

Resonances for 1d Stark operators

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*Dedicated to the memory of Professor Viktor Havin
(St. Petersburg, 1933-2015)*

Abstract. We consider the Stark operator perturbed by a compactly supported potential (of a certain class) on the real line. We prove the following results: (a) upper and lower bounds on the number of resonances in complex discs with large radii, (b) the trace formula in terms of resonances only, (c) all resonances determine the potential uniquely.

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1. Introduction and main results

1.1. Introduction. We consider the operator $H = H_0 + V$ acting on $L^2(\mathbb{R})$, where the unperturbed operator H_0 is the Stark operator given by

$$H_0 = -\frac{d^2}{dx^2} + x.$$

Here x is an external electric field and the potential $V = V(x)$, $x \in \mathbb{R}$ is real and satisfies the following condition.

Condition V. The potential $V \in L^2_{\text{real}}(\mathbb{R})$ and $\text{supp } V \subset [0, \gamma]$ for some $\gamma > 0$.

Our main results devote to the asymptotics of the number of resonances in large discs and an inverse problem in terms of resonances (all resonances determine the potential uniquely). Under Condition V the operator $V(H_0 - i)^{-1}$ is compact (see Lemma 2.1). Then the operators H_0 and H are self-adjoint on the same domain and $C_0^\infty(\mathbb{R})$ is a core for both H_0 and H . The spectrum of both H_0 and H is purely absolutely continuous and covers the real line \mathbb{R} (see Avron and Herbst [3] and Herbst [16]).

The Stark effect is the shifting and splitting of spectral lines of atoms and molecules due to presence of an external electric field. The effect is named after Stark, who discovered it in 1913. The Stark effect has been of marginal benefit in the analysis of atomic spectra, but has been a major tool for molecular rotational spectra. The perturbation theory for the Stark effect has some problems. In absence of an electric field, states of atoms and molecules are square-integrable. In the presence of an electric field, states of atoms and molecules are not square-integrable and they becomes resonances of finite width. For weak fields low lying states can be regarded as bound, but for all other cases we need to calculate resonances and the corresponding states, which are not square-integrable.

It is well known that the wave operators W_\pm for the pair H_0, H given by

$$W_\pm = s - \lim e^{itH} e^{-itH_0} \quad \text{as } t \rightarrow \pm\infty,$$

exist and are unitary (even under much less restrictive assumptions on the potential than considered here, see [3] and [16]). Thus the scattering operator $S = W_+^* W_-$ is unitary. The operators H_0 and S commute and thus are simultaneously diagonalizable:

$$L^2(\mathbb{R}) = \int_{\mathbb{R}}^\oplus \mathcal{H}_\lambda d\lambda, \quad H_0 = \int_{\mathbb{R}}^\oplus \lambda I_\lambda d\lambda, \quad S = \int_{\mathbb{R}}^\oplus S(\lambda) d\lambda; \quad (1.1)$$

here I_λ is the identity in the fiber space $\mathcal{H}_\lambda = \mathbb{C}$ and $S(\lambda)$ is the scattering matrix (which is a scalar function in $\lambda \in \mathbb{R}$ for our case) for the pair H_0, H (see Yajima [47]).

1.2. Determinants. The main objects studied in the present paper are the resonances of H and the scattering matrix $S(\lambda) = e^{-2\pi i \phi_{sc}(\lambda)}$, $\lambda \in \mathbb{R}$ for the pair H_0, H , where $\phi_{sc}(\lambda)$ is the scattering phase (or the spectral shift function in the terminology associated with the trace formula). In order to study resonances we chose an approach where a central role is played by the *Fredholm determinant*. More precisely, we set

$$V = |V|^{\frac{1}{2}} V^{\frac{1}{2}}, \quad V^{\frac{1}{2}} = |V|^{\frac{1}{2}} \text{sign } V,$$

$$R_0(\lambda) = (H_0 - \lambda)^{-1}, \quad Y_0(\lambda) = |V|^{\frac{1}{2}} R_0(\lambda) V^{\frac{1}{2}}, \quad \lambda \in \mathbb{C}_\pm.$$

Here $\mathbb{C}_\pm = \{\lambda \in \mathbb{C} : \pm \text{Im } \lambda > 0\}$ denote the upper and lower half plane and λ is a spectral parameter. We shortly describe standard properties of the operator-valued function Y_0 , which we will prove in Section 2. We shall show that each operator $Y_0(\lambda)$, $\text{Im } \lambda \neq 0$, is trace class and thus we can define the determinant:

$$D_\pm(\lambda) = \det(I + Y_0(\lambda)), \quad \lambda \in \mathbb{C}_\pm.$$

Moreover, we show that the function $D_\pm(\lambda)$, $\lambda \in \mathbb{C}_\pm$ is analytic in \mathbb{C}_\pm , continuous up to the real line and $D_\pm(\lambda) \neq 0$ for all $\lambda \in \bar{\mathbb{C}}_\pm$. Furthermore, the function D_\pm satisfies

$$\bar{D}_\pm(\lambda) = 1 + O(\lambda^{-a}) \quad \text{as } |\lambda| \rightarrow \infty, \quad \lambda \in \bar{\mathbb{C}}_\pm,$$

for any fixed $a \in (0, \frac{1}{2})$, uniformly with respect to $\arg \lambda \in [0, \pm\pi]$. Thus we can define the branch $\log D_\pm(\lambda)$, $\lambda \in \mathbb{C}_\pm$ by $\log D_\pm(\lambda) = O(\lambda^{-a})$ as $|\lambda| \rightarrow \infty$, $\lambda \in \bar{\mathbb{C}}_\pm$. For each $\lambda \in \mathbb{R}$ the following identities hold true:

$$S(\lambda) = \frac{\bar{D}_+(\lambda + i0)}{D_+(\lambda + i0)} = \frac{D_-(\lambda - i0)}{D_+(\lambda + i0)} = e^{-2\pi i \phi_{sc}(\lambda)}, \tag{1.2}$$

where $\phi_{sc} = \frac{1}{\pi} \arg D_+(\lambda + i0)$ is the scattering phase. Thus the standard arguments give that the function $S(\lambda)$, defined by (1.1), is continuous in $\lambda \in \mathbb{R}$. The basic properties of determinants (see below (2.3) and (2.1)) give the identity

$$\bar{D}_+(\lambda) = \det(I + Y_0(\lambda)^*) = \det(I + Y_0(\bar{\lambda})) = D_-(\bar{\lambda}), \quad \text{for all } \lambda \in \mathbb{C}_+. \tag{1.3}$$

Due to this identity it is enough to consider D_+ or D_- . Our first preliminary theorem describes the Fredholm determinant $D_+(\lambda)$ and its asymptotics at high energy.

Condition C. The potential V satisfies Condition **V** and the restriction of V on the interval $(0, \gamma)$ is absolutely continuous.

Theorem 1.1. *Let V satisfy Condition **C** and let $\tau < 1$. Then the function $\log D_+(\lambda)$ is analytic in \mathbb{C}_+ , continuous up to the real line and satisfies*

$$\log D_+(\lambda) = \frac{iV_0}{2\sqrt{\lambda}} + \frac{O(1)}{\lambda^\tau} \quad \text{as } |\lambda| \rightarrow \infty, \lambda \in \bar{\mathbb{C}}_+, \tag{1.4}$$

where $V_0 = \int_{\mathbb{R}} V(x)dx$, uniformly with respect to $\arg \lambda \in [0, \pi]$. In particular,

$$\phi_{\text{sc}}(\lambda) = \frac{V_0}{2\pi\sqrt{\lambda}} + \frac{O(1)}{\lambda^\tau} \quad \text{as } \lambda \rightarrow +\infty, \tag{1.5a}$$

$$\phi_{\text{sc}}(\lambda) = O(1/\lambda^\tau) \quad \text{as } \lambda \rightarrow -\infty. \tag{1.5b}$$

Remark 1.2. Under our assumptions on V the proof of the asymptotic expansion (1.4) is a bit technical. If V is, e.g. in the Schwartz class, then the expansion becomes much easier and higher order terms can be derived as well (see [19]), similar to the 3-dimensional case in [30].

1.3. Resonances. In order to describe resonances we recall the definition of order and type.

Definition. The entire function f is of order β if

$$\limsup_{r \rightarrow \infty} \frac{\log \log M(r)}{\log r} = \beta,$$

where $M(r) = \sup_{|z|=r} |f(z)|$. The function f of positive order $\beta > 0$ is of type $a \geq 0$ if

$$\limsup_{r \rightarrow \infty} \frac{\log M(r)}{r^\beta} = a.$$

Under Condition **V** we will obtain an analytic continuation of $D_+(\lambda)$, $\lambda \in \mathbb{C}_+$ to the entire complex plane and information on its zeros and obtain upper bounds on the number of resonances of the operator H . We denote by $(\lambda_n)_1^\infty$ the sequence of zeros in \mathbb{C}_- of D_+ (counting multiplicities), arranged such that

$$0 < |\lambda_1| \leq |\lambda_2| \leq |\lambda_2| \leq \dots \tag{1.6}$$

By definition, a zero $\lambda_n \in \mathbb{C}_-$ of D_+ is called a resonance. The multiplicity of the resonance is the multiplicity of the corresponding zero of D_+ . In order to obtain lower bounds on the number of resonances we assume that the potential V satisfies Condition **C** with $V \neq 0$. More precisely

Theorem 1.3. *Let V satisfy Condition **V**. Then $D_{\pm}(\lambda), \lambda \in \mathbb{C}_{\pm}$ has an analytic extension into the whole complex plane and satisfies*

$$|D_{\pm}(\lambda)| \leq C_0 e^{\frac{4}{3}|\lambda|^{\frac{3}{2}}}, \quad \text{for all } \lambda \in \mathbb{C}, \tag{1.7}$$

for some constant C_0 . Furthermore, by (1.2), the S -matrix $S(\lambda), \lambda \in \mathbb{R}$ has an analytic extension into the whole upper half plane \mathbb{C}_+ and a meromorphic extension into the whole lower half plane \mathbb{C}_- . The zeros of $S(\lambda), \lambda \in \mathbb{C}_+$ coincide with the zeros of D_- and the poles of $S(\lambda), \lambda \in \mathbb{C}_-$ are precisely the zeros of D_+ . Let, in addition, V satisfy Condition **C** and $V(0) \neq 0$. Then D_{\pm} is an entire function of order $\frac{3}{2}$ and type $\frac{4}{3}$.

Remark. 1) From (1.7) we deduce that $D_+(\lambda)$ has the Hadamar factorization:

$$D_+(\lambda) = D_+(0)e^{p\lambda} \lim_{r \rightarrow +\infty} \prod_{|\lambda_n| \leq r} \left(1 - \frac{\lambda}{\lambda_n}\right) e^{\frac{\lambda}{\lambda_n}}, \quad \lambda \in \mathbb{C}, \tag{1.8}$$

uniformly on any compact subset of \mathbb{C} , where the constant p satisfies

$$p = \frac{D'_+(0)}{D_+(0)}, \quad \text{Im } p = \pi \phi'_{\text{sc}}(0),$$

with $\phi_{\text{sc}}(\cdot)$ defined in (1.2).

2) By (1.8), the operator H has an infinite number of resonances.

3) Due to (1.2) the resonances are the zeros $\lambda_n \in \mathbb{C}_-, n \geq 1$ of D_+ (and the poles of $S(\lambda)$ with the same multiplicity) in \mathbb{C}_- labeled according to (1.6). The zeros of the S -matrix $S(\lambda)$ are the zeros of D_- in \mathbb{C}_- given by $\bar{\lambda}_n \in \mathbb{C}_-, n \geq 1$.

Denote by $\mathcal{N}(r, f)$ the number of zeros (counted according to multiplicity) of f having modulus $\leq r$. The Lindelöf Theorem jointly with Theorem 1.3 applied to the Fredholm determinant of the perturbed Stark operator gives

Corollary 1.4. *Let V satisfy Condition **V**. Then the entire function D_+ satisfies*

$$\mathcal{N}(r, D_+) \leq C_1 r^{\frac{3}{2}} \tag{1.9}$$

for $r > 0$ sufficiently large and for some positive constant C_1 .

Let in addition V satisfy Condition **C** and $V(0) \neq 0$. Then there is a sequence of positive numbers $r_j, j \in \mathbb{N}$, tending to ∞ and a positive constant $C_0 > 0$ such that

$$\mathcal{N}(r_j, D_+) \geq C_0 r_j^{\frac{3}{2}}, \quad \text{for all } j \in \mathbb{N}. \tag{1.10}$$

Remark. We emphasize that the Hadamard factorization of $D_+(\lambda)$ in (1.8) crucially depends on determining its order and type (which are equal to the order and type of the squared Airy function, which gives the generalized eigenfunctions for the unperturbed Stark operator). In structure the Hadamard factorization looks similar to the factorization for the Schrödinger operator $-\Delta + V$ in \mathbb{R}^3 , see e.g. [50], [45], and [4]. In fact, this is closely connected to counting the number of resonances, by a result of Lindelöf (see [35]), which is contained in Boas's book, see p. 25 in [5] and Section 5.

