Y_e) is $\varepsilon^{(1-\gamma)/n} \rightarrow 0$, while the longitudinal length scale (the one of the metric graph edges I_e) is $\varepsilon^{-(1-1/n)(1-\gamma)} \rightarrow \infty$ as $\varepsilon \rightarrow 0$.

Unfortunately, we cannot extend the result of [17] to the case $n = 3$ and 1-forms (as the exact 1-form spectrum converges).

6.3. Families of manifolds with special spectral properties. Let us now consider families of graph-like manifolds, constructed according to a sequence of graphs $\{G^i\}_{i \in \mathbb{N}}$. We assume for simplicity that the vertex degree is uniformly bounded, say by $k_0 \in \mathbb{N}$. Then we have (if there are no isolated vertices)

$$|V(G^i)| \leq \sum_{v \in V(G^i)} \deg_{G^i} v = 2|E(G^i)| \leq 2k_0|V(G^i)|,$$

i.e., $v_i := |V(G^i)| \simeq |E(G^i)|$ as $i \rightarrow \infty$. We first start with a general statement about the spectral convergence. Assume that $\{G^i\}_{i \in \mathbb{N}}$ is a family of discrete graphs and that $\{X_0^i\}_{i \in \mathbb{N}}$ is the family of associated equilateral metric graphs, each graph X_0^i having edge lengths equal to ℓ_i .

Assume now that we construct accordingly a family of graph-like manifolds $\{X_\varepsilon^i\}_{i \in \mathbb{N}}$ where the building blocks X_v and Y_e are isometric to a given number of prototypes (independent of i), such that Y_e all have trivial cohomology for all $1 \leq p \leq n - 2$ (see Example 4.1), so that all graph-like manifolds X_ε^i are transversally trivial and hence our estimates (26)–(28) are uniform in the building blocks and (27) holds for the *first* exact eigenvalue. We call such a family of graph-like manifolds *uniform*.

Let us now specify ε_i and ℓ_i in dependence of the number of vertices v_i of G^i . We assume that

$$\varepsilon_i = v_i^{-\alpha} \quad \text{and} \quad \ell_i = v_i^{-\beta} \tag{29}$$

for some $\alpha > 0$ and $\beta \in \mathbb{R}$ (negative values for β are not excluded). In particular, X_0^i now also depends on ε , and we write $v_i^{-\beta} X_0^i$ for the metric graph X_0^i with all edge lengths being $v_i^{-\beta}$.

- – For the 0-form eigenvalue convergence in (26) to hold we need $\max\{\beta, 0\} < \alpha/2$, as the error term is of order $\varepsilon_i^{1/2} / \min\{\ell_i, 1\} = v_i^{-\alpha/2 + \max\{\beta, 0\}}$.
- For the metric graph eigenvalue, we have $\lambda_j(v_i^{-\beta} X_0^i) = v_i^{2\beta} \lambda_j(X_0^i)$.
- For the metric graph eigenvalue (of order $v_i^{2\beta}$) to be dominant with respect to the error (of order $v_i^{-\alpha/2 + \max\{\beta, 0\}}$), we need $\beta \geq -\alpha/2, \beta \geq 0$ or $\beta \geq -\alpha/4, \beta \leq 0$. Hence

$$\lambda_j^0(X_\varepsilon^i) \begin{cases} \approx v_i^{2\beta} \lambda_j(X_0^i), & (\beta \geq -\alpha/2, \beta \geq 0) \text{ or } (\beta \geq -\alpha/4, \beta \leq 0), \\ \lesssim v_i^{-\alpha/2}, & \text{otherwise.} \end{cases} \tag{26''}$$

- For the divergence in (27) to hold we need $\alpha > 1/2$ (resp. $2\alpha > 1 + \beta$). In particular, we have

$$\bar{\lambda}_1^p(X_{\varepsilon_i}^i) \gtrsim \begin{cases} v_i^{2\alpha-1}, & \alpha \geq \beta, \\ v_i^{4\alpha-2\beta-1}, & \alpha \leq \beta \end{cases} \quad (27'')$$

for $2 \leq p \leq n - 1$.

