J. Spectr. Theory 7 (2017), 985–1022
DOI 10.4171/JST/183

Journal of Spectral Theory
© European Mathematical Society

On sums of eigenvalues of elliptic operators on manifolds

Ahmad El Soufi (†), Evans M. Harrell II, Saïd Ilias, and Joachim Stubbe

Dedicated to the memory of Ahmad El Soufi, 1960–2016

Abstract. We use the averaged variational principle introduced in a recent article on graph spectra [10] to obtain upper bounds for sums of eigenvalues of several partial differential operators of interest in geometric analysis, which are analogues of Kröger's bound for Neumann spectra of Laplacians on Euclidean domains [15]. Among the operators we consider are the Laplace-Beltrami operator on compact subdomains of manifolds. These estimates become more explicit and asymptotically sharp when the manifold is conformal to homogeneous spaces (here extending a result of Strichartz [26] with a simplified proof). In addition we obtain results for the Witten Laplacian on the same sorts of domains and for Schrödinger operators with confining potentials on infinite Euclidean domains. Our bounds have the sharp asymptotic form expected from the Weyl law or classical phase-space analysis. Similarly sharp bounds for the trace of the heat kernel follow as corollaries.

Mathematics Subject Classification (2010). 58J50, 35P15, 47A75.

Keywords. Manifold with density, weighted Laplacian, Schrödinger operator, Witten Laplacian, eigenvalue, upper bound, phase space, Weyl law, homogeneous space, conformal.

Contents

1 Introduction 986
2 The averaged variational principle 989
3 Bounds for Neumann eigenvalues on domains of Riemannian manifolds 994
4 Sums of Neumann eigenvalues on domains conformal to Euclidean sets, and phase-space volumes 1004
5 Bounds for Neumann eigenvalues on subdomains of compact homogeneous spaces 1009
References 1020

† Deceased on December 29, 2016.

1. Introduction

In this article we consider the eigenvalues of self-adjoint, second-order elliptic partial differential operators defined on a subdomain of a Riemannian manifold (M, g) of dimension $\nu \geq 2$. The model for the operators we are able to treat is the Laplacian on a domain with Neumann boundary conditions, defined in the weak sense, i.e. via the Laplacian energy

$$\frac{\int_\Omega (|\nabla^g \varphi(\mathbf{x})|^2 dv_g}{\int_\Omega |\varphi(\mathbf{x})|^2 dv_g}$$

on functions $\varphi \in H^1(\Omega)$, but the class treated includes a large variety of Schrödinger operators, even with weights. Specifically, the eigenvalues we shall discuss are operationally defined by the min-max procedure applied to expressions of the general form

$$\mathcal{R}(\varphi) := \frac{\int_\Omega (|\nabla^g \varphi(\mathbf{x})|^2 + V(\mathbf{x})|\varphi(\mathbf{x})|^2)e^{-2\theta(\mathbf{x})} dv_g}{\int_\Omega |\varphi(\mathbf{x})|^2 e^{-2\rho(\mathbf{x})} dv_g}.$$

For convenience we set $w = e^{2(\rho-\theta)}$ so that $\mathcal{R}$ takes on the form

$$\mathcal{R}(\varphi) = \frac{\int_\Omega (|\nabla^g \varphi(\mathbf{x})|^2 + V(\mathbf{x})|\varphi(\mathbf{x})|^2)w(\mathbf{x})e^{-2\rho(\mathbf{x})} dv_g}{\int_\Omega |\varphi(\mathbf{x})|^2 e^{-2\rho(\mathbf{x})} dv_g}. \tag{1}$$

Here $\rho \in C^1(\Omega)$, $0 < C \leq w(\mathbf{x}) \in C^0(\Omega)$, and $V \in \mathrm{Lip}(\Omega)$ are real-valued functions. We define the Neumann eigenvalues of (1) by the min-max principle [3, 27], *i.e.*,

$$\mu_\ell := \max_{\{\text{subspace } \mathfrak{S}:\dim(\mathfrak{S})=\ell\}} \min_{\{\varphi\in H^1(\Omega):\varphi\perp\mathfrak{S},\|\varphi\|_{L^2}=1\}} \mathcal{R}(\varphi). \tag{2}$$

Of course, μ_ℓ depends on the domain Ω as well as the choice of the metric g, the density $e^{-2\rho}$ and weight w, and the potential V, but dependence on these quantities will not be indicated explicitly unless necessary.

Under suitable regularity assumptions on Ω and V, the sequence $\{\mu_\ell\}$ is nothing but the spectrum of the eigenvalue problem

$$H\varphi = \mu\varphi \quad \text{in } \Omega, \tag{3}$$

with Neumann boundary conditions if $\partial\Omega \neq \emptyset$, where

$$\begin{aligned} H\varphi &= -e^{2\rho}\mathrm{div}_g(we^{-2\rho}\nabla^g \varphi) + Vw\varphi \\ &= w\{\Delta_g\varphi + 2(\nabla^g\theta, \nabla^g\varphi)_g + V\varphi\} \end{aligned} \tag{4}$$

and $\Delta_g\varphi := -\mathrm{div}_g(\nabla^g\varphi)$ is the Laplace Beltrami operator associated with g.