Thus we see that the perturbed Stark operator H on the real line has much more resonances than the corresponding Schrödinger operator on the real line. In fact, the number of resonances (i.e., $\mathcal{N}(r, D_+) \leq Cr^{\frac{3}{2}}$ in the disc $\{|\lambda_n| \leq r\}$) of the perturbed Stark operator H on the real line corresponds to the one for the Schrödinger operator on \mathbb{R}^3 . Recall that for the Schrödinger operator on \mathbb{R}^3 the number \mathcal{N}_ρ of resonances λ_n in the disc $\{|\lambda_n| \leq \rho^2\}$ has the bound $\mathcal{N}_\rho \leq C\rho^3 = Cr^{\frac{3}{2}}$ at $r = \rho^2$. This explains the similarity in the Hadamard factorization.

Our next corollary concerns the trace formula in terms of resonances. So far, trace formulas for one-dimensional Schrödinger operators in terms of resonances have only been determined in [27]. Here we also follow the approach in [27].

Corollary 1.5. *Let V satisfy Condition V and let $R(\lambda) = (H - \lambda)^{-1}$, $\text{Im } \lambda \neq 0$. Then the following identity (the trace formula) holds true:*

$$\text{Tr} (R_0(\lambda) - R(\lambda)) = p + \sum_{n \geq 1} \frac{\lambda}{\lambda_n(\lambda - \lambda_n)}, \quad (1.11)$$

where the series converges absolutely and uniformly on any compact set of $\mathbb{C} \setminus \{\lambda_n, n \geq 1\}$.

Remark. We discuss trace formulas in Section 5 and we will show the following identity:

$$\phi'_{\text{sc}}(\lambda) = \phi'_{\text{sc}}(0) + \frac{\lambda}{\pi} \text{Im} \sum_{n \geq 1} \frac{1}{\lambda_n(\lambda - \lambda_n)}, \quad \text{for all } \lambda \in \mathbb{R}, \quad (1.12)$$

uniformly on any compact subset of \mathbb{R} . Note that the identity (1.12) is a Breit-Wigner type formula for resonances (see p. 53 of [39]).

We discuss now inverse resonance problems. We show that all resonances determine the potential uniquely. It is a first result about inverse resonance problems for perturbed Stark operators.

Theorem 1.6. *Let the perturbed Stark operators $H_j = H_0 + V_j$, $j = 1, 2$ act on $L^2(\mathbb{R})$ and let each potential V_j , $j = 1, 2$ satisfy Condition **V**. Assume that H_1 and H_2 have the same resonances. Then $V_1 = V_2$.*

Remark. In the case of Schrödinger operator with a compactly supported potential on the half-line all resonances determine the potential [27]. In the case of the real line all resonances do not determine the potential [28].

1.4. Brief overview. Concerning previous results on resonances, we recall that from a physicists point of view, they were first studied by Regge [41]. Since then, properties of resonances have been the object of intense study and we refer to [45] for the mathematical approach in the multi-dimensional case and references given there.

A lot of papers are devoted to resonances of the one-dimensional Schrödinger operator, see Froese [12], Korotyaev [27], Simon [43], Zworski [50], and references therein. We recall that Zworski [50] obtained the first results about the asymptotic distribution of resonances for the Schrödinger operator with compactly supported potentials on the real line (this result is sharper than Corollary 1.4 in the present paper). Inverse problems (characterization, recovering, uniqueness) in terms of resonances were solved by Korotyaev for a Schrödinger operator with a compactly supported potential on the real line [28] and the half-line [27], see also Zworski [51], Brown, Knowles, and Weikard [6] concerning the uniqueness.

Next, we mention some results for one-dimensional perturbed Stark operators. The one-dimensional scattering theory was considered by Rejto and Sinha [42], Jensen [21], and Liu [37]. The one-dimensional inverse scattering problem is studied by Calogero and Degasperis [7], Graffi and Harrell [14], Kachalov and Kurylev [23], Kristensson [32], and Lin, Qian, and Zhang [36]. There are a lot of results about the resonances of the one-dimensional perturbed Stark operator, where the dilation analyticity techniques are used, see e.g., [17] and [22] and [8] and references therein. Note that compactly supported potentials are not treated in these papers.

We mention also interesting results about resonances for one-dimensional Stark-Wannier operators $-\frac{d^2}{dx^2} + \varepsilon x + V_\pi$, where the constant $\varepsilon > 0$ is the electric field strength and V_π is the real periodic potential: Agler and Froese [1], Grecchi and Sacchetti [15], Herbst and Howland [18], and Jensen [20]. In the case $\varepsilon = 0$, the resonances for one-dimensional operators $-\frac{d^2}{dx^2} + V_\pi + V$, where V is a compactly supported potential were considered by Firsova [11], Korotyaev [29], and Korotyaev and Schmidt [31].

1.5. Plan of the paper. In Section 2 we recall well known results on the spectral representation of the Stark operator in a form useful for our approach and obtain basic estimates on $Y_0(\lambda)$, using Privalov's Lemma. Section 3 contains the stationary representation of the scattering matrix and the proof of Theorem 1.1. Section 4 establishes the analytic continuation of $D_{\pm}(\lambda)$, $\lambda \in \mathbb{C}_{\pm}$ and the meromorphic continuation of $S(\lambda)$ and gives the crucial estimates on order and type leading to Corollary 1.4. In Section 5 we prove Theorem 1.3 and Theorem 1.5. The Appendix contains technical estimates needed in the proof of Lemma 2.4 which is crucial to obtain the asymptotic expansion (1.4).

2. Unperturbed Stark operators

2.1. The well-known facts. We denote by C various possibly different constants whose values are immaterial in our constructions. By \mathcal{B} and \mathcal{B}_{∞} we denote the classes of bounded and compact operators, respectively. Let \mathcal{B}_1 and \mathcal{B}_2 be the trace and the Hilbert–Schmidt class equipped with the norm $\|\cdot\|_{\mathcal{B}_1}$ and $\|\cdot\|_{\mathcal{B}_2}$, respectively. We recall some well known facts. Let $A, B \in \mathcal{B}$ and $AB, BA \in \mathcal{B}_1$. Then

$$\operatorname{Tr} AB = \operatorname{Tr} BA,$$

$$\det(I + AB) = \det(I + BA), \quad (2.1)$$

$$\text{the mapping } X \rightarrow \det(I + X) \text{ is continuous on } \mathcal{B}_1, \quad (2.2)$$

$$|\det(I + X)| \leq e^{\|X\|_{\mathcal{B}_1}},$$

$$\overline{\det(I + X)} = \det(I + X^*), \quad (2.3)$$

$$|\det(I + X) - \det(I + Y)| \leq \|X - Y\|_{\mathcal{B}_1} e^{1 + \|X\|_{\mathcal{B}_1} + \|Y\|_{\mathcal{B}_1}}, \quad (2.4)$$

for all $X, Y \in \mathcal{B}_1$, see e.g., Section 3 in the book [44]. Let the operator-valued function $\Omega: \mathcal{D} \rightarrow \mathcal{B}_1$ be analytic for some domain $\mathcal{D} \subset \mathbb{C}$ and $(I + \Omega(z))^{-1} \in \mathcal{B}$ for any $z \in \mathcal{D}$. Then the function $F(z) = \det(I + \Omega(z))$ satisfies

$$F'(z) = F(z) \operatorname{Tr}(I + \Omega(z))^{-1} \Omega'(z), \quad z \in \mathcal{D}.$$

Recall that the kernels of the operators $(-\Delta - \lambda)^{-1}$ and $e^{it\Delta}$ on $L^2(\mathbb{R})$ have the form

$$(-\Delta - \lambda)^{-1}(x, y) = \frac{i}{2\sqrt{\lambda}} e^{i\sqrt{\lambda}|x-y|}, \quad \lambda \in \mathbb{C} \setminus \bar{\mathbb{R}}_+, \quad \sqrt{\lambda} \in \mathbb{C}_+,$$

$$(e^{it\Delta})(x, y) = \frac{e^{-i\pi/4}}{\sqrt{4\pi t}} e^{i|x-y|^2/4t}, \quad t \neq 0,$$

$x, y \in \mathbb{R}^1$. We need the identities for the Stark operator $H_0 = -\frac{d^2}{dx^2} + x$ from [3] given by

$$e^{-itH_0} = e^{-itx} e^{it\Delta} e^{-i\partial t^2} e^{-i\frac{t^3}{3}}, \quad \text{for all } t \in \mathbb{R},$$

where $\partial = -i\frac{d}{dx}$, $\Delta = \frac{d^2}{dx^2}$. The free resolvent $R_0(\lambda) = (H_0 - \lambda)^{-1}$ and its kernel $R_0(x, y, \lambda)$ satisfy

$$R_0(\lambda) = i \int_0^\infty e^{-it(H_0 - \lambda)} dt = i \int_0^\infty e^{-itx} e^{-i\partial t^2} e^{it\Delta} e^{it\lambda - i\frac{t^3}{3}} dt, \quad (2.5a)$$

$$R_0(x, y, \lambda) = \frac{e^{i\frac{\pi}{4}}}{\sqrt{4\pi}} \int_0^\infty e^{-itx} e^{\frac{i}{4t}|x-t^2-y|^2} e^{-i\frac{t^3}{3} + it\lambda} \frac{dt}{t^{1/2}}, \quad (2.5b)$$

for $x, y \in \mathbb{R}$ and $\lambda \in \mathbb{C}_+$. We introduce the resolvent $R(\lambda)$ for H and operators Y, J by

$$\begin{aligned} R(\lambda) &= (H - \lambda)^{-1}, & Y(\lambda) &= |V|^{\frac{1}{2}} R(\lambda) V^{\frac{1}{2}}, \\ J(\lambda) &= I - Y(\lambda), & J_0(\lambda) &= I + Y_0(\lambda), \end{aligned}$$

for $\lambda \in \mathbb{C}_\pm$ and recall that $Y_0(\lambda) = |V|^{\frac{1}{2}} R_0(\lambda) V^{\frac{1}{2}}$. Below we will use the identities

$$R = R_0 - R_0 V R, \quad J(\lambda) J_0(\lambda) = I, \quad \lambda \in \mathbb{C}_\pm. \quad (2.6)$$

2.2. The spectral representation for H_0 . We will need some facts concerning the spectral decomposition of the Stark operator H_0 , which we denote by

$$E_0(\lambda) = \chi(\lambda - H_0), \quad \lambda \in \mathbb{R}, \quad \text{where } \chi(\lambda) = \begin{cases} 1, & \lambda \geq 0, \\ 0, & \lambda < 0, \end{cases}$$

Now we recall formulae for $E_0(\lambda)$ due to [3]. Let φ be the multiplication operator by the function $\varphi(k) = e^{ik^3/3}, k \in \mathbb{R}$. Then

$$H_0 = -\frac{d^2}{dx^2} + x = \varphi(\partial)^*(x)\varphi(\partial), \quad \partial = -i\frac{d}{dx}.$$

Let $U: f \mapsto \tilde{f}$ be the unitary transformation on $L^2(\mathbb{R})$, which can be defined on $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ by the explicit formula

$$\tilde{f}(p) = (Uf)(p) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \text{Ai}(x - p) f(x) dx, \quad (2.7)$$

where $\text{Ai}(\cdot)$ is the Airy function:

$$\text{Ai}(z) = \frac{1}{\pi} \int_0^\infty \cos\left(\frac{t^3}{3} + tz\right) dt, \quad \text{for all } z \in \mathbb{R}.$$

The unitary transformation (2.7) carries H_0 over into multiplication by p in $L^2(\mathbb{R}, dp)$:

$$(UH_0U^* \tilde{f})(p) = p\tilde{f}(p), \quad \tilde{f} \in \mathcal{D}(p).$$

Thus, for any $f \in L^2(\mathbb{R})$, the quadratic form of $E_0(\lambda)$ can be presented as

$$\langle E_0(\lambda)f, f \rangle = \int_{p<\lambda} |\tilde{f}(p)|^2 dp, \quad \text{for all } \lambda \in \mathbb{R},$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $L^2(\mathbb{R})$. Differentiation with respect to λ gives

$$\frac{d}{d\lambda} \langle E_0(\lambda)f, f \rangle = |G(\lambda)f|^2, \quad f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}),$$

where $G(\lambda): L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \rightarrow \mathbb{C}$ is given by

$$G(\lambda)f := U(\lambda)f = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \text{Ai}(x - \lambda) f(x) dx, \quad \text{for all } \lambda \in \mathbb{R}. \quad (2.8)$$

The Airy function $\text{Ai}(z)$, $z \in \mathbb{C}$ is entire and satisfies (see (4.01)–(4.05) in [38]):

$$\text{Ai}''(z) = z \text{Ai}(z),$$

$$\text{Ai}(-z) = e^{i\pi/3} \text{Ai}(ze^{i\pi/3}) + e^{-i\pi/3} \text{Ai}(ze^{-i\pi/3}),$$

and it obeys the following asymptotics, as $|z| \rightarrow \infty$ uniformly in $\arg z$ for any fixed $\varepsilon > 0$:

$$\text{Ai}(z) = \frac{1}{2} z^{-1/4} e^{-\frac{2}{3}z^{3/2}} (1 + O(z^{-3/2})), \quad \text{if } |\arg z| < \pi - \varepsilon, \quad (2.9a)$$

$$\text{Ai}(-z) = z^{-1/4} [\sin \vartheta + O(z^{-3/2} e^{|\text{Im} \vartheta|})], \quad \text{if } |\arg z| \leq \varepsilon, \quad \vartheta = \frac{2}{3} z^{3/2} + \frac{\pi}{4}. \quad (2.9b)$$

Introduce the space $L^p(\mathbb{R})$, $p \geq 1$, equipped with the norm

$$\|f\|_p = \left(\int_{\mathbb{R}} |f(x)|^p dx \right)^{1/p} \geq 0.$$

For the scalar product $\langle f, f \rangle$ in $L^2(\mathbb{R})$ and we have $\langle f, f \rangle = \|f\|^2 = \|f\|_2^2$. Define the linear functional $\Psi(\lambda): L^2(\mathbb{R}) \rightarrow \mathbb{C}$, by

$$\Psi(\lambda)f = \langle f, G(\lambda)|V|^{1/2} \rangle = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \text{Ai}(x - \lambda) |V(x)|^{1/2} f(x) dx, \quad \text{for all } \lambda \in \mathbb{R}, \quad (2.10)$$

which is bounded, since $\text{Ai}(\cdot) \in L^\infty(\mathbb{R})$ by (2.9) and $|V|^{1/2} \in L^2(\mathbb{R})$ under Condition V.