- For the volume, we have

$$\text{vol } X_\varepsilon \approx v_i^{-n\alpha+1} + v_i^{-(n-1)\alpha-\beta+1} \approx \begin{cases} v_i^{-(n-1)\alpha-\beta+1}, & \alpha \geq \beta, \\ v_i^{-n\alpha+1}, & \alpha \leq \beta. \end{cases} \quad (28'')$$

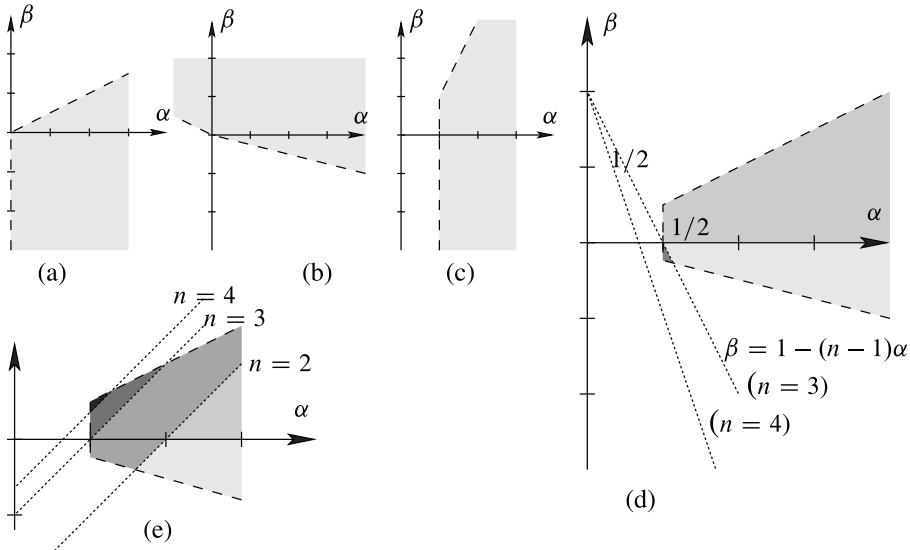


Figure 1. Parameter regions.

- (a) Where the 0-form eigenvalue convergence in (26) holds ($\max\{\beta, 0\} < \alpha/2$).
- (b) Where $\lambda_j^0(X_{\varepsilon_i}^i) \approx v_i^{2\beta} \lambda_j(X_0^i)$ ($\beta > -\alpha/2, \beta \geq 0$ or $\beta > -\alpha/4, \beta \leq 0$).
- (c) Where $\bar{\lambda}_1^p(X_{\varepsilon_i}^i)$ diverges ($\alpha > 1/2, \alpha \geq \beta$ or $4\alpha - 2\beta - 1 > 0, \alpha \leq \beta$).
- (d) *Dark grey*: region where all eigenvalues diverge. *Light grey*: region where form eigenvalues diverge, function eigenvalues converge to 0. *Dotted line*: volume is constant; above: volume tends to 0; below: volume tends to ∞ .
- (e) Regions of divergence of the rescaled 0-form eigenvalue. *Very dark grey* for $n = 4$ and *very dark, darker grey* for $n = 3$ and $n = 2$; dashed lines are $\beta = \alpha - 1/(n - 1)$ for $n \in \{2, 3, 4\}$.

Ramanujan graphs. Let us now conclude from the above and the diagram some examples. We assume that the underlying sequence of metric graphs X_0^i (with all lengths equal to 1) is Ramanujan, i.e., the (metric) graph Laplacians have a common spectral gap $(0, h)$. If we choose (α, β) from the dark grey area ($\alpha > 1/2$, $\beta \geq 0$, $\beta < \alpha/2$) we have:

Proposition 6.2. *There is a sequence of graph-like manifolds $(X_{\varepsilon_i}^i)_i$ with underlying Ramanujan graphs G^i with $v_i = |V(G^i)|$ many vertices, such that the Hodge Laplacian of all degrees has an arbitrarily large spectral gap, i.e., there exists $h_i \approx v_i^{\min\{2\beta, 2\alpha-1\}} \rightarrow \infty$ such that*

$$\sigma(\Delta_{X_{\varepsilon_i}^i}^\bullet) \cap (0, h_i) = \emptyset$$

and such that the volume shrinks to 0, more precisely, $\text{vol } X_{\varepsilon_i}^i \approx v_i^{-(n-1)\alpha-\beta+1}$.

In particular, if $\beta = 0$, then there exists a common spectral gap $(0, h)$ of the Hodge Laplacian. If, additionally, $n = 3$, then the volume decay can be made arbitrarily small as $\alpha \searrow 1/2$, i.e., of order $v_i^{-2\alpha+1}$.

