

Quantum Cayley graphs for free groups

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Abstract. Differential operators $\Delta + q$ are considered on metric Cayley graphs of the finitely generated free groups \mathbb{F}_M . The function q and the graph edge lengths may vary with the M edge types. Using novel methods, a set of M multipliers $\mu_m(\lambda)$ depending on the spectral parameter is found. These multipliers are used to construct the resolvent and characterize the spectrum.

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1. Introduction

The interplay between a group action and the spectral analysis of a differential operator invariant under the action is a popular theme in analysis. If the group acts on a metric graph, the operators $-D^2 + q$ with invariant q are obvious candidates for a spectral theoretic analysis. This work treats operators $-D^2 + q$ on a metric Cayley graph \mathcal{T}_M of the nonabelian free group \mathbb{F}_M on M generators. These Cayley graphs are regular trees, with each vertex having degree $2M$. In the present work the M edge types of \mathcal{T}_M associated to the generators of \mathbb{F}_M may have different lengths, with even functions q varying over the M edge types. Remarkably, novel

techniques show that there is a system of M multipliers $\mu_m(\lambda)$, resembling those of Hill's equation [20], which can be used to construct the resolvents of the operators. Echoing the Hill's equation analysis, the location of the spectrum is encoded in the behavior of the multipliers on the real axis.

There is a large literature treating various aspects of analysis on symmetric infinite graphs. Homogeneous trees were considered as discrete graphs in [6]. Anisotropic random walks on homogeneous trees received an extensive treatment in [11]. The quantum graph spectral theory of $-D^2 + q$ on homogeneous trees was studied in [7], assuming that each edge had length 1, and that $q(x)$ was the same even function on each edge. These assumptions meant that the graph admitted radial functions, a structure which facilitated a Hill's equation type analysis of the spectral theory. The spectral theory of radial tree graphs was considered in [8] and [23]. A sampling of work exploiting this structure includes [5, 10, 12, 16, 26]. Certain physical models can also lead to graphs of lattices in Euclidean space where the group (e.g. \mathbb{Z}^2) is abelian [19, 22]. When graphs have symmetric trees as their universal covering spaces, there are explicit connections linking the spectral theories of their differential or difference operators. These connections, developed for manifolds in [2, 9], are used in [6, 7]. There is extensive related work in [27, 28].

The quantum Cayley graph analysis begins in the second section with a review of quantum graphs and the definition of the self-adjoint Hilbert space operator $\Delta + q$ which acts by sending f in its domain to $-D^2 f + qf$. The third section reviews basic material on Cayley graphs, and in particular the Cayley graphs \mathcal{T}_M of the free groups \mathbb{F}_M . For each edge e of the Cayley graph and each $\lambda \in \mathbb{C} \setminus [0, \infty)$, a combination of operator theoretic and differential equations arguments identifies a one dimensional space of 'exponential type' functions which are initially defined on half of \mathcal{T}_M . The translational action of generators of \mathbb{F}_M on subtrees of \mathcal{T}_M induces linear maps on the one-dimensional spaces of exponential functions, thus producing multipliers $\mu_m(\lambda)$ for $m = 1, \dots, M$. A square integrability condition shows that $|\mu_m(\lambda)| \leq 1$ for all $\lambda \in \mathbb{C} \setminus [0, \infty)$.

The fourth section starts by linking the multipliers and rather explicit formulas for the resolvent of $\Delta + q$. Recall that the multipliers for the classical Hill's equation satisfy quadratic polynomial equations with coefficients which are entire functions of the spectral parameter λ . In this work the multipliers $\mu_m(\lambda)$ satisfy a coupled system of quadratic equations with coefficients that are entire functions of λ . An elimination procedure shows that the equations can be decoupled, leading to higher order polynomial equations with entire coefficients for individual multipliers $\mu_m(\lambda)$. The multipliers $\mu_m(\lambda)$ have extensions from above and below

to real σ . The extension is generally holomorphic, but as in the classical Hill's equation the difference $\delta_m(\sigma)$ of the limits from above and below can be nonzero. Except for a discrete set, the spectrum of $\Delta + q$ is characterized by the condition $\delta_m(\sigma) \neq 0$ for some m .

In the final section the system of multiplier equations is explicitly decoupled for the case $M = 2$. Computer based calculations are used to generate several spectral plots.

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2. Quantum graphs

Suppose Γ is a locally finite graph with a finite or countably infinite vertex set \mathcal{V} and a directed edge set \mathcal{E} . In the usual manner of metric graph construction [4], a collection of intervals $\{[0, l_e], e \in \mathcal{E}\}$ is indexed by the graph edges. Consistent with the directions of the graph edges (v, w) , the initial endpoint v is associated with 0, and w is associated with l_e . Assume that each unordered pair of distinct vertices is joined by at most one edge. As a result, the map from the directed graph to the undirected graph which simply replaces a directed edge (u, v) with an undirected edge $[u, v]$ is one-to-one on the edges. A topological graph results from the identification of interval endpoints associated to a common vertex.

The Euclidean metric on the intervals is extended to a metric on this topological graph by defining the length of a path joining two points to be the sum of its (partial) edge lengths. The (geodesic) distance between two points is the infimum of the lengths of the paths joining them. The resulting metric graph will also be denoted Γ .

To extend the topological graph Γ to a quantum graph, function spaces and differential operators are included. A function $f: \Gamma \rightarrow \mathbb{C}$ has restrictions to components $f_e: [0, l_e] \rightarrow \mathbb{C}$. Let $L^2(\Gamma)$ denote the complex Hilbert space $\oplus_e L^2[0, l_e]$ with the inner product

$$\langle f, g \rangle = \int_{\Gamma} f \bar{g} = \sum_e \int_0^{l_e} f_e(x) \overline{g_e(x)} dx.$$

Given a bounded real-valued function q on Γ , measurable on each edge, differential operators $-D^2 + q$ are defined to act component-wise on functions $f \in L^2(\Gamma)$ in their domains. The functions q are also assumed to be even on each edge, $q_e(l_e - x) = q_e(x)$. This assumption plays an important role as the analysis becomes more detailed.

Self-adjoint operators acting by $-D^2 + q$ can be defined using standard vertex conditions. The construction of the operator begins with a domain \mathcal{D}_{com} of compactly supported continuous functions $f \in L^2(\Gamma)$ such that f'_e is absolutely continuous on each edge e , and $f''_e \in L^2[0, l_e]$. In addition, functions in \mathcal{D}_{com} are required to be continuous at the graph vertices, and to satisfy the derivative condition

$$\sum_{e \sim v} \partial_v f_e(v) = 0, \tag{2.1}$$

where $e \sim v$ means the edge e is incident on the vertex v , and $\partial_v = \partial/\partial x$ in outward pointing local coordinates.

Since the addition of a constant will make the potential nonnegative, but have only a trivial effect on the spectral theory, the assumption

$$q \geq 0 \tag{2.2}$$

is made for convenience. With the domain \mathcal{D}_{com} , the operators $-D^2 + q$ are symmetric and bounded below, with quadratic form

$$\langle (-D^2 + q)f, f \rangle = \int_{\Gamma} |f'|^2 + q|f|^2. \tag{2.3}$$

These operators always have a self-adjoint Friedrich’s extension, denoted $\Delta + q$, whose spectrum is a subset of $[0, \infty)$. When the edge lengths of Γ have a positive lower bound the Friedrich’s extension is the unique self adjoint extension [4, p. 30].

Say that an edge $e = (v_-, v_+) \in \mathcal{E}$ of a connected graph Γ is a bridge if the removal of (the interior of) e separates the graph into two disjoint connected subgraphs. If e is a bridge, let Γ_e^+ denote the closure of the connected component of $\Gamma \setminus v_-$ which contains the vertex v_+ . That is, Γ_e^+ consists of the v_+ side of Γ together with the edge e , including the vertex v_- . Γ_e^- may be defined similarly.

For $\lambda \in \mathbb{C} \setminus [0, \infty)$, the resolvents $R(\lambda) = (\Delta + q - \lambda I)^{-1}$ of the self adjoint operators $\Delta + q$ provide special solutions of $-y'' + qy = \lambda y$ on Γ_e^\pm . Let \mathbb{X}_e^\pm denote the space of functions $y_\pm: \Gamma_e^\pm \rightarrow \mathbb{C}$ which (i) satisfy

$$-y'' + qy = \lambda y \tag{2.4}$$

on each edge $e \in \Gamma_\epsilon^\pm$, (ii) are continuous and square integrable on Γ_ϵ^\pm , and (iii) which satisfy the derivative conditions (2.1) at each vertex of Γ_ϵ^\pm except possibly v_\mp .

Lemma 2.1. *Suppose e is a bridge of the connected graph Γ , and $\lambda \in \mathbb{C} \setminus [0, \infty)$. Then \mathbb{X}_e^\pm is one dimensional.*

Proof. Suppose two linearly independent functions g_1, g_2 on Γ_e^+ satisfy (i)–(iii). Then a nontrivial linear combination $h = \alpha_1 g_1 + \alpha_2 g_2$ would satisfy $h(v_-) = 0$. Consider altering the domain of $\Delta + q$ by replacing the vertex conditions at v_- by the Dirichlet condition $f(v_-) = 0$ for each edge incident on v_- . The resulting operator is still self-adjoint and nonnegative on $L^2(\Gamma)$, and restricts to a self-adjoint operator on $L^2(\Gamma_e^+)$. Let $R_D(\lambda)$ denote its resolvent. The function h is then a square integrable eigenfunction with eigenvalue λ , which is impossible. A similar argument applies to Γ_e^- . Thus \mathbb{X}_e^\pm is at most one dimensional.

Let z be the nontrivial solution of the equation

$$-z'' + qz = \bar{\lambda}z, \quad z(0) = 0, \quad z'(0) = 1, \tag{2.5}$$

on the interval $[0, l_e]$. Extend z by 0 to obtain an element of $L^2(\Gamma_e^+)$. Using the Dirichlet resolvent above, define $f = R_D(\lambda)z$. Note that $f(v_-) = 0$. Integration by parts gives

$$0 = \int_0^{l_e} f[-z'' + qz - \bar{\lambda}z] dx = [-fz' + f'z] \Big|_0^{l_e} + \int_0^{l_e} |z|^2 dx.$$

Since the boundary terms vanish at 0, either $f(v_+) \neq 0$ or $f'(v_+) \neq 0$.

Since z vanishes outside of e , the function $f = R_D(\lambda)z$ satisfies $-f'' + qf = \lambda f$ on each edge of Γ_e^+ other than e . Define $g \in \mathbb{X}_e^+$ which agrees with f on $\Gamma_e^+ \setminus e$, but which satisfies (2.4) on e , with initial data $g(v_+) = f(v_+)$ and with $\partial_v g(v_+)$ chosen so the derivative conditions (2.1) at v_+ are satisfied for g . The space \mathbb{X}_e^+ is then the span of g , and the case of \mathbb{X}_e^- is similar. \square

The construction of Lemma 2.1 also provides the next result.