Lemma 2.1. *Let the potential V satisfy Condition **V** and let $\tau \in [0, 1]$, $\lambda, \mu \in \mathbb{R}$, $|\lambda - \mu| \leq 1$.*

i) *The linear functional $\Psi(\lambda): L^2(\mathbb{R}) \rightarrow \mathbb{C}$, defined by (2.10) is bounded and satisfies*

$$\|\Psi(\lambda)\| \leq \frac{C}{(1 + |\lambda|)^{\frac{1}{4}}}, \tag{2.11}$$

$$\|\Psi(\lambda) - \Psi(\mu)\| \leq \frac{C|\lambda - \mu|^\tau}{(1 + |\lambda|)^{\frac{1}{4} - \frac{\tau}{2}}}. \tag{2.12}$$

ii) *Moreover, $\Omega(\lambda) = \Psi(\lambda)^*\Psi(\lambda)$, $\lambda \in \mathbb{R}$ is a rank one operator on $L^2(\mathbb{R})$ satisfying*

$$\begin{aligned} \|\Omega(\lambda)\|_{\mathcal{B}_1} &\leq \frac{C}{(1 + |\lambda|)^{\frac{1}{2}}}, \\ \|\Omega(\lambda) - \Omega(\mu)\|_{\mathcal{B}_1} &\leq \frac{C|\lambda - \mu|^\tau}{(1 + |\lambda|)^{\frac{1-\tau}{2}}}. \end{aligned} \tag{2.13}$$

Here the constant C in (2.11)-(2.13) depends on V only.

iii) *Let q, q_1 be multiplication operators by functions $q, q_1 \in L^2(\mathbb{R})$ respectively. Then for all $\text{Im } z \neq 0$ the following holds true:*

$$qR_0(z), qR(z) \in \mathcal{B}_2, \quad \text{and} \quad qR_0(z)q_1, qR(z)q_1 \in \mathcal{B}_1, \tag{2.14}$$

and in particular, $VR_0(z) \in \mathcal{B}_2$.

Proof. In order to prove the lemma we need a following simple estimate:

$$\int_{\mathbb{R}} \frac{|V(x)|dx}{(1 + |x - \lambda|)^a} \leq \|V\|_2 \frac{\gamma(1 + \gamma)^a}{\left(1 + \frac{|\lambda|}{2}\right)^a}, \quad \text{for all } (a, \lambda) \in [0, \frac{1}{2}] \times \mathbb{C}. \tag{2.15}$$

We have

$$\int_{\mathbb{R}} \frac{|V(x)|dx}{(1 + |x - \lambda|)^a} \leq \|V\|_2 J(\lambda), \quad J(\lambda)^2 = \int_0^\gamma \frac{dx}{(1 + |x - \lambda|)^{2a}}.$$

We will estimate $J(\lambda)$: firstly, if $|\lambda| \leq 2\gamma$, then $J^2(\lambda) \leq \gamma$; secondly, if $|\lambda| \geq 2\gamma$, then

$$J(\lambda)^2 \leq \int_0^\gamma \frac{dx}{\left(1 + \frac{|\lambda|}{2}\right)^{2a}} = \frac{\gamma}{\left(1 + \frac{|\lambda|}{2}\right)^{2a}} \leq \frac{2\gamma}{(1 + |\lambda|)^{2a}},$$

which gives (2.15).

i) Let $f \in L^2(\mathbb{R})$. The asymptotics (2.9) and the estimate (2.15) imply

$$\begin{aligned} |\Psi(\lambda)f|^2 &= \frac{1}{\pi} \left| \int_{\mathbb{R}} \text{Ai}(x - \lambda) |V(x)|^{\frac{1}{2}} f(x) dx \right|^2 \\ &\leq C \left(\int_{\mathbb{R}} \frac{|V(x)|^{\frac{1}{2}} |f(x)| dx}{(1 + |x - \lambda|)^{\frac{1}{4}}} \right)^2 \\ &\leq C \|f\|^2 \int_{\mathbb{R}} \frac{|V(x)| dx}{(1 + |x - \lambda|)^{\frac{1}{2}}} \\ &\leq \frac{C_1 \|f\|^2}{(1 + |\lambda|)^{\frac{1}{2}}}, \end{aligned}$$

for some constants C, C_1 . This yields (2.11). Next, we show (2.12). Asymptotics (2.9) entails

$$|\text{Ai}(\lambda) - \text{Ai}(\mu)| \leq \frac{C|\lambda - \mu|^\tau}{(1 + |\lambda|)^{\frac{1}{4} - \frac{\tau}{2}}}, \quad \text{for all } \lambda, \mu \in \mathbb{R}, |\lambda - \mu| \leq 1. \quad (2.16)$$

Then similar arguments as above give (with $t = \frac{1}{4} - \frac{\tau}{2}$)

$$\begin{aligned} |\Psi(\lambda)f - \Psi(\mu)f| &= \frac{1}{\sqrt{\pi}} \left| \int_{\mathbb{R}} (\text{Ai}(x - \lambda) - \text{Ai}(x - \mu)) |V(x)|^{\frac{1}{2}} f(x) dx \right| \\ &\leq C_0 |\lambda - \mu|^\tau \int_{\mathbb{R}} \frac{|V(x)|^{\frac{1}{2}} |f(x)| dx}{(1 + |x - \lambda|)^{\frac{1}{4} - \frac{\tau}{2}}} \\ &\leq \frac{C_2 |\lambda - \mu|^\tau}{(1 + |\lambda|)^{\frac{1}{4} - \frac{\tau}{2}}} \|f\|, \end{aligned} \quad (2.17)$$

for some constants C, C_1 . This yields (2.12). The results of ii) follow from i).

iii) For $\text{Im } z \neq 0$ and $q \in L^2(\mathbb{R})$ due to (2.9) we obtain

$$\begin{aligned} \| |R_0(z)|^{\frac{1}{2}} q \|_{\mathcal{B}_2}^2 &= \int_{\mathbb{R}^2} \frac{\text{Ai}^2(x - p) |q(x)|^2}{\pi |p - z|} dx dp \\ &< \int_{\mathbb{R}^2} \frac{C |q(x)|^2 dx dp}{|p - z| (1 + |p - x|)^{\frac{1}{2}}} \\ &< \infty, \end{aligned} \quad (2.18)$$

which yields $qR_0(z), VR_0(z) \in \mathcal{B}_2$ and the equality of the domains $\mathcal{D}(H_0) = \mathcal{D}(H)$. From (2.18) we deduce that

$$qR_0(z)q_1 = (q|R_0(z)|^{\frac{1}{2}})(R_0(z)^{\frac{1}{2}}q_1) \in \mathcal{B}_1.$$

Combining these results with the identity $R = R_0 - R_0VR$ we arrive at (2.14). \square

2.3. Estimates on Y_0 . In order to estimate the operator-valued functions $Y(\lambda), Y_0(\lambda), \lambda \in \bar{\mathbb{C}}_{\pm}$ in terms of the trace class norm \mathcal{B}_1 we need some additional definitions.

Let \mathcal{H} be a Banach space equipped with the norm $\|\cdot\|_{\mathcal{H}}$. For any $\vartheta > 0, \tau \in (0, 1)$ we introduce the Banach space $\mathfrak{X}_{\vartheta, \tau}(\mathbb{R}) = \mathfrak{X}_{\vartheta, \tau}(\mathbb{R}, \mathcal{H})$ of the functions $f: \mathbb{R} \rightarrow \mathcal{H}$ equipped with the norm

$$\|f\|_{\mathfrak{X}_{\vartheta, \tau}(\mathbb{R})} = \sup_{t \in \mathbb{R}, |h| \leq 1} (1 + |t|)^{\vartheta} \left(\|f(t)\|_{\mathcal{H}} + \frac{\|f(t+h) - f(t)\|_{\mathcal{H}}}{|h|^{\tau}} \right) < \infty,$$

and the Banach space $\mathfrak{X}_{\vartheta, \tau}(\mathbb{C}_{\pm}) = \mathfrak{X}_{\vartheta, \tau}(\mathbb{C}_{\pm}, \mathcal{H})$ of the functions $F: \mathbb{C}_{\pm} \rightarrow \mathcal{H}$ equipped with the norm

$$\|F\|_{\mathfrak{X}_{\vartheta, \tau}(\mathbb{C}_{\pm})} = \sup_{\lambda, \mu \in \mathbb{C}_{\pm}, |\lambda - \mu| < 1} (1 + |\lambda|)^{\vartheta} \left(\|F(\lambda)\|_{\mathcal{H}} + \frac{\|F(\lambda) - F(\mu)\|_{\mathcal{H}}}{|\lambda - \mu|^{\tau}} \right) < \infty.$$

We recall Privalov’s Lemma. Privalov actually proved his lemma for a certain contour and for scalar functions. Faddeev (see Lemma 3.1 in [10]) proved a version for Hilbert space valued functions, where the contour is the real line, see also [2] about the Hilbert transformation.

Lemma 2.2 (Privalov). *Assume that $f: \mathbb{R} \rightarrow \mathcal{H}$ belongs to the Banach space $\mathfrak{X}_{\vartheta, \tau}(\mathbb{R}, \mathcal{H})$ for some $\vartheta > 0, \tau \in (0, 1)$ and for some Banach space \mathcal{H} . Then the function F given by*

$$F(z) = \int_{\mathbb{R}} \frac{f(t)}{t - z} dt, \quad z \in \mathbb{C}_{+},$$

is analytic in \mathbb{C}_{+} and continuous up to the real line. Moreover, it is bounded as a map $f \rightarrow F$ from $\mathfrak{X}_{\vartheta, \tau}(\mathbb{R}, \mathcal{H})$ into $\mathfrak{X}_{\vartheta_1, \tau}(\mathbb{C}_{\pm}, \mathcal{H})$, for any $\vartheta_1 < \vartheta$, and satisfies

$$\|F\|_{\mathfrak{X}_{\vartheta_1, \tau}(\mathbb{C}_{\pm}, \mathcal{H})} \leq C \|f\|_{\mathfrak{X}_{\vartheta, \tau}(\mathbb{R}, \mathcal{H})}, \quad \text{for all } \vartheta_1 < \vartheta,$$

where the constant $C = C(\vartheta_1, \vartheta, \tau)$ depends on ϑ_1, ϑ and τ only.

We apply Privalov’s Lemma 2.2 to study the sandwiched resolvents $Y_0(\cdot)$ and $Y(\cdot)$.

Lemma 2.3. *Let the potential V satisfy Condition \mathbf{V} and let $\vartheta < \frac{1-\tau}{2}$, $0 < \tau < 1$.*

- i) *The operator-valued functions $I + Y_0$ and $(I + Y_0)^{-1}$ are uniformly Hölder on $\bar{\mathbb{C}}_+$ and*

$$(I + Y_0(\lambda))^{-1} \in \mathcal{B}, \quad \text{for all } \lambda \in \bar{\mathbb{C}}_+,$$

Moreover, let $F = Y_0$ or $F = Y$. Then the operator-valued function F is analytic on \mathbb{C}_\pm and continuous up to the real line in the \mathcal{B}_1 -norm and satisfies

$$\|F\|_{\mathfrak{X}_{\vartheta,\tau}(\mathbb{C}_\pm, \mathcal{B}_1)} < \infty. \tag{2.19}$$

- ii) *Moreover, the function D_\pm is analytic in \mathbb{C}_\pm and satisfies*

$$D_\pm - 1 \in \mathfrak{X}_{\vartheta,\tau}(\mathbb{C}_\pm). \tag{2.20}$$

Proof. i) Introduce the Hilbert space $\mathfrak{S}_a, a \geq 0$ of Hilbert–Schmidt operators X equipped with the norm

$$\|X\|_{\mathfrak{S}_a}^2 = \sum_{n \geq 1} n^{2a} \sigma_n^2,$$

where the non-negative numbers $\sigma_1 \geq \sigma_2 \geq \sigma_3 \dots$ are eigenvalues of the operator $(X^* X)^{\frac{1}{2}} \geq 0$. Note that $\|X\|_{\mathcal{B}_1} \leq C_a \|X\|_{\mathfrak{S}_a}$ for all $a > \frac{1}{2}$, where the constant is given by $C_a^2 = \sum_{n \geq 1} n^{-2a}$.

