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Cluster expansion of the resolvent for the Schrödinger operator on non-percolating graphs with applications to Simon–Spencer type theorems and localization

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Abstract. The paper contains a generalization of the well-known 1D results on the absence of the a.c. spectrum (in the spirit of the Simon–Spencer theorem) and localization to the wide class of "non-percolating" graphs, which include the Sierpiński lattice and quasi 1D trees. The main tools are cluster expansion of the resolvent and real analytic techniques (Kolmogorov's lemma and similar estimates).

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Contents

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1. Introduction

The idea of a possible connection between the spectral phase transition for the Anderson Hamiltonian on the lattices Z^d, $d \geq 3$ and percolation has always been popular in the physical literature. In one direction it is definitely correct. Consider on the lattice Z^1 the random Schrödinger operator

$$H\phi(x) = \Delta\phi(x) + \sigma V(x,\omega), \quad \Delta\phi(x) = \phi(x+1) + \phi(x-1)$$

where σ is a coupling constant and the potentials $V(x,\omega)$ are unbounded i.i.d., say, $N(0,1)$ random variables. Then, $P - a.s.$, for arbitrarily small σ and for arbitrarily large M there are infinitely many points $x\colon |V(x,\omega)| > M$, i.e. the set $\{x\colon |V(x,\omega)| \leq M\}$ contains only finite connected components, that is, the components do not percolate, although their concentration can be arbitrarily close to 1. More formally, let Γ be an abstract graph and $X(x,\omega)$, $x \in \Gamma$ be the Bernoulli field such that $P\{X(\cdot) = 1\} = \epsilon$, $P\{X(\cdot) = 0\} = 1 - \epsilon$. We call Γ a non-percolating graph if for arbitrary $\epsilon > 0$ the set $\{x\colon X(x) = 0\}$ contains $P - a.s.$ only bounded connected components. The class of non-percolating graphs is very rich. It includes, for instance, all nested fractal lattices, among them the Sierpiński lattice.

Let's consider for such graphs the Anderson operators

$$H\phi(x) = \sum_{x'\colon x\sim x'} \phi(x') + \sigma V(x,\omega)$$

where $\{x'\colon x' \sim x\}$ is the set of nearest neighbours of x and the $V(x,\omega)$ are unbounded i.i.d. random variables. At the level of physical intuition we have the following picture: the realization of $\sigma V(\cdot)$ contains a sequence of higher and higher "walls" and the quantum particle can't avoid interaction with such walls in its attempts to reach infinity. Since tunnelling through higher and higher walls has a smaller and smaller probability, one can expect here some kind of localization phenomena, say, the absence of the a.c. spectrum. In the case of the 1D lattice Z^1 these a bit fuzzy arguments were transformed into a mathematical theorem in the famous Simon–Spencer paper [6]. Let

$$H\phi(x) = \phi(x+1) + \phi(x-1) + V(x)\phi(x), x \in Z^1.$$

If the potential $V(x)$ is unbounded near $\pm\infty$, i.e.

$$\limsup_{x\longrightarrow+\infty} |V(x)| = \limsup_{x\longrightarrow-\infty} |V(x)| = +\infty,$$

then $\Sigma_{\text{a.c.}} = \emptyset$.

Of course, for random i.i.d. unbounded potentials this result provides the absence of the a.c. spectrum $P - a.s.$. The Simon–Spencer theorem can be extended on a class of (non-random) potentials even in $R^d, d \geq 2$, see [19]. All results of this kind include very strong assumptions about the existence of an infinite system of "rings" or "belts" around the origin, where the potential V is higher and higher (i.e. $\min_{x \in b_n} V(x) = h_n \to +\infty$, $n \to \infty$). They also require an additional condition on how fast the "heights" h_n need to increase. B. Simon [5] constructed (for $\Gamma = Z^2$) such a Schrödinger operator, where $V(x) \geq 0$ contains a system of higher and higher walls (i.e. the set $\{x : V(x) \leq M\}$ doesn't percolate for any M), but the spectrum of H contains a.c. components.

Unfortunately the lattices Z^d, $d > 1$ "percolate." There exist critical thresholds $h_{cr}, \tilde{h}_{cr}$ depending on the distribution of $V(x, \omega)$: if $h > h_{cr}$ then the set $\{x : V(x, \omega) > h\}$ where the $V(x, \omega)$ are i.i.d., unbounded random variables doesn't percolate, but for $h < \tilde{h}_{cr}$ it contains an infinite connected component.

The goal of this paper is to give sufficient conditions for the absence of the a.c.spectrum or the existence of the pure point (p.p.) spectrum for deterministic or random Schrödinger operators on some classes of graphs. For the particular situations of "non-percolating" graphs we will prove Simon–Spencer type results and a localization theorem for Anderson Hamiltonians. Technical tools here are extensions of the real-analytic methods presented for the 1D lattice Z^1 and corresponding Schrödinger operators in [23]. The central moment is the cluster expansion of the resolvent with respect to appropriate partitions of Γ.

The general theory can be illustrated by the following particular results.

Theorem (B). *Consider the Quasi-1 Dimensional Tree T (see Figure 2) with Laplacian $\Delta\psi(x) = \sum_{x' \sim x} \psi(x')$. Let $H = \Delta + V(x, \omega)$ be the Anderson Hamiltonian and the $V(x, \omega)$ be i.i.d. unbounded random variables with bounded distribution density $f(t)$. Then the spectrum of H is pure point with probability* 1.

Theorem (A). *Consider the Sierpiński lattice S^∞(see Figure 1) with Laplacian $\Delta\psi(x) = \sum_{x' \sim x} \psi(x')$. Let $H = \Delta + V(x, \omega)$ be the Anderson Hamiltonian and the $V(x, \omega)$ be i.i.d. unbounded random variables. Then $P - a.s.$ $H = H(\omega)$ has no absolutely continuous spectrum.*

This theorem is a particular case of a much more general result, see Corollary 6.2.

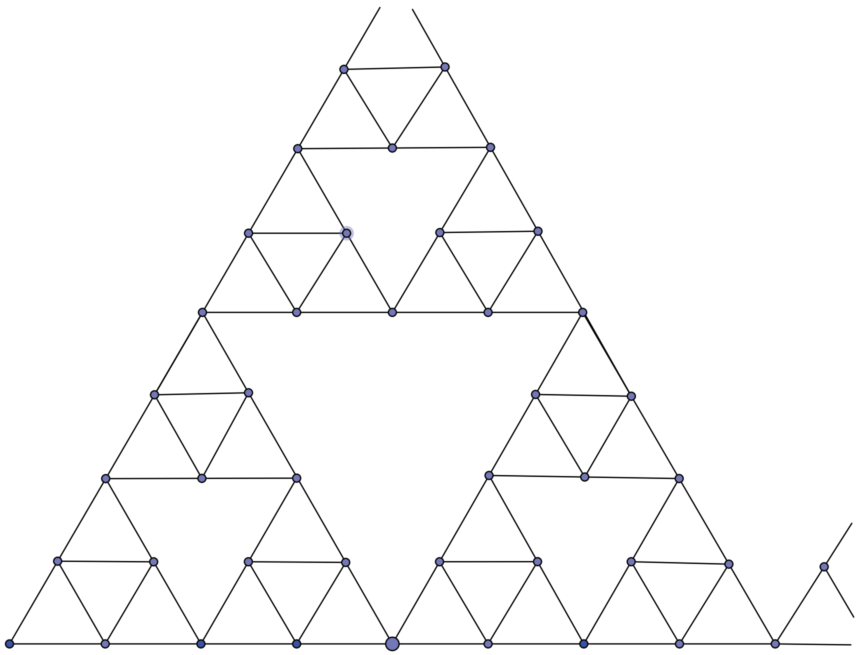

Figure 1. Sierpiński lattice S^{∞}.

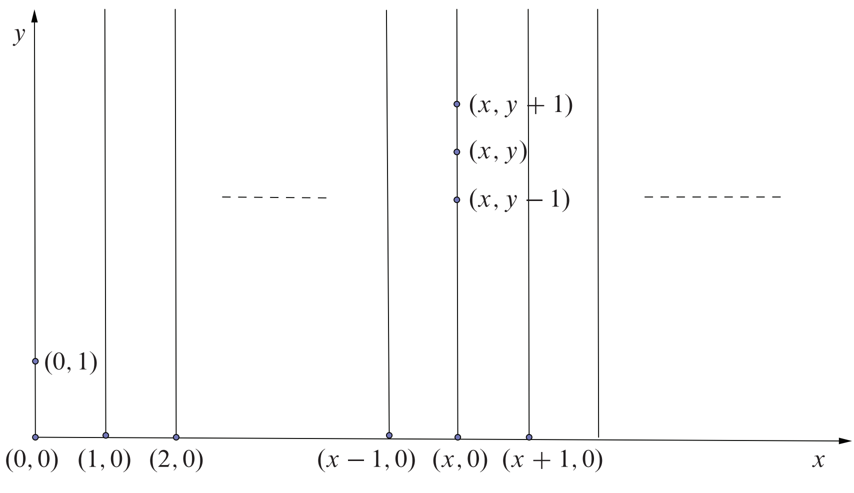

Figure 2. Quasi-1 dimensional tree T.

Remark 1. For the physical interpretation of Theorems (A) and (B), we need the information on the spectral properties of the underlying Laplacians. Let's present without proof several results in this direction.

The spectrum of Δ on $\ell_2(T)$ is the closed interval $\left[-\frac{5}{2}, \frac{5}{2}\right]$, i.e. $\|\Delta\|_{\ell_2} = \frac{5}{2}$. The spectral measure $\mu_f(d\lambda) = (E(d\lambda)f, f)$ is pure absolutely continuous. Here $E(d\lambda)$ is the operator-valued measure from the spectral decomposition $\Delta = \int \lambda E(d\lambda)$. The generalized eigenfunctions have different structures for different energies λ. If $\lambda \in [-2, 2]$, then the eigenfunctions are sinusoidal waves along the x-axis and each "vertical" line $(x, y), y \geq 0$. For $\lambda \in (2, 5/2]$ or $\lambda \in [-5/2, -2)$, the eigenfunctions are exponentially decreasing as functions of y (for each fixed x) and have sinusoidal structure along the x-axis, i.e. they propagate along the "boundary" $y = 0$ of T.The spectral dimension (as well as Hausdorff dimension) of T equals 2.

The transition from the a.c. spectrum of Δ to the p.p. spectrum of H has the same nature as 1-D Anderson localization (the destruction of resonances between adjacent potential walls).

The spectrum of the operator Δ in $\ell^2(S^\infty)$ is exotic. It is a Cantor-like closed subset of interval $[-1, 4]$ with (Lebesgue) measure zero. Specifically, it is the Julia set of the mapping $z \to (z+1)(z-5)$ of the complex plane C. The spectral measure $\mu_f(d\lambda)$, $f \in \ell^2(S^\infty)$, contains both point and singular continuous components (but not the a.c. ones). All eigenvalues have infinite multiplicity (see [10]).

A fundamental corollary is the boundedness of the resolvent $R_\lambda = (\lambda I - \Delta)^{-1}$ for a.e. real λ. We will use this fact in future studies of the Anderson Hamiltonian on $\ell^2(S^\infty)$ for bounded r.v. $\sigma V(x, \cdot)$, $x \in S^\infty$.

The P-a.s. absence of the a.c. spectrum for $H = H(\omega)$ (Theorem (A)) is not a direct corollary of the similar fact for Δ. Although the Sierpiński lattice S^∞ has the spectral dimension $S = \frac{\ln 9}{\ln 5} > 1$, the mechanism of the localization (see Section 7) here has a 1-D nature related to the existence of the pairs $(2^n \vec{\imath}, 2^n \vec{\omega})$, $n \geq 1$ where $\vec{\imath} = (1, 0)$ and $\vec{\omega} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, which separate S^∞ into disjoint parts.

2. Basic definitions and models

Let (Γ, A) be an undirected infinite graph with symmetric adjacency matrix $A(\Gamma) = A^*(\Gamma) = [a(x, y)]$. If $a(x, y) = 1$, we will call x and y nearest neighbors, denoted by $x \sim y$. Suppose that for each $x \in \Gamma$, the number $\nu(x)$ of nearest neighbors x': $x' \sim x$ is bounded: $\nu(x) \leq K$ for some fixed K: $2 \leq K < \infty$.