Lemma 2.2. *Suppose e is a bridge of the connected graph Γ , and $\lambda \in \mathbb{C} \setminus [0, \infty)$. A basis $g(\lambda)$ of \mathbb{X}_e^\pm may be chosen holomorphically in an open disc centered at λ , and real valued if $\lambda \in (-\infty, 0)$. If $x \in \Gamma_e^\pm$ then $g(x, \lambda)$ is holomorphic.*

Proof. For $\lambda \in \mathbb{C} \setminus [0, \infty)$ the resolvent $R(\lambda)$ is a holomorphic operator valued function, so the functions $f = R(\lambda)z$ are holomorphic with values in the domain of $\Delta + q$. For $x \in \Gamma^\pm \setminus e$ the evaluations $f(x), f'(x)$ are continuous functionals [18, p. 191-194] on the domain of $\Delta + q$, so the values $g(v_+)$ and $\partial_v g_i(v_+)$ from Lemma 2.1 are holomorphic, as are the $L^2(\Gamma^\pm)$ functions $g(\lambda)$ and the values $g(x, \lambda)$ for $x \in e$. All of these functions can be chosen to be real valued if $\lambda \in (-\infty, 0)$. \square

3. Quantum Cayley graphs

3.1. General remarks. Suppose \mathbb{G} is a finitely generated group with identity ι . Let $\mathbb{S} \subset \mathbb{G}$ be a finite generating set for \mathbb{G} , meaning that every element of \mathbb{G} can be expressed as a product of elements of \mathbb{S} and their inverses. Following [21], the Cayley graph $\Gamma_{\mathbb{G},\mathbb{S}}$ for the group \mathbb{G} with generating set \mathbb{S} is the directed graph whose vertex set \mathcal{V} is the set of elements of \mathbb{G} . The edge set of $\Gamma_{\mathbb{G},\mathbb{S}}$ is the set \mathcal{E} of directed edges (v, vs) with $s \in \mathbb{S}$, *initial vertex* $v \in \mathbb{G}$ and *terminal vertex* vs . When confusion is unlikely we will simply write Γ for $\Gamma_{\mathbb{G},\mathbb{S}}$. Assume that if $s \in \mathbb{S}$, then $s^{-1} \notin \mathbb{S}$. This condition avoids loops $(v, v\iota)$, and insures that at most one directed edge connects any (unordered) pair of vertices. We will often consider Γ to have undirected edges $[v, vs]$, with the directions given above available when needed.

\mathbb{G} acts transitively by left multiplication on the vertices of Γ ; that is, for every $v, w \in \mathcal{V}$ there is a $g \in \mathbb{G}$ such that $w = gv$. If $e = (v, vs) \in \mathcal{E}$, then $ge = (gv, gvs) \in \mathcal{E}$, so \mathbb{G} also acts on \mathcal{E} , although this action is not generally transitive. Say that two directed edges e_1, e_2 are equivalent if there is a $g \in \mathbb{G}$ such that $e_2 = ge_1$. The equivalence classes will be called edge orbits of the \mathbb{G} action on \mathcal{E} .

Proposition 3.1. *A set B of directed edges is an edge orbit if and only if there is a unique $s \in \mathbb{S}$ such that $B = \{(v, vs), v \in \mathbb{G}\}$. It follows that the number of edge orbits is the cardinality of \mathbb{S} .*

Proof. If $v, w \in \mathbb{G}$, then $wv^{-1}(v, vs) = (w, ws)$, so for a fixed $s \in \mathbb{S}$ all edges of the form (v, vs) are in the same orbit. If $g(v, vs_1) = (w, ws_2)$, then $gv = w$ and $gvs_1 = ws_2$, so $ws_1 = ws_2$ and $s_1 = s_2$. \square

Proposition 3.2. *If \mathbb{G} is generated by the finite set \mathbb{S} , then the undirected graph $\Gamma_{\mathbb{G},\mathbb{S}}$ is path connected.*

Proof. If our requirements on generating sets are momentarily relaxed and \mathbb{S} is extended to the set $\mathbb{S}_1 = \{s, s^{-1}, s \in \mathbb{S}\}$, then the Cayley graph $\Gamma_{\mathbb{G},\mathbb{S}_1}$ will have a directed path from every element of \mathbb{G} to ι . An edge of this graph has one of the forms (v, vs) or (v, vs^{-1}) . As an undirected edge, $[v, vs^{-1}] = [vs^{-1}, v] = [vs^{-1}, vs^{-1}s]$, so for every directed edge of $\Gamma_{\mathbb{G},\mathbb{S}_1}$ there is an undirected edge of $\Gamma_{\mathbb{G},\mathbb{S}}$ with the same vertices. Consequently, the undirected graph $\Gamma_{\mathbb{G},\mathbb{S}}$ is path connected. \square

Cayley graphs Γ can be linked with the spectral theory of differential operators. To maintain a strong connection with the group \mathbb{G} , the edges of Γ in the same \mathbb{G} orbit will have the same length. The action of \mathbb{G} on the combinatorial edges may then be extended to the edges of the metric graph by taking $x \in (0, l_{(v,vs)})$ to $gx = x \in (0, l_{(gv,gv_s)})$. This group action also provides a \mathbb{G} action on the functions f on Γ . The action simply moves the edge index, so that in terms of function components $gf_{ge}(x) = f_e(x)$. Functions are \mathbb{G} -invariant if $f_{ge}(x) = f_e(x)$ for all directed edges e and all $g \in \mathbb{G}$. A quantum Cayley graph will be a quantum graph whose underlying combinatorial graph is the Cayley graph of a finitely generated group, whose edge lengths are constant on edge orbits, and whose differential operator $\Delta + q$ commutes with the group action on functions. Since there is little chance of confusion, the same notation, e.g. Γ , will be used for the corresponding quantum, metric, and combinatorial graphs.

3.2. Free groups and their graphs \mathcal{T}_M . This work treats the Cayley graphs with $\mathbb{G} = \mathbb{F}_M$, the free group [21] with rank M . Recall that the elements of \mathbb{F}_M are equivalence classes of finite length words generated by M distinct symbols s_1, \dots, s_M and their inverse symbols $s_1^{-1}, \dots, s_M^{-1}$. Two words are equivalent if they have a common reduction achieved by removing adjacent symbol pairs $s_m s_m^{-1}$ or $s_m^{-1} s_m$. The group identity is the empty word class, the group product of words w_1, w_2 is the class of the concatenation $w_1 w_2$, and inverses are formed by using inverse symbols in reverse order, e.g. $(s_2 s_3^{-1} s_1)^{-1} = s_1^{-1} s_3 s_2^{-1}$.

Given a free group \mathbb{F}_M with generating set $\mathbb{S} = \{s_1, \dots, s_M\}$, let \mathcal{T}_M denote the corresponding Cayley graph. These (undirected) graphs (see Figure 1) have a simple structure [21, p. 56].

Proposition 3.3. *The undirected graph \mathcal{T}_M is a tree whose vertices have degree $2M$.*

Proof. Suppose \mathcal{T}_M had a cycle with distinct vertices w_1, \dots, w_K and edges $[w_K, w_1]$ and $[w_k, w_{k+1}]$ for $k = 1, \dots, K - 1$. In the undirected graph \mathcal{T}_M edges extend from w_k by some s_m or s_m^{-1} , so that $w_{k+1} = w_k s_m$ or $w_{k+1} = w_k s_m^{-1}$, and each vertex has degree $2M$. The sequence of visited vertices w_1, \dots, w_K, w_1 is described by a word of right multiplications by the generators and their inverse symbols equal to ι in \mathbb{F}_M . Since this word can be reduced to the empty word, it must have adjacent symbols $s_m s_m^{-1}$ or $s_m^{-1} s_m$. This means the vertices w_1, \dots, w_K are not distinct, so no such cycle exists. Since \mathcal{T}_M is connected by Proposition 3.2 and has no cycles, \mathcal{T}_M is a tree. □

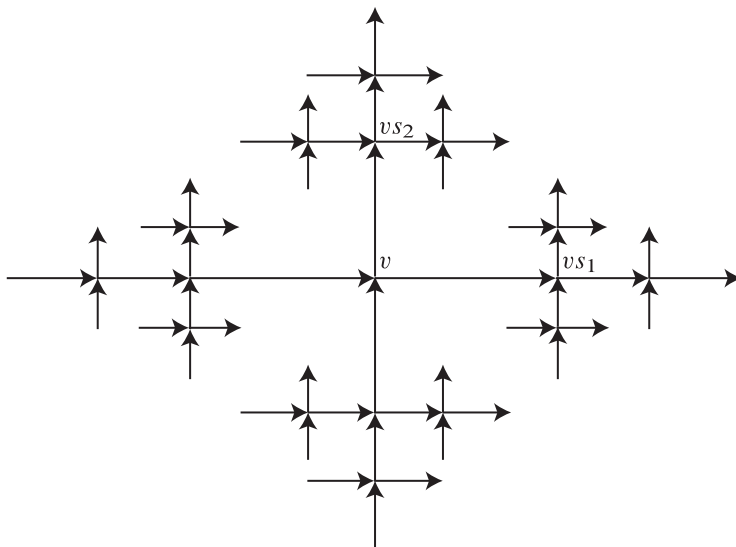


Figure 1. A rescaled graph \mathcal{T}_2 .

3.3. Abelian subgroups and multipliers for \mathcal{T}_M . Each edge $e = (v, vs) \in \mathcal{T}_M$ is a bridge. With $v = v_-$ and $vs = v_+$, the subgraphs Γ_e^\pm described above will be subtrees of \mathcal{T}_M , denoted by \mathcal{T}_e^\pm . The vector spaces \mathbb{X}_e^\pm are as in Lemma 2.1.

Lemma 3.4. *Suppose $e = (v, vs)$ is an edge of \mathcal{T}_M and $\lambda \in \mathbb{C} \setminus [0, \infty)$. If y^\pm is a nontrivial element of \mathbb{X}_e^\pm , then y^\pm is nowhere vanishing on \mathcal{T}_e^\pm .*

Proof. Suppose $y^\pm(x_0) = 0$ for some $x_0 \in \mathcal{T}_e^\pm$. First notice that y^\pm must then vanish identically on the subtree \mathcal{T}_0 consisting of points x_1 of \mathcal{T}_e^\pm with the property that paths from x_1 to v must include x_0 . Otherwise, a nonnegative self-adjoint operator $\Delta + q$ could be obtained on $L^2(\mathcal{T}_0)$ by using the boundary condition $f(x_0) = 0$. This operator would have a nontrivial square integrable eigenfunction, the restriction of y^\pm to \mathcal{T}_0 , with the eigenvalue $\lambda \in \mathbb{C} \setminus [0, \infty)$, which is impossible.

Since solutions of $-y'' + qy = \lambda y$ are identically zero on an edge e if $y(x_0) = y'(x_0) = 0$ for some $x_0 \in e$, we may assume x_0 is a vertex. Since the function y^\pm vanishes identically on \mathcal{T}_0 , the continuity and derivative conditions at x_0 force y to vanish on all the edges with x_0 as a vertex. The function y^\pm must now be identically zero on \mathcal{T}_e^\pm , contradicting the assumption that the function was nontrivial. □

The structure of the elements of \mathbb{X}_e^\pm is strongly constrained by the symmetries of \mathcal{T}_e^\pm combined with the fact that \mathbb{X}_e^\pm is one dimensional. A simple observation is the following.

Lemma 3.5. *Suppose $e_1 = (v_1, v_1 s_m)$, $g \in \mathbb{G}$, and $e_2 = (v_2, v_2 s_m) = (g v_1, g v_1 s_m)$. For $j = 1, 2$ let $y_j \in \mathbb{X}_{e_j}^\pm$ with $y_j(v_j) = 1$. Then $g y_1 = y_2$.*

Proof. The action by g is an isomorphism of $\mathcal{T}_{e_1}^\pm$ and $\mathcal{T}_{e_2}^\pm$. Since $\mathbb{X}_{e_2}^\pm$ is one dimensional, $g y_1$ is a scalar multiple of y_2 . These two functions agree at v_2 , so are equal. \square

For each vertex v and integers k , left multiplication by the abelian subgroup of elements $v s_m^k v^{-1}$ acts on \mathcal{T}_M . These maps carry the edge $e(0) = (v, v s_m)$ to the edges $e(k) = (v s_m^k, v s_m^{k+1})$. The key role of these group actions is related to the following geometric observation.