In the free case the operator $Y_0(\lambda)$ has the form

$$Y_0(\lambda) = \int_{\mathbb{R}} \frac{\Omega(t)}{t - \lambda} dt, \quad \lambda \in \mathbb{C}_+,$$

$$\Omega(t) = \Psi(t)^* \Psi(t) V_S, \quad V_S = \text{sign } V.$$

Here, due to Lemma 2.1, the function $\Omega(t), t \in \mathbb{R}$ is a rank one operator-valued function and, in particular, $\Omega(t) \in \mathfrak{S}_1, t \in \mathbb{R}$. Thus Lemma 2.1 shows that $\Omega \in \mathfrak{X}_{\vartheta,\tau}(\mathbb{R}, \mathfrak{S}_1)$ for any $\vartheta = \frac{1-\tau}{2}, 0 < \tau < 1$. Then Privalov’s Lemma 2.2 yields (2.19) for Y_0 .

The operator $I + Y_0(\lambda)$ is invertible for all $\lambda \in \mathbb{C}_\pm$, since H_0 is self-adjoint and satisfies (2.14). Moreover, it is standard fact that the operator $I + Y_0(\lambda \pm i0)$ is invertible for all $\lambda \in \mathbb{R}$. In fact, this follows from (2.11), (2.12). These remarks and the properties of Y_0 imply that the operator-valued functions $I + Y_0$ and $(I + Y_0)^{-1}$ are analytic in \mathbb{C}_+ , continuous up to the real line and uniformly Hölder continuous in $\bar{\mathbb{C}}_+$.

Consider the case $F = Y$. Using the identity $(I + Y_0)Y = Y_0$ and the properties of Y_0 and $I + Y_0$ described above we obtain the proof of i).

ii) Due to (2.19) the function $D_{\pm}(\lambda) = \det(I + Y_0(\lambda))$ is well-defined and analytic in \mathbb{C}_{\pm} . Moreover, using (2.4) we have

$$|\det(I + Y_0(\lambda)) - 1| \leq \|Y_0(\lambda)\|_{\mathcal{B}_1} e^{1+\|Y_0(\lambda)\|_{\mathcal{B}_1}}$$

and adding (2.19), we obtain (2.20). □

Lemma 2.4. *Let V satisfy Condition V and let $\tau < \frac{1}{2}$. Then*

$$|\operatorname{Tr} Y_0^n(\lambda)| \leq \|Y_0^n(\lambda)\|_{\mathcal{B}_1} \leq \frac{C}{|\lambda|^{n\tau}}, \quad \text{for all } n \geq 1,$$

where $\lambda \in \bar{\mathbb{C}}_+$, $|\lambda| > 1$ and $C = C(\tau, V)$ depending on τ, V . Let, in addition, V satisfy Condition C. Then

$$\operatorname{Tr} Y_0(\lambda) = \frac{i}{2\sqrt{\lambda}} V_0 + \frac{O(1)}{\lambda} \tag{2.21}$$

as $|\lambda| \rightarrow \infty, \lambda \in \bar{\mathbb{C}}_+$, where $V_0 = \int_{\mathbb{R}} V(x) dx$ uniformly in $\arg \lambda \in [0, \pi]$.

Proof. For $\lambda \in \bar{\mathbb{C}}_+$, due to (2.19) we have

$$|\operatorname{Tr} Y_0^n(\lambda)| \leq \|Y_0^n(\lambda)\|_{\mathcal{B}_1} \leq \|Y_0(\lambda)\|_{\mathcal{B}_1}^n \leq \frac{C_{\tau}}{|\lambda|^{n\tau}},$$

where the constant $C = C(\varepsilon, V)$ does not depend on λ . We show (2.21) in Lemma 6.1. □

3. Determinants and S-matrix

3.1. The Determinants. We discuss the determinant $D_{\pm}(\lambda), \lambda \in \bar{\mathbb{C}}_{\pm}$, when the potential V satisfies Condition V. In this case we have (2.14) and this gives the identity

$$\frac{D'_{\pm}(\lambda)}{D_{\pm}(\lambda)} = \operatorname{Tr} R_0(\lambda) V R(\lambda) = \operatorname{Tr}(R_0(\lambda) - R(\lambda)), \quad \lambda \in \mathbb{C}_{\pm}, \tag{3.1}$$

which is well-known for large class of operators. Due to (2.19) the operator-valued function $Y_0(\lambda)$ attains value in \mathcal{B}_1 and belongs to the class $\mathfrak{X}_{\vartheta, \tau}(\mathbb{C}_{\pm}, \mathcal{B}_1)$ for any $\vartheta < \frac{1-\tau}{2}$ and $0 < \tau < 1$. Recall that we define $\log D_{\pm}(\lambda)$, by $\log D_{\pm}(\lambda) = o(1)$ as $|\lambda| \rightarrow \infty, \lambda \in \mathbb{C}_{\pm}$, since $D_{\pm}(\lambda) \neq 0$ for all $\lambda \in \bar{\mathbb{C}}_{\pm}$ and $\|Y_0(\lambda)\|_{\mathcal{B}_1} = o(1)$ as $|\lambda| \rightarrow \infty$ and there exists $r_0 > 0$ such that

$$\sup_{\lambda \in \bar{\mathbb{C}}_+, |\lambda| \geq r_0} \|Y_0(\lambda)\|_{\mathcal{B}_1} < \frac{1}{2}. \tag{3.2}$$

Then using (2.2)–(2.4) and (2.19) we obtain

$$\log D_{\pm} \in \mathfrak{X}_{\vartheta, \tau}(\mathbb{C}_{\pm}), \quad \text{for all } \vartheta < \frac{1-\tau}{2}, 0 < \tau < 1. \tag{3.3}$$

It is well-known (see [39]) that, under condition (3.2), the function $\log D_{\pm}(\lambda)$ satisfies

$$-\log D_{\pm}(\lambda) = \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr}(-Y_0(\lambda))^n, \tag{3.4}$$

for any $\lambda \in \bar{\mathbb{C}}_{\pm}$, $|\lambda| > r_0$, where the series converges absolutely and uniformly. Then using (3.4) and (3.2) for some $r_0 > 0$ and any $\tau < \frac{1}{2}$ we obtain

$$|\log D_{\pm}(\lambda) + \sum_{n=1}^N \frac{1}{n} \text{Tr}(-Y_0(\lambda))^n| \leq \frac{\|Y_0(\lambda)\|_{\mathfrak{B}_1}^{N+1}}{N+1} \leq \frac{C}{|\lambda|^{(N+1)\tau}}, \quad \text{for all } N \geq 0, \tag{3.5}$$

3.2. The scattering matrix. Recall that the S-matrix $S(\lambda)$ is a scalar function of $\lambda \in \mathbb{R}$, acting as multiplication in the fiber spaces $\mathbb{C} = \mathcal{H}_{\lambda}$. Thus $|S(\lambda)| = 1$ for all $\lambda \in \mathbb{R}$ we have

$$S(\lambda) = e^{-2\pi i \phi_{\text{sc}}(\lambda)}, \quad \lambda \in \mathbb{R}. \tag{3.6}$$

The stationary representation for the scattering matrix has the form (see e.g. [47]):

$$S(\lambda) = I - 2\pi i \mathcal{A}(\lambda), \quad \lambda \in \mathbb{R}, \tag{3.7a}$$

$$\mathcal{A} = \mathcal{A}_0 - \mathcal{A}_1, \tag{3.7b}$$

$$\mathcal{A}_0(\lambda) = \Psi(\lambda) V_S \Psi^*(\lambda), \quad \mathcal{A}_1(\lambda) = \Psi(\lambda) V_S Y(\lambda + i0) \Psi^*(\lambda), \tag{3.7c}$$

$$\Psi(\lambda) = G(\lambda) |V|^{\frac{1}{2}}, \quad V_S = \text{sign } V, \tag{3.7d}$$

where G is given by (2.8). Note that due to (2.19) the operator $Y(\lambda \pm i0)$ is continuous in $\lambda \in \mathbb{R}$. We shall represent $S(\lambda)$ in terms of $D_{\pm}(\lambda)$.

Lemma 3.1. *Let V satisfy Condition V. Then the scattering amplitude $\mathcal{A}(\lambda)$ is a continuous scalar function of $\lambda \in \mathbb{R}$ and satisfies*

$$\mathcal{A}_0(\lambda) = \int_{\mathbb{R}} \text{Ai}(x - \lambda)^2 V(x) dx, \quad \text{for all } \lambda \in \mathbb{R}, \tag{3.8}$$

$$\mathcal{A}_0(\cdot) \in \mathfrak{X}_{\frac{1-\tau}{2}, \tau}(\mathbb{R}, \mathbb{C}), \tag{3.9}$$

$$\mathcal{A}_1(\cdot) \in \mathfrak{X}_{\varepsilon, \tau}(\mathbb{R}, \mathbb{C}), \quad \text{for all } \varepsilon < 1 - \tau, \tag{3.10}$$

for any $\tau \in (0, 1)$. Moreover, the functions $S(\lambda), \phi_{sc}(\lambda)$ are continuous in $\lambda \in \mathbb{R}$ and satisfy asymptotics

$$S(\lambda) - 1 = O(\lambda^{-\frac{1}{2}}), \quad \phi_{sc}(\lambda) = O(\lambda^{-\frac{1}{2}}) \quad \text{as } \lambda \rightarrow \pm\infty, \tag{3.11}$$

$$\phi_{sc}(\lambda) = \frac{1}{\pi} \arg D_+(\lambda), \quad \lambda \in \bar{\mathbb{C}}_+, \tag{3.12}$$

and the identities (1.2) which uniquely defines ϕ_{sc} by (3.6), continuity and the asymptotics (3.11).

Proof. The definitions of \mathcal{A}_0 and Ψ (see (3.7)) give (3.8). Relation (3.9) follows from Lemma 2.1. Relation (3.10) follows from Lemma 2.1 and (2.19), since $Y(\cdot) \in \mathfrak{X}_{\vartheta, \tau}(\mathbb{C}_{\pm}, \mathcal{B}_1)$, for any $\vartheta < \frac{1-\tau}{2}$, $\tau \in (0, 1)$ due to (2.19).

Next, we show (1.2). Recall that $S(\lambda)$ satisfies (3.7) and that we have the standard identity

$$Y_0(\lambda + i0) - Y_0(\lambda - i0) = 2\pi i \Psi(\lambda)^* \Psi(\lambda) V_S, \quad \lambda \in \mathbb{R}.$$

Then (3.7) and (2.1) give

$$\begin{aligned} \det S(\lambda) &= \det(I - 2\pi i \Psi(\lambda) V_S J(\lambda + i0) \Psi(\lambda)^*) \\ &= \det(I - 2\pi i \Psi(\lambda)^* \Psi(\lambda) V_S J(\lambda + i0)) \\ &= \det J(\lambda + i0) \det(J_0(\lambda + i0) - 2\pi i \Psi(\lambda)^* \Psi(\lambda) V_S) \\ &= \det J(\lambda + i0) \det(J_0(\lambda + i0) - Y_0(\lambda + i0) + Y_0(\lambda - i0)) \\ &= \det J(\lambda + i0) \det J_0(\lambda - i0), \end{aligned}$$

which together with (2.6) yields (1.2) since

$$\begin{aligned} \det S(\lambda) &= \det J(\lambda + i0) \det J_0(\lambda - i0) \\ &= \frac{\det J_0(\lambda - i0)}{\det J_0(\lambda + i0)} \\ &= \frac{D_-(\lambda - i0)}{D_+(\lambda + i0)} \\ &= \frac{\bar{D}_+(\lambda + i0)}{D_+(\lambda + i0)}. \end{aligned}$$

This yields (3.12) and adding the relation (3.3) we obtain $\phi_{sc}(\lambda) = O(|\lambda|^{-a})$ as $\lambda \rightarrow \pm\infty$ for any $a < \frac{1}{2}$. Substituting estimates (2.19) and (2.11) into (3.7) we obtain $S(\lambda) - 1 = O(\lambda^{-\frac{1}{2}})$ as $\lambda \rightarrow \pm\infty$. As both $\phi_{sc}(\lambda)$ and $S(\lambda)$ are continuous in λ , formula (3.6) determines $\phi_{sc}(\lambda)$ by $\phi_{sc}(\lambda) = \frac{i}{2\pi} \log S(\lambda)$ and the asymptotics $\phi_{sc}(\lambda) = O(\lambda^{-\frac{1}{2}})$ as $\lambda \rightarrow \pm\infty$. All together this proves Lemma 3.1. \square

Proof of Theorem 1.1. Due to Lemma 2.3 the function D_{\pm} is analytic in \mathbb{C}_{\pm} , continuous up to the real line. Asymptotics (3.5) and (2.21) yield (1.4), which gives (1.5). \square