For a homogeneous graph (with appropriate group of symmetries) $\nu(x) \equiv K$. In this case, K is called the branching index of the graph (Γ, A). Typical examples of homogeneous graphs are: lattices $\mathbb{Z}^d$ with branching index equal to $2d$, groups with a finite number of generators, homogeneous trees, etc.

The path $[\gamma]$ of length $|[\gamma]| = n$, from point x to point y on Γ, denoted by $[\gamma]: x \to y$ is defined to be the sequence of points

$$[\gamma] = \{x_i\}_{i=0}^n$$

such that $x_i \sim x_{i-1}$ for $i = 1, 2, \ldots, n$ and $x_0 = x$, $x_n = y$. Here $x_0 = x$ is called the start point of the path and $x_n = y$ is called the end point of the path. All other points are called internal points of the path. We denote by $(\gamma): x \to y$ the internal part of $[\gamma]: x \to y$, that is,

$$(\gamma) = \{x_i\}_{i=1}^{n-1}$$

with $x_1 \sim x_0 = x$ and $x_{n-1} \sim x_n = y$. Any non-empty set $B \subset V$ with more than one point is called connected if for all $x, y \in B$ there exist $[\gamma]: x \to y$ and $[\gamma] \subset B$. The boundary of B is defined as

$$\partial B = \{y : y \notin B, \text{ and } y \sim x, \text{ for some } x \in B\}.$$

In this paper, we assume that graph Γ is connected.

A metric $d(x, y)$ on Γ and distances $d(x, B)$, $d(B_1, B_2)$ are defined in the standard way. The volume of the ball, centered at point $x_0 \in \Gamma$, can not increase faster than an exponential:

$$|B_R(x_0)| = |\{x : d(x, x_0) \le R\}| \le K^R + 1.$$

Recall that $K = \max_{x \in \Gamma} \nu(x)$.

Let $\ell_2(\Gamma)$ be the Hilbert space of square-summable functions $f(x): \Gamma \to \mathbb{C}$ with the inner product and norm

$$(f, g) = \sum_{x \in \Gamma} f(x)\bar{g}(x), \ \|f\|^2 = \sum_{x \in \Gamma} |f(x)|^2.$$

for $f, g \in \ell_2(\Gamma)$.

The lattice Laplacian in the space $\ell_2(\Gamma)$ is given by the usual formula

$$\Delta f(x) = \sum_{x' : d(x', x) = 1} f(x')$$

As is easy to see,

$$\|\Delta\| = \sup_{f : \|f\| = 1} \|\Delta f\| \le K,$$

i.e. the lattice Laplacian is bounded.

The Schrödinger operator (Hamiltonian), by definition, has the form

$$H = \Delta + V(x)$$

where $V(x)$ is an arbitrary real-valued potential. In the most interesting case, $V(x) = \sigma\xi(x,\omega)$ and $\xi(x,\omega)$ will be a family of i.i.d.r.v.'s with bounded continuous density $p_\xi(\cdot)$. Here $x \in \Gamma$, $\omega \in (\Omega, \mathcal{F}, P)$(a basic probability space), and $\sigma > 0$ is the coupling constant (a measure of disorder). In this case, we will call H the Anderson Hamiltonian on $l^2(\Gamma)$. A fundamental and still unsolved problem is to determine the spectral type of H for general graphs (or at least for lattices $\mathbb{Z}^d$, $d \geq 2$). It is known ([14])that for an arbitrary graph Γ and very general symmetric bounded operators L, the spectral measure is pure point(p.p) for large disorder, $\sigma > \sigma_0$, where σ_0 can be effectively determined by the geometry of the graph Γ. If $\Gamma = \mathbb{Z}^1$ (or $\Gamma = \mathbb{Z}^1 \times A, \mathrm{Card}(A) < \infty$) the spectrum of H is p.p. P-a.s. and the corresponding eigenfunctions are exponentially decaying for arbitrarily small coupling parameter σ (see details in [20]). The second case where the spectral picture is well understood is the homogeneous tree T^N, see [13]. The last case demonstrates an Anderson type phase transition from a p.p. spectrum for large σ to a mixed spectrum (a.c. component plus p.p. component) for small σ. For similar results on such transitions, see [4], [8], [15], [17], and [22].

3. Expansion theory of the resolvent kernel

The resolvent kernel of the operator H

$$R_\lambda(x,y) = (H - \lambda I)^{-1}(x,y)$$

is well defined at least for complex λ. We will use the following "exact" formula for $R_\lambda(x,y)$ (the so-called path expansion) see [23].

Proposition 3.1. *Let $V(x)$ be the potential of the Schrödinger operator on Γ with* $\mathrm{Range}(V) = \{V(x)\colon x \in \Gamma\} \subset R$. *Then*

$$R_\lambda(x,y) = \frac{\delta_y(x)}{\lambda - V(x)} + \sum_{[\gamma]:x\longrightarrow y}\Big(\prod_{z\in\gamma}\frac{1}{\lambda - V(z)}\Big). \tag{1}$$

where $\delta_y(x) = 1$ if $x = y$ and 0 otherwise. This formula holds, at least for λ's such that $d(\lambda, \overline{\mathrm{Range}(V)}) \geq K + \delta$, for some $\delta > 0$.

Proof. Since $H\psi(x) - \lambda\psi(x) = \delta_y(x)$, we have

$$\sum_{x':x'\sim x} \psi(x') + V(x)\psi(x) - \lambda\psi(x) = \delta_y(x).$$

Thus,

$$\psi(x) = \frac{\delta_y(x)}{\lambda - V(x)} + \sum_{x':x'\sim x} \frac{\psi(x')}{\lambda - V(x)}.$$

For each x', one can use the same formula and continue these iterations to get (1).

The number of paths $[\gamma]$ with fixed start point x and length n is at most K^n and

$$\left| \prod_{z\in[\gamma]} \frac{1}{\lambda - V(z)} \right| \le \frac{1}{(K+\delta)^{|[\gamma]|+1}}.$$

These facts lead to the convergence of the series (1). □

A similar construction works for the restriction of H on subsets B of Γ. Consider $H_B = \triangle + V(x)$ with the Dirichlet boundary condition:

$$\psi(x) = 0$$

for $x \in \Gamma - B$.

Proposition 3.2. *Let* $R_\lambda^{(B)}(x, y) = (H_B - \lambda I)^{-1}(x, y)$ *be the resolvent kernel of* H_B *(which is well defined at least for* $|Im\lambda| \ge K + \delta$*,* $\delta > 0$ *). Then for* $x, y \in B$*,*

$$R_\lambda^{(B)}(x, y) = \frac{\delta_y(x)}{\lambda - V(x)} + \sum_{\substack{[\gamma]:x\longrightarrow y\\ [\gamma]\subset B}} \Big(\prod_{z\in[\gamma]} \frac{1}{\lambda - V(z)} \Big).$$

Note that for the bounded set B, the spectrum of H_B contains $|B|$ real eigenvalues (which can be multiple), i.e. $R_\lambda^{(B)}(x, y)$ is real and finite for all real λ, except for the finite set of eigenvalues of H_B.

Let us fix some point $x_0 \in \Gamma$, called a *reference point*. The finite subset $b_1 \subset \Gamma$ is called a *belt* with respect to x_0, if the difference $\Gamma - b_1$ is the union of a single bounded connected component B_1^- containing x_0 (we assume that $x_0 \notin b_1$) and finitely many unbounded connected components $B_{1,i}^+$, $i = 1, 2, \ldots, k_1$. We assume that the set $E_{b_1} = B_1^- \cup b_1$ (called the enclosure of belt b_1) is connected.

We will call $B_1^+ = \bigcup_{i=1}^{k_1} B_{1,i}^+$ the *outer complement* of b_1 and B_1^- the *inner complement* of b_1. The *inner boundary* of the belt b_1 is defined to be the set $\partial b_1^- = \partial b_1 \cap B_1^-$ and the *outer boundary* of the belt b_1 is defined to be the set $\partial b_1^+ = \partial b - \partial b^- \subset B_1^+$.

The assumption of the existence of only one single bounded connected component B_1^- in $\Gamma - b_1$ exclude the existence of such bounded components (lakes) inside b_1. For instance, an 8-shaped subset is not considered to be an belt according to our definition.

Remark 2. Graphs S^∞ and T in Figures 1 and 2 show that B_1^+ typically contains several separate unbounded connected components.

To introduce the second belt b_2, we select a reference point $x_{1,i} \in \partial b_1^+ \cap B_{1,i}^+$ in each unbounded connected component $B_{1,i}^+$. Consider sub-graph $B_{1,i}^+$ with the reference point $x_{1,i}$. A belt $b_{2,i}$ is defined in $B_{1,i}^+$ with the same properties as the belt b_1 has in Γ. It divides $B_{1,i}^+$ into an inner part consisting of a single bounded connected component $B_{1,i}^{+-}$ and an outer part $B_{1,i}^{++}$ consisting of several unbounded connected outer components. We have $B_{1,i}^+ = B_{1,i}^{+-} \cup b_{2,i} \cup B_{1,i}^{++}$. The second belt on Γ is defined to be the set $b_2 = \bigcup_{i=1}^{k_1} b_{2,i}$. Let $B_2^- = (0) \cup \left(\bigcup_{i=1}^{k_1} B_{1,i}^{+-} \right)$ and $E_{b_2} = B_2^- \cup b_2$. Here E_{b_2} is the enclosure of belt b_2.

One can apply the same algorithm to define $b_3, E_{b_3}, B_3^-, B_3^+$ and so on.

We can define now the following quantity (similar to the resolvent of b): for $x \in \partial b^-,\ y \in \partial b^+$, put

$$\beta_\lambda^b(x, y) = \sum_{\substack{[\gamma]: x \longrightarrow y \\ (\gamma) \subset b}} \prod_{z \in (\gamma)} \frac{1}{\lambda - V(z)}.$$

(and a similar expression if $x \in \partial b^+,\ y \in \partial b^-$). Note that path (γ) stays inside belt b. In the future the potential $V(\cdot)$ will be large on b, i.e. for $\lambda \in I$ (I is a fixed interval) the quantity $\beta_\lambda^b(x, y)$ will be small. Physically it means that tunnelling of the quantum particle through the belt is very unlikely.

Now assume that $\{b_i : i \geq 1\}$ is the sequence of belts that exist with respect to the reference point x_0 and the corresponding sequence of the enclosures is $\{E_{b_i}\}_{i=1}^\infty$. It is easy to see that $E_{b_i} \subset E_{b_{i+1}}$, for any $i = 1, 2, 3, \ldots$. Let $S_0, S_1, \ldots$ be subsets of Γ between successive belts(S_0 contains x_0). That is, $S_i = E_{b_i} - E_{b_{i-1}} - b_{i-1}$ for $i = 1, 2, \ldots$. We define the main blocks $(0), (1), \ldots$ as

$$(0) = S_0 \cup b_1, \qquad \partial(0) = \partial b_1^+,$$

$$(1) = b_1 \cup S_1 \cup b_2, \quad \partial(1) = \partial b_1^- \cup \partial b_2^+,$$

and so on. Note that $(i-1) \cap (i) = b_i$ for $i \geq 1$. See Figure 3.

We call ∂b_i^- the inner boundary of b_i and ∂b_i^+ the outer boundary of b_i for $i \geq 1$.

Now we will study the cluster expansion of $R_\lambda(x_0, x)$. Consider the path expansion of H:

$$R_\lambda(x_0, x) = \frac{\delta_{x_0}(x)}{\lambda - V(x_0)} + \sum_{[\gamma]:x_0 \longrightarrow x} \Big(\prod_{z \in [\gamma]} \frac{1}{\lambda - V(z)} \Big)$$

and introduce the procedure of the union of paths into special groups(clusters).

Consider a particle following some path $[\gamma]\colon x_0 \to x$ (Figure 4) . It will stay in the main block (0) for a certain amount of time. And then at some moment it will reach ∂b_1^+. By definition, this is the moment of transition from main block (0) to main block (1). The particle again can stay for a certain amount of time in (0) $\cup$ (1) doing transferring between (0) and (1) until at some moment it reaches ∂b_2^+, and this moment, by definition, is the transition from (1) to (2), etc.