Lemma 3.6. *The trees $\mathcal{T}_{e(k)}^+$ are nested, with $\mathcal{T}_{e(k+1)}^+ \subset \mathcal{T}_{e(k)}^+$. In addition,*

$$\mathcal{T}_M = \bigcup_{k=0}^{\infty} \mathcal{T}_{e(-k)}^+.$$

Proof. Other than $v s_m^k$, the vertices of the trees $\mathcal{T}_{e(k)}^+$ are those elements of \mathbb{F}_M which have a representation $v s_m^k s_m g$, where $s_m g$ is a reduced word in \mathbb{F}_M . If $w = v s_m^{k+1} s_m g \in \mathcal{T}_{e(k+1)}^+$ is a vertex with $s_m g$ reduced, then $s_m^2 g$ is reduced and $w = v s_m^k s_m s_m g \in \mathcal{T}_{e(k)}^+$. Thus the trees $\mathcal{T}_{e(k)}^+$ are nested.

More generally, for any integer k , a word $w \in \mathbb{F}_M$ may be represented as $w = v s_m^k s_m g$ with $s_m g = s_m^{-k} v^{-1} w$. First take a reduced representative u for $v^{-1} w$. Suppose u begins on the left with a string s_m^j , followed by an element of \mathbb{S} different from $s_m^{\pm 1}$. Taking $k = j - 1$ gives the desired form, and every vertex w is in some $\mathcal{T}_{e(k)}^+$. \square

Suppose $y \in \mathbb{X}_{e(0)}^+$ satisfies $y(v) = 1$, and $z \in \mathbb{X}_{e(1)}^+$ satisfies $z(v s_m) = 1$. Since $\mathcal{T}_{e(1)}^+ \subset \mathcal{T}_{e(0)}^+$, the restriction of y to $\mathcal{T}_{e(1)}^+$ is an element of $\mathbb{X}_{e(1)}^+$. Because y is nonvanishing, there is a nonzero multiplier $\mu_m(\lambda) \in \mathbb{C}$ associated to each generator s_m such that $y = \mu_m(\lambda) z$ on $\mathcal{T}_{e(1)}^+$. In particular $\mu_m(\lambda) = y(l_m)/y(0)$.

Lemma 3.7. *The multipliers $\mu_m(\lambda)$ are holomorphic for $\lambda \in \mathbb{C} \setminus [0, \infty)$, with $\mu_m(\lambda) = \overline{\mu_m(\bar{\lambda})}$.*

Proof. By Lemma 2.2 the formula $\mu_m(\lambda) = y(l_m)/y(0)$ shows that $\mu_m(\lambda)$ is holomorphic when $\lambda \in \mathbb{C} \setminus [0, \infty)$. If $\lambda \in (-\infty, 0)$ and y is chosen real, then $\mu(\lambda)$ is real. The two functions $\mu_m(\lambda)$ and $\overline{\mu_m(\bar{\lambda})}$ are holomorphic and agree for $\lambda \in (-\infty, 0)$, so agree for all $\lambda \in \mathbb{C} \setminus [0, \infty)$. \square

Because the function q is even on each edge, that is $q(l_m - x) = q(x)$, the same multipliers will arise when comparing elements of \mathbb{X}_e^- if the edge directions are reversed by using the generators s_m^{-1} of \mathbb{F}_M instead of s_m . These multipliers provide a global extension of functions in $\mathbb{X}_{e(0)}^\pm$.

Lemma 3.8. *Suppose $\mathcal{T}_{e(0)}^+ \subset \mathcal{T}_{e(j)}^+ \subset \mathcal{T}_{e(k)}^+$. If $y_j \in \mathbb{X}_{e(j)}^+$ with $y_j(v) = 1$, and $y_k \in \mathbb{X}_{e(k)}^+$ with $y_k(v) = 1$, then $y_j = y_k$ on $\mathcal{T}_{e(j)}$. Elements y^\pm of $\mathbb{X}_{e(0)}^\pm$ may be extended via the multipliers to functions defined on all of \mathcal{T}_M .*

Proof. The function y_k restricts to an element of $\mathbb{X}_{e(j)}^+$. Since nontrivial elements of $\mathbb{X}_{e(j)}^+$ never vanish, but $y_k(v) - y_j(v) = 0$, the difference is the zero element of $\mathbb{X}_{e(j)}^+$. Since these extensions of $\mathbb{X}_{e(0)}^+$ are consistent, elements y^\pm of $\mathbb{X}_{e(0)}^\pm$ extend via the multipliers to functions defined on all of the trees $\mathcal{T}_{e(k)}^+$. □

Lemma 3.9. *For $\lambda \in \mathbb{C} \setminus [0, \infty)$, the multipliers satisfy $|\mu_m(\lambda)| < 1$.*

Proof. Recall that y is nowhere vanishing, so

$$\int_v^{vs_m} |y|^2 \neq 0.$$

A nontrivial element y of \mathbb{X}_e^\pm is square integrable on \mathcal{T}_e^+ , so in particular

$$\sum_{k=0}^\infty \int_{vs_m^k}^{vs_m^{k+1}} |y|^2 = \int_v^{vs_m} |y|^2 \sum_{k=0}^\infty |\mu_m(\lambda)|^{2k} < \infty,$$

and $|\mu_m(\lambda)| < 1$. □

4. Analysis of the multipliers

On each edge $[0, l_m]$ the space of solutions to the eigenvalue equation (2.4) has a basis $C_m(x, \lambda), S_m(x, \lambda)$ satisfying $C_m(0, \lambda) = 1 = S'_m(0, \lambda)$ and $S_m(0, \lambda) = 0 = C'_m(0, \lambda)$. These solutions satisfy the Wronskian identity

$$C_m(x, \lambda)S'_m(x, \lambda) - C'_m(x, \lambda)S_m(x, \lambda) = 1. \tag{4.1}$$

If $q = 0$ and $\omega = \sqrt{\lambda}$, these functions are simply $C_m(x, \lambda) = \cos(\omega x)$, $S_m(x, \lambda) = \sin(\omega x)/\omega$.

If $S_m(l_m, \lambda) = 0$ then λ is an eigenvalue for a classical Sturm–Liouville problem, implying $\lambda \in [0, \infty)$. For $\lambda \in \mathbb{C} \setminus [0, \infty)$ there is a unique solution of (2.4) with boundary values $y_m(0, \lambda) = \alpha$, $y_m(l_m, \lambda) = \beta$ given by

$$y_m(x, \lambda) = \alpha C_m(x, \lambda) + \frac{\beta - \alpha C_m(l_m, \lambda)}{S_m(l_m, \lambda)} S_m(x, \lambda). \tag{4.2}$$

Because $q_m(x) = q_m(l_m - x)$ for each edge, there is an identity

$$C_m(l_m - x, \lambda) = S'_m(l_m, \lambda) C_m(x, \lambda) - C'_m(l_m, \lambda) S_m(x, \lambda),$$

since both sides of the equation are solutions of (2.4) with the same initial data at $x = l_m$. Setting $x = 0$ leads to the identity

$$C_m(l_m, \lambda) = S'_m(l_m, \lambda). \tag{4.3}$$

In addition to the coordinates originally given to the edges of \mathcal{T}_M , it will be helpful to also consider local coordinates for \mathcal{T}_e^+ which identify edges with the same intervals $[0, l_k]$, but with the local coordinate increasing with distance from a given vertex v . Since q is assumed even on each edge, the operators $\Delta + q$ are unchanged despite the coordinate change.

4.1. Multipliers and the resolvent. The next results show that edges in the same orbit have the same multipliers.

Theorem 4.1. *Assume $e = (v, vs_m)$, $\lambda \in \mathbb{C} \setminus [0, \infty)$, and $y \in \mathbb{X}_e^+$ with $y(v) = 1$. Suppose the edge ϵ in \mathcal{T}_e^+ is in the same edge orbit as e , with the local coordinate for ϵ increasing with the distance from v . Using the identifications of e and ϵ with $[0, l_m]$, the restriction y_1 of y to ϵ satisfies*

$$\frac{y_1(l_m)}{y_1(0)} = \frac{y(l_m)}{y(0)} = \mu_m(\lambda).$$

Proof. If w is the vertex of the edge ϵ closest to v (see Figure 2), then ϵ has one of the forms (w, ws_m) or (ws_m^{-1}, w) . In the first case, where $wv^{-1}e = \epsilon$, the tree \mathcal{T}_e^+ is a subtree of \mathcal{T}_e^+ , and translation by wv^{-1} carries \mathbb{X}_e^+ to \mathbb{X}_ϵ^+ . As functions on $[0, l_m]$, y_1 is a nonzero multiple of y since \mathbb{X}_e^+ and \mathbb{X}_ϵ^+ are one dimensional.

In the second case, when $\epsilon = (ws_m^{-1}, w)$, the tree \mathcal{T}_e^+ is generally not a subtree of \mathcal{T}_e^+ , but $\mathcal{T}_{(ws_m^{-1}, w)}^-$ is. A different argument will reduce the second case to the first. As undirected graphs there are isomorphisms between the trees $\mathcal{T}_{(w, ws_m)}^+$ and $\mathcal{T}_{(ws_m^{-1}, w)}^-$. One such is obtained by interchanging the roles of s_m and s_m^{-1} . There is a corresponding involution of \mathbb{X}_e^+ obtained by interchanging function values on

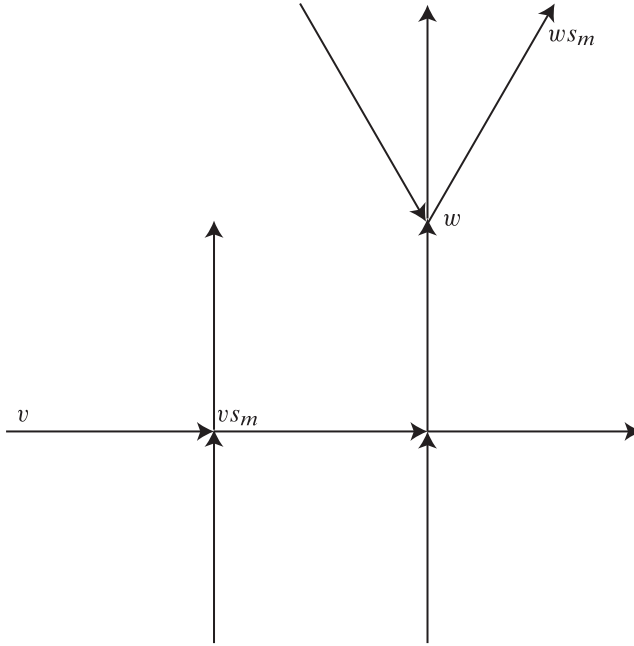


Figure 2. Edges in a common orbit.

the isomorphic trees. Since \mathbb{X}_e^+ is one dimensional, this involution is given by a constant factor. The nonzero value of y at the vertex w is fixed by the involution, so the tree interchange must leave the functions fixed. \square

Corollary 4.2. *Assume $e = (v, vs_m)$, $\lambda \in \mathbb{C} \setminus [0, \infty)$, and $y \in \mathbb{X}_e^+$ with $y(v) = 1$. Suppose that for $j = 1, 2$, the edges $\epsilon_j = (w_j, w_js_k)$ in \mathcal{T}_e^+ are in the same edge orbit, with the local coordinates for ϵ_j increasing with the distance from v . The restrictions y_j of y to ϵ_j satisfy*

$$\frac{y_1(l_m)}{y_1(0)} = \frac{y_2(l_m)}{y_2(0)} = \mu_k(\lambda).$$

Proof. If $e_1 = (vs_k^{-1}, v)$, then \mathcal{T}_e^+ is a subtree of $\mathcal{T}_{e_1}^+$. Since e_1 and ϵ_j lie in the same edge orbit, the previous theorem may now be applied. \square

As a consequence of Theorem 4.1 and Corollary 4.2 the functions $y \in \mathbb{X}_e^+$ have the following description.