Proposition 3.2. *Let V satisfy Condition C. Then the following trace formulas hold true:*

$$\int_{\mathbb{R}} V(x)dx = \frac{2}{\pi} \int_{\mathbb{R}} \operatorname{Re} \frac{\log D_+(\lambda + i0)}{\sqrt{\lambda + i0}} d\lambda, \tag{3.13}$$

$$\lim_{r \rightarrow \infty} \int_{-r}^r \operatorname{Im} \frac{\log D_+(\lambda + i0)}{\sqrt{\lambda + i0}} d\lambda = 0. \tag{3.14}$$

Proof. Define a contour: $\Gamma_r = c_r \cup (-r, r)$, where $c_r = \{|\lambda| = r\} \cap \mathbb{C}_+$ for large $r \rightarrow +\infty$. The function $f(\lambda) = i \frac{\log D_+(\lambda)}{\sqrt{\lambda}}$ is analytic in the upper-half-plane and continuous up to the real line without zero. This gives

$$0 = \int_{\Gamma_r} f(\lambda)d\lambda = I_r + I_r^+, \quad I_r^+ = \int_{c_r} f(\lambda)d\lambda, \quad I_r = \int_{-r}^r f(\lambda)d\lambda.$$

Due to asymptotics (1.4) $f(\lambda) = \frac{iV_0 + O(\lambda^{-\varepsilon})}{2\lambda}$ as $|\lambda| \rightarrow \infty$ uniformly in $\arg \lambda \in [0, \pi]$ for some $\varepsilon > 0$, we obtain

$$I_r^+ = \int_{c_r} f(\lambda)d\lambda = \frac{iV_0}{2} \int_{c_r} \frac{d\lambda}{\lambda} + o(1) = -\frac{\pi}{2}V_0 + o(1) \tag{3.15}$$

and

$$I_r = \int_{-r}^r f(\lambda)d\lambda = \frac{\pi}{2}V_0 + o(1)$$

as $r \rightarrow \infty$, and here

$$\operatorname{Re} f(\lambda+i0) \in L^1(\mathbb{R}), \quad \operatorname{Im} f(\lambda+i0) = \frac{V_0}{2\lambda} + O(\lambda^{-\varepsilon-1}) \quad \text{as } \pm\lambda \rightarrow \infty. \tag{3.16}$$

Combining all relations (3.15)–(3.16) we obtain (3.13)–(3.14). \square

Remark. We recall that trace formulas are important to study non linear equations, inverse problems, spectral theory, etc. see [9], [48], [24], [34] and references therein. The complete asymptotic expansion of the scattering phase (the spectral shift function) ϕ_{sc} at high energies and a sequence of trace formulas for 3-dim perturbed Stark operators were determined by Korotyaev-Pushnitski [30].

4. Analyticity of \mathcal{A}_0

4.1. Estimates on Airy functions. In this section we assume that the potential V satisfies Condition **V**. Recall that by (3.7), the functional $\Psi(\lambda): L^2(\mathbb{R}) \rightarrow \mathbb{C}$ and its adjoint $\Psi^*(\lambda): \mathbb{C} \rightarrow L^2(\mathbb{R})$ for $\lambda \in \mathbb{R}$ are given by

$$\Psi(\lambda)f = \int_{\mathbb{R}} \text{Ai}(x-\lambda)|V(x)|^{\frac{1}{2}} f(x)dx, \quad \Psi^*(\lambda)c = \text{Ai}(x-\lambda)|V(x)|^{\frac{1}{2}}c, \quad \lambda \in \mathbb{R}, \tag{4.1}$$

where $(f, c) \in L^2(\mathbb{R}) \times \mathbb{C}$. Assuming that V has compact support, these mappings are bounded on the real line and have analytic extensions from \mathbb{R} onto the whole complex plane. We remark that $\text{supp } V$ being bounded from below suffices to render $\Psi(\lambda)$ analytic. To prove our estimates $\text{supp } V$ being compact is, however, helpful. Thus $\Psi^*(\bar{\lambda})$ can be identified with the function $\text{Ai}(\cdot - \lambda)|V(\cdot)|^{\frac{1}{2}}$ in $L^2(\mathbb{R})$, which is analytic in $\lambda \in \mathbb{C}$.

In order to estimate Ψ, Ψ^* , we need the asymptotics of the Airy function from (2.9). Furthermore we have

$$\begin{cases} (x - \lambda)^{-\frac{1}{4}} = (-\lambda)^{-\frac{1}{4}}(1 + O(|\lambda|^{-1}), \\ (x - \lambda)^{\frac{3}{2}} = (-\lambda)^{\frac{3}{2}} + \frac{3}{2}x(-\lambda)^{\frac{1}{2}} + O(|\lambda|^{-\frac{1}{2}}), \end{cases} \quad |\arg \lambda| \geq \varepsilon, \tag{4.2}$$

and

$$\begin{cases} (\lambda - x)^{-\frac{1}{4}} = \lambda^{-\frac{1}{4}}(1 + O(|\lambda|^{-1}), \\ (\lambda - x)^{\frac{3}{2}} = \lambda^{\frac{3}{2}} - \frac{3}{2}x\lambda^{\frac{1}{2}} + O(|\lambda|^{-\frac{1}{2}}), \end{cases} \quad |\arg \lambda| \leq \varepsilon, \tag{4.3}$$

locally uniformly in $x \in \mathbb{R}$, as $|\lambda| \rightarrow \infty$. We will use these estimates in order to determine asymptotics of Airy functions. Asymptotics (4.2), (2.9) with $z = x - \lambda$ and straightforward calculation give the following asymptotics (4.4), (4.5):

i) Let $|\arg \lambda| \geq \varepsilon$ and let $\zeta = \sqrt{-\lambda}$, $|\arg \zeta| \leq \frac{\pi-\varepsilon}{2}$. Then as $|\lambda| \rightarrow \infty$ one has

$$\text{Ai}(x - \lambda)^2 = \frac{1}{4\zeta} e^{-(\frac{4}{3}\zeta^3 + 2x\zeta)} \left(1 + \frac{O(1)}{\zeta}\right), \tag{4.4}$$

and, in particular,

$$|\text{Ai}(x - \lambda)|^2 = \frac{1}{4|\lambda|^{\frac{1}{2}}} e^{-\text{Re}(\frac{4}{3}\zeta^3 + 2x\zeta)} \left(1 + \frac{O(1)}{|\lambda|^{1/2}}\right). \tag{4.5}$$

Moreover, for the case $|\arg \lambda| \leq \varepsilon$ using (4.3), we set $X = \frac{4}{3}(\lambda - x)^{\frac{3}{2}}$ and obtain after short calculation

$$X = \frac{4}{3}\lambda^{\frac{3}{2}} - 2x\lambda^{\frac{1}{2}} + O(|\lambda|^{-\frac{1}{2}}) = \eta + O(|\lambda|^{-\frac{1}{2}}) \quad \text{as } |\lambda| \rightarrow \infty, \quad |\arg \lambda| \leq \varepsilon.$$

where $\eta = \frac{4}{3}\zeta^3 - 2x\zeta$. Substituting this into (2.9) for $|\arg \lambda| \leq \varepsilon$, we obtain the following asymptotics and the estimate:

ii) Let $|\arg \lambda| \leq \varepsilon, \zeta = \sqrt{\lambda}, |\arg \zeta| \leq \frac{\varepsilon}{2}$ and let $|\lambda|$ be sufficiently large. Then

$$\text{Ai}(x - \lambda)^2 = \frac{1 + \sin \eta}{2\zeta} + \frac{O(e^{|\text{Im} \eta|})}{\lambda},$$

where $\eta = \frac{4}{3}\zeta^3 - 2x\zeta$, and in particular, the following slightly weaker estimate holds true:

$$|\text{Ai}(x - \lambda)|^2 \leq \frac{1}{|\lambda|^{\frac{1}{2}}} e^{|\text{Im} \eta|} \left(1 + \frac{O(1)}{|\lambda|^{\frac{1}{2}}}\right). \tag{4.6}$$

Note that all estimates (4.4)-(4.6) are locally uniform in x on bounded intervals.

4.2. Estimates on the Born term \mathcal{A}_0 . Now we are ready to study the Born term \mathcal{A}_0 .

Lemma 4.1. *Let V satisfy Condition V and let $\varepsilon > 0$. Then the Born term $\mathcal{A}_0(\lambda), \lambda \in \mathbb{R}$ given by (3.8) has an analytic extension from the real line into the whole complex plane and satisfies as $|\lambda| \rightarrow \infty$.*

i) Let $|\arg \lambda| \geq \varepsilon$ and let $-i\zeta = \sqrt{-\lambda}, |\arg \zeta - \frac{\pi}{2}| \leq \varepsilon$. Then

$$\mathcal{A}_0(\lambda) = \frac{i}{4\zeta} e^{-i\frac{4}{3}\zeta^3} \int_0^\gamma e^{i2x\zeta} V(x) \left(1 + \frac{O(1)}{\zeta}\right) dx, \tag{4.7}$$

and

$$|\mathcal{A}_0(\lambda)| \leq \frac{C}{|\zeta|} e^{\frac{4}{3}\text{Re} \zeta^3} \|V\|_{L^1(0,\gamma)}, \tag{4.8}$$

for some absolute constants C .

ii) Let $|\arg \lambda| \leq \varepsilon$ and let $\zeta = \sqrt{\lambda}, |\arg \zeta| \leq \varepsilon$. Then

$$\mathcal{A}_0(\lambda) = \frac{1}{2\zeta} \int_0^\gamma V(x) \left[1 + \sin \eta + \frac{O(e^{|\text{Im} \eta|})}{\zeta}\right] dx, \tag{4.9}$$

where $\eta = \frac{4}{3}\zeta^3 - 2x\zeta$ and

$$|\mathcal{A}_0(\lambda)| \leq \frac{1}{2|\zeta|} \int_0^\infty |V(x)| \left[1 + |\sin \eta| + \frac{O(e^{|\text{Im} \eta|})}{|\zeta|}\right] dx, \tag{4.10}$$

Let, in addition, $|\zeta| \geq 1 + \gamma$ and $|\arg \zeta| \leq \frac{\pi}{6}$. Then

$$|\mathcal{A}_0(\lambda)| \leq \frac{e^{\frac{4}{3}|\text{Im} \zeta^3|}}{2|\zeta|} \int_0^\gamma |V(x)| \left(1 + \frac{O(1)}{|\zeta|}\right) dx. \tag{4.11}$$

Proof. i) Substituting (4.2) and (4.4) in (3.8) we obtain (4.7) and (4.8).

ii) Using (4.5) we obtain (4.9), which yields (4.10). Let $\zeta = re^{i\phi}$, where $|\zeta| \geq 1 + \gamma$ and $|\phi| \leq \frac{\pi}{6}$. Then we have

$$|\operatorname{Im} \eta| = \left| \operatorname{Im} \left(\frac{2}{3} \zeta^3 - x \zeta \right) \right| = r \left| \frac{2}{3} r^2 \sin 3\phi - x \sin \phi \right| \leq \frac{2}{3} r^3 \sin 3|\phi|.$$

Substituting the last estimate into (4.9) we obtain (4.11). □

5. Resonances and S-matrix

5.1. Analyticity of S-matrix. We discuss a meromorphic continuation of the S-matrix $S(\lambda)$ from the real line onto the whole complex plane.

Lemma 5.1. *Let V satisfy Condition V and let $\varepsilon > 0$.*

i) *The functionals $\Psi(\lambda): L^2(\mathbb{R}) \rightarrow \mathbb{C}$ given by (4.1), and the mapping $\Psi^*(\bar{\lambda}) = \Psi^*(\lambda)$, for all $\lambda \in \mathbb{R}$ have analytic extensions from the real line into the whole complex plane and satisfy*

$$\|\Psi(\lambda)\|^2 = \|\Psi^*(\bar{\lambda})\|^2 = \int_0^\gamma |\operatorname{Ai}(x - \lambda)|^2 |V(x)| dx, \tag{5.1}$$

$$\|\Psi(\lambda)\|^2 \leq \frac{C}{(1 + |\lambda|)^{\frac{1}{2}}} e^{\frac{4}{3}|\lambda|^{\frac{3}{2}}}, \tag{5.2}$$

for all $\lambda \in \mathbb{C}$ and for some constant $C = C(V)$.

ii) *The scattering amplitude $\mathcal{A}(\lambda) = \mathcal{A}_0(\lambda) - \mathcal{A}_1(\lambda)$, defined in (3.7) for $\lambda \in \mathbb{R}$, has an analytic extension from the real line into the whole upper half-plane satisfying*

$$|\mathcal{A}_1(\lambda)| \leq \frac{C_a}{(1 + |\lambda|)^a} \|\Psi(\lambda)\|^2, \quad \text{for all } \lambda \in \bar{\mathbb{C}}_+, \tag{5.3}$$

for any $a \in (0, \frac{1}{2})$ and some constant $C = C_a$ depending on a .