Therefore, for any $[\gamma]\colon x_0 \to x$, one can define a sequence of the main blocks with which $[\gamma]$ successively intersects. Let's introduce the graph $\widetilde{\Gamma}$ with the vertices being the main blocks: (0), (1), $\ldots$ and possible transitions $(n) \to (n \pm 1), n > 0$ and $(0) \to (1)$. For each path $[\tilde{\gamma}]$ on $\widetilde{\Gamma}$, one can consider the cluster of elementary paths $[\gamma]$ following $[\tilde{\gamma}]$ between the main blocks. If so, we say that $[\gamma] \in [\tilde{\gamma}]$. The class of all paths which stays in (0) forever corresponds to the $[\tilde{\gamma}] = (0)$.

Put

$$R_\lambda^{[\tilde{\gamma}]}(x_0, x) = \sum_{[\gamma] \in [\tilde{\gamma}]} \prod_{z \in [\gamma]} \Big(\frac{1}{\lambda - V(z)} \Big).$$

Here the paths $[\gamma]$ under summation are from x_0 to x in the main block (k).

By definition, we have

$$R_\lambda(x_0, x) = \frac{\delta_{x_0}(x)}{\lambda - V(x_0)} + \sum_{[\tilde{\gamma}]:(0) \longrightarrow (k)} R_\lambda^{[\tilde{\gamma}]}(x_0, x).$$

Let's provide a deeper analysis of an arbitrary path $[\gamma] \in [\tilde{\gamma}]$. Assume that $[\gamma]$ contains, say, transitions $(0) \to (1) \to (2)$. Then at moment τ_1^+ the path enters some point z_2 on ∂b_1^+ for the first time (the first entrance to (1)). Let τ_1^- be the last moment before τ_1^+ when the path enters the belt b_1 through ∂b_1^- and stays in b_1 until moment τ_1^+. After moment τ_1^+, the trajectory $[\gamma]$ moves inside (1) and at moment τ_2^+ enters ∂b_2^+ (the first entrance to (2)). τ_2^- is defined similarly to τ_1^-, such that in time interval $[\tau_2^-, \tau_2^+]$ it stays inside b_2, etc.

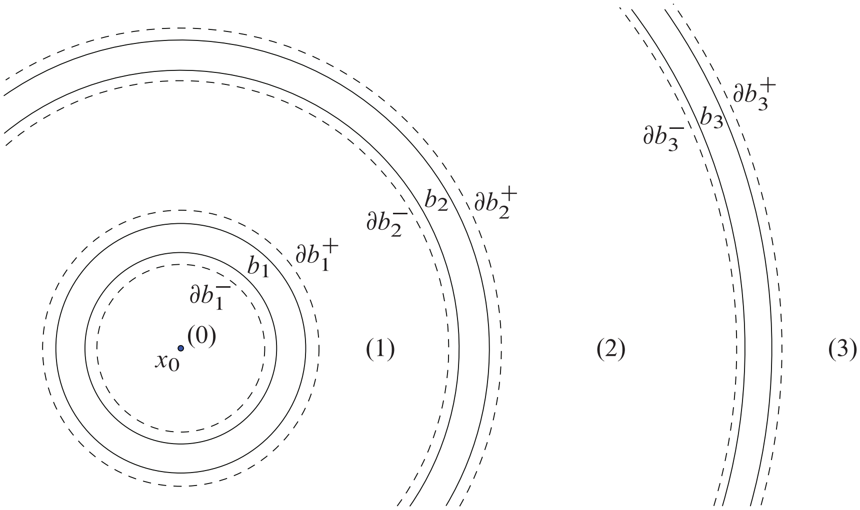

Figure 3. Main blocks and belts.

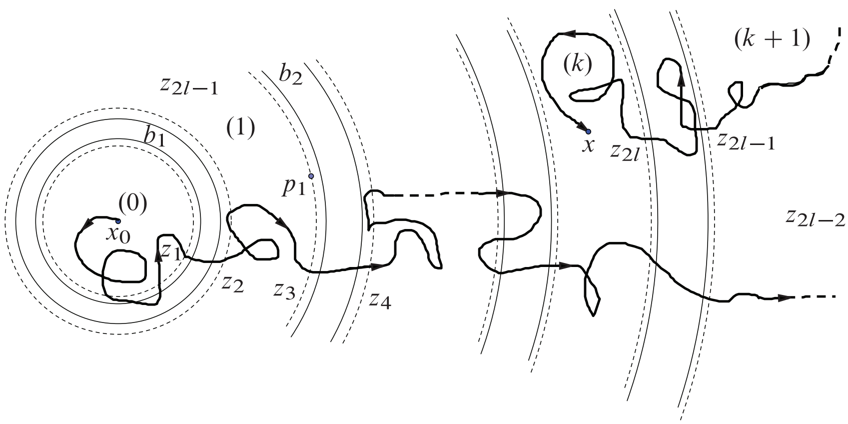

Figure 4. The path $[\gamma]$.

This classification leads to a simple combinatorial representation of $R_\lambda^{[\tilde\gamma]}(x_0,x)$. For instance, consider the special case when $x = x_0$. If $[\tilde\gamma] \equiv (0)$, $R_\lambda^{[\tilde\gamma]}(x_0,x_0) = R_\lambda^{(0)}(x_0,x_0)$.

For other paths $[\tilde\gamma]$, say, $[\tilde\gamma] = (0) \to (1) \to \ldots \to (2) \to (1) \to (0)$, their contribution to $R_\lambda(x_0,x_0)$ equals

$$\begin{aligned} R_\lambda^{[\tilde\gamma]}(x_0,x_0) = \sum R_\lambda^{(0)}(x_0,z_1)\beta_\lambda^{b_1}(z_1,z_2)R_\lambda^{(1)}(z_2,\cdot)\ldots \\ R_\lambda^{(2)}(\cdot,z_{2l-3})\beta_\lambda^{b_2}(z_{2l-3},z_{2l-2})R_\lambda^{(1)}(z_{2l-2},z_{2l-1}) \\ \beta_\lambda^{b_1}(z_{2l-1},z_{2l})R_\lambda^{(0)}(z_{2l},x_0) \end{aligned}$$

where the summation is over $z_1 \in \partial b_1^-$, $z_2 \in \partial b_1^+,\ldots,z_{2l-3} \in \partial b_2^+$, $z_{2l-2} \in \partial b_2^-$, $z_{2l-1} \in \partial b_1^+$, $z_{2l} \in \partial b_1^-$ with l being the number of times the path goes through the belts until it returns to x_0. For instance, consider the path $[\tilde\gamma]$: $(0) \to (0)$:

$$(0) \longrightarrow (1) \longrightarrow (2) \longrightarrow (1) \longrightarrow (2) \longrightarrow (3) \longrightarrow (2) \longrightarrow (1) \longrightarrow (0)$$

. The corresponding contribution is

$$\begin{aligned} R_\lambda^{[\tilde\gamma]}(x_0,x_0) = \sum R_\lambda^{(0)}(x_0,z_1)\beta_\lambda^{b_1}(z_1,z_2)R_\lambda^{(1)}(z_2,z_3)\beta_\lambda^{b_2}(z_3,z_4) \\ R_\lambda^{(2)}(z_4,z_5)\beta_\lambda^{b_2}(z_5,z_6)R_\lambda^{(1)}(z_6,z_7)\beta_\lambda^{b_2}(z_7,z_8) \\ R_\lambda^{(2)}(z_8,z_9)\beta_\lambda^{b_3}(z_9,z_{10})R_\lambda^{(3)}(z_{10},z_{11})\beta_\lambda^{b_3}(z_{11},z_{12}) \\ R_\lambda^{(2)}(z_{12},z_{13})\beta_\lambda^{b_2}(z_{13},z_{14}) \\ R_\lambda^{(1)}(z_{14},z_{15})\beta_\lambda^{b_1}(z_{15},z_{16})R_\lambda^{(0)}(z_{16},x_0) \end{aligned}$$

where the summation is over $z_1 \in \partial b_1^-$, $z_2 \in \partial b_1^+,\ldots,z_{16} \in \partial b_1^-$ with $l = 8$ being the number of times the path goes through the belts until it returns to x_0.

In each factor $\beta_\lambda^{b_k}$, the arguments $z_i,\ z_j$ belong to different parts $\partial b_k^\pm$ of the boundary ∂b_k. However, the arguments in $R_\lambda^{(k)}(z_i,z_j)$ can be both on ∂b_k^+, both on ∂b_{k+1}^-, or one on ∂b_k^+ and the other on ∂b_{k+1}^-.

For each term $R_\lambda^{(0)}(\cdot,\cdot)$, there are two possibilities:

$$R_\lambda^{(0)}(x_0,z_i), z_i \in \partial b_1^-;\ \ R_\lambda^{(0)}(z_j,x_0), z_j \in \partial b_1^-.$$

Similar formulas exist for $R^{[\tilde\gamma]}(x_0,x),\ x \in (k)$. For example,

$$[\tilde\gamma] = (0) \longrightarrow (1) \longrightarrow \cdots \longrightarrow (k+1) \longrightarrow (k),$$

$$R_\lambda^{[\tilde\gamma]}(x_0,x)=\sum R_\lambda^{(0)}(x_0,z_1)\beta_\lambda^{b_1}(z_1,z_2)R_\lambda^{(1)}(z_2,z_3)\beta_\lambda^{b_2}(z_3,z_4)R_\lambda^{(2)}(z_4,\cdot)$$
$$R_\lambda^{(k+1)}(\cdot,z_{2l-1})\beta_\lambda^{b_{k+1}}(z_{2l-1},z_{2l})R_\lambda^{(k)}(z_{2l},x),$$

where the summation is over all $z_1 \in \partial b_1^-$, $z_2 \in \partial b_1^+, z_3 \in \partial b_2^-$, $z_4 \in \partial b_2^+, \dots,$ $z_{2l-1} \in \partial b_{k+1}^+$, $z_{2l} \in \partial b_{k+1}^-$ with l being the number of times the path goes through the belts until it reaches x.

For convenient reference, let's formulate the main result of this section as

Theorem 3.3 (the cluster expansion theorem). *Let x_0 be the reference point and $x \in (k)$, then*

$$R_\lambda(x_0,x)=\frac{\delta_{x_0}(x)}{\lambda-V(x_0)}+\sum_{[\tilde\gamma]:(0)\longrightarrow(k)}R_\lambda^{[\tilde\gamma]}(x_0,x). \tag{2}$$

Here

$$\begin{aligned}R_\lambda^{[\tilde\gamma]}(x_0,x)=\sum R_\lambda^{(0)}(x_0,z_1)\beta_\lambda^{b_1}(z_1,z_2)&\\ R_\lambda^{(1)}(z_2,z_3)\beta_\lambda^{b_2}(z_3,z_4)R_\lambda^{(2)}(z_4,\cdot)&\\ R_\lambda^{(k\pm1)}(\cdot,z_{2l-1})\beta_\lambda^{b_{k\pm1}}(z_{2l-1},z_{2l})R_\lambda^{(k)}(z_{2l},x)&\end{aligned} \tag{3}$$

where the summations are as described above.

4. Technical lemmas based on the resolvent kernel $R_{\tilde\lambda}(x_0,x)$ where $\mathrm{Im}(\tilde\lambda)=\epsilon>0$

In this section we will formulate and prove several criteria for the absence of the a.c. spectrum, localization, etc. All results here will be based on complex analysis. For a real analytic approach, see Section 5.

Due to general theory,

$$H=\int_{\mathrm{Sp}(H)}\lambda E(d\lambda),$$

where $E(d\lambda)$ is the operator-valued spectral measure. If $f \in l^2(\Gamma)$, then $\mu_f(d\lambda)=(E(d\lambda)f,f)$ is the spectral measure of the element f and

$$\int_{\mathrm{Sp}(H)}\mu_f(d\lambda)=\int_{R^1}\mu_f(d\lambda)=\|f\|_2^2.$$

The resolvent $R_\lambda = (H - \lambda I)^{-1}$ is a bounded operator for $\lambda \notin \mathrm{Sp}(H)$, for instance, for $\tilde{\lambda} = \lambda + i\epsilon, \lambda \in R^1$,

$$(R_{\tilde{\lambda}} f, f) = \int_{\mathrm{Sp}(H)} \frac{\mu_f(dz)}{z - \tilde{\lambda}}.$$

The function $f_0(x) \in l^2(\Gamma)$ is of the *maximal spectral type* if $\mu_{f_0}(d\lambda) \gg \mu_g(d\lambda)$ for any $g \in l^2(\Gamma)$. The dense set of functions $f_0 \in l^2(\Gamma)$ are of the maximal spectral type.