Theorem 4.3. Assume $\lambda \in \mathbb{C} \setminus [0, \infty)$, $e = (v, vs_m)$ and $y_+ \in \mathbb{X}_e^+$ with $y_+(v) = 1$. Suppose w is a vertex in \mathcal{T}_e^+ , and the path from v to w is given by the reduced word $s_m s_{k(1)}^{\pm 1} \dots s_{k(n)}^{\pm 1}$. Then

$$y_+(w) = \mu_m \mu_{k(1)} \dots \mu_{k(n)}. \tag{4.4}$$

Using (4.2), the vertex values of y_+ can be interpolated to the edges.

Because the functions in \mathbb{X}_e^+ are continuous, the multipliers $\mu_k(\lambda)$ are simply the value at l_k of a solution y_k in \mathbb{X}_e^+ with initial value 1 at $x = 0$ on edges of type k . That is,

$$\mu_k(\lambda) = C_k(l_k, \lambda) + y'_k(0)S_k(l_k, \lambda). \tag{4.5}$$

Theorem 4.3 may also be used to describe the functions $y_- \in \mathbb{X}_e^-$. The functions y_+, y_- can be used to construct the resolvent $R(\lambda) = [\Delta + q - \lambda]^{-1}$ on $\mathbb{C} \setminus [0, \infty)$. This construction begins with consideration of the Wronskian $W_k(\lambda) = y_+ y'_- - y'_+ y_-$.

If the (nonvanishing) functions y_- and y_+ were linearly dependent on e , then there would be a nonzero constant c such that $y_-(x) = c y_+(x)$ for $x \in e$, and the function

$$\begin{cases} y_-(x), & x \in \mathcal{T}_e^- \\ c y_+(x), & x \in \mathcal{T}_e^+ \end{cases}$$

would be a square integrable eigenfunction for $\Delta + q$. Consequently, the functions y_- and y_+ must be linearly independent on e if $\lambda \in \mathbb{C} \setminus [0, \infty)$.

In particular for each $\lambda \in \mathbb{C} \setminus [0, \infty)$ the Wronskian $W_k(\lambda) = y_+ y'_- - y'_+ y_-$ for edges of each type k is nonzero, and independent of x . By using (4.5) the Wronskian $W_k(\lambda)$ can be expressed in terms of the multipliers. Consider evaluation of $W_k(\lambda)$ at $x = l_k$. Compared to y_+ , which satisfies (4.5), with $y_+(0, \lambda) = 1$, the function y_- would have the edge direction reversed. This function has $y_-(l_k) = 1$, and because of the reversed edge direction,

$$y_-(x, \lambda) = C_k(l_k - x, \lambda) - y'_-(l_k)S_k(l_k - x, \lambda),$$

so that

$$\mu_k(\lambda) = y_-(0, \lambda) = C_k(l_k, \lambda) - y'_-(l_k)S_k(l_k, \lambda).$$

Evaluation at $x = l_k$ gives

$$\begin{aligned} W_k(\lambda) &= (y_+ y'_- - y'_+ y_-)(l_k) \\ &= \mu_k(\lambda) \frac{C_k(l_k, \lambda) - \mu_k(\lambda)}{S_k(l_k, \lambda)} - \left[C'_k(l_k, \lambda) + \frac{\mu_k(\lambda) - C_k(l_k, \lambda)}{S_k(l_k, \lambda)} S'_k(l_k, \lambda) \right] \end{aligned}$$

and the identities (4.1) and (4.3) give the simplification

$$W_k(\lambda) = \frac{1 - \mu_k^2(\lambda)}{S_k(l_k, \lambda)}. \tag{4.6}$$

The resolvent $R(\lambda) = [\Delta + q - \lambda]^{-1}$ has a simple representation when it acts on functions f_e with support in a single edge e . Assuming that $t \in e$, and using the identification of e with $[0, l_e]$, interpret the inequality $x \leq t$ to mean $x \in \mathcal{T}_e^-$, and $x \leq t$ with the usual meaning if $x \in e$. Interpret $x \geq t$ similarly when $x \in \mathcal{T}_e^+$. For $\lambda \in \mathbb{C} \setminus [0, \infty)$ and $t \in e$, define the (variation of parameters) kernel

$$R_e(x, t, \lambda) = \begin{cases} y_-(x, \lambda)y_+(t, \lambda)/W_k(\lambda), & x \leq t \leq l_e, \\ y_-(t, \lambda)y_+(x, \lambda)/W_k(\lambda), & 0 \leq t \leq x. \end{cases} \tag{4.7}$$

If f_e has support in the interior of e , the function

$$h_e(x) = \int_0^{l_e} R_e(x, t, \lambda) f_e(t) dt$$

satisfies the equation $[\Delta + q - \lambda]h_e = f_e$, the continuity and derivative vertex conditions (2.1), and is square integrable on \mathcal{T}_M . Thus $h_e = R(\lambda)f_e$, the image of the resolvent acting on f_e . Since the linear span of functions f_e is dense in $L^2(\mathcal{T}_M)$, and the resolvent is a bounded operator for $\lambda \in \mathbb{C} \setminus [0, \infty)$, this discussion implies the next result.

Theorem 4.4. For $\lambda \in \mathbb{C} \setminus [0, \infty)$,

$$R(\lambda)f = \sum_e \int_0^{l_e} R_e(x, t, \lambda) f_e(t) dt, \quad f \in L^2(\mathcal{T}_M),$$

the sum converging in $L^2(\mathcal{T}_M)$.

4.2. Equations for the multipliers

Theorem 4.5. For $\lambda \in \mathbb{C} \setminus [0, \infty)$ and $m = 1, \dots, M$, the multipliers $\mu_m(\lambda)$ satisfy the system of equations

$$\frac{\mu_m^2(\lambda) - 1}{S_m(l_m, \lambda)\mu_m(\lambda)} - 2 \sum_{k=1}^M \frac{\mu_k(\lambda) - C_k(l_k, \lambda)}{S_k(l_k, \lambda)} = 0. \tag{4.8}$$

Proof. Suppose $e = (v, vs_m)$ is an edge of type m . The vertex $w = vs_m$ has $2M - 1$ other incident edges: a type m edge (w, ws_m) and for $k \in \{1, \dots, M\} \setminus \{m\}$ the type k edges (w, ws_k) and (w, ws_k^{-1}) . If $y \in \mathbb{X}_e^+$ satisfies $y(w) = 1$, then by Theorem 4.3 the values of y at the vertices adjacent to w are

$$y(v) = \mu_m^{-1}, \quad y(ws_m) = \mu_m, \quad y(ws_k) = y(ws_k^{-1}) = \mu_k, \quad k \neq m.$$

For each edge incident on w choose coordinates identifying w with 0. Using $y(w) = 1$, the interpolation formula (4.2) provides the following values for $y'(0)$ on the incident edges:

$$\begin{aligned} & \frac{\mu_m^{-1} - C_m(l_m, \lambda)}{S_m(l_m, \lambda)} \text{ on } (v, w), \quad \frac{\mu_m - C_m(l_m, \lambda)}{S_m(l_m, \lambda)} \text{ on } (w, ws_m), \\ & \frac{\mu_k - C_k(l_k, \lambda)}{S_k(l_k, \lambda)} \text{ on } (w, ws_k) \text{ and } (w, ws_k^{-1}), \quad k \neq m. \end{aligned}$$

The sum of these derivative values is 0 by (2.1); (4.8) follows by simple algebra. □

The next result characterizes the solutions $\mu_1(\lambda), \dots, \mu_M(\lambda)$ of (4.8) coming from the resolvent of $\Delta + q$.

Theorem 4.6. *For $\lambda \in \mathbb{C} \setminus [0, \infty)$, assume that $\mu_1(\lambda), \dots, \mu_M(\lambda)$ satisfy (4.8). This M -tuple is the set of the multipliers for $\Delta + q$ if and only if all eigenvalues η of the matrix*

$$A = \begin{pmatrix} |\mu_1|^2 & 2|\mu_1|^2 & 2|\mu_1|^2 & \dots & 2|\mu_1|^2 \\ 2|\mu_2|^2 & |\mu_2|^2 & 2|\mu_2|^2 & \dots & 2|\mu_2|^2 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 2|\mu_M|^2 & 2|\mu_M|^2 & \dots & 2|\mu_M|^2 & |\mu_M|^2 \end{pmatrix} \tag{4.9}$$

satisfy $|\eta| < 1$.

Proof. The proof takes advantage of several equivalent formulations of the result. Suppose $e = (v, vs_m)$ is a fixed edge, and let w denote any vertex in \mathcal{T}_e^+ . Define a function $y_+ : \mathcal{T}_e^+ \rightarrow \mathbb{C}$ by taking $y_+(v) = 1$, defining $y_+(w)$ by (4.4), and interpolating the vertex values of y_+ to the edges using (4.2). The three conditions (i) $|\eta| < 1$ for all eigenvalues η of A , (ii) the function y_+ is square integrable on \mathcal{T}_e^+ , and (iii)

$$\sum_{w \in \mathcal{T}_e^+} |y_+(w)|^2 < \infty, \tag{4.10}$$

will be shown to be equivalent. In addition, if these conditions hold and the map $y_-: \mathcal{T}_e^- \rightarrow \mathbb{C}$ is defined similarly using these multipliers, then the formula (4.7) gives the resolvent of $\Delta + q$ as in Theorem 4.4.

The first step will show that the function y_+ is square integrable on \mathcal{T}_e^+ if and only if (4.10) holds. The solutions of (4.8) satisfy $\mu_m(\lambda) \neq 0$. For $j = 1, 2$, two edges $(w_j, w_j s_m) \in \mathcal{T}_e^+$ have vertex values satisfying $y_+(w_j s_m)/y_+(w_j) = \mu_m(\lambda)$, so the interpolated edge values y_j given by (4.2) satisfy $y_2(0)y_1(x) = y_1(0)y_2(x)$, and

$$\int_0^{l_m} |y_2(x)|^2 dx = \frac{|y_2(0)|^2}{|y_1(0)|^2} \int_0^{l_m} |y_1(x)|^2 dx.$$

Fixing a reference edge of type m for y_1 which is incident on vs_m , the first step is then proven by writing $\int_{\mathcal{T}_e^+} |y_+|^2$ as sums of integrals over edges of the M types.

Running the argument of Theorem 4.5 in reverse shows that the continuity and derivative conditions (2.1) hold at the vertices of \mathcal{T}_e^+ except possibly at v . If (4.10) holds, then $y_+ \in \mathbb{X}_e^+$ and the claims about the resolvent formula follow.