Proof. i) Since the Airy function is entire, the functional $\Psi(\lambda): L^2(\mathbb{R}) \rightarrow \mathbb{C}$ given by (4.1), and the mapping $\Psi(\lambda)^*$, for all $\lambda \in \mathbb{R}$ have analytic extensions from the real line into the whole complex plane. The identities in (5.1) are obvious from the definitions of Ψ, Ψ^* . The proof of (5.2) is a repetition of the proof of (4.11) and (4.7). In fact, substituting (4.5), (4.6) into (5.1) we obtain (5.2).

ii) The operators Ψ, Ψ_1 have analytic extensions from the real line into the whole complex plane and operator-valued function $Y(\lambda + i0)$ also has an analytic extension from the real line into the upper-half plane. Then $\mathcal{A}_1(\lambda) = \Psi(\lambda)V_S Y(\lambda + i0)\Psi^*(\lambda)$ has an analytic extension from the real line into the upper-half plane and thus \mathcal{A} has so. Moreover, (2.19) gives

$$|\mathcal{A}_1(\lambda)| \leq \|\Psi(\lambda)\|^2 \|Y(\lambda)\| \leq C(1 + |\lambda|)^{-a} \|\Psi(\lambda)\|^2, \quad \text{for all } \lambda \in \bar{\mathbb{C}}_+,$$

where $C = C(V)$ is some constant. \square

Lemma 5.2. *Let V satisfy Condition C with $V(0) \neq 0$ and let $\lambda = t^2 e^{i\frac{\pi}{3}}$, $\zeta = e^{i\frac{\pi}{6}} t$ as $t \rightarrow +\infty$. Then*

$$\begin{aligned} \mathcal{A}_0(\lambda) &= \frac{e^{i\frac{\pi}{3} + \frac{4}{3}t^3}}{4t} \int_0^\gamma e^{(-1+i\sqrt{3})tx} V(x) \left(1 + \frac{O(1)}{t}\right) dx \\ &= \frac{e^{i\frac{2\pi}{3} + \frac{4}{3}t^3}}{8t^2} (V(0) + o(1)), \end{aligned} \quad (5.4)$$

$$\|\Psi(\lambda)\|^2 \leq e^{\frac{4}{3}t^3} \frac{C}{t^2} \|V\|, \quad (5.5)$$

$$\mathcal{A}(\lambda) = \frac{e^{i\frac{2\pi}{3} + \frac{4}{3}t^3}}{8t^2} (V(0) + o(1)). \quad (5.6)$$

Proof. Let $\sigma = 1 - i\sqrt{3} = 2e^{-i\frac{\pi}{3}}$ and let $\zeta = e^{i\frac{\pi}{6}} t$ as $t \rightarrow +\infty$. From (4.7) we have

$$\begin{aligned} \mathcal{A}_0(\lambda) &= \frac{ie^{-i\frac{\pi}{6}}}{4t} e^{\frac{4}{3}t^3} \mathcal{J}(\lambda), \quad \mathcal{J}(\lambda) = \int_0^\gamma e^{-\sigma tx} V(x) \left(1 + \frac{O(1)}{t}\right) dx \\ &= \mathcal{J}_1(\lambda) + \mathcal{J}_2(\lambda), \\ \mathcal{J}_1(\lambda) &= \int_0^{\gamma_1} e^{-\sigma tx} V(x) \left(1 + \frac{O(1)}{t}\right) dx, \\ \mathcal{J}_2(\lambda) &= \int_{\gamma_1}^\gamma e^{-\sigma tx} V(x) \left(1 + \frac{O(1)}{t}\right) dx \\ &= e^{-t\gamma_1} \|V\|_1 O(1). \end{aligned}$$

An integration by parts yields

$$\begin{aligned} \int_0^\gamma e^{-\sigma tx} V(x) dx &= -\frac{1}{\sigma t} e^{-\sigma tx} V(x) \Big|_0^\gamma + \frac{1}{\sigma t} \int_0^\gamma e^{-\sigma tx} V'(x) dx \\ &= \frac{1}{\sigma t} \left(V(0) - e^{-\sigma t\gamma} V(\gamma) + \int_0^\gamma e^{-\sigma tx} V'(x) dx \right) \\ &= \frac{V(0) - o(1)}{\sigma t}, \end{aligned}$$

since since V is absolutely continuous, its derivative is in $L^1(\mathbb{R})$ and for $u \in L^1(0, \infty)$ the following asymptotics hold true

$$\int_0^\gamma e^{-tx}u(x)dx = o(1) \quad \text{as } t \rightarrow \infty. \tag{5.7}$$

Moreover, (5.7) gives $\int_0^\gamma e^{-\sigma tx}V(x)dx = o(1)$. Combining all these estimates we obtain (5.4).

We consider $\|\Psi(\lambda)\|^2$ and we show (5.5). Let $\zeta = te^{i\frac{\pi}{6}}$ as $t \rightarrow +\infty$. From (4.7) we have

$$\|\Psi(\lambda)\|^2 = \int_0^\gamma |\text{Ai}(x - \lambda)|^2|V(x)|dx = \frac{1}{4t}e^{\frac{4}{3}t^3} \int_0^\gamma e^{-tx}|V(x)|(1 + o(1))dx,$$

and

$$\begin{aligned} \int_0^\gamma e^{-tx}|V(x)|dx &= \int_0^{\gamma_1} e^{-tx}|V(x)|dx + \int_{\gamma_1}^\gamma e^{-tx}|V(x)|dx \\ &\leq \|V\|_{L^\infty(0,\gamma_1)} \int_0^{\gamma_1} e^{-tx}dx + e^{-t\gamma_1} \int_{\gamma_1}^\gamma |V(x)|dx \\ &\leq \frac{1}{t}\|V\|_\infty + e^{-t\gamma_1}\|V\|_1, \end{aligned}$$

which yields (5.5). By (3.7), the S-matrix has the form

$$S(\lambda) = I - 2\pi i\mathcal{A}(\lambda), \quad \mathcal{A} = \mathcal{A}_0 - \mathcal{A}_1.$$

Using (5.3),(5.5) we obtain

$$\mathcal{A}(\lambda) = \mathcal{A}_0(\lambda) - \mathcal{A}_1(\lambda) = \frac{e^{i\frac{2\pi}{3} + \frac{4}{3}t^3}}{8t^2}[(V(0) + o(1)) + O(t^{-a})],$$

which permits to get (5.6). □

We are now ready to prove the main theorems.

Proof of Theorem 1.3. It follows from Lemma 5.1 and 4.1 that $S(\lambda)$, $\lambda \in \mathbb{R}$ has an analytic extension from \mathbb{R} into the upper half-plane \mathbb{C}_+ and satisfies

$$|S(\lambda)| \leq C_S e^{\frac{4}{3}|\lambda|^{\frac{3}{2}}}, \quad \text{for all } \lambda \in \bar{\mathbb{C}}_+,$$

for some constant C_S . Invoking the identity (1.2) we obtain $D_-(\lambda - i0) = S(\lambda)D_+(\lambda + i0)$, $\lambda \in \mathbb{R}$. Thus it follows from Lemma 5.1, 4.1 and 2.3 that

$D_-(\lambda), \lambda \in \mathbb{C}_-$ has an analytic extension from $\overline{\mathbb{C}}_-$ into the whole complex plane \mathbb{C} and satisfies

$$|D_-(\lambda)| \leq |S(\lambda)||D_+(\lambda)| \leq C_S e^{\frac{4}{3}|\lambda|^{\frac{3}{2}}} \sup_{\lambda \in \mathbb{C}_+} |D_+(\lambda)|, \quad \text{for all } \lambda \in \overline{\mathbb{C}}_+,$$

$$\sup_{\lambda \in \mathbb{C}_{\pm}} |D_{\pm}(\lambda)| < \infty,$$

which together with (1.3) yields (1.7). Then, by (1.2), the S-matrix $S(\lambda), \lambda \in \mathbb{R}$ has an analytic extension into the whole upper half plane \mathbb{C}_+ and a meromorphic extension into the whole lower half plane \mathbb{C}_- satisfying

$$S(\lambda) = \frac{D_-(\lambda)}{D_+(\lambda)}, \quad \text{for all } \lambda \in \mathbb{C},$$

where the functions D_{\pm} do not vanish in $\overline{\mathbb{C}}_{\pm}$. The zeros of $S(\lambda), \lambda \in \mathbb{C}_+$ coincide with the zeros of D_- and the poles of $S(\lambda), \lambda \in \mathbb{C}_-$ are precisely the zeros of D_+ .

Let, in addition, V satisfy Condition C with $V(0) \neq 0$. Then asymptotics (5.6), (1.4) give

$$D_-(t^2 e^{i\frac{\pi}{3}}) = \frac{e^{i\frac{2\pi}{3} + \frac{4}{3}t^3}}{8t^2} (V(0) + o(1)) \quad \text{as } t \rightarrow +\infty.$$

Thus D_{\pm} is an entire functions of order $\frac{3}{2}$ and type $\frac{4}{3}$. □

Now we discuss the remarks after Theorem 1.3. By Theorem 1.3, the determinants $D_{\pm}(\lambda), \lambda \in \mathbb{C}_{\pm}$ extend to entire functions of order $\frac{3}{2}$ and type $\frac{4}{3}$. It is well known that in this case D_+ has the Hadamard factorization (1.8) and (5.8) for some p , see p. 22 in [5]. Moreover, we have

$$\frac{D'_+(\lambda)}{D_+(\lambda)} = p + \sum_{n \geq 1} \frac{\lambda}{\lambda_n(\lambda - \lambda_n)}, \tag{5.8}$$

where the constant $p = \frac{D'_+(0)}{D_+(0)}$ and the series converges absolutely and uniformly on any compact set of $\mathbb{C} \setminus \{\lambda_n, n \geq 1\}$. Recall that if an entire function F has order $m > 0$ and has zeros $z_n, n \geq 1$, then (see p. 17 in [5])

$$\sum_{z_n \neq 0, n \geq 1} |z_n|^{-m_1} < \infty, \quad \text{for all } m_1 > m.$$

Using (1.8), (5.8) and differentiating

$$S(\lambda) = \left(\frac{\bar{D}_+(\lambda + i0)}{D_+(\lambda + i0)} \right) = e^{-2\pi i \phi_{sc}(\lambda)}, \quad \lambda \in \mathbb{R},$$

we obtain

$$S(\lambda)(-2\pi i \phi'_{sc}(\lambda)) = -S(\lambda) \left(\frac{D'_+(\lambda + i0)}{D_+(\lambda + i0)} - \frac{\bar{D}'_+(\lambda + i0)}{\bar{D}_+(\lambda + i0)} \right), \quad \text{as } \lambda \in \mathbb{R},$$

which yields

$$2\pi i \phi'_{sc}(\lambda) = p - \bar{p} + \lambda \sum_{n \geq 1} \left(\frac{1}{\lambda_n(\lambda - \lambda_n)} - \frac{1}{\bar{\lambda}_n(\lambda - \bar{\lambda}_n)} \right), \quad \text{for all } \lambda \in \mathbb{R},$$

and then $2\pi i \phi'_{sc}(0) = p - \bar{p}$. Thus we obtain (1.12).

Finally, we prove Corollary 1.4 and 1.5.

Proof of Corollary 1.4. Under Condition V, Theorem 1.3 gives that the determinant $D_+(\lambda)$, $\lambda \in \mathbb{C}_+$ extends to an entire function and satisfies (1.7), which yields the standard upper bound (1.9) (see page 16 in [5]).

Let V satisfy Condition C and $V(0) \neq 0$. Then by Theorem 1.3, determinant $D_+(\lambda)$, $\lambda \in \mathbb{C}_+$ is an entire function of order $\frac{3}{2}$ and type $\frac{4}{3}$. Since the order of D_+ is not integer, (1.9) follows from the Lindelöf Theorem [35]. We recall this theorem.

Theorem (Lindelöf, 1903). *Let f be an entire function of order β , which is not an integer. Then*

- i) f is of zero type if and only if $\mathcal{N}(r, f) = o(r^\beta)$;
- ii) f is of finite type if and only if $\mathcal{N}(r, f) = O(r^\beta)$.

Moreover, since the type of D_+ is different from zero, $\mathcal{N}(r, D_+)$ is not $o(r^{3/2})$. This is (1.10). □

Proof of Corollary 1.5. Due to (3.1) we have

$$\text{Tr}(R(\lambda) - R_0(\lambda)) = \frac{D'_+(\lambda)}{D_+(\lambda)}$$

for each $\lambda \in \mathbb{C}_+$ and adding (5.8) we obtain (1.11). □

Proposition 5.3. *Let V satisfy Condition V. Then for any $f \in C_0^\infty(\mathbb{R})$ the following identity*

$$\text{Tr}(f(H) - f(H_0)) = -\frac{1}{\pi} \int_{\mathbb{R}} f(t) \left(\phi'_{\text{sc}}(0) + \frac{\lambda}{\pi} \text{Im} \sum_{n \geq 1} \frac{1}{\lambda_n(t - \lambda_n)} \right) dt, \quad (5.9)$$

holds true, furthermore,

$$\phi_{\text{sc}}^{(m)}(0) = \frac{(m-1)!}{\pi} \text{Im} \sum_{n \geq 1} \frac{1}{\lambda_n^m}, \quad (5.10)$$

where the first series converges absolutely and uniformly on any compact set of $\mathbb{C} \setminus \{\lambda_n, n \geq 1\}$.