The measure $\mu_f(d\lambda)$ has a Lebesgue decomposition into an a.c. component $\mu_{f,a.c.}$, a singular continuous component $\mu_{f,s.c.}$, and finally a point component $\mu_{f,p}$.

The a.c. component is responsible for the transport of quantum particles (electric conductivity, scattering, etc). The point component is related to localization.

Theorem 4.1. *The limit*

$$\pi \lim_{\epsilon \longrightarrow 0} \mathrm{Im}(R_{\lambda+\epsilon i} f,\ f)$$

exists for a.e. $\lambda \in R$ *and equals* $\rho_f(\lambda)$. *Here* $\rho_f(\lambda)$ *is the density of the a.c. part of the spectral measure* $\mu_f(d\lambda)$. *See details in* [5].

Corollary 4.2. *For a given energy interval* $I \subset R$, *the a.c. part of the spectral measure* $\mu_{f,a.c.}(d\lambda)$ *is equal to 0 if*

$$\lim_{\epsilon \longrightarrow 0} \mathrm{Im}(R_{\lambda+\epsilon i} f, f) = 0$$

a.e. on $\lambda \subset I$.

Assume that $\mu_{\mathrm{a.c.}}(d\lambda) = 0$ (operator H has no a.c. spectrum). How can μ_p and $\mu_{s.c.}$ be separated? The following theorem (see [7]) gives a simple criterion for the (p.p) spectrum.

Theorem 4.3 (Simon–Wolff). *Assume that for real* λ *and any* $x_0 \in \Gamma$,

$$\lim_{\epsilon \longrightarrow 0} \sum_{y \in \Gamma} |R_{\lambda+i\epsilon}(x_0, y)|^2 = \sum_{y \in \Gamma} |R_{\lambda+0i}(x_0, y)|^2 < \infty.$$

Then the operator H *"typically" has a p.p. spectrum.*

Remark 3. It is not difficult to prove the existence of the limit above.

The last sentence "typically" means that, in the subspace $\ell_2(\delta_{x_0}) = \mathrm{Span}(R_\lambda \delta_{x_0})$, the perturbed operator $H_a = H + a\delta_{x_0}(\cdot)$, $x_0 \in \Gamma$ (rank-one perturbation) has a p.p. spectrum for a.e. $a \in R$. At the same time(in a wide class of situations), the operator H_a has a pure singular spectrum for some a from the appropriate G_δ set (with measure 0), see [1] and [3].

This result is especially convenient for random Schrödinger operators (Anderson Hamiltonians).

As we already mentioned, for large σ (large disorder) the operator $H(\omega)$ has a p.p. spectrum P-a.s., see [14].

Let's formulate and prove the result, closely related to the Simon–Spencer approach to the theorems on the absence of the a.c. spectrum.

Theorem 4.4. *Let Γ be a graph with estimate $\nu(x) \leq K$, Δ be the Laplacian, and $H = \Delta + V(x)$ be the Schrödinger operator with potential $V(\cdot)$. Assume that for some sequence of points $\mathcal{D} = \{x_n : n = 1, 2, \dots\}$*

$$\sum_{n=1}^{\infty} \frac{1}{|V(x_n)|} < \infty.$$

Consider the new operator $\widetilde{H} = \Delta + V(x)$ with boundary condition $\phi(x_n) = 0$, $n = 1, 2, \dots$, i.e. the restriction of H on $\Gamma - \mathcal{D}$ with the Dirichlet boundary condition on $\mathcal{D}$. Then

$$\mathrm{Sp}_{\text{a.c.}}(H) = \mathrm{Sp}_{\text{a.c.}}(\widetilde{H})$$

and for any $f \in L^2(\Gamma)$, the a.c. components of the spectral measures of f for H and $\widetilde{H}$ are mutually a.c.. In particular, $\mathrm{Sp}_{\text{a.c.}}(\widetilde{H}) = \emptyset \Longleftrightarrow \mathrm{Sp}_{\text{a.c.}}(H) = \emptyset$.

Proof. The proof is based on the following criterion.

Lemma 4.5. *Let $A = [A(x, y)]$, $x, y \in \Gamma$ be a linear operator acting on $\ell^2(\Gamma)$ and $\sum_{x,y\in\Gamma} |A(x, y)| < \infty$. Then A belongs to the trance-class $\mathcal{B}_1(\ell^2(\Gamma))$.*

The proof of the Lemma is easy. In fact, the matrix A can be presented as a sum: $A = [A(x, y)] = \sum_{z,v\in\Gamma}[\alpha_{z,v}(x, y)]$, where

$$\alpha_{z,v\in\Gamma}(x, y) = \begin{cases} A(z, v), & \text{if } x = z, y = v, \\ 0, & \text{otherwise,} \end{cases}$$

Note that $\alpha_{z,v}$ is a rank-one operator, i.e. $\|\alpha_{z,v}\|_1 = \|\alpha_{z,v}\|_{\ell^2(\Gamma)} = |A(z, v)|$. It gives

$$\|A\|_1 \leq \sum_{z,v\in\Gamma} \|\alpha_{z,v}\|_1 = \sum_{z,v\in\Gamma} |A(z, v)| < \infty.$$

Here $\|\cdot\|_1$ is the trace norm: the sum of the singular numbers of the corresponding matrix.

Let

$$A_{\lambda_0}(x,y) = R_{\lambda_0}(x,y) - \widetilde{R}_{\lambda_0}(x,y)$$

where

$$R_{\lambda_0}(x,y) = (H-\lambda_0 I)^{-1}(x,y) \quad \text{and} \quad \widetilde{R}_{\lambda_0}(x,y) = (\widetilde{H}-\lambda_0 I)^{-1}(x,y).$$

To prove Theorem 4.4, it is sufficient now to check that for some λ_0

$$\sum_{x,y\in\Gamma} |A_{\lambda_0}(x,y)| < \infty.$$

It follows from the path expansion (Equation (1)) that

$$A_{\lambda_0}(x,y) = \sum_{z\in\Gamma} A_{\lambda_0,z}(x,y)$$

where

$$A_{\lambda_0,z}(x,y) = \sum_{[\gamma]:x\longrightarrow z\longrightarrow y} \Big(\prod_{\nu\in[\gamma]} \frac{1}{\lambda_0 - V(\nu)} \Big).$$

Here z is the entrance point for path $[\gamma]$ to the set $\mathcal{D}$ at some moment $\tau: 0 \le \tau \le \infty$(i.e. the path $[\gamma]$ never visits $\mathcal{D}$ before moment τ). After moment τ, the path can visit $\mathcal{D}$ (and y) infinitely many times before its final stop at y.

Let's now estimate $|A_{\lambda_0,z}(x,y)|,\ z\in\mathcal{D}$. Put

$$\lambda_0 = iK(K+1) \quad \text{and} \quad \kappa = \frac{1}{|\lambda_0|} = \frac{1}{K(K+1)}.$$

Consider three cases.

a) $x = y = z \in \mathcal{D}$. Then,

$$|A_{\lambda_0,z}(z,z)| = \frac{1}{|\lambda_0 - V(z)|}\Big(1 + \sum_{\ell=1}^{\infty} K^\ell \kappa^\ell\Big) \le \frac{2}{|\lambda_0 - V(z)|}.$$

b) $x = z, y \ne z$. Then

$$|A_{\lambda_0,z}(z,z)| \le \frac{2}{|\lambda_0 - V(z)|}(K\kappa)^{d(z,y)}.$$

c) $x \ne z$. Then

$$|A_{\lambda_0,z}(z,z)| \le \frac{1}{|\lambda_0 - V(z)|}(K\kappa)^{d(x,z)+d(z,y)}.$$

Adding these estimates, we get

$$\sum_{x,y\in\Gamma} |A_{\lambda_0,z}(z,z)| \le \frac{C(\kappa)}{|\lambda_0 - V(z)|} \le \frac{C(\kappa)}{|V(z)|}, \quad z \in \mathcal{D}.$$

It gives

$$\sum_{x,y\in\Gamma} |A_{\lambda_0}(x,y)| \le \sum_{z\in\mathcal{D}} \frac{C(\kappa)}{|V(z)|} = C_2(\kappa) < \infty. \qquad \square$$

Theorem 4.4 can be used both ways: to prove the existence of the a.c. spectrum and to prove the absence of the a.c. spectrum. Let's illustrate the first option.

Consider the Sierpiński lattice S^∞. It is well known that the spectrum of the corresponding Laplacian as a minimal closed set supporting the spectral measure $\mu_f(d\lambda)$, $f \in \ell^2(s^\infty)$ belongs to $[-1, 4]$ and has Lebesgue measure zero. The maximal spectral type contains infinitely many eigenvalues of the infinite multiplicity and a singular continuous component. The eigenfunctions of the point spectrum can be selected to be compactly supported.

Let's perturb the operator $-\Delta$ by the potential $V(x)$ supported on the system of points $\{x_k = (1,0) + (k-1)\vec{\omega}; k = 1, 2, \ldots\}$, where $\vec{\omega} = (1/2, \sqrt{3}/2)$, see Figure 5.

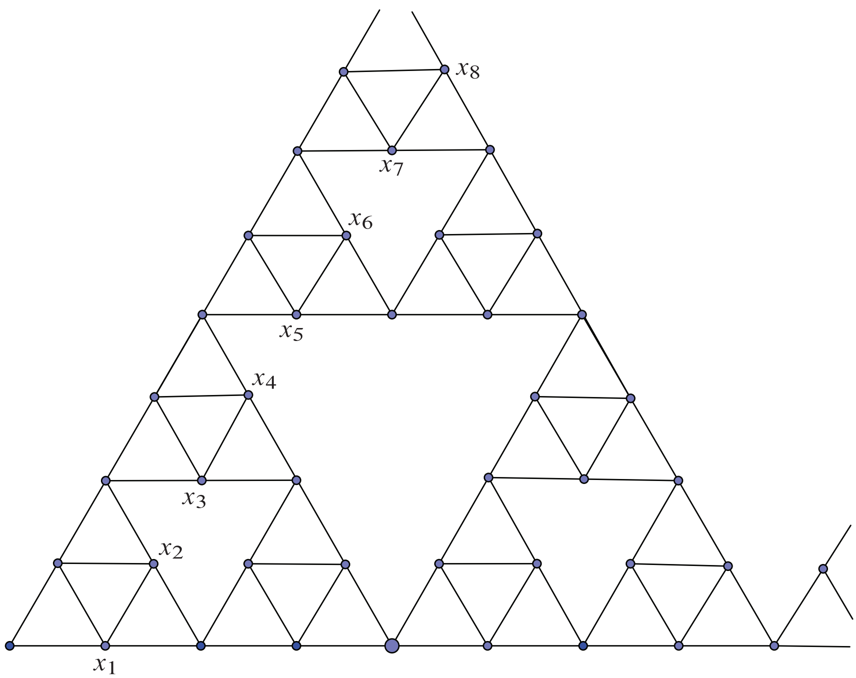

Figure 5. Sierpiński lattice S^∞.

Assume that $\sum_{k=1}^{\infty} \frac{1}{|V(x_k)|} < \infty$. The set of points $\{x_k : k = 1, 2, \dots\}$ separates the 1-D semi-axis, the left boundary of S^∞, from the rest. Due to Theorem 4.4, we have

Corollary 4.6. *The operator $H = -\Delta + V(x)$ introduced above has an a.c. spectrum supported on the interval $[-2, 2]$, the spectrum of the 1-D lattice Laplacian.*

Corollary 4.7. *Assume that for the Schrödinger operator $H = \Delta + V(x)$, one can find a set $\mathcal{D} = \{x_n; n = 1, 2, \dots\}$ such that $\Gamma - \mathcal{D}$ is the union of disjoint finite sets (in our terminology, $\mathcal{D}$ is the union of belts). Then under the assumption of the above theorem ($\sum_n \frac{1}{|V(x_n)|} < \infty$), H has no a.c. spectrum.*

The idea of Corollary (4.7) goes to Simon and Spencer [6].