The condition (4.10) has a description in terms of the multipliers $\mu_k(\lambda)$. If the path from v to w is given by the reduced word $s_m s_{k(1)}^{\pm 1} \dots s_{k(n)}^{\pm 1}$, then $y_+(w) = \mu_m \mu_{k(1)} \dots \mu_{k(n)}$ by Theorem 4.3. For $j = 1, \dots, M$ let $\rho_j(n) = \sum_{j, d(w,v)=n+1} |y_+(w)|^2$ denote the partial sum taken over vertices $w \in \mathcal{T}_e^+$ whose combinatorial distance to v is equal to $n + 1$, with the last edge on the path from v to w having type $k(n) = j$. Then

$$\sum_{w \in \mathcal{T}_e^+} |y_+(w)|^2 = \sum_{n=0}^{\infty} \sum_{j=1}^M \rho_j(n). \tag{4.11}$$

A path given by the reduced word $s_m s_{k(1)}^{\pm 1} \dots s_{k(n)}^{\pm 1}$, may be extended by one edge to a vertex more distant from v in one way if the new edge has the previous edge type $k(n)$ and in two ways for each of the other edge types. Thus

$$\rho_j(n + 1) = |\mu_j|^2 [\rho_j(n) + 2 \sum_{k \neq j} \rho_k(n)], \quad j = 1, \dots, M, \tag{4.12}$$

or

$$\begin{pmatrix} \rho_1(n + 1) \\ \vdots \\ \rho_M(n + 1) \end{pmatrix} = A \begin{pmatrix} \rho_1(n) \\ \vdots \\ \rho_M(n) \end{pmatrix}, \tag{4.13}$$

where A is the matrix of (4.9). Given that the first edge has type m , the initial condition for these sums is

$$\rho_m(0) = |\mu_m|^2, \quad \rho_j(0) = 0, \quad j \neq m.$$

Suppose V_m denotes the vector which is zero except for the m -th component, which is $|\mu_m|^2$. The condition (4.10) thus becomes

$$\sum_{n=1}^{\infty} (1, \dots, 1) A^n V_m < \infty. \tag{4.14}$$

The matrix geometric series $\sum_{n=1}^{\infty} A^n$ converges [3, p. 491] if and only if all eigenvalues η of A satisfy $|\eta| < 1$, so this condition is sufficient for convergence of (4.14).

To see that the condition is necessary, first observe that \mathcal{T}_e^+ has subtrees $\mathcal{T}_{e^{(k)}}^+$ whose initial edge may be any of the M edge types. Thus (4.14) must hold for $m = 1, \dots, M$. If the series $\sum_{n=1}^{\infty} A^n$ diverges and E_m denotes the m -th standard basis vector, then $\sum_{n=1}^{\infty} A^n E_m$ is divergent for some $m = 1, \dots, M$. Because the entries of A are all positive, the components of the partial sums $\sum_{n=1}^N A^n E_m$ are increasing nonnegative sequences, at least one of which is unbounded. Consequently, if $|\eta| \geq 1$ for any eigenvalue η of A , then the series (4.14) is divergent for some m . \square

The equations (4.8) have implications for the decay of $\mu_m(\lambda)$.

Theorem 4.7. *For $0 < \sigma < \pi$, let Ω denote the set of λ with $|\arg(\lambda) - \pi| \leq \sigma$ and $|\lambda| \geq 1$. For $\lambda \in \Omega$,*

$$\lim_{|\lambda| \rightarrow \infty} |\mu_m(\lambda)| e^{|\Im(\sqrt{\lambda})|l_m} = \frac{1}{2M}. \tag{4.15}$$

Proof. Rewrite (4.8) as

$$\frac{\mu_m(\lambda)}{\sqrt{\lambda} S_m(\lambda, l_m)} - \frac{1}{\sqrt{\lambda} S_m(\lambda, l_m) \mu_m(\lambda)} = 2 \sum_{k=1}^M \frac{\mu_k(\lambda) - C_k(\lambda, l_k)}{\sqrt{\lambda} S_k(\lambda, l_k)}. \tag{4.16}$$

Take $\Im(\sqrt{\lambda}) > 0$ and recall that $|\mu_k(\lambda)| \leq 1$. The functions $C_k(\lambda, l_k)$, $S_k(\lambda, l_k)$ satisfy the estimates [24, p. 13]

$$|C_k(\lambda, l_k) - \cos(\sqrt{\lambda} l_k)| \leq \frac{C}{|\sqrt{\lambda}|} \exp(|\Im(\sqrt{\lambda})|l_k),$$

$$\left| S_k(\lambda, l_k) - \frac{\sin(\sqrt{\lambda} l_k)}{\sqrt{\lambda}} \right| \leq \frac{C}{|\lambda|} \exp(|\Im(\sqrt{\lambda})|l_k),$$

while

$$\cos(\sqrt{\lambda} x) = \frac{1}{2} e^{-i\sqrt{\lambda} x} (1 + e^{2i\sqrt{\lambda} x}), \quad \sin(\sqrt{\lambda} x) = \frac{i}{2} e^{-i\sqrt{\lambda} x} (1 - e^{2i\sqrt{\lambda} x}).$$

For $\lambda \in \Omega$, taking $|\lambda| \rightarrow \infty$ in (4.16) gives

$$\lim_{|\lambda| \rightarrow \infty} \frac{1}{\sqrt{\lambda} S_m(l_m) \mu_m(\lambda)} = -2iM,$$

which implies (4.15). □

For each $\lambda \in \mathbb{C}$ and $m = 1, \dots, M$, the equations (4.8) are a system $P_m(\xi_1, \dots, \xi_M, \lambda) = 0$ which is satisfied by the multipliers μ_1, \dots, μ_M . The independence of the equations and local structure of the solutions may be determined by computing the gradients ∇P_m with respect to ξ_1, \dots, ξ_M , with λ treated as a parameter. Recall from Lemma 3.9 that the multipliers satisfy $|\mu_m(\lambda)| < 1$ if $\lambda \in \mathbb{C} \setminus [0, \infty)$.

Theorem 4.8. *Suppose $\lambda \in \mathbb{C}$, $S_m(\lambda, l_m) \neq 0$, and $\xi_m(\lambda)^2 + 1 \neq 0$ for $m = 1, \dots, M$. Then the complex gradients ∇P_m are linearly independent if*

$$\sum_{m=1}^M \frac{\xi_m^2}{\xi_m^2 + 1} \neq 1/2. \tag{4.17}$$

Proof. The relevant partial derivatives are

$$\frac{\partial P_m}{\partial \xi_m} = \frac{1}{S_m(l_m)} \left[\frac{1}{\xi_m^2(\lambda)} - 1 \right]$$

and for $j \neq m$

$$\frac{\partial P_m}{\partial \xi_j} = -2 \frac{1}{S_j(l_j)}.$$

That is, there is an m -independent vector function W such that

$$\nabla P_m = V_m + W, \quad W = -2 \begin{pmatrix} 1/S_1(l_1) \\ \vdots \\ 1/S_M(l_M) \end{pmatrix},$$

with V_m having m -th component equal to

$$\partial P_m / \partial \xi_m + 2/S_m(l_m) = \frac{1}{S_m(l_m)} \left[\frac{1}{\xi_m^2(\lambda)} + 1 \right]$$

and all other components zero.

If the vectors $V_k + W$ are linearly dependent, then there are constants α_k not all zero such that $\sum_k \alpha_k (V_k + W) = 0$. If $S_m(l_m) \neq 0$ the component equations can be written as

$$\alpha_m \left[\frac{1}{\xi_m(\lambda)^2} + 1 \right] = 2 \sum_{k=1}^M \alpha_k.$$

This linear system is

$$\text{diag} \left[\frac{\xi_1^2 + 1}{\xi_1^2}, \dots, \frac{\xi_M^2 + 1}{\xi_M^2} \right] \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_M \end{pmatrix} = 2 \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \dots & \vdots \\ 1 & \dots & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_M \end{pmatrix}.$$

If none of the ξ_m^2 have the value -1 , this system says $[\alpha_1, \dots, \alpha_M]$ is an eigenvector with eigenvalue $1/2$ for the matrix

$$\begin{pmatrix} \frac{\xi_1^2}{\xi_1^2 + 1} & \dots & \frac{\xi_1^2}{\xi_1^2 + 1} \\ \vdots & \dots & \vdots \\ \frac{\xi_M^2}{\xi_M^2 + 1} & \dots & \frac{\xi_M^2}{\xi_M^2 + 1} \end{pmatrix}.$$

Vectors with $\sum \alpha_k = 0$ are in the null space of this matrix, and the remaining eigenvalue is the trace, with eigenvector $[\frac{\xi_1^2}{\xi_1^2 + 1}, \dots, \frac{\xi_M^2}{\xi_M^2 + 1}]$. Thus the condition for dependent gradients is

$$\sum_{m=1}^M \frac{\xi_m^2}{\xi_m^2 + 1} = 1/2. \quad \square$$

By applying the inverse and implicit function theorems for holomorphic functions [14, p. 18-19] we obtain the following corollary.

Corollary 4.9. *Suppose, as in Theorem 4.8, that $\xi_1(\lambda), \dots, \xi_M(\lambda)$ is a solution of (4.8). Assume that $\lambda \in \mathbb{C}$, $S_m(\lambda, l_m) \neq 0$, and*

$$\xi_m(\lambda)^2 + 1 \neq 0, \quad \sum_{m=1}^M \frac{\xi_m^2}{\xi_m^2 + 1} \neq 1/2, \quad m = 1, \dots, M.$$

Then the solutions of the system (4.8) are locally given in $\mathbb{C}^M \times \mathbb{C}$ by a holomorphic \mathbb{C}^M -valued function of λ .

Theorem 4.10. *There is a discrete set $Z_0 \subset \mathbb{R}$ and a positive integer N such that for all $\lambda \in \mathbb{C} \setminus Z_0$ the equations (4.8) satisfied by the multipliers $\mu_m(\lambda)$ have at most N solutions $\xi_1(\lambda), \dots, \xi_M(\lambda)$. For $\lambda \in \mathbb{C} \setminus Z_0$, the functions $\mu_m(\lambda)$ are solutions of polynomial equations $p_m(\xi_m) = 0$ in the one variable ξ_m of positive degree, with coefficients which are entire functions of λ .*

Proof. The polynomial equation in the single variable $\xi_1(\lambda)$ will be considered; the equations satisfied by the other functions $\mu_m(\lambda)$ may be treated in the same manner.

Notice that the system (4.8) has the form

$$F_1(\xi_1, \lambda) = g(\xi_1, \dots, \xi_M, \lambda), \dots, F_M(\xi_M, \lambda) = g(\xi_1, \dots, \xi_M, \lambda),$$

with

$$F_m(\xi_m, \lambda) = \frac{\xi_m^2(\lambda) - 1}{S_m(l_m)\xi_m(\lambda)}, \tag{4.18}$$

$$g(\xi_1, \dots, \xi_M, \lambda) = 2 \sum_{k=1}^M \frac{\xi_k(\lambda) - C_k(l_k)}{S_k(l_k)}.$$

Subtraction of successive equations eliminates the right hand sides from $M - 1$ equations, giving a system of M equations, the equations indexed by the value of $j = 1, \dots, M$,

$$\begin{aligned} j = 1 & \quad F_1(\xi_1, \lambda) = g(\xi_1, \dots, \xi_M, \lambda), \\ j = 2 & \quad F_1(\xi_1, \lambda) - F_2(\xi_2, \lambda) = 0, \\ & \quad \vdots \\ j = M & \quad F_{M-1}(\xi_{M-1}, \lambda) - F_M(\xi_M, \lambda) = 0. \end{aligned}$$

For $m > 1$, the m -th equation can be written as

$$\xi_m(\lambda)^2 = \left(\frac{\xi_{m-1}^2(\lambda) - 1}{S_{m-1}\xi_{m-1}(\lambda)} \right) S_m \xi_m + 1, \tag{4.19}$$

or by using the quadratic formula,

$$2\xi_m(\lambda) - \left(\frac{\xi_{m-1}^2(\lambda) - 1}{S_{m-1}\xi_{m-1}(\lambda)} \right) S_m = \left[\left(\frac{\xi_{m-1}^2(\lambda) - 1}{S_{m-1}\xi_{m-1}(\lambda)} \right)^2 S_m^2 + 4 \right]^{1/2}. \tag{4.20}$$

The variables ξ_M, \dots, ξ_2 can be successively eliminated from the first equation. Starting with $k = M$ and continuing up the list of indices, the first equation can be written as a polynomial equation for ξ_k with coefficients which are polynomials in ξ_1, \dots, ξ_{k-1} and the entire functions $S_m(l_k, \lambda)$. Repeated use of the substitution (4.19), followed by clearing of the denominators, reduces the first equation to degree one in ξ_k . These substitutions result in an equivalent system of equations as long as $\lambda \notin Z_0$, where

$$Z_0 = \bigcup_m \{S_m(l_m, \lambda) = 0\}. \tag{4.21}$$

Now solve for $2\xi_k$, subtract $[\xi_{m-1}^2(\lambda) - 1]S_m/[S_{m-1}\xi_{m-1}(\lambda)]$, use the substitution (4.20), and square both sides. Since squaring is a two-to-one map, it will not change the dimension of the set of solutions. After applying these substitutions, the variable ξ_k has been eliminated, and after clearing the denominators, the modified $j = 1$ equation is a polynomial in $\xi_1(\lambda), \dots, \xi_{k-1}(\lambda)$ with entire coefficients. The substitution process provides a common bound N for the degrees of the polynomials p_m .