Proof. Due to (2.14) we obtain $R(\lambda) - R_0(\lambda) \in \mathcal{B}_1$ for each $\text{Im } \lambda \neq 0$. Then the Krein formula [33] for the operators H, H_0 and for any $f \in C_0^\infty(\mathbb{R})$ gives

$$\text{Tr}(f(H) - f(H_0)) = - \int_{\mathbb{R}} f(t) \phi'_{\text{sc}}(t) dt \quad (5.11)$$

and substituting (1.12) into (5.11) we obtain (5.9).

Due to (1.12) we have the identity

$$\phi'_{\text{sc}}(\lambda) = \phi'_{\text{sc}}(0) + \frac{1}{\pi} \text{Im} \sum_{n \geq 1} \left(\frac{1}{\lambda_n - \lambda} - \frac{1}{\lambda_n} \right), \quad \text{for all } \lambda \in \mathbb{C},$$

uniformly on any compact subset of $\mathbb{C} \setminus \{\lambda_1, \lambda_2, \lambda_3, \dots\}$. Differentiating $\phi'_{\text{sc}}(\lambda)$ we then arrive at (5.10). □

Proof of Theorem 1.6. There are results about inverse problems for perturbed Stark operators $H = H_0 + V$ on $L^2(\mathbb{R})$. For example, Kachalov and Kurylev [23] consider inverse scattering problem, when the potential V satisfies

$$\int_{\mathbb{R}} (1 + |x|)^4 |V(x)| dx < \infty. \quad (5.12)$$

They prove the recovering problem using the Gelfand–Levitan equation: for given S-matrix $S(\lambda)$ to determine a potential V , which satisfies (5.12) (Theorem 1 and 2 in [23]).

Let V satisfy Condition V. Then due to Theorem (1.2) and (1.8) the S-matrix $S(\lambda)$ is given by

$$S(\lambda) = S(0)S_1(\lambda), \quad S_1(\lambda) = e^{-2i\lambda\phi'_{\text{sc}}(0)} \lim_{r \rightarrow +\infty} \prod_{|\lambda_n| \leq r} \left(\frac{1 - \frac{\lambda}{\lambda_n}}{1 - \frac{\lambda}{\lambda_n}} \right) e^{\frac{\lambda}{\lambda_n} - \frac{\lambda}{\lambda_n}}, \quad \lambda \in \mathbb{C}, \quad (5.13)$$

uniformly on any compact subset of \mathbb{C} , where $S(0) = \frac{\bar{D}_+(0)}{D_+(0)} = e^{-2i\phi_{sc}(0)}$. Asymptotics (3.11) implies $S(\lambda) \rightarrow 1$ as $\lambda \rightarrow \pm\infty$. Then we deduce that there exists a following limit:

$$S(0) = S(\lambda)\bar{S}_1(\lambda) = \lim_{\lambda \rightarrow -\infty} \bar{S}_1(\lambda). \tag{5.14}$$

Thus due to (5.13) and (5.14) the S-matrix is expressed in terms of resonances only.

Now we consider two perturbed Stark operators $H_j = H_0 + V_j, j = 1, 2$ on $L^2(\mathbb{R})$, where the potential V_j satisfies Condition V. We denote the S-matrix for H_0, H_j by $S_j(\lambda)$. Assume that H_1 and H_2 have the same resonances. Then by the above results, the S-matrices for H_1 and H_2 coincide, i.e., $S_1 = S_2$. After this the Kachalov and Kurylev results [23] give $V_1 = V_2$. \square

6. Appendix

Introduce the Fourier transformation

$$\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)e^{-ixt} dx, \quad t \in \mathbb{R}.$$

Lemma 6.1. *Let V satisfy the conditions in Theorem 1.1 and $\lambda \in \bar{\mathbb{C}}_+$. Then the following identity and asymptotics*

$$\begin{aligned} \text{Tr } Y_0(\lambda) &= \frac{e^{i\frac{\pi}{4}}}{(4\pi)^{1/2}} \int_{\mathbb{R}} V(x)dx \int_0^\infty e^{-it^3/12} e^{it(\lambda-x)} \frac{dt}{t^{1/2}} \\ &= \frac{e^{i\frac{\pi}{4}}}{\sqrt{2}} \int_0^\infty e^{it\lambda - \frac{i}{12}t^3} \hat{V}(t) \frac{dt}{t^{1/2}}, \end{aligned} \tag{6.1}$$

$$\text{Tr } Y_0(\lambda) = i \frac{V_0}{2\sqrt{\lambda}} + \frac{O(1)}{\lambda} \quad \text{as } |\lambda| \rightarrow \infty, \tag{6.2}$$

hold true, uniformly with respect to $\arg \lambda \in [0, \pi]$.

Proof. We show (6.1). In view of (2.19) we have $Y_0(\lambda) \in \mathcal{B}_1$ for all $\lambda \in \mathbb{C}_+$. Identity (2.5) for $\lambda \in \mathbb{C}_+$ with $c := \frac{e^{i\frac{\pi}{4}}}{\sqrt{2}}$ gives

$$\begin{aligned} \text{Tr } Y_0(\lambda) &= \int_{\mathbb{R}} V(x)R_0(x, x, \lambda)dx \\ &= \frac{c}{\sqrt{2\pi}} \int_{\mathbb{R}} V(x)dx \int_0^\infty e^{it(\lambda-x) - i\frac{t^3}{12}} \frac{dt}{t^{\frac{1}{2}}} \\ &= c \int_0^\infty e^{it\lambda - \frac{i}{12}t^3} \hat{V}(t) \frac{dt}{t^{\frac{1}{2}}}. \end{aligned}$$

We characterize the properties of V in terms of its Fourier transform \widehat{V} :

$$\widehat{V}, (\widehat{V})', (\widehat{V})'' \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R}), \quad (6.3a)$$

$$\begin{aligned} \widehat{V}(t) &= \frac{V(0) - e^{-it\gamma} V(\gamma)}{it\sqrt{2\pi}} + \frac{1}{it\sqrt{2\pi}} \int_0^\gamma e^{-itx} V'(x) dx \\ &= \frac{O(1)}{t} \quad \text{as } t \rightarrow \pm\infty. \end{aligned} \quad (6.3b)$$

Let us show (6.2). Define cut-off functions $w_0, w_1 \in C^\infty(\mathbb{R}_+)$ by

$$w_0(t) = \begin{cases} 1 & t \in [0, 1], \\ 0 & t > 2, \end{cases} \quad w_1 = 1 - w_0.$$

Thus (6.1) yields the decomposition for $\lambda \in \bar{\mathbb{C}}_+$:

$$\text{Tr } Y_0(\lambda) = c \int_0^\infty e^{it\lambda - \frac{i}{12}t^3} \widehat{V}(t) \frac{dt}{t^{1/2}} = cI_0(\lambda) + cI_1(\lambda), \quad c = \frac{e^{i\pi/4}}{\sqrt{2}},$$

where

$$I_j = \int_0^\infty w_j(t) e^{i\phi(t)} \widehat{V}(t) \frac{dt}{t^{1/2}}, \quad j = 0, 1,$$

$$\phi(t) = t\lambda - \frac{t^3}{12}.$$

Let us first consider I_0 . Recall that $\widehat{V}(t)$ and has the decomposition:

$$e^{-\frac{i}{12}t^3} \widehat{V}(t) = \widehat{V}(0) + t\beta(t) \quad \text{as } t \rightarrow 0,$$

where $\widehat{V}(0) = \frac{V_0}{\sqrt{2\pi}}$ and the function β is entire. Substituting this asymptotics into I_0 we obtain

$$I_0 = \int_0^2 w_0(t) e^{it\lambda - \frac{i}{12}t^3} \widehat{V}(t) \frac{dt}{t^{1/2}} = I_{01} + I_{02}, \quad (6.4a)$$

$$I_{01} = \widehat{V}(0) \int_0^2 w_0(t) e^{it\lambda} \frac{dt}{t^{1/2}}. \quad (6.4b)$$

The stationary phase method gives

$$I_{01} = \frac{V_0}{\sqrt{2\pi}} \int_0^2 w_0(t) e^{it\lambda} \frac{dt}{t^{1/2}} = \frac{e^{i\pi/4} V_0}{\sqrt{2\pi}} \sqrt{\frac{\pi}{\lambda}} + \frac{O(1)}{\lambda^{3/2}} = \frac{cV_0}{\sqrt{\lambda}} + \frac{O(1)}{\lambda^{3/2}}, \quad (6.5)$$

and an integration by parts further implies

$$I_{02} = \int_0^2 w_0 e^{it\lambda} (e^{-\frac{i}{12}t^3} \widehat{V}(t) - \widehat{V}(0)) \frac{dt}{t^{1/2}} = \frac{O(1)}{\lambda}.$$

Thus, if we assume that $I_1 = O(1/\lambda)$, then (6.4) and (6.5) yield (6.2). Hence, it remains to show that $I_1 = O(1/\lambda)$. To this end we distinguish three cases.

- First, let $\lambda = \mu + i\nu, \nu \geq K|\mu|$ for some $K > 0$ and let $\nu \rightarrow \infty$. We have

$$|I_1| \leq \int_1^\infty e^{-t\nu} |\widehat{V}(t)| dt \leq \|\widehat{V}\|_\infty \int_1^\infty e^{-t\nu} dt = \frac{\|\widehat{V}\|_\infty}{\nu} e^{-\nu}. \quad (6.6)$$

- Second, let $\text{Re } \lambda \rightarrow -\infty$. We put $f = w_1(1)\widehat{V}(t)/t^{1/2}$ and here $\phi'(t) = \lambda - \frac{t^2}{4}$. Then due to (6.3) we have

$$\begin{aligned} I_1 &= \int_1^\infty e^{i\phi(t)} f(t) dt \\ &= \int_1^\infty \frac{f(t)}{i\phi'(t)} de^{i\phi(t)} \\ &= i \int_1^\infty e^{i\phi(t)} \left(\frac{f(t)}{\phi'(t)}\right)' dt \\ &= \frac{O(1)}{\lambda} \end{aligned} \quad (6.7)$$

since (6.3) yields $f', f = O(t^{-3/2}), t \rightarrow \infty$ and we have the following estimate

$$\left| \left(\frac{f(t)}{i(\lambda - \frac{t^2}{4})}\right)' \right| \leq \frac{C}{|\lambda| t^{\frac{3}{2}}}, \quad \text{for all } t \geq 1,$$

for some constant C

- Third, let $\lambda = \frac{k^2 + i\nu}{4} \in \bar{\mathbb{C}}_+, k \rightarrow +\infty$ and $Kk^2 \geq \nu$. Define the cut-off functions $g_0, g_1 \in C_0^\infty(\mathbb{R})$ by

$$g(s) = \begin{cases} 1 & s \in (-1/2, 1/2), \\ 0 & s \in \mathbb{R}_+ \setminus (-1, 1), \end{cases} \quad g_1 = 1 - g(t - k).$$

Then we have

$$\begin{aligned} I_1 &= \int_1^\infty e^{it\lambda - \frac{i}{12}t^3} f(t) dt = I_{11} + I_{12}, \\ I_{11} &= \int_1^\infty e^{it\lambda - \frac{i}{12}t^3} g(t - k) f(t) dt, \\ f &= \widehat{V}(t) \frac{w_1(t)}{t^{\frac{1}{2}}}. \end{aligned}$$

Let $F(t) = g_1(t)f(t)$. Similar to (6.7) we obtain

$$\begin{aligned}
 I_{12} &= \int_1^\infty e^{i\phi(t)} F(t) dt \\
 &= \int_1^\infty \frac{F(t)}{i\phi'(t)} d e^{i\phi(t)} \\
 &= - \int_1^\infty e^{i\phi(t)} \left(\frac{F(t)}{i\phi'(t)} \right)' dt \\
 &= - \int_1^\infty e^{i\phi(t)} \left(\frac{1}{\phi'(t)} \left(\frac{F(t)}{\phi'(t)} \right)' \right)' dt \\
 &= \frac{O(1)}{|\lambda|}
 \end{aligned} \tag{6.8}$$

since due to (6.3) we have

$$\left| \left(\frac{1}{\phi'(t)} \left(\frac{F(t)}{\phi'(t)} \right)' \right)' \right| \leq \frac{C}{|\lambda| t^{\frac{3}{2}}}, \quad \text{for all } t \geq 1.$$

Let us now consider the main term I_{11} . We have for $\lambda = \frac{k^2 + iv}{4}$ and $t = k + s$:

$$\begin{aligned}
 t\lambda - \frac{i}{12}t^3 &= \frac{1}{12}(3(k+s)(k^2 + iv) - (k+s)^3) \\
 &= \frac{1}{12}(3(k+s)k^2 + i3tv - (k+s)^3) \\
 &= \frac{1}{12}(2k^3 + i3tv - 3s^2k - s^3) \\
 &= \frac{k^3}{6} + iv\frac{k+s}{4} + -\frac{k}{4}s^2 - \frac{s^3}{12}.
 \end{aligned}$$

Define $\psi(s, \lambda) = e^{-i\frac{s^3}{12}} e^{-\frac{v}{4}(k+s)} g(s) f(k+s)$ and note that $f(k+s) = O(k^{-\frac{3}{2}})$ as $k \rightarrow \infty$, uniformly in $s \in [-1, 1]$. Due to (6.3) the stationary phase method gives

$$\begin{aligned}
 I_{11} &= \int_{k-1}^{k+1} e^{it\lambda - \frac{i}{12}t^3} g(t-k) f(t) dt \\
 &= e^{\frac{i}{6}k^3} \int_{-1}^1 e^{-\frac{i}{4}ks^2} \psi(s, \lambda) ds \\
 &= \frac{O(1)}{k^2} \\
 &= \frac{O(1)}{|\lambda|}.
 \end{aligned} \tag{6.9}$$