In the case of S^∞, we have the following result.

Corollary 4.8. *Consider the pairs of the points $(2^n\vec{\imath}, 2^n\vec{\omega})$, $n = 1, 2, \dots$, with $\vec{\imath} = (1, 0)$, separating the triangle with size 2^n of S^∞ from the rest. Let $h_n = \min(|V(2^n\vec{\imath})|, |V(2^n\vec{\omega})|)$ and $\limsup_{n\to\infty} h_n = +\infty$. Then the operator $H = -\Delta + V(x)$ on $\ell^2(S^\infty)$ has no a.c. spectrum.*

In fact, one can select a sub-sequence n' such that $\sum_{n'} \frac{1}{h_{n'}} < \infty$.

Corollary 4.8 contains Theorem (A) from the introduction. For more general results of this type, see Section 6.

5. The real analytic approach to the absence of the a.c. spectrum and localization

In all future constructions, the belts $\{b_l,\ l = 1, 2, \dots\}$ will be selected based on the assumption that for the fixed energy interval I on the λ-axis:

$$\max_{z_1\in\partial b_l^-, z_2\in\partial b_l^+} |\beta_\lambda^{b_l}(z_1, z_2)| \le \delta_l, \quad \delta_l \to 0,\ l \to \infty \tag{4}$$

for all $\lambda \in I$.

There are different ways to guarantee that the $\beta_\lambda^{b_l}(\cdot,\cdot)$ take small values. For instance, assume that $|V(x)| \ge h_l,\ x \in b_l$ and h_l is sufficiently large. Then, for fixed energy interval I,

$$\left|\frac{1}{\lambda - V(z)}\right| \le \frac{2}{h_l}, \quad \lambda \in I, z \in b_l,$$

i.e.

$$\beta_\lambda^{b_l}(\cdot,\cdot) \le C\left(\frac{2K}{h_l}\right)^{t_l} = \delta_l,$$

with $t_l = d(\partial b_l^-, \partial b_l^+)$, for some constant C.

In the following, to prove the localization theorem or the absence of the a.c. spectrum, we will use cluster expansion Theorem 3.3 (formulas(2) and (3)). For the contribution to the resolvent kernel $R_\lambda^{[\tilde\gamma]}(x_0, x)$ for a fixed path on the graph $\widetilde\Gamma$ of the main blocks, we will use the following estimate, which is a combination of (3) and (4):

$$\begin{aligned}
|R_\lambda^{[\tilde\gamma]}(x_0,x_0)| &= \Big| \sum R_\lambda^{(0)}(x_0,z_1)\beta_\lambda^{b_1}(z_1,z_2)R_\lambda^{(1)}(z_2,\cdot) \\
&\qquad R_\lambda^{(2)}(\cdot,z_{2l-3})\beta_\lambda^{b_2}(z_{2l-3},z_{2l-2}) \\
&\qquad\quad R_\lambda^{(1)}(z_{2l-2},z_{2l-1})\beta_\lambda^{b_1}(z_{2l-1},z_{2l})R_\lambda^{(0)}(z_{2l},x_0)\Big| \\
&\le \Big(\sum_{z_1}|R_\lambda^{(0)}(x_0,z_1)|\Big)|\beta_\lambda^{b_1}(z_1,z_2)|\Big(\sum_{z_2,z_3}|R_\lambda^{(1)}(z_2,z_3)|\Big) \\
&\qquad \Big(\sum_{\substack{z_{2n-4},\\ z_{2L-3}}}|R_\lambda^{(2)}(z_{2n-4},z_{2n-3})|\Big)|\beta_\lambda^{b_2}(z_{2n-3},z_{2n-2})| \\
&\qquad \Big(\sum_{\substack{z_{2n-2},\\ z_{2n-1}}}|R_\lambda^{(1)}(z_{2n-2},z_{2n-1})|\Big)|\beta_\lambda^{b_1}(z_{2n-1},z_{2n})| \\
&\qquad \Big(\sum_{z_{2n}}|R_\lambda^{(0)}(z_{2n},x_0)|\Big) \\
&\le \Big(\sum_{z_1}|R_\lambda^{(0)}(x_0,z_1)|\Big)\delta_1\Big(\sum_{z_2,z_3}|R_\lambda^{(1)}(z_2,z_3)|\Big) \\
&\qquad \Big(\sum_{\substack{z_{2n-4},\\ z_{2L-3}}}|R_\lambda^{(2)}(z_{2n-4},z_{2n-3})|\Big) \\
&\qquad \delta_2\Big(\sum_{\substack{z_{2n-2},\\ z_{2n-1}}}|R_\lambda^{(1)}(z_{2n-2},z_{2n-1})|\Big) \\
&\qquad \delta_1\Big(\sum_{z_{2n}}|R_\lambda^{(0)}(z_{2n},x_0)|\Big)
\end{aligned}$$

where the summations are over $z_1 \in \partial b_1^-$, $z_2 \in \partial b_1^+, \dots, z_{2L-3} \in \partial b_2^+$, $z_{2L-2} \in \partial b_2^-$, $z_{2L-1} \in \partial b_1^+$, $z_{2n} \in \partial b_1^-$ with L being the number of times the path goes through the belts until it returns to x_0.

The complicated multiple sum for $R_\lambda^{[\tilde{\gamma}]}(\cdot,\cdot)$ is now the product of the simple local sums over $\partial b_l^{\pm}$ and factors δ_l. The factors δ_l are small due to our assumption, but the resolvent kernels of the main blocks $R_\lambda^{(l)}(\cdot,\cdot)$ can be large. Lemma 5.1 shows that these kernels are not too large except, perhaps, for a set of small measure on the energy axis (see the proof of this result in [23] in a convenient form). This is the key for the proof of our main results.

Lemma 5.1 (Kolmogorov Lemma). *Let* $M > 0$, *then*

$$m\Big(\lambda\colon |F(\lambda)| = \Big|\sum_{l=1}^{N} \frac{\alpha_l}{\lambda-\lambda_l}\Big| \geq M\Big) \leq \frac{4\sum_l |\alpha_l|}{M}.$$

Traditionally this Lemma is attributed to Kolmogorov, but this fact was known earlier [9] and [11].

The following proposition is the central point of our approach.

Proposition 5.2. *Let* $H_B = \Delta + V(x)$ *with Dirichlet boundary condition*: $\phi(x) = 0$, $x \in \Gamma - B$, *where* $B \subset \Gamma$. *Consider* $(R_\lambda^{(B)} f, g)$, $f, g \in l^2(B)$, *then in the obvious notation,*

$$(R_\lambda^{(B)} f, g) = \sum_{k=1}^{|B|} \frac{(f,\psi_k)(g,\psi_k)}{\lambda-\lambda_k^B}$$
$$= \|f\|_2 \|g\|_2 \sum_k \frac{[(f,\psi_k)/\|f\|_2]\,[(g,\psi_k)/\|g\|_2]}{\lambda-\lambda_k^B}.$$

where $\lambda_k^B, k = 1, 2, \ldots, |B|$ *are the eigenvalues of* H_B *and* ψ_k *are the corresponding orthonormal eigenfunctions. Due to the Cauchy-Schwartz inequality and Kolmogorov's lemma, we have*

$$m(\lambda \in R\colon |(R_\lambda^{(B)} f, g)| > M) \leq \frac{4\|f\|_2\|g\|_2}{M}.$$

In particular, for $f = \delta_{z_1}(\cdot)$ and $g = \delta_{z_2}(\cdot)$,

$$m(\lambda \in R\colon |R_\lambda^{(B)}(z_1, z_2)| > M) \leq \frac{4}{M}.$$

Corollary 5.3. *We have*

$$m(\lambda \in R\colon \max_{z_1,z_2 \in \partial b_l^+} |R_\lambda^{(l)}(z_1,z_2)| > M_l^+) \le \frac{4|\partial b_l^+|^2}{M_l^+}.$$

Similarly,

$$m(\lambda \in R\colon \max_{\substack{z_1 \in \partial b_l^+, \\ z_2 \in \partial b_{l+1}^-}} |R_\lambda^{(l)}(z_1,z_2)| > \sqrt{M_l^+ M_{l+1}^-}\,) \le \frac{4|\partial b_l^+|\,|\partial b_{l+1}^-|}{\sqrt{M_l^+ M_{l+1}^-}},$$

$$m(\lambda \in R\colon \max_{z_1,z_2 \in \partial b_{l+1}^-} |R_\lambda^{(l)}(z_1,z_2)| > M_{l+1}^-) \le \frac{4|\partial b_{l+1}^-|^2}{M_{l+1}^-},$$

$$m(\lambda \in R\colon \max_{\substack{z_1 \in \partial b_{l+1}^-, \\ z_2 \in \partial b_l^+}} |R_\lambda^{(l)}(z_1,z_2)| > \sqrt{M_l^+ M_{l+1}^-}\,) \le \frac{4|\partial b_l^+|\,|\partial b_{l+1}^-|}{\sqrt{M_l^+ M_{l+1}^-}}.$$

where $M_l^+, M_{l+1}^- > 0$ *are constants.*

The following theorem (again due to Simon and Wolff) gives a sufficient condition for the square summability of $R_{\lambda+i0}(x_0,\cdot)$ for a.e. $\lambda \in R^1$ in real analytic terms.

Theorem 5.4. *Assume that* $Q_n \uparrow \Gamma$ *is an increasing family of connected sets and* $R_{n,\lambda}(x,y)$, $n = 1,2,\ldots$ *are the resolvents of the operators* H_n*: the restrictions of* H *on* Q_n *with Dirichlet boundary condition* ($\psi \equiv 0, x \notin Q_n$)*. Assume also that for any* $x_0 \in \Gamma$ *and for a.e.* $\lambda \in \mathbb{R}^1$,

$$\limsup_{n \longrightarrow \infty} \sum_{y \in Q_n} (R_{n,\lambda}(x_0,y))^2 \le c(\lambda) < \infty.$$

Then

$$\lim_{\epsilon \longrightarrow 0} \sum_{y \in \Gamma} |R_{\lambda+i\epsilon}(x_0,y)|^2 = \sum_{y \in \Gamma} (R_{\lambda+i0}(x_0,y))^2 \le c(\lambda) < \infty,$$

i.e. one can apply Theorem 4.3 on localization (*with appropriate randomization*).

The future proof of the localization theorem will use the cluster expansion, Lemma 5.1 in the form of Corollary 5.3, and the estimate for the ℓ^2-norm of the resolvent of the main blocks:

Lemma 5.5. *Let*

$$L_n = |(n)|(|\partial b_n^+| + |\partial b_{n+1}^-|) \quad and \quad h_2^{(n)}(\lambda) = \sum_{a\in\partial b_n^+}\sum_{x\in(n)} R_\lambda^2(a,x) \quad for\ n \geq 1.$$

Then

$$m\{\lambda : h_2^{(n)}(\lambda) > M\} \leq \frac{4\sqrt{|(n)|\,|\partial b_n^+|}}{M} \leq \frac{4\sqrt{L_n}}{M}.$$

This Lemma is an extension of Lemma 2.8 in [23]. Note that in obvious notation

$$R_\lambda^{(n)}(\xi, x) = \sum_{k=1}^{|(n)|} \frac{\psi_{n,k}(a)\psi_{n,k}(x)}{\lambda - \lambda_{n,k}},$$

i.e.

$$\|R_\lambda^{(n)}(\xi,\cdot)\|_2^2 = \sum_{k=1}^{|(n)|} \frac{\psi_{n,k}^2(\xi)}{(\lambda-\lambda_{n,k})^2}, \quad \sum_{k=1}^{|(n)|}\psi_{n,k}^2(\xi) = 1$$

Then,

$$h_2^{(n)}(\lambda) = \sum_{a\in\partial b_n^+} \|R_\lambda^{(n)}(\xi,\cdot)\|_2^2 = \sum_{k=1}^{|(n)|} \frac{\zeta_{n,k}}{(\lambda-\lambda_{n,k})^2}.$$

Here

$$\sum_{k=1}^{|(n)|} \zeta_{n,k} = \|I_{\partial b_n^+}\|.$$

Also, $\{\psi_{n,k}\}$, $\{\lambda_{n,k}\}$, $k = 1,\ldots,|(n)|$ are the eigenfunctions and eigenvalues of the restriction of H to the main block (n).