Proof. The proof follows from the above considerations of eigenvalue asymptotics. Note that for a sequence of Ramanujan graphs, there exists $h > 0$ such that the first non-zero eigenvalue of the metric graph Laplacian with unit edge length fulfils $\lambda_1(X_0^i) \geq h$ for all i , hence we can conclude divergence from the first line of (26''). □

Note that the length scale of the underlying metric graphs is of order $v_i^{-\beta}$, but the radius is of order $\varepsilon_i = v_i^{-\alpha}$, which is smaller; hence the injectivity radius of $X_{\varepsilon_i}^i$ is of order $\varepsilon_i = v_i^{-\alpha}$, and the curvature is of order $\varepsilon_i^{-2} = v_i^{2\alpha}$.

Rescaling the metric. Let us now rescale the metric to have fixed volume (i.e., set $\tilde{g}_i := (\text{vol}(X_{\varepsilon_i}^i, g_{\varepsilon_i}))^{-2/n} g_{\varepsilon_i}$) and consider $\tilde{X}^i := (X_{\varepsilon_i}^i, \tilde{g}_i)$. Then the latter manifold has volume 1. Unfortunately, we cannot have divergence at all degrees at the same time; e.g. for $n = 3$ the conditions are $\beta > \alpha - 1/2$ for divergence of the eigenvalues of degree 0, while $\beta < \alpha - 1/2$ is needed for divergence of exact 2-forms. But we can have divergence of 0-forms and higher degree forms separately.

Corollary 6.3. *For all $n \geq 2$ there exists a family of graph-like manifolds \tilde{X}^i of volume 1 with underlying Ramanujan graphs such that the first non-zero eigenvalue on functions (0-forms) diverges.*

Proof. The rescaling factor $\tau_i = (\text{vol}(X_{\varepsilon_i}^i, g_{\varepsilon_i}))^{-1/n}$ is of order $v_i^{(1-1/n)\alpha + \beta/n - 1/n}$. The rescaled eigenvalue on functions fulfils

$$\lambda_1(\tilde{X}^i) = \tau_i^{-2} \lambda_1(X_{\varepsilon_i}^i) \approx \tau_i^{-2} v_i^{2\beta} \lambda_1(X_0^i) \approx v_i^{2/n - 2(1-1/n)(\alpha - \beta)} \lambda_1(X_0^i) \quad (30)$$

and the latter exponent is positive if and only if $\beta > \alpha - 1/(n - 1)$. The allowed parameters (α, β) lie inside the triangle $(0, 0), (4, -1)/(5(n - 1)), (2, 1)/(n - 1)$ such that $\lambda_1(\tilde{X}^i) \approx v_i^{2/n - \delta}$ (see the differently grey coloured regions in Figure 2 (right) for different n). The difference $\beta - \alpha$ approaches its maximum on this triangle at the vertex $(0, 0)$. Hence for any $\delta > 0$ there exists (α, β) inside the triangle such that $\lambda_1(\tilde{X}^i) \approx v_i^{2/n - \delta}$. \square

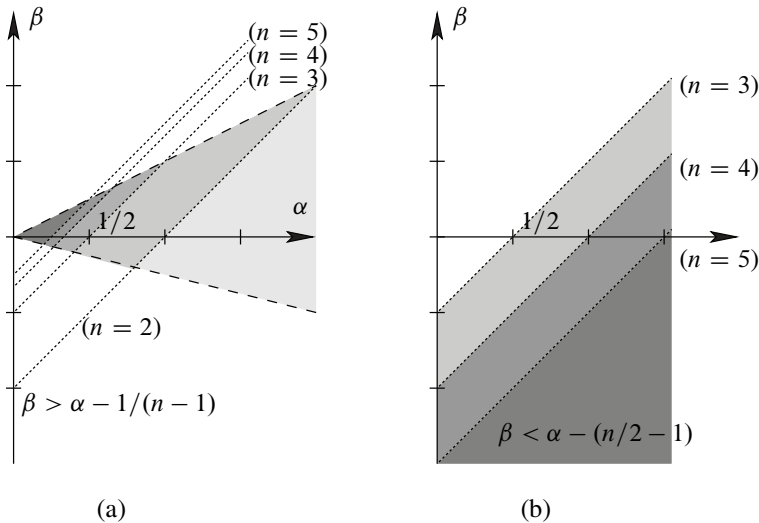


Figure 2. Parameter regions for rescaled metric (volume is 1).

- (a) Where the 0-form eigenvalue $\lambda_1^0(\tilde{X}_{\varepsilon_i}^i)$ diverges.
- (b) Where $\bar{\lambda}_1^p(\tilde{X}_{\varepsilon_i}^i)$ diverges ($2 \leq p \leq n - 1$).