Combining (6.6)–(6.9) we obtain $I_1 = O(1/\lambda)$ and (6.2). □

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References

- [1] J. Agler and R. Froese, Existence of Stark ladder resonances. *Comm. Math. Phys.* **100** (1985), no. 2, 161–171. [MR 0804457](#) [Zbl 0651.47006](#)
- [2] A. B. Aleksandrov, Norm of the Hilbert transformation in a space of Hölder functions. *Funkts. Anal. Prilozh.* **9** (1975), no. 2, 1–4. In Russian. English translation, *Funct. Anal. Appl.* **9** (1975), 94–96. [Zbl 0318.44008](#)
- [3] J. Avron and I. Herbst, Spectral and scattering theory of Schrödinger operators related to the Stark effect. *Comm. Math. Phys.* **52** (1977), no. 3, 239–254. [MR 0468862](#) [Zbl 0351.47007](#)
- [4] A. Barreto, Remarks on the distribution of resonances in odd dimensional Euclidean scattering. *Asymptot. Anal.* **27** (2001), no. 2, 161–170. [MR 1852004](#) [Zbl 1116.35344](#)
- [5] R. Boas, Jr., *Entire functions*. Academic Press, New York, 1954. [MR 0068627](#) [Zbl 0058.30201](#)
- [6] B. Brown, I. Knowles, and R. Weikard, On the inverse resonance problem. *J. London Math. Soc.* (2) **68** (2003), no. 2, 383–401. [MR 1994689](#) [Zbl 1050.34004](#)
- [7] F. Calogero and A. Degasperis, Inverse spectral problem for the one-dimensional Schrödinger equation with an additional linear potential. *Lett. Nuovo Cimento* (2) **23** (1978), no. 4, 143–149. [MR 0514526](#)
- [8] H. Cycon, R. Froese, W. Kirsch, and B. Simon, *Schrödinger operators with applications to quantum mechanics and global geometry*. Texts and Monographs in Physics. Springer Study Edition. Springer-Verlag, Berlin, 1987. [MR 0883643](#) [Zbl 0619.47005](#)
- [9] P. Deift, and R. Killip, On the absolutely continuous spectrum of one-dimensional Schrödinger operators with square summable potentials. *Comm. Math. Phys.* **203** (1999), no. 2, 341–347. [MR 1697600](#) [Zbl 0934.34075](#)
- [10] L. D. Faddeev, On a model of Friedrichs in the theory of perturbations of the continuous spectrum. *Trudy Mat. Inst. Steklov* **73** (1964) 292–313. In Russian. English translation, *Amer. Math. Soc. Transl. Ser. 2* **62** (1967), 177–203. [MR 0178362](#) [Zbl 0183.41902](#)

- [11] W. Firsova, Resonances of a Hill operator, perturbed by an exponentially decreasing additive potential. *Mat. Zametki* **36** (1984), no. 5, 711–724, 798. In Russian. English translation, *Math. Notes* **36** (1984), no. 5-6, 854–861. [MR 0773807](#) [Zbl 0566.34013](#) (Russian) [Zbl 0598.34017](#) (English transl.)
- [12] R. Froese, Asymptotic distribution of resonances in one dimension. *J. Differential Equations* **137** (1997), no. 2, 251–272. [MR 1456597](#) [Zbl 0955.35057](#)
- [13] I. Gohberg and M. Kreĭn, Introduction to the theory of linear nonselfadjoint operators. Translated from the Russian by A. Feinstein. Translations of Mathematical Monographs, 18. American Mathematical Society, Providence, R.I., 1969. [MR 0246142](#) [Zbl 0181.13504](#)
- [14] S. Graffi and E. Harrell, Inverse scattering for the one-dimensional Stark effect and application to the cylindrical KdV equation. *Ann. Inst. H. Poincaré Sect. A (N.S.)* **36** (1982), no. 1, 41–58. [MR 0653017](#) [Zbl 0506.35079](#)
- [15] V. Grecchi and A. Sacchetti, Lifetime of the Wannier–Stark resonances and perturbation theory. *Comm. Math. Phys.* **185** (1997), no. 2, 359–378. [MR 1463046](#) [Zbl 0899.35086](#)
- [16] I. W. Herbst, Unitary equivalence of Stark Hamiltonians. *Math. Z.* **155** (1977), no. 1, 55–70. [MR 0449318](#) [Zbl 0338.47009](#)
- [17] I. W. Herbst, Dilation analyticity in constant electric field. I. The two body problem. *Comm. Math. Phys.* **64** (1979), no. 3, 279–298. [MR 0520094](#) [Zbl 0447.47028](#)
- [18] I. W. Herbst and J. S. Howland, The Stark ladder and other one-dimensional external field problems. *Comm. Math. Phys.* **80** (1981), no. 1, 23–42. [MR 0623150](#) [Zbl 0473.47037](#)
- [19] A. Its and V. Sukhaviy, Riemann–Hilbert approach for the inverse problem for the Stark operator on the line. *Inverse Problems* **32** (2016), no. 5, article id. 055003.
- [20] A. Jensen, Asymptotic completeness for a new class of Stark effect Hamiltonians. *Comm. Math. Phys.* **107** (1986), no. 1, 21–28. [MR 0861882](#) [Zbl 0606.34020](#)
- [21] A. Jensen, Commutator methods and asymptotic completeness for one-dimensional Stark effect Hamiltonians. In E. Balslev. (ed.), *Schrödinger operators, Aarhus 1985*. Papers from the symposium held at Aarhus University, Aarhus, October 2–4, 1985. Springer-Verlag, Berlin etc., 1986, 151–166. [MR 0869600](#) [Zbl 0608.35013](#)
- [22] A. Jensen, Perturbation results for Stark effect resonances. *J. Reine Angew. Math.* **394** (1989), 168–179. [MR 0977439](#) [Zbl 0655.35061](#)
- [23] A. P. Kachalov and Ya. V. Kurylëv, The method of transformation operators in the inverse scattering problem. The one-dimensional Stark effect. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **179** (1989), Mat. Vopr. Teor. Rasprostr. Voln. 19, 73–87, 189. In Russian. English translation, *J. Soviet Math.* **57** (1991), no. 3, 3111–3122. [MR 1039596](#) [Zbl 0715.34139](#) (Russian) [Zbl 0745.34076](#) (English transl.)
- [24] P. P. Kargaev and E. L. Korotyaev, The inverse problem for the Hill operator, a direct approach. *Invent. Math.* **129** (1997), no. 3, 567–593. [MR 1465335](#) [Zbl 0878.34011](#)

- [25] P Koosis, *The logarithmic integral*. I. Cambridge Studies in Advanced Mathematics, 12. Cambridge University Press, Cambridge (U.K.) etc., 1988. [MR 0961844](#) [Zbl 0665.30038](#)
- [26] E. L. Korotyaev, *Mat. Sb. (N.S.)* **132(174)** (1987), no. 2, 182–201. In Russian. English translation, *Math. USSR-Sb.* **60** (1988), no. 1, 177–196. [MR 0882833](#)
- [27] E. L. Korotyaev, Inverse resonance scattering on the half line. *Asymptot. Anal.* **37** (2004), no. 3–4, 215–226. [MR 2047740](#) [Zbl 1064.34007](#)
- [28] E. L. Korotyaev, Inverse resonance scattering on the real line. *Inverse Problems* **21** (2005), no. 1, 325–341. [MR 2146179](#) [Zbl 1074.34081](#)
- [29] E. L. Korotyaev, Resonance theory for perturbed Hill operator. *Asymptot. Anal.* **74** (2011), no. 3–4, 199–227. [MR 2868052](#) [Zbl 1259.34088](#)
- [30] E. L. Korotyaev and A. Pushnitski, Trace formulae and high energy asymptotics for the Stark operator. *Comm. Partial Differential Equations* **28** (2003), no. 3–4, 817–842. [MR 1978316](#) [Zbl 1106.35305](#)
- [31] E. L. Korotyaev and K. Schmidt, On the resonances and eigenvalues for a 1D half-crystal with localised impurity. *J. Reine Angew. Math.* **670** (2012), 217–248. [MR 2982697](#) [Zbl 1251.47066](#)
- [32] G. Kristensson, The one-dimensional inverse scattering problem for an increasing potential. *J. Math. Phys.* **27** (1986), no. 3, 804–815. [MR 0825167](#) [Zbl 0594.47006](#)
- [33] G. M. Kreĭn, On perturbation determinants and the trace formula for unitary and self-adjoint operators. *Dokl. Akad. Nauk SSSR* **144** (1962), 268–271. In Russian. English translation, *Soviet Math. Dokl.* **3** (1962), 707–710. [MR 0139006](#)
- [34] A. Laptev, and T. Weidl, Sharp Lieb-Thirring inequalities in high dimensions. *Acta Math.* **184** (2000), no. 1, 87–111. [MR 1756570](#) [Zbl 1142.35531](#)
- [35] E. Lindelöf, Sur les fonctions entières d'ordre entier. *Ann., Sci., Ecole Norm. Sup.* **31,22** (1903), 369–395.
- [36] Y. Lin, M. Qian, and Q. Zhang, Inverse scattering problem for one-dimensional Schrödinger operators related to the general Stark effect. *Acta Math. Appl. Sinica (English Ser.)* **5** (1989), no. 2, 116–136. [MR 1004049](#) [Zbl 0699.35200](#)
- [37] Y. Liu, Scattering and spectral theory for Stark Hamiltonians in one dimension. *Math. Scand.* **72** (1993), no. 2, 265–297. [MR 1241819](#) [Zbl 0808.35093](#)
- [38] F. W. J. Olver, *Asymptotics and special functions*. Computer Science and Applied Mathematics. Academic Press, New York and London, 1974. [MR 0435697](#) [Zbl 0303.41035](#)
- [39] M. Reed and B. Simon, *Methods of modern mathematical physics*. IV. Analysis of operators. Academic Press, New York and London, 1978. [MR 0493421](#) [Zbl 0401.47001](#)
- [40] M. Reed and B. Simon, *Methods of modern mathematical physics*. III. Scattering theory. Academic Press, New York and London, 1979. [MR 0529429](#) [Zbl 0405.47007](#)

- [41] T. Regge, Analytic properties of the scattering matrix. *Nuovo Cimento* (10) **8** (1958), 671–679. [MR 0095702](#) [Zbl 0080.41903](#)
- [42] P. A. Rejto and K. Sinha, Absolute continuity for a 1-dimensional model of the Stark–Hamiltonian. *Helv. Phys. Acta* **49** (1976), no. 3, 389–413. [MR 0416356](#)
- [43] B. Simon, Resonances in one dimension and Fredholm determinants. *J. Funct. Anal.* **178** (2000), no. 2, 396–420. [MR 1802901](#) [Zbl 0977.34075](#)
- [44] B. Simon, *Trace ideals and their applications*. Second edition. Mathematical Surveys and Monographs, 120. American Mathematical Society, Providence, R.I., 2005. [MR 2154153](#) [Zbl 1074.47001](#)
- [45] J. Sjöstrand and M. Zworski, Complex scaling and the distribution of scattering poles. *J. Amer. Math. Soc.* **4** (1991), no. 4, 729–769. [MR 1115789](#) [Zbl 0752.35046](#)
- [46] K. Yajima, Spectral and scattering theory for Schrödinger operators with Stark effect. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **26** (1979), no. 3, 377–390. [MR 0560003](#) [Zbl 0429.35027](#)
- [47] K. Yajima, Spectral and scattering theory for Schrödinger operators with Stark effect. II. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **28** (1981), no. 1, 1–15. [MR 0617860](#) [Zbl 0465.35024](#)
- [48] V. Zakharov and L. Faddeev, The Korteweg–de Vries equation is a completely integrable Hamiltonian system. *Funkcional. Anal. i Priložen.* **5** (1971), no. 4, 18–27. In Russian. English translation, *Functional Anal. Appl.* **5** (1971), 280–287. Reprint in L. Faddeev et al., *Fifty years of mathematical physics*. Selected works of Ludwig Faddeev. Edited by M. Ge and A. J. Niemi. World Scientific Series in 21entury Mathematics, 2. World Scientific, Hackensack, N.J., 277–284. [Zbl 0257.35074](#) [MR 0303132](#) [Zbl 1347.81016](#) (reprint, collection)
- [49] M. Zworski, Sharp polynomial bounds on the number of scattering poles of radial potentials. *J. Funct. Anal.* **82** (1989), no. 2, 370–403. [MR 0987299](#) [Zbl 0681.47002](#)
- [50] M. Zworski, Distribution of poles for scattering on the real line. *J. Funct. Anal.* **73** (1987), no. 2, 277–296. [MR 0899652](#) [Zbl 0662.34033](#)
- [51] M. Zworski, A remark on isopolar potentials. *SIAM J. Math. Anal.* **32** (2001), no. 6, 1324–1326. [MR 1856251](#) [Zbl 0988.34067](#)

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