Then

$$m(\lambda : h_2^{(n)}(\lambda) > M) = m\Big(\lambda : \tilde{h}_2(\lambda) > \frac{M}{|\partial b_n^+|}\Big),$$

$$\tilde{h}_2(\lambda) = \sum_{k=1}^{|(n)|} \frac{\beta_{n,k}}{(\lambda-\lambda_{n,k})^2}, \quad \beta_{n,k} = \frac{\zeta_{n,k}}{|\partial b_n^+|}, \quad \sum_{k=1}^{|(n)|}\beta_{n,k} = 1.$$

Now one can apply Lemma 2.6 b) on page 283 in [23].

Remark 4. The localization theorems (on the pure point spectrum of the Hamiltonian H) include as a central ingredient the square summability of the resolvent kernel: $\sum_{y\in\Gamma} |R_{\lambda+i0}(x,y)|^2 < \infty$ for any $x_0 \in \Gamma$ and a.e. $\lambda \in R^1$. One can expect that the absence of the a.c. spectrum requires weaker conditions on $R_\lambda(x,y)$. Let's formulate the real analytic result, which is based on the recent elegant result by A. Gordon. His paper [2] was instantly accepted by Proceedings of the AMS and will appear soon.

Theorem 5.6 (A.Gordon). *Let $\mu_n, n \geq 1$ be a family of discrete probability measures on R^1 supported on the finite set $\{\lambda_{ni}, i = 1, 2, \ldots, N_n\}$ with atoms α_{ni}, $\sum_i \alpha_{ni} = 1$ for each n. Assume that $\mu_n \xrightarrow{w} \mu$ (in $C(R^1)$) and $\frac{d\mu}{d\lambda} = \pi(\lambda)$ a.e. with $\int_{R^1} \pi(\lambda)d\lambda > 0$. The function $\pi(\lambda)$ is the density of the a.c. component of the limiting measure μ. Consider the Hilbert transforms $h_n(\lambda) = \int \frac{\mu_n(dz)}{\lambda - z} = \sum_{i=1}^{N_n} \frac{\alpha_{ni}}{\lambda - \lambda_{ni}}$. Then*

$$\limsup_{n\longrightarrow\infty} h_n(\lambda) = +\infty,$$

a.e. on the set $\{\lambda : \pi(\lambda) > 0\}$.

Remark 5. Theorem 5.6 and its corollary Theorem II in Section 6 will not be used below. For our applications in Section sec: main results, it is sufficient to use the weaker Theorem 4.4. Theorem 5.6 is crucial in case of bounded random potentials, which are not covered by Theorem 4.4.

Remark 6. As was pointed out in [2], the result of Theorem 5.6 was conjectured by the first author of the present paper. The conjecture was based on the analysis of the particular example which we present here as an illustration of Gordon's Theorem.

Consider Chebyshev's polynomial $T_n(\lambda) = \cos(n \arccos \lambda), \lambda \in [-1, 1]$ and the function

$$h_n(\lambda) = \frac{T_n'(\lambda)}{nT_n(\lambda)} = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{\lambda - \lambda_{n,i}}$$

where $\lambda_{n,i} = \cos\frac{\pi(1+2i)}{2n}$, $i = 1, \ldots, n$.

The function $h_n(\lambda)$ is the Hilbert transform of the measure μ_n which has atoms $\alpha_{n,i} = \frac{1}{n}$ at the points $\lambda_{n,i}$. As $n \to \infty$, this measure converges weakly to the a.c. measure μ with density $\frac{1}{\pi}\frac{1}{\sqrt{1-\lambda^2}}$, $|\lambda| < 1$. The substitution $\lambda = \cos\varphi, \varphi \in [0, \pi]$ gives

$$h_n(\cos\varphi) = \tilde{h}_n(\varphi) = \frac{\sin n\varphi}{\sin\varphi\cos n\varphi} = \frac{\tan n\varphi}{\sin\varphi}.$$

In the probability space $(\Omega = [0,\pi], \mathcal{B}([0,\pi]), \frac{d\varphi}{\pi}$, let's consider the events

$$\widetilde{\Gamma}_n = \left\{\varphi \in [0,\pi]\colon |\tilde{h}_n(\lambda) > \frac{M}{\sin\varphi}\right\} = \{\varphi\colon |\tan n\varphi| > M\}.$$

The set $\widetilde{\Gamma}_n$ has a very simple structure: it is the union of the intervals $\Delta_{n,i}$ of length $\frac{2}{n}\arctan\frac{1}{M} \sim \frac{2}{Mn}(M \gg 1)$. Consider a fast increasing subsequence n', say $n' = 2^{k^2}, k = 1, 2, \dots$. Then the events $\widetilde{\Gamma}_{n'}$ will be almost independent in our probability space. Since $\mu(\widetilde{\Gamma}_{n'}) \sim \frac{2}{M}$, one can easily prove that for any M, a.e. φ on $[0,\pi]$ belongs to infinitely many $\widetilde{\Gamma}_n$. It yields that $\limsup |h_n(\lambda)| = \infty$, a.e. on $[-1,1]$. A similar idea for the study of the lacunar and super lacunar functional series goes to A. Kolmogorov, see [24] containing a martingale approach to problems of this kind.

From Theorem 5.6, one can deduce the following result.

Theorem 5.7. *If for any $x_0 \in \Gamma$ the cluster expansion for $R_\lambda(x_0, x_0)$ converges absolutely for a.e. $\lambda \in R^1$ (i.e. $\lim_{N\to\infty} R_\lambda^{(N)}(x_0, x)$ is finite a.e., where $R_\lambda^{(N)}(\cdot,\cdot)$ is the resolvent of H on $D_N = \bigcup_{n=0}^{N}(n)$ with Dirichlet Boundary condition on $\Gamma - D_N$), then the spectrum of H is pure singular.*

6. Theorems on the absence of an a.c. spectrum and localization

In this section, we will prove the major results of this paper. They will be illustrated by examples in sections 6 and 7.

Theorem I. *Consider the graph Γ introduced in section 1 and the Hamiltonian $H = \Delta + V(x)$. Assume that one can find a system of counters b_n, $n = 1, 2, \dots$ (the belts of thickness 1) such that for $h_n = \min_{x\in b_n} |V(x)|$,*

$$|b_n|/h_n \longrightarrow 0, \quad n \to \infty.$$

Then $\mu_{\text{a.c.}}(H, f) \equiv 0$.

This result is a trivial corollary of Theorem 4.4. In fact if $h_n^{-1}|b_n| \to 0$, then one can find a sequence $\{n'\}$:

$$\sum_{n'} h_{n'}^{-1}|b_{n'}| < \infty$$

but

$$\sum_{z \in \cup b_{n'}} \frac{1}{|V(x)|} \leq \sum_{n'} \frac{|b_{n'}|}{h_{n'}} < \infty.$$

The following example illustrates the above theorem.

Example 6.1. Let $\Gamma = S^\infty$ be the Sierpiński lattice and $H = \Delta + V(x, \omega)$, $x \in S^\infty$, where $V(x, \omega)$ is a system of unbounded i.i.d. random variables. Then P-a.s. $\mu_{\text{a.c.}}(H) = 0$.

Proof. Consider the following events

$$A_n = \{|V(2^n \vec{\imath}, \omega)| > h_n,\ |V(2^n \vec{w}, \omega)| > h_n\},$$

where $\vec{\imath} = (1, 0)$, $\vec{w} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Events A_n are independent and $P(A_n) = P^2(|V| > h_n) > 0$. For appropriate $h_n \to \infty$, $\sum_n P(A_n) = \infty$. The second Borel–Cantelli lemma provides the existence P-a.s. of infinitely many events $A_{n'}$. One can apply now the above Theorem I to the counters containing only two points: $b_n = \{2^n \vec{\imath}, 2^n \vec{w}\}$. □

In fact, we didn't use here the specific structure of S^∞ and proved the following result.

Corollary 6.2. *Suppose for the general graph Γ with conditions $\nu(x) \leq K$, one can find an infinite system of belts b_n: $|b_n| \leq C_0 < \infty$ for appropriate C_0. Assume that the i.i.d. potential $V(x, \omega)$ is unbounded ($P(|V| > A) = P(A) > 0$ for any A). Then the Anderson Hamiltonian $H = \Delta + V(x, \omega)$ has no a.c. spectrum, P-a.s.*

We will return to the spectral theory of self-similar fractal graphs (like the Sierpiński lattice or snow flake) in another paper to cover cases with bounded random potentials, where again $\mu_{\text{a.c.}}(H) = 0$.

The following result is more general than Theorem I. It is based on Theorem 5.6 (Gordon's Theorem).

Theorem II. *The condition*

$$\delta_l |\partial b_l^+|^2 |\partial b_l^-|^2 \longrightarrow 0, \quad l \to \infty$$

implies that $\mathrm{Sp}_{\text{a.c.}}(H) = \emptyset$. *Here* $\delta_l = \max\limits_{\substack{z_1 \in \partial b_l^-, \\ z_2 \in \partial b_1^+}} |\beta_\lambda^{b_l}(z_1, z_2)|$.

Proof. One can assume without loss of generality that

$$\sum_{l'} \delta_{l'} |\partial b_{l'}^{+}|^2 |\partial b_{l'}^{-}|^2 < \infty$$

for a subsequence of belts $\{b_{l'}\}$.

Let's now apply the Borel–Cantelli lemma, which will show that the resolvent kernels $R_\lambda^{(l)}(\cdot,\cdot)$ are not "too large."

Put

$$M_l^- = \frac{|\partial b_l^-|}{\alpha_l}, \quad M_l^+ = \frac{|\partial b_l^+|}{\alpha_l}, \quad l = 1, 2, \dots,$$

where $\alpha_l > 0$ is any sequence such that $\sum_l \alpha_l < \infty$ (and as a result, $\sum_l \sqrt{\alpha_l \alpha_{l+1}} < \infty$). Then the formulas in Corollary 5.3 will have the form

$$m\,(\lambda \in R\colon \max|\cdot| > \cdot) \le 4\alpha_l$$

or

$$m\,(\lambda \in R\colon \max|\cdot| > \cdot) \le 4\sqrt{\alpha_{l+1}\alpha_l}.$$

And the Borel–Cantelli lemma gives that for a.e. $\lambda \in R$,

$$\max_{z_1, z_2 \in \partial b_l^+} |R_\lambda^{(l)}(z_1, z_2)| \le \frac{|\partial b_l^+|^2}{\alpha_l}, \quad l \ge L_0(\lambda) \tag{5}$$

Similarly,

$$\max_{z_1, z_2 \in \partial b_{l+1}^-} |R_\lambda^{(l)}(z_1, z_2)| \le \frac{|\partial b_{l+1}^-|^2}{\alpha_{l+1}}; \tag{6}$$

$$\max_{\substack{z_1 \in \partial b_l^+, \\ z_2 \in \partial b_{l+1}^-}} |R_\lambda^{(l)}(z_1, z_2)| \le \frac{|\partial b_l^+|\,|\partial b_{l+1}^-|}{\sqrt{\alpha_l \alpha_{l+1}}}; \tag{7}$$

$$\max_{\substack{z_1 \in \partial b_{l+1}^-, \\ z_2 \in \partial b_l^+}} |R_\lambda^{(l)}(z_1, z_2)| \le \frac{|\partial b_l^+|\,|\partial b_{l+1}^-|}{\sqrt{\alpha_l \alpha_{l+1}}}. \tag{8}$$

In the case $\sum_{z_1 \in \partial b_1^-} R_\lambda^{(0)}(x_0, z_1)$ one can put

$$\max_{z_1 \in \partial b_1^-} |R_\lambda^{(0)}(x_0, z_1)| \le \frac{|\partial b_1^-|}{\sqrt{\alpha_1}}.$$