Suppose the final version of the first equation does not have positive degree for ξ_1 . Define $Q_m = F_m(\xi_m, \lambda) - F_{m+1}(\xi_{m+1}, \lambda)$ for $m = 1, \dots, M - 1$. The system of equations for ξ_1, \dots, ξ_M is then the system

$$Q_1 = 0, \dots, Q_{M-1} = 0,$$

where

$$\frac{\partial Q_m}{\partial \xi_m} = \frac{1}{S_m} \left[\frac{1}{\xi_m^2(\lambda)} - 1 \right], \quad \frac{\partial Q_m}{\partial \xi_{m+1}} = \frac{-1}{S_{m+1}} \left[\frac{1}{\xi_{m+1}^2(\lambda)} - 1 \right],$$

and all other partial derivatives are zero. Suppose $S_m(\lambda, l_m) \neq 0$, and $\xi_m(\lambda)^2 + 1 \neq 0$ for $m = 1, \dots, M$. Then the $M - 1$ gradients ∇Q_m are linearly independent, so outside of a discrete set of λ the functions ξ_1, \dots, ξ_{K-1} would be holomorphic functions of λ, ξ_M ; that is, the solution set would have dimension 2, contradicting Corollary 4.9 which showed the dimension is 1. \square

4.3. Extension of multipliers to $[0, \infty)$ and the spectrum. The mapping $z \rightarrow \lambda$ given by

$$\lambda = \left(\frac{1-z}{1+z} \right)^2, \quad z = \frac{1-\sqrt{\lambda}}{1+\sqrt{\lambda}}$$

is a conformal map from the unit disc $\{|z| < 1\}$ onto $\lambda \in \mathbb{C} \setminus [0, \infty)$. By using this conformal map and Lemma 3.9 the functions $\mu_m(\lambda(z))$ may be considered as bounded holomorphic functions on the unit disc. Classical results in function theory [17, p. 38] insure that $\mu_m(\lambda(z))$ has nontangential limits almost everywhere as a function of z , and so the limits

$$\mu_m^\pm(\sigma) = \lim_{\epsilon \rightarrow 0^+} \mu_m(\sigma \pm i\epsilon) \tag{4.22}$$

exist almost everywhere on $[0, \infty)$. By (4.8) and (4.21), the values $\mu_m^\pm(\sigma)$ are bounded away from zero uniformly on compact subsets of $\mathbb{C} \setminus Z_0$.

Since the functions $\mu_m(\lambda)$ satisfy the polynomial equations $p_m(\xi_m) = 0$, more information about $\mu_m^\pm(\sigma)$ is available. The equations $p_m(\xi_m) = 0$ have entire coefficients and positive degree for $\lambda \in \mathbb{C} \setminus [0, \infty)$. Let $Z_m \subset \mathbb{C}$ denote the discrete set where the leading coefficient vanishes. A contour integral computation which is a variant of the Argument Principle, [1, p. 152] or Problem 2 of [15, p. 174], shows that for $\lambda \in \mathbb{C} \setminus Z_m$ the roots of $p_m(\lambda)$, in particular $\mu_m(\lambda)$, are holomorphic as long as the root is simple. For $\lambda \in \mathbb{C} \setminus Z_m$ the roots extend continuously to $\lambda \in [0, \infty)$ even if the roots are not simple. The limiting values $\mu^\pm(\sigma)$ need not agree; let

$$\delta_m(\sigma) = \mu_m^+(\sigma) - \mu_m^-(\sigma), \quad \sigma \in [0, \infty).$$

Proposition 4.11. *If $\mu_m^+(\sigma) = \mu_m^-(\sigma)$ for $\sigma \in (\alpha, \beta) \setminus Z_m$, then $\mu_m(\lambda)$ extends holomorphically across (α, β) .*

Proof. On any subinterval $(\alpha_1, \beta_1) \subset (\alpha, \beta)$ where $\mu_m(\lambda)$ extends continuously to the common value $\mu_m^\pm(\sigma)$, the extension is holomorphic by Morera’s Theorem [13, p. 121]. The points in the discrete set $Z_m \cap (\alpha, \beta)$ appear to be possible obstacles to the existence of a holomorphic extension, but since the extended function $\mu_m(\lambda)$ is bounded the extension can be continued holomorphically across $Z_m \cap (\alpha, \beta)$ too by Riemann’s Theorem on removable singularities. □

Theorem 4.12. *Assume $\sigma \notin Z_0$. For $m = 1, \dots, M$ suppose $\mu_m^\pm(\sigma) \neq \pm 1$ and $\mu_m(\lambda)$ extends holomorphically (resp. continuously) to $\sigma \in \mathbb{R}$ from above (resp. below). Then the kernel function $R_e(x, t, \lambda)$ of (4.7) extends holomorphically (resp. continuously) from above (resp. below) to*

$$R_e^\pm(x, t, \sigma) = \lim_{\epsilon \rightarrow 0^+} R_e(x, t, \sigma + i\epsilon), \quad \sigma \in \mathbb{R}.$$

Proof. The Wronskian formula (4.6) shows that $1/W_m(\lambda)$ extends holomorphically (resp. continuously) if $\mu_m(\lambda)$ does and $\mu_m^\pm(\sigma) \neq \pm 1$. Theorem 4.3 shows that the vertex values $y_\pm(w)$ extend in the same fashion as the multipliers μ_m . Finally, the interpolation formula (4.2) provides a holomorphic extension of y_\pm from the vertex values as long as $\sigma \notin Z_0$, that is $S_m(l_m, \sigma) \neq 0$ for $m = 1, \dots, M$. □

Recall [25, p. 237,264] that if P denotes the family of spectral projections for a self adjoint operator, in this case $\Delta + q$, then for any $f \in L^2(\mathcal{T})$

$$\frac{1}{2}[P_{[a,b]} + P_{(a,b)}]f = \lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \int_a^b [R(\sigma + i\epsilon) - R(\sigma - i\epsilon)]f \, d\sigma. \quad (4.23)$$

Theorem 4.13. *Suppose $(\alpha, \beta) \cap Z_0 = \emptyset$. For $m = 1, \dots, M$ also assume that $(\alpha, \beta) \cap Z_m = \emptyset$ and that $\mu_m^\pm(\sigma) \neq \pm 1$ for all $\sigma \in (\alpha, \beta)$. If $[a, b] \subset (\alpha, \beta)$, e is an edge of type m , and $f \in L^2(e)$, then*

$$P_{[a,b]}f = \frac{1}{2\pi i} \int_a^b [R_e^+(\sigma) - R_e^-(\sigma)]f \, d\sigma. \tag{4.24}$$

If $\sigma_1 \in (\alpha, \beta)$, then σ_1 is not an eigenvalue of $\Delta + q$.

Proof. As noted above, the assumption that $(\alpha, \beta) \cap Z_m = \emptyset$ means the multipliers μ_m extend continuously to $[a, b]$ from above and below. Since $\mu_m^\pm(\sigma) \neq \pm 1$ the function $1/W_m(\lambda)$ extends continuously to $[a, b]$. Based on Theorem 4.3 and the interpolation formula (4.2), the kernel $R_e(x, t, \lambda)$ described in (4.7) extends continuously to $[a, b]$ from above and below. The convergence of $R_e(x, t, \sigma \pm i\epsilon)$ to $R_e(x, t, \sigma)$ is uniform for t, x coming from a finite set of edges.

If the support of $g \in L^2(\mathcal{T}_M)$ is contained in a finite set of edges, then (4.23) and the uniform convergence of $R_e(x, t, \sigma \pm i\epsilon)$ to $R_e(x, t, \sigma)$ gives

$$\left\langle \frac{1}{2} [P_{[a,b]} + P_{(a,b)}]f, g \right\rangle = \frac{1}{2\pi i} \int_a^b [R_e^+(\sigma) - R_e^-(\sigma)] \langle f, g \rangle \, d\sigma.$$

The set of g with support in a finite set of edges is dense in $L^2(\mathcal{T}_M)$, so the restriction on the support of g may be dropped. Suppose g is an eigenfunction with eigenvalue $\sigma_1 \in (a, b)$ and with $\|g\| = 1$, while f is the restriction of g to the edge e . Then the continuity of $R_e(x, t, \sigma)$ means there is a C_e such that

$$\left| \frac{1}{2\pi i} \int_a^b [R_e^+(\sigma) - R_e^-(\sigma)] \langle f, g \rangle \, d\sigma \right| \leq C_e |b - a|.$$

This implies $\langle P_{\sigma_1}g, g \rangle = 0$, so the eigenfunction g doesn't exist. Finally, the absence of point spectrum in (α, β) means that $P_{[a,b]} = P_{(a,b)}$, giving the formula (4.24). □

Theorem 4.14. *Assume $\sigma \in [0, \infty) \setminus Z_0$ and for $m = 1, \dots, M$ suppose $\mu_m^\pm(\sigma) \neq \pm 1$. Then σ is in the resolvent set of $\Delta + q$ if and only if $\delta_m(\sigma) = 0$ in an open neighborhood of σ for $m = 1, \dots, M$.*

Proof. If σ is in the resolvent set then the kernels described in (4.7) will have a common holomorphic extension to σ from above and below. Evaluation gives

$$R_m(l_m, 0, \lambda) = \frac{\mu_m^2(\lambda)}{1 - \mu_m^2(\lambda)} S_m(l_m, \lambda) = \left[\frac{1}{1 - \mu_m^2(\lambda)} - 1 \right] S_m(l_m, \lambda)$$

so that $(\mu_m^+)^2(\sigma) = (\mu_m^-)^2(\sigma)$. A second evaluation,

$$R_m(0, 0, \lambda) = \frac{\mu_m(\lambda)}{1 - \mu_m^2(\lambda)} S_m(l_m, \lambda),$$

shows $\mu_m^+(\sigma) - \mu_m^-(\sigma) = \delta_m(\sigma) = 0$.