In particular, for $n = 2$ we have:

Corollary 6.4. *There exists a sequence of graph-like surfaces \tilde{X}^i of area 1 and genus $\gamma(\tilde{X}^i)$ with underlying Ramanujan graphs such that the first non-zero eigenvalue on functions diverges. Moreover, for any $\delta > 0$ there exists a sequence $(\tilde{X}^i)_i$ such that*

$$\lambda_1(\tilde{X}^i) \approx \gamma(\tilde{X}^i)^{1-\delta},$$

i.e., the bound in (5) is asymptotically almost optimal.

Proof. We have to choose $Y_e = \mathbb{S}^1$ here, moreover we let the vertex neighbourhood be a sphere with k discs removed (as in Example 4.1). In this case, the genus of the surface \tilde{X}^i is given by $1 - \chi(G^i)$ where $\chi(G^i)$ is the Euler characteristic of the graph G^i , and hence

$$\gamma(\tilde{X}^i) = 1 - |V(G^i)| + |E(G^i)| = 1 - v_i + \frac{k}{2}v_i = 1 + \left(\frac{k}{2} - 1\right)v_i \rightarrow \infty$$

as $i \rightarrow \infty$ as $k \geq 3$ for a Ramanujan graph. In particular, $\gamma(\tilde{X}^i) \asymp v_i$. \square

Arbitrarily large differential form spectrum, constant volume and arbitrary graphs. Let us now assume that (G^i) is any sequence of graphs with $v_i = |V(G^i)| \rightarrow \infty$ as $i \rightarrow \infty$ and with degrees bounded by k . As we want the form spectrum to diverge, we do not need that the underlying graphs are Ramanujan.

Proposition 6.5. *For all $n \geq 3$ there exists a family of graph-like manifolds \tilde{X}^i of volume 1 such that the first eigenvalue on exact p -forms diverges ($2 \leq p \leq n-1$). Moreover, the first non-zero eigenvalue on functions converges.*

Proof. The rescaled eigenvalue on p -forms fulfils

$$\bar{\lambda}_1^p(\tilde{X}^i) = \tau_i^{-2} \bar{\lambda}_1^p(X_{\varepsilon_i}^i) \gtrsim \tau_i^{-2} v_i^{2\alpha-1} \asymp v_i^{2(\alpha-\beta+1)/n-1}$$

(as $\alpha \geq \beta$, see (27'')) and the latter exponent is positive if and only if $\beta < \alpha - (n/2 - 1)$. The allowed parameters (α, β) lie below this line (see Figure 2 (left)).

For the first non-zero eigenvalue on functions, note first that $\lambda_1(X_0^i)$ (the first non-zero eigenvalue of the unilateral metric graph X_0^i) can be bounded from above by π^2 , this follows immediately from the spectral relation (9). Therefore, we conclude from (30) that $\lambda_1(\tilde{X}^i) \rightarrow 0$ as $i \rightarrow \infty$ as $\beta < \alpha - (n/2 - 1)$ implies that $2/n - 2(1-n)(\alpha - \beta) < 0$. \square

Actually, comparing the speed of divergence and convergence, we obtain

$$\bar{\lambda}_1^p(\tilde{X}^i) \gtrsim v_i^{\frac{n^2}{2(n-1)}} \lambda_1(\tilde{X}^i)^{-\frac{n^2}{4(n-1)}},$$

confirming again that we cannot have divergence for both eigenvalues with our construction.

If we choose the family of graphs $(G^i)_i$ to consist of trees only, we can modify any given manifold X to become a graph-like manifold with underlying tree graph (“growing a tree on X ”, see Remark 4.2). In particular, we can show the following.

Corollary 6.6. *On any compact manifold X of dimension $n \geq 3$, there exists a sequence of metrics g_i of volume 1 such that the infimum of the (non-zero) function spectrum converges to 0, while the exact p -form eigenvalues ($2 \leq p \leq n - 1$) diverge.*

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Michela Egidi, Department of Mathematical Sciences, Durham University,
Science Laboratories, South Road, DH1 3LE, Durham UK

e-mail: michela.egidi@durham.ac.uk

Olaf Post, Fachbereich 4 – Mathematik, Universität Trier, 54286, Trier, Germany

ON LEAVE FROM: Department of Mathematical Sciences, Durham University,
Science Laboratories, South Road, DH1 3LE, Durham UK

e-mail: olaf.post@durham.ac.uk