First, assume $L_0(\lambda) = 0$ and choose $\alpha_l = 3\delta_l|\partial b_l^-|^2|\partial b_l^+|^2$. Then, due to Theorem 3.3, using the fact that the number of of paths $[\tilde{\gamma}]$ of length n is no more than 2^n,

$$\begin{aligned}
|R_\lambda^{[\tilde{\gamma}]}(x_0, x_0)| = \Big| \sum R_\lambda^{(0)}(x_0, z_1)\beta_\lambda^{b_1}(z_1, z_2)R_\lambda^{(1)}(z_2, \cdot) \\
R_\lambda^{(2)}(\cdot, z_{2l-3})\beta_\lambda^{b_2}(z_{2l-3}, z_{2l-2}) \\
R_\lambda^{(1)}(z_{2l-2}, z_{2l-1})\beta_\lambda^{b_1}(z_{2l-1}, z_{2l})R_\lambda^{(0)}(z_{2l}, x_0)\Big| \\
\leq \Big(\sum_{z_1} |R_\lambda^{(0)}(x_0, z_1)|\Big)|\beta_\lambda^{b_1}(z_1, z_2)|\Big(\sum_{z_2, z_3} |R_\lambda^{(1)}(z_2, z_3)|\Big) \\
\Big(\sum_{\substack{z_{2n-4},\\ z_{2L-3}}} |R_\lambda^{(2)}(z_{2n-4}, z_{2n-3})|\Big)|\beta_\lambda^{b_2}(z_{2n-3}, z_{2n-2})| \\
\Big(\sum_{z_{2n-2}, z_{2n-1}} |R_\lambda^{(1)}(z_{2n-2}, z_{2n-1})|\Big)|\beta_\lambda^{b_1}(z_{2n-1}, z_{2n})| \\
\Big(\sum_{z_{2n}} |R_\lambda^{(0)}(z_{2n}, x_0)|\Big),
\end{aligned} \tag{9}$$

where the summations are over $z_1 \in \partial b_1^-$, $z_2 \in \partial b_1^+, \ldots, z_{2L-3} \in \partial b_2^+$, $z_{2L-2} \in \partial b_2^-$, $z_{2L-1} \in \partial b_1^+$, $z_{2n} \in \partial b_1^-$ with L being the number of times the path goes through the belts until it returns to x_0. Let n_l^+ be the number of times the path goes through belt b_l in the direction away from x_0 and n_l^- be the number of times the path goes through belt b_l in the direction toward x_0. Then we have $n_l^+ = n_l^- = n_l$, since the path starts from x_0 and ends at x_0. For each time the path goes through belt b_l in the direction away from x_0, the contribution to the left part of the inequality (9) is bounded by

$$\frac{|\partial b_l^-|^2}{\sqrt{\alpha_l}}\delta_l\frac{|\partial b_l^+|^2}{\sqrt{\alpha_l}} = \frac{|\partial b_l^-|^2|\partial b_l^+|^2}{\alpha_l}\delta_l \tag{10}$$

Similarly, for each time the path goes through the belt b_l in the direction toward x_0, the contribution to the left part of the inequality 9 is bounded by

$$\frac{|\partial b_l^+|^2}{\sqrt{\alpha_l}}\delta_l\frac{|\partial b_l^-|^2}{\sqrt{\alpha_l}} = \frac{|\partial b_l^-|^2|\partial b_l^+|^2}{\alpha_l}\delta_l \tag{11}$$

Therefore, since $n_l^+ = n_l^- = n_l$, we have the following

$$|R_\lambda^{[\tilde{\gamma}]}(x_0, x_0)| \leq \prod_{l=1}^{L}\Big(\frac{|\partial b_l^-|^2|\partial b_l^+|^2}{\alpha_l}\delta_l\Big)^{2n_l} \leq \Big(\frac{1}{3}\Big)^{2n} \tag{12}$$

where $\sum_l 2n_l = 2n$ and $\alpha_l = 3\delta_l|\partial b_l^-|^2|\partial b_l^+|^2$.

The number of such paths can be estimated as 2^n. The same estimates are applicable to $R_\lambda^{(N)}(\cdot,\cdot)$.

Therefore

$$|R_\lambda(x_0,x_0)| \le \sum_{n=0}^{\infty} \frac{2^n}{3^n} = 3.$$

Let $S_k = \{\lambda \in R\colon L_0(\lambda) = k\}$, $k \ge 0$. We proved that on the set S_0 where $L_0(\lambda) = 0$

$$R_\lambda(x_0,x_0) = \lim_{N\longrightarrow\infty} R_\lambda^{(N)}(x_0,x_0)$$

and A. Gordon's Theorem 5.5 states that the a.c. component of the spectral measure $\rho_f(d\lambda)$, $f = \delta_{x_0}(x)$, equals 0 on S_0.

Assume that $L_0(\lambda) > 0$. If $L_0(\lambda) = k \ge 1$ (i.e. inequalities (5)-(8) hold for $l \ge L_0(\lambda) = k$)) , we introduce the new operator

$$\widetilde{H}_N = \Delta + \widetilde{V}$$

where

$$\widetilde{V} = \begin{cases} V(x) & \text{if } x \in (l)\colon l \ge k, \\ A_k & \text{if } x \in (l)\colon l < k. \end{cases}$$

We select constants A_k large enough that

$$\frac{|\partial b_l^-|^2 |\partial b_l^+|^2 \delta_l}{\alpha_l} < \frac{1}{3}$$

on belts $b_l\colon l < k$.

Then for the resolvent kernel

$$\widetilde{R}_{\lambda,k}(x,x_0) = ((\widetilde{H}_k - \lambda I)^{-1}\delta_{x_0}(x), \delta_{x_0}),$$

we can repeat for any $k \ge 1$ the previous consideration, which gives that on the set $S_k\{\lambda \in R\colon L_0(\lambda) = k\}$, there is no a.c. spectrum of $\widetilde{H}_k$ for $f = \delta_{x_0}(x)$. Due to the Kato–Birman theorem [25], the same is true for H (the transition from H to $\widetilde{H}_k$ is a finite rank perturbation of H). This proves the theorem. □

The last theorem in this section gives sufficient conditions for localization.

Let's stress that Theorems I and II about the absence of an a.c. spectrum don't contain information about the volumes of the main blocks. This information is crucial for localization. It is clear that in the 1-D case(graph Z^1) for the potentials presented as a sum of very sparse bumps, where the heights are increasing not very fast, the spectral measure is pure singular continuous.

Theorem III. *Assume that for some sequence $A_n \geq 3, n = 1, 2, \ldots,$ we have*

$$\sum_n \frac{|(n)|}{A_1^2 A_2^2 \ldots A_n^2} < \infty \tag{13}$$

and

$$\sum_{n=1}^{\infty} A_n \delta_n |\partial b_n^+|^2 |\partial b_n^-|^2 < \infty. \tag{14}$$

Then for any $x_0 \in \Gamma$, the resolvent kernel $R_{\lambda+0i}(x_0, \cdot)$ belongs to $\ell^2(\Gamma)$ for a.e. $\lambda \in R^1$.

Corollary 6.3. *Consider the random Schrödinger operator $H = \Delta + V(x, \omega)$. The potential $V(x, \omega)$ is a system of i.i.d.r.v.'s with bounded distribution density $p(v)$. If P-a.s. one can find a system of belts $\{b_n, n \geq 1\}$ satisfying the conditions (13) and (14) in Theorem III, then the operator H has a pure point spectrum (P-a.s.).*

Corollary 6.3 is a standard application of Simon–Wolff theorem 4.3.

Corollary 6.4 (delocalization). *In the situation of Corollary 6.3, the spectrum of H(as a closed subset of λ-axis) is a union of disjoint intervals. In this case the well-known result in* [3] *and* [21] *states that there exists a subset Λ of the G_δ-class such that the spectral measure of H is pure singular continuous, i.e. the delocalization is generic topologically. Of course, $m(\Lambda) = 0$.*

Proof of the Theorem III. The future calculations will be formal. We will work with the resolvent kernel $R_\lambda(x_0, x)$ which is not defined for real λ. Of course, we have to start from the resolvent of the operator H_N on $\bigcup_{n=1}^{N}(n) = Q_N$ and pass to the limit $N \to \infty$. For brevity, we will not use this additional index N but assume its existence.

Let's introduce the following partition of the resolvent kernel $R_\lambda(x_0, x)$:

$$R_\lambda(x_0, x) = \sum_{n=0}^{\infty} \widetilde{R}_{\lambda,n}(x_0, x)$$

where

$$\widetilde{R}_{\lambda,n}(x_0, x) = \sum_{\tilde{\gamma}:(0)\longrightarrow(n)} R_\lambda^{(\tilde{\gamma})}(x_0, x).$$

Note that $\widetilde{R}_{\lambda,n}(x_0, x) = 0,\ x \notin (n)$. It gives

$$\sum_\Gamma R_\lambda^2(x_0, x) \leq 2 \sum_{n=0}^{\infty} \sum_{x \in (n)} \widetilde{R}_{\lambda,n}^2(x_0, x)$$

(each x belongs to at most two main blocks).

Let's analyze the function $\widetilde{R}_{\lambda,n}(\cdot,\cdot)$ and its ℓ^2-norm. To compensate for the possibly fast growth of $|(n)|$, in the inequalities (5)-(8), we will put

$$\alpha_l = A_l \delta_l |\partial b_l^-|^2 |\partial b_l^+|^2, \quad A_l \geq 3$$

and assume that $\sum_l \alpha_l < \infty$. Also in Lemma 5.5, we will put

$$M_n = \frac{|(n)|(|\partial b_n^+| + |\partial b_{n+1}^-|)}{\beta_n^2}, \quad \sum_n \beta_n < \infty.$$

One can take $\beta_n = \frac{1}{n^2}$. Due to the Borel–Cantelli lemma

$$\sum_{\xi \in \partial b_n^+ \cup \partial b_{n+1}^-} \sum_{x \in (n)} [R_\lambda^{(n)}(\xi, x)]^2 \leq M_n, \quad n \geq n_0(\omega). \tag{15}$$

Let $L_0(\lambda)$ be the set of $\lambda \in R^1$ such that inequalities (5)–(8) and (15) are true for any $n \geq 0$.

Then one can repeat with small changes the proof of the Theorem II.

Consider the particular term

$$\widetilde{R}_{\lambda,n}(x_0, x) = \sum_{\tilde{\gamma}:(0)\longrightarrow(n)} R_\lambda^{(\tilde{\gamma})}(x_0, x).$$

The path $\tilde{\gamma}$ can enter (n) from $(n-1)$ or $(n+1)$. Consider the first possibility. The shortest path $[\tilde{\gamma}]_n^{(-)}$ in this class has length n and is of the form:

$$(0) \longrightarrow (1) \longrightarrow \cdots \longrightarrow (n-1) \longrightarrow (n).$$

Its contribution $R_\lambda^{([\tilde{\gamma}]_n^{(-)})}(x_0, x)$ can be estimated (as in Theorem II) by the expression

$$\begin{aligned}
&\frac{|\partial b_1^-|^2 \delta_1 |\partial b_1^+|^2}{\alpha_1} \cdots \frac{|\partial b_{n-1}^-|^2 \delta_{n-1} |\partial b_{n-1}^+|^2}{\alpha_{n-1}} \\
&\qquad |\partial b_n^-|^2 \delta_n \frac{|\partial b_n^+|^2}{|\partial b_n^+|^2} \frac{\alpha_n}{\alpha_n} \sum_{\xi \in \partial b_n^-} \left| R_\lambda^{(n)}(\xi, x) \right| \\
&\leq \frac{1}{A_1} \cdots \frac{1}{A_n} \frac{\alpha_n}{|\partial b_n^+|^2} \sqrt{|\partial b_n^+| \sum_{\xi \in \partial b_n^-} |R_\lambda^{(n)}(\xi, x)|^2} \\
&\leq \frac{1}{A_1 A_2 \ldots A_n} \frac{\alpha_n}{|\partial b_n^+|^{3/2}} \sqrt{\sum_{\xi \in \partial b_n^-} |R_\lambda^{(n)}(\xi, x)|^2}
\end{aligned}$$

Summation over all paths $[\tilde{\gamma}]$: $(0) \to \cdots \to (n-1) \to (n)$ that enter (n) from $(n-1)$ can be estimated by the expression:

$$\frac{1}{A_1 A_2 \ldots A_n} \frac{\alpha_n}{|\partial b_n^+|^{3/2}} \sqrt{\sum_{\xi \in \partial b_n^+} |R_\lambda^{(n)}(\xi, x)|^2} \Big(1 + \frac{2^{n+2}}{3^2} + \frac{2^{n+4}}{3^4} + \cdots\Big)$$
$$\leq \frac{C 2^n}{A_1 A_2 \ldots A_n} \frac{\sqrt{\sum_{\xi \in \partial b_n^+} |R_\lambda^{(n)}(\xi, x)|^2}}{|\partial b_n^+|^{3/2}}$$