Now suppose $\delta_m(\sigma) = 0$ in open neighborhood of σ . Theorem 4.12 notes that $R_m(x, t, \lambda)$ extends holomorphically to a neighborhood of σ . If $g \in L^2(\Gamma)$ with support in a finite set of edges, the function $\langle R_m(\lambda) f, g \rangle$ also extends holomorphically as a single valued function in an interval (α, β) containing σ . If $[a, b] \subset (\alpha, \beta)$, then for any $f \in L^2(e)$

$$\left\langle \frac{1}{2}[P_{[a,b]} + P_{(a,b)}]f, g \right\rangle = 0.$$

The set of g with with support in a finite set of edges is dense in $L^2(\Gamma)$, so $\frac{1}{2}[P_{[a,b]} + P_{(a,b)}]f = 0$ for all $f \in L^2(e)$. By linearity $[P_{[a,b]} + P_{(a,b)}]h = 0$ for any $h \in L^2(\Gamma)$ with support in a finite set of edges, and since the projections are bounded we conclude that $P_{[a,b]} + P_{(a,b)} = 0$ and (α, β) is in the resolvent set [18, p. 357]. □

Corollary 4.15. *Assume $\sigma \in [0, \infty) \setminus Z_0$ and for $m = 1, \dots, M$ suppose $\mu_m^\pm(\sigma) \neq \pm 1$. Then σ is in the resolvent set of $\Delta + q$ if and only if $\mu_m^+(\sigma)$ is real valued in open neighborhood of σ for $m = 1, \dots, M$.*

Proof. If $\mu_m^+(\sigma)$ is real valued, then the symmetry $\mu_m(\bar{\lambda}) = \overline{\mu_m(\lambda)}$ established in Lemma 3.7 means $\mu_m^-(\sigma)$ has the same real value. The same symmetry also implies that $\delta_m(\sigma) \neq 0$ if $\mu_m^+(\sigma)$ is not real valued. □

Theorem 4.16. *For $q \geq 0$ and $M \geq 2$ the spectrum of $\Delta + q$ has a strictly positive lower bound.*

Proof. Since

$$\langle (\Delta + q)f, f \rangle = \langle \Delta f, f \rangle + \int_{\mathcal{T}_M} q|f|^2,$$

it suffices to verify the result when $q = 0$.

Consider the case when the edge lengths l_m are all equal to 1. Then the system (4.8) reduces to

$$(2M - 1)\mu^2(\lambda) - 2MC(1, \lambda)\mu(\lambda) + 1 = 0.$$

The quadratic formula gives

$$\mu(\lambda) = \frac{2MC(1, \lambda) \pm \sqrt{4M^2C^2(1, \lambda) - 4(2M - 1)}}{2(2M - 1)}.$$

Since $C(1, 0) = \cos(0) = 1$, the discriminant has the positive value $4M^2 - 4(2M - 1) = 4(M - 1)^2$ when $\lambda = 0$, and $\mu(\sigma)$ is real as long as $\cos^2(\sqrt{\sigma}) \geq (2M - 1)/M^2$.

Returning to the general case of a graph Γ with unconstrained edge lengths, recall (2.3) that the quadratic form for Δ is

$$\langle \Delta f, f \rangle = \int_{\mathcal{T}_M} (f')^2.$$

Let x be a coordinate for intervals $[0, l_m]$ and t for the interval $[0, 1]$. For $m = 1, \dots, M$ let $x = \phi_m(t)$ be a smooth change of variables. Assume $\phi' \geq C_1 > 0$ and $\phi'(t) = 1$ for t in neighborhoods of 0 and 1. If f is in the domain of Δ for the graph Γ , then $f \circ \phi$ will be in the domain of Δ for a graph Γ_1 whose edge lengths are all 1.

The chain rule and the change of variables formula for integrals give

$$\int_0^{l_m} |f(x)|^2 dx = \int_0^1 |f(\phi_m(t))|^2 \phi'_m(t) dt,$$

and

$$\int_0^{l_m} \left| \frac{df(x)}{dx} \right|^2 dx = \int_0^1 \left| \frac{d}{dt} f(\phi_m(t)) \right|^2 \frac{1}{\phi'_m(t)} dt.$$

As a consequence there is a constant $C > 0$ such that

$$\frac{\int_{\Gamma} \left| \frac{df(x)}{dx} \right|^2 dx}{\int_{\Gamma} |f(x)|^2 dx} \geq C \frac{\int_{\Gamma_1} \left| \frac{df(\phi(t))}{dt} \right|^2 dt}{\int_{\Gamma_1} |f(t)|^2 dt}.$$

The calculation for graphs with edge lengths 1 shows that the expression on the right has a strictly positive lower bound. □

Corollary 4.17. *Suppose $M \geq 2$ and the lengths l_m are rational. Then the resolvent set of Δ includes an unbounded subset of $[0, \infty)$.*

Proof. Assume $\lambda \in \mathbb{C} \setminus (-\infty, 0)$ so that $\sqrt{\lambda}$ may be taken to be continuous and positive for $\lambda > 0$. In case $q = 0$,

$$C(l_m, \lambda) = \cos(l_m \sqrt{\lambda}), \quad \sqrt{\lambda} S(l_m, \lambda) = \sin(l_m \sqrt{\lambda}),$$

and these functions are periodic in $\sqrt{\lambda}$ with period $2\pi/l_m$. If $l_m = \tau_m/\eta_m$ with τ_m, η_m positive integers, then the functions have a common period $p = 2\pi \prod_{m=1}^M \eta_m$.

The functions $C(l_m, \lambda)$ and $S(l_m, \lambda)$ appear as coefficients in (4.8). After multiplication by $1/\sqrt{\lambda}$, the equations (4.8) exhibit the same periodicity, so have

identical solutions for λ and λ_1 whenever $\sqrt{\lambda_1} = \sqrt{\lambda} + 2np$ for any positive integer n .

By Theorem 4.6 the solutions of (4.8) which are multipliers are determined by the summability condition if $\lambda \in \mathbb{C} \setminus [0, \infty)$, so $\mu_m([\sqrt{\lambda} + 2np]^2) = \mu_m(\lambda)$ for non-real λ . This identity extends by continuity to $\lambda \in [0, \infty)$. By Theorem 4.16 there is a $\sigma_0 > 0$ such that $[0, \sigma_0)$ is in the resolvent set of Δ . Corollary 4.15 shows that except possibly at a discrete set of points, the points $\sigma \in [0, \infty)$ which are in the resolvent set are characterized by real values of the multipliers $\mu_m(\sigma)$, so except for a discrete set of possible exceptions, $(4n^2 p^2, [\sqrt{\sigma_0} + 2np]^2)$ is a subset of the resolvent set for Δ . □

5. Sample computations

In this section some sample spectral computations are carried out for the case $M = 2$. The first step is to reduce the system of equations (4.8) to equations for individual multipliers. Two equations of degree four with entire coefficients are obtained. For $q = 0$ these equations are solved numerically (using Matlab) for positive values σ of the spectral parameter. After eliminating spurious solutions, the multiplier data is displayed in several figures.

5.1. Elimination step. When $M = 2$ the system of equations (4.8) may be written as

$$\begin{aligned} \frac{\mu_1^2(\lambda) - 1}{S_1(l_1, \lambda)\mu_1(\lambda)} &= 2\frac{\mu_1(\lambda) - C_1(l_1, \lambda)}{S_1(l_1, \lambda)} + 2\frac{\mu_2(\lambda) - C_2(l_2, \lambda)}{S_2(l_2, \lambda)}, & (5.1) \\ \frac{\mu_2^2(\lambda) - 1}{S_2(l_2, \lambda)\mu_2(\lambda)} &= 2\frac{\mu_1(\lambda) - C_1(l_1, \lambda)}{S_1(l_1, \lambda)} + 2\frac{\mu_2(\lambda) - C_2(l_2, \lambda)}{S_2(l_2, \lambda)}. \end{aligned}$$

Subtracting the second equation from the first gives

$$\frac{\mu_1^2(\lambda) - 1}{S_1(l_1, \lambda)\mu_1(\lambda)} - \frac{\mu_2^2(\lambda) - 1}{S_2(l_2, \lambda)\mu_2(\lambda)} = 0,$$

Solving this quadratic equation for $2\mu_2(\lambda)$ gives

$$2\mu_2(\lambda) - \frac{\mu_1^2(\lambda) - 1}{S_1(l_1, \lambda)\mu_1(\lambda)} S_2(l_2, \lambda) = \pm \left[\frac{(\mu_1^2(\lambda) - 1)^2}{S_1^2(l_1, \lambda)\mu_1^2(\lambda)} S_2^2(l_2, \lambda) + 4 \right]^{1/2}. \quad (5.2)$$

Equation (5.1) is already first order in $\mu_2(\lambda)$, and may be rewritten as

$$2\mu_2(\lambda) - \frac{\mu_1^2(\lambda) - 1}{S_1(l_1, \lambda)\mu_1(\lambda)} S_2(l_2, \lambda) = -2S_2(l_2, \lambda) \frac{\mu_1(\lambda) - C_1(l_1, \lambda)}{S_1(l_1, \lambda)} + 2C_2(l_2, \lambda).$$

Replacing the left hand side using (5.2) and squaring gives

$$\begin{aligned} & \frac{(\mu_1^2(\lambda) - 1)^2}{S_1^2(l_1, \lambda)\mu_1^2(\lambda)} S_2^2(l_2, \lambda) + 4 \\ &= 4S_2^2(l_2, \lambda) \left[\frac{\mu_1(\lambda) - C_1(l_1, \lambda)}{S_1(l_1, \lambda)} \right]^2 \\ & \quad - 8C_2(l_2, \lambda)S_2(l_2, \lambda) \frac{\mu_1(\lambda) - C_1(l_1, \lambda)}{S_1(l_1, \lambda)} + 4C_2^2(l_2, \lambda) \end{aligned}$$

After some clean-up we get

$$\begin{aligned} & 3S_2^2(l_2, \lambda)\mu_1^4(\lambda) - 8[S_1(l_1, \lambda)C_2(l_2, \lambda)S_2(l_2, \lambda) + S_2^2(l_2, \lambda)C_1(l_1, \lambda)]\mu_1^3(\lambda) \\ & \quad + [2S_2^2(l_2, \lambda) - 4S_1^2(l_1, \lambda) + 4S_2^2(l_2, \lambda)C_1^2(l_1, \lambda) + 4C_2^2(l_2, \lambda)S_1^2(l_1, \lambda) \\ & \quad + 8S_1(l_1, \lambda)C_2(l_2, \lambda)S_2(l_2, \lambda)C_1(l_1, \lambda)]\mu_1^2(\lambda) - S_2^2(l_2, \lambda) \\ &= 0. \end{aligned}$$

The equation satisfied by $\mu_2(\lambda)$ is obtained by interchanging the subscripts 1 and 2.

5.2. Numerical work. Figures 3, 4, and 5 display multiplier data for three cases. In all cases $q = 0$ and $l_1 = 1$. The values of l_2 are (i) $l_2 = 1$, (ii) $l_2 = 0.89$, and (iii) $l_2 = 2$.

For a range of positive values of σ , solutions of the degree four polynomial equations for $\mu_1(\sigma)$ and $\mu_2(\sigma)$ are computed. Actual multiplier pairs (μ_1, μ_2) must satisfy the system (4.8), as well as the eigenvalue bounds of Theorem 4.6. To eliminate spurious solutions, the expressions in (4.8) were evaluated, and candidate pairs (μ_1, μ_2) were rejected if either equation had an expression with magnitude greater than 10^{-8} . Pairs were also rejected if the eigenvalue bounds of Theorem 4.6 were exceeded.

Each figure contains two parts, the multiplier arguments and the logarithm of the magnitudes. Figure 3 is the case with $l_1 = l_2 = 1$. In this case the two multipliers are equal. By Corollary 4.15, real points in the resolvent set can be recognized by real values for both multipliers, except when σ lies in a discrete exceptional set. Eigenvalues in these sets are possible, as discussed in [7].