(where C is a constant) due to the facts that the number of paths with length $n+k$ is bounded by 2^{n+k} and

$$\frac{|\partial b_l^-|^2 |\partial b_l^+|^2 \delta_l}{\alpha_l} < \frac{1}{3}.$$

Similar estimates exist for the $\tilde{\gamma}$: $(0) \to (1) \to \cdots \to (n+1) \to (n)$ but with the last term being

$$\frac{\sqrt{\sum_{\xi \in \partial b_{n+1}^-} |R_\lambda^{(n)}(\xi, x)|^2}}{|\partial b_{n+1}^-|^{3/2}}.$$

Taking squares of the two estimations above and considering summation over $x \in (n)$, we get

$$\sum_{x \in (x)} R_{\lambda,n}^2(x_0, x) \leq \frac{C^2 4^n}{A_1^2 A_2^2 \ldots A_n^2} \Big(\frac{\sum_{\xi \in \partial b_n^+, x \in (x)} (R_\lambda^{(n)}(\xi, x))^2}{|\partial b_n^+|^3} + \frac{\sum_{\xi \in \partial b_{n+1}^-, x \in (x)} (R_\lambda^{(n)}(\xi, x))^2}{|\partial b_{n+1}^-|^3} \Big) \qquad (16)$$
$$\leq C_0 \frac{4^n |(n)| n^4}{A_1^2 A_2^2 \ldots A_n^2},$$

where C_0 is a constant. Since we can modify A_n by any constant factor $c \geq 3$ preserving the convergence of the series (14), we can neglect the factors 2^n and n^4 in (16).

If we have our estimates for $L_0(\lambda) = k \geq 1$, we can consider a new Hamiltonian $\widetilde{H}$ which has "very large potentials" inside the first $L_0(\lambda) = k$ main blocks and our initial potential $V(x, \omega)$ inside blocks (l) with $l \geq L_0(\lambda)$.

The operator $\widetilde{H}$ is a finite rank perturbation of H, which preserves the square integrability of the resolvent, see [5]. But for $\widetilde{H}$ we have desirable estimations for all $l \geq 0$. This completes the proof of Theorem III. □

The idea of a transition using a finite rank perturbation from a general $L_0(\lambda)$ to $L_0(\lambda) = 0$ is the same as in the proof of the absence of an a.c. spectrum (Theorems I and II).

Corollary 6.5. *Assume that one can find a sequence of belts $\{b_l; l \geq 1\}$ with $|b_l| \leq M$ for some constant M and the corresponding main blocks (n) satisfy $|(n)| \leq C_1^n$ for some constant $C_1 > 1$. Then the condition $\sum_l \delta_l < \infty$ is sufficient for the square integrability of $R_\lambda(x_0, \cdot)$ for a.e. $\lambda \in R^1$.*

Corollary 6.5 implies the 1-D localization theorem in [23].

7. Examples

We will illustrate Theorem III with several examples. In all of those examples, the belts will be "relatively short." The belt factors in these examples will be balanced by large values of the potential on the belts.

Example 7.1 (Localization on Sierpiński lattice S^∞). Let's start from the fractal (nested) lattice and consider as a typical example the Sierpiński Lattice S^∞. Let S^n be the part of S^∞ with vertices $\vec{0}$, $2^n\vec{\imath}$, and $2^n\vec{w}$. The volume of S^n is given as

$$|S^n| = \frac{3^{n+1} + 3}{2}.$$

See Figure 1.

Consider the Anderson Hamiltonian $H = \Delta + V(\vec{x}, \omega)$, where $\vec{x} \in S^\infty$ and $V(\cdot, \omega)$ are unbounded i.i.d. random variables with bounded density function $f(x)$ on R. Consider for fixed A the sequence of independent events

$$B_{A,n} = \{|V(2^n\vec{\imath}, \omega)| > A,\ |V(2^n\vec{w}, \omega)| > A\},$$

where $\vec{\imath} = (1, 0)$ and $\vec{w} = (1/2, \sqrt{3}/2)$. Then

$$P(B_{A,n}) = p^2(A) = \left(\int_{|x|>A} f(x)dx\right)^2.$$

Let τ_A be the moment of the first occurrence of $B_{A,n}$ in the sequence $B_{A,0}$, $B_{A,1}, \ldots$. Then τ_A has a geometric distribution

$$P(\tau_A = k) = (1 - p^2(A))^{k-1} p^2(A) \quad \text{for} k = 1, 2, \ldots,$$

with

$$E\tau_A = \frac{1}{p^2(A)}.$$

It is easy to see that

$$p^2(A)\tau_A \xrightarrow{\text{law}} \text{Exp}(1) \quad \text{as } A \to \infty.$$

Consider the increasing sequence $\{A_n = n\}: n \to \infty$ and the moments τ_n. Since

$$\sum_n P(p^2(n)\tau_n > (1+\epsilon)\ln n) \le \sum_n \frac{c_0}{n^{1+\epsilon}} < \infty$$

for some constant c_0 and any $\epsilon > 0$, we have (due to the Borel–Cantelli lemma) P-a.s.

$$\tau_n \le \frac{(1+\epsilon)\ln n}{p^2(n)}, \quad \text{for } n \ge n_0(\omega).$$

The successive belts $b_n, n \ge 1$ contain the pairs of points $\{2^{\tau_{A_n}}\vec{\imath}, 2^{\tau_{A_n}}\vec{w}\}$. Of course, we have

$$|(n)| \le c_1 3^{\tau_n} \le c_1 \exp\Big((1+\epsilon)\ln 3 \frac{\ln n}{p^2(n)}\Big) < \infty$$

and for fixed energy interval I,

$$\beta_\lambda^{b_k} \le \frac{c_2(I)}{n},$$

where c_1, c_2 are some constants.

Assume that $P\{|V(\cdot)| > A\} = p(A) \ge \frac{c_3}{A^\theta}$ for any $A > 0$, where c_3 is a constant. Then Corollary 6.5 provides the P-a.s. localization with certainty if $\theta < \frac{1}{2}$, i.e.

$$|(n)| \le \exp((1+\epsilon)n^{2\theta}\ln n)$$

for some $\epsilon > 0$.

Theorem 7.2. *Condition*

$$P\{|V(\cdot)| > A\} = p(A) \ge \frac{c}{A^\theta}, \quad \theta < \frac{1}{2},$$

is sufficient for P-a.s. localization on S^∞.

The same proof works for all nested fractal lattices.

Let's stress that we didn't use here the fundamental properties of self-similarity of S^∞. The spectral analysis of the Laplacian on S^∞ can provide much better localization results and cover the case when $V(\cdot,\omega)$ has "light" tails. We will return to this subject in other publications and prove localization for cases when $(A) \ge \frac{C}{A^\theta}$ for any $\theta > 0$.

Example 7.3. Consider the Quasi-1 dimensional tree as shown in Figure 2, denoted by T. The set of vertices is

$$\{\vec{x} = (x_1, x_2) \colon x_1, x_2 \text{ are nonnegative integers}\} \cup \{(-1, 0)\}.$$

Consider the Anderson Hamiltonian $H = \Delta + V(\vec{x}, \omega)$ on T, where the $V(\cdot, \omega)$ are i.i.d. random variables with density $f(x)$ such that

$$P(V(\vec{x}; w) > A) = \int_A^\infty f(x)dx = p(A) > 0 \quad \text{for all } A \in R.$$

For a fixed energy interval I, let's select a constant A such that $\left|\frac{\lambda}{A}\right| \leq \frac{1}{2}$, $\lambda \in I$ and introduce the following points on the $x - axis$ and on the vertical lines $\{(x, y) \colon y > 0\}$ for positive integers x. Put

$$\begin{aligned}
\tau_1 &= \min\{x_1 > 0 \colon |V(x_1, 0, \omega)| > A\}, \\
\tau_2 &= \min\{x_1 \colon |V(x_1, 0, \omega)| > A, |V(x_1 + 1, 0, \omega)| > A\}, \\
&\vdots \\
\tau_n &= \min\{x_1 > \tau_{n-1} \colon |V(x_1, 0, \omega)| > A, \ldots, |V(x_1 + n - 1, 0, \omega)| > A\}, \\
&\vdots
\end{aligned}$$

Similarly, for fixed x, on the vertical line $\{(x, y) \colon y > 0\}$, we define

$$\begin{aligned}
\tau_{x,1} &= \min\{y > 0 \colon |V(x, y, \omega)| > A\}, \\
\tau_{x,2} &= \min\{y > \tau_{x,1} \colon |V(x, y, \omega)| > A, |V(x, y + 1, \omega)| > A\}, \\
&\vdots \\
\tau_{x,n} &= \min\{y > \tau_{x,n-1} \colon |V(x, y, \omega)| > A, \ldots, |V(x, y + n - 1, \omega)| > A\}, \\
&\vdots
\end{aligned}$$

The random variables

$$\tau_1, \tau_2 - \tau_1, \ldots, \tau_n - \tau_{n-1}, \ldots,$$

$$\tau_{x,1}, \tau_{x,2} - \tau_{x,1}, \ldots, \tau_{x,n} - \tau_{x,n-1}, \ldots, \qquad x = 0, 1, 2, \ldots$$

are independent. As is easy to see, $\tau_n - \tau_{n-1}$ or $\tau_{x,n} - \tau_{x,n-1}$ are majorated by the random variable $n\theta_n^*$ or $n\theta_{x,n}^*$, where θ_n^* and $\theta_{x,n}^*$ are geometrically distributed with parameter $p^n(A)$. As in the previous example $p^n(A)\theta_n^* \to \text{Exp}(1)$. The Borel–Cantelli lemma gives P-a.s.

$$\theta_n^* \leq \frac{(1 + \epsilon) \ln n}{p^n(A)}, \quad n \geq n_0(\omega)$$

i.e.

$$\tau_n - \tau_{n-1} \le \frac{(1+\epsilon)n \ln n}{p^n(A)}, \quad n \ge n_0(\omega)$$

The same calculations show that $p^n(A)\theta^*_{x,n} \le (1+\epsilon)(\ln n + \ln x)$ except for finitely many pairs (x, n), i.e.

$$\tau_{x,n} \le \frac{(1+\epsilon)\ln(nx)}{p^n(A)}, \quad x + n \ge n_1(\omega).$$

The belt b_n consists of the points $\{(x_1, 0), \tau_n \le x_1 \le \tau_n + n - 1\}$ on the x-axis and for any fixed x the points $\{(x, y), \tau_{x,n} \le y \le \tau_{x,n} + n - 1\}$. As a result,

$$\begin{aligned} |(n)| &\le \frac{(1+\epsilon)n \ln n}{p^n(A)} \frac{(1+\epsilon)n \ln\left(n \cdot \frac{(1+\epsilon)n \ln n}{p^n(A)}\right)}{p^n(A)} \\ &\le \frac{c(A)n^3 \ln n}{p^{2n}(A)} \\ &\le c(A)n^3 \ln n e^{\vartheta(A)n} \end{aligned}$$

for some $c(A), \vartheta(A) > 0$. Also we have

$$\beta_\lambda^{b_n} \le \left(\frac{1}{2}\right)^n$$

and

$$\left(\prod_{k=1}^{n} \beta_\lambda^{b_k}\right)^2 \le e^{-\vartheta n^2},$$

for some $\vartheta > 0$.

Applying the general result of Theorem III, we get the following result.

Theorem 7.4. *Consider the Anderson Hamiltonian on the graph T (see Figure 2), where the $V(\vec{x}, \omega)$ are i.i.d. random variables with bounded distribution density $f(v)$ such that*

$$\int_{|v|>A} f(v)dv = p(A) > 0 \quad \textit{for any } A > 0.$$

The spectrum of H is pure point with probability 1.

Remark 7. One can prove that the spectrum of the pure Laplacian Δ on the graph T is a.c.

Remark 8. The Hausdorff dimension of the graph T equals 2: it is simply the lattice Z^2 after removing some edges.

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