Figure 4 illustrates the case $l_2 = 0.89$. When the multipliers are not real they will appear in conjugate pairs; in this case the multipliers have distinct extensions $\mu_m^\pm(\sigma)$ to the real axis, as discussed before Proposition 4.11. One then sees four multipliers in the figure. Unlike the classical Hill’s equation, multipliers may vary in magnitude when they are not real valued. The multiplier arguments may exhibit occasional discontinuities.

Figure 5 illustrates the case $l_2 = 2$. The multiplier argument discontinuities are clearly visible. Notice that the horizontal axis displays $\sigma^{1/2}$; the predicted periodicity from the proof of Corollary 4.17 is evident.

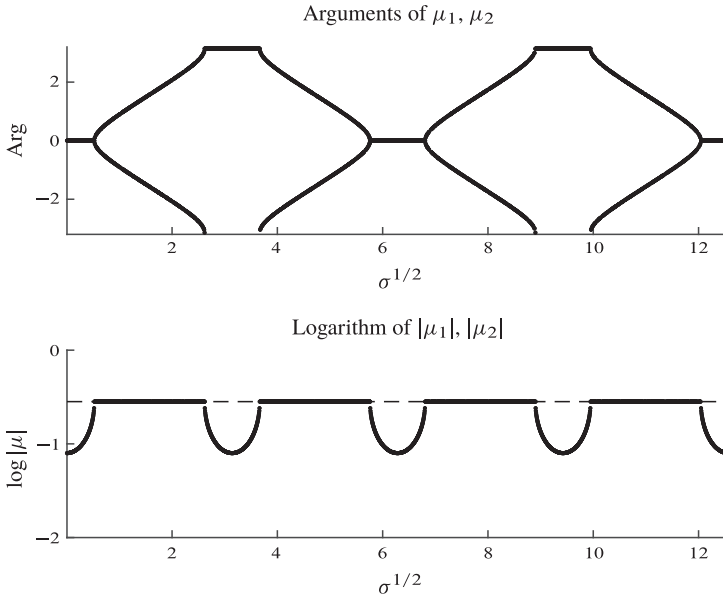


Figure 3. Case $l_1 = 1, l_2 = 1$.

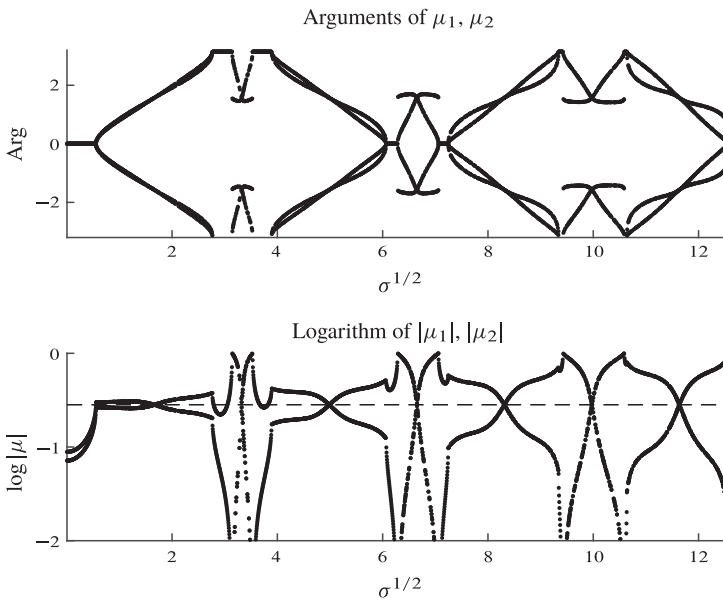
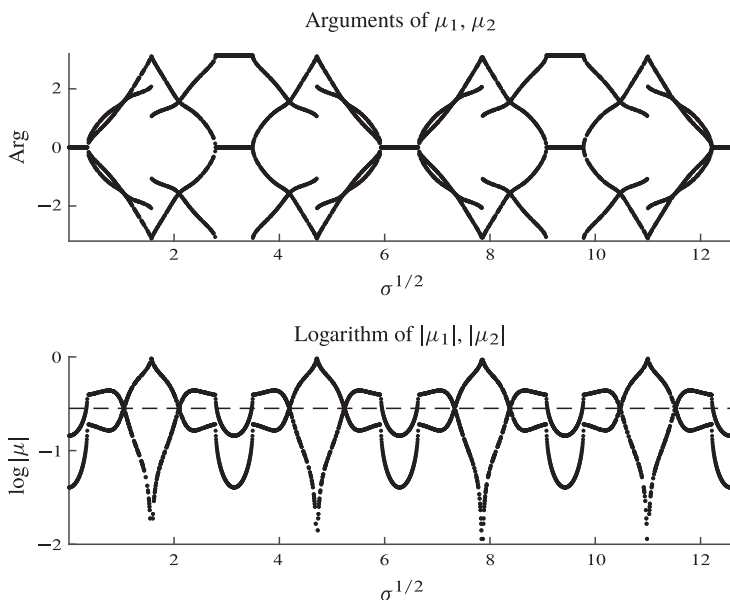


Figure 4. Case $l_1 = 1, l_2 = 0.89$.

Figure 5. Case $l_1 = 1, l_2 = 2$.

References

- [1] L. Ahlfors, *Complex analysis*. An introduction of the theory of analytic functions of one complex variable. Second edition. McGraw-Hill Book Co., New York etc., 1966. [MR 0188405](#) [Zbl 0154.31904](#)
- [2] M. Atiyah, Elliptic operators, discrete groups and von Neumann algebras. In *Colloque "Analyse et Topologie" en l'Honneur de Henri Cartan*. Tenu le 17–20 juin 1974 à Orsay. Astérisque, 32–33. Société Mathématique de France, 1976. 43–72. [MR 0420729](#) [Zbl 0323.58015](#)
- [3] K. E. Atkinson, *An introduction to numerical analysis*. Second edition. John Wiley & Sons, New York, 1989. [MR 1007135](#) [Zbl 0718.65001](#)
- [4] G. Berkolaiko and P. Kuchment, *Introduction to quantum graphs*. Mathematical Surveys and Monographs, 186. American Mathematical Society, Providence, R.I., 2013. [MR 3013208](#) [Zbl 1318.81005](#)
- [5] J. Breuer and R. Frank, Singular spectrum for radial trees. *Rev. Math. Phys.* **21** (2009), no. 7, 929–945. [MR 2553430](#) [Zbl 1177.05069](#)
- [6] R. Brooks, The spectral geometry of k -regular graphs. *J. Anal. Math.* **57** (1991), 120–151. [MR 1191744](#) [Zbl 0772.58061](#)
- [7] R. Carlson, Hill's equation for a homogeneous tree. *Electron. J. Differential Equations* **1997**, no. 23, 30 pp. [MR 1487199](#) [Zbl 0890.34066](#)

- [8] R. Carlson, Nonclassical Sturm–Liouville problems and Schrödinger operators on radial trees. *Electron. J. Differential Equations* **2000**, no. 71, 24 pp. [MR 1800836](#) [Zbl 0960.34070](#)
- [9] H. Donnelly, On the spectrum of towers. *Proc. Amer. Math. Soc.* **87** (1983), no. 2, 322–329. [MR 0681842](#) [Zbl 0512.58038](#)
- [10] P. Exner and J. Lipovsky, On the absence of absolutely continuous spectra for Schrödinger operators on radial tree graphs. *J. Math. Phys.* **51** (2010), no. 12, article id. 122107, 19 pp. [MR 2779168](#) [Zbl 1314.81086](#)
- [11] A. Figa-Talamanca and T. Steger, Harmonic analysis for anisotropic random walks on homogeneous trees. *Mem. Amer. Math. Soc.* **110** (1994), no. 531. [MR 1219707](#) [Zbl 0836.43019](#)
- [12] R. Frank and H. Kovarik, Heat kernels of metric trees and applications. *SIAM J. Math. Anal.* **45** (2013), no. 3, 1027–1046. [MR 3049649](#) [Zbl 1278.47048](#)
- [13] T. Gamelin, *Complex analysis*. Undergraduate Texts in Mathematics. Springer-Verlag, New York, 2001. [MR 1830078](#) [Zbl 0978.30001](#)
- [14] P. Griffiths and J. Harris, *Principles of algebraic geometry*. Pure and Applied Mathematics. Wiley-Interscience, New York, 1978. [MR 0507725](#) [Zbl 0408.14001](#)
- [15] R. Greene and S. Krantz, *Function theory of one complex variable*. Third edition. Graduate Studies in Mathematics, 40. American Mathematical Society, Providence, R.I., 2006. [MR 2215872](#) [Zbl 1114.30001](#)
- [16] P. Hislop and O. Post, Anderson localization for radial tree-like quantum graphs. *Waves Random Complex Media* **19** (2009), no. 2, 216–261. [MR 2536455](#) [Zbl 1267.81165](#)
- [17] K. Hoffman, *Banach spaces of analytic functions*. Prentice-Hall Series in Modern Analysis. Prentice-Hall, Englewood Cliffs, N.J., 1962. [MR 0133008](#) [Zbl 0117.34001](#)
- [18] T. Kato, *Perturbation theory for linear operators*. Reprint of the 1980 edition. Classics in Mathematics. Springer-Verlag, Berlin, 1995. [MR 1335452](#) [Zbl 0836.47009](#)
- [19] P. Kuchment and O. Post, On the spectra of carbon nano-structures. *Comm. Math. Phys.* **275** (2007), no. 3, 805–826. [MR 2336365](#) [Zbl 1145.81032](#)
- [20] W. Magnus and S. Winkler, *Hill's equation*. Corrected reprint of the 1966 edition. Dover Publications, New York, 1979. [MR 0559928](#) [Zbl 1141.34002](#)
- [21] J. Meier, *Groups, graphs and trees*. An introduction to the geometry of infinite groups. London Mathematical Society Student Texts, 73. Cambridge University Press, Cambridge, 2008. [MR 2498449](#) [Zbl 1276.20052](#)
- [22] E. Montroll, Quantum theory on a network. I. A solvable model whose wavefunctions are elementary functions. *J. Math. Phys.* **11** (1970), 635–648.
- [23] K. Naimark and M. Solomyak, Eigenvalue estimates for the weighted Laplacian on metric trees. *Proc. London Math. Soc.* (3) **80** (2000), no. 3, 690–724. [MR 1744781](#) [Zbl 1046.34092](#)

- [24] J. Pöschel and E. Trubowitz, *Inverse spectral theory*. Pure and Applied Mathematics, 130. Academic Press, Boston, MA, 1987. [MR 0894477](#) [Zbl 0623.34001](#)
- [25] M. Reed and B. Simon, *Methods of modern mathematical physics*. I. Functional analysis. Academic Press, New York and London, 1972. [MR 0493419](#) [Zbl 0242.46001](#)
- [26] A. Sobolev and M. Solomyak, Schrödinger operators on homogeneous metric trees: spectrum in gaps. *Rev. Math. Phys.* **14** (2002), no. 5, 421–467. [MR 1912093](#) [Zbl 1038.81023](#)
- [27] H. M. Stark and A. Terras, Zeta functions of finite graphs and coverings. *Adv. Math.* **121** (1996), no. 1, 124–165. [MR 1399606](#) [Zbl 0874.11064](#)
- [28] 0972.11086 and A. Terras, Zeta functions of finite graphs and coverings. II. *Adv. Math.* **154** (2000), no. 1, 132–195. [MR 1780097](#) [Zbl 0972.11086](#)

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