In the following sections we derive semiclassically sharp phase-space upper bounds for the sums of the first k eigenvalues associated with (1). We also obtain bounds for the corresponding Riesz means and heat trace. The following inequalities, which are valid for any bounded domain $\Omega \subset \mathbb{R}^\nu$, provide a sampling of these bounds:

$$\frac{1}{k}\sum_{j=0}^{k-1}\mu_j \leq \frac{4\pi^2\nu}{\nu+2}\Big(\frac{k}{|\Omega|\omega_\nu}\Big)^{\frac{2}{\nu}} \fint_\Omega w(\mathbf{x}) d^\nu x + \fint_\Omega \widetilde{V}(\mathbf{x}) w(\mathbf{x}) d^\nu x$$

and

$$\sum_{j\geq 0} e^{-t\mu_j} \geq \frac{|\Omega|}{(4\pi t)^{\frac{\nu}{2}}}\Big(\fint_\Omega w(\mathbf{x})\, d^\nu x\Big)^{-\frac{\nu}{2}} e^{-t \fint_\Omega \widetilde{V}(\mathbf{x}) w(\mathbf{x})\, d^\nu x},$$

where $\widetilde{V}(\mathbf{x}) := V(\mathbf{x}) + |\nabla\rho|^2(\mathbf{x})$, $|\Omega|$ is the volume of Ω, ω_ν is the volume of the unit ball in $\mathbb{R}^\nu$, and, for every $f \in L^1(\Omega)$, $\fint_\Omega f(\mathbf{x}) d^\nu x = \frac{1}{|\Omega|}\int_\Omega f(\mathbf{x}) d^\nu x$ is the mean value of f with respect to Lebesgue measure.

When appropriate we remark on the simpler consequences that apply under assumptions on ρ, w, and V. The case where $V = \rho = 0$, $w = 1$ identically, and $M = \mathbb{R}^\nu$ reduces to the situation treated by Kröger in his ground-breaking work [15], which was extended to subdomains of general homogeneous spaces by Strichartz [26] (see also [8]). The upper bounds in [15, 26] are notable for being sharp in the sense of agreeing with the "semiclassical" Weyl law, with the optimal constant. For the background and context of Weyl-sharp bounds on sums of Laplacian eigenvalues, we refer to [16, 17].

In this article a new, simplified proof is used, and we considerably enlarge the family of self-adjoint elliptic operators for which semiclassical upper bounds are proved. Even when $V = 0$, new cases of interest that are treated include the Witten Laplacian, for which $w = 1$; the Laplacian of a conformal metric $\tilde{g} = \alpha^{-2}g$, for which $e^{-2\rho} = \alpha^{-n}$ and $w = \alpha^2$; and the vibrating membrane with variable density $\gamma(\mathbf{x})$, for which $\gamma(\mathbf{x}) = e^{-2\rho}$ and $w = e^{2\rho}$.

In the last part of the paper, we focus on domains of compact homogeneous Riemannian spaces. We revisit the inequality of Strichartz ([26, Theorem 2.2]) in the light of this new approach and obtain extensions of Strichartz's inequality to the case where the Laplace operator is penalized by a potential in the presence of weights. For example, we prove that if Ω is a domain of a compact homogeneous Riemannian manifold (M, g), then the eigenvalues μ_l associated with (1) on Ω satisfy

$$\sum_{j\geq 0}\big(z-\mu_j\big)_+ \geq \frac{|\Omega|_g}{|M|_g}\sum_{j\geq 0}(z-\tilde{\lambda}_j)_+$$

for all $z \in \mathbb{R}$, and

$$\sum_{j\geq 0} e^{-\mu_j t} \geq \frac{|\Omega|_g}{|M|_g} \sum_{j\geq 0} e^{-\tilde{\lambda}_j t}$$

for all $t > 0$, where $\tilde{\lambda}_j = \lambda_j ⨍_\Omega w\, dv_g + ⨍_\Omega \widetilde{V} w\, dv_g$, and where the $\lambda_j's$ are the eigenvalues of the Laplacian on the whole manifold M (see Theorem 5.1 and Corollary 5.2). The extension (stricto sensu) of Strichartz's inequality is given in Theorem 5.2 and takes the following form when $\Omega = M$:

$$\sum_{j=0}^{k-1} \mu_j \leq \sum_{j=0}^{k-1} \tilde{\lambda}_j .$$

It is known that without assumptions of regularity, these variationally defined Neumann eigenvalues for the Laplacian may have finite points of accumulation of a quite arbitrary sort, as entertainingly discussed in [12]. In this case the definition (2) would yield $\mu_\ell = \inf(\sigma_{\text{ess}})$ for all ℓ greater than some value, and the bounds we shall provide would become uninteresting. We note that, for example, the spectrum of the Neumann Laplacian is guaranteed to have no finite points of accumulation if the boundary is piecewise smooth [12].

Remark 1.1. Before closing this section, we make some further technical remarks about how to define the Dirichlet and Neumann problems for these elliptic operators in the weak, or quadratic-form, sense. In this regard we follow Edmunds and Evans [5], where in Chapter VII it is shown that uniformly elliptic quadratic forms, on arbitrary open sets in Euclidean space, determine unique operators via the Friedrichs extension, which, when the domain is sufficiently regular, reduce to the classically defined operators for the Dirichlet and Neumann problems (see also [28, 24]). In particular, defining the quadratic form (1) initially on the Sobolev space $W_0^{1,2}(\Omega)$ corresponds to Dirichlet boundary conditions, whereas defining it initially on the restrictions to Ω of functions in the space $W_0^{1,2}(\mathbb{R}^\nu)$ corresponds to Neumann conditions. (For domains allowing a Sobolev extension property the latter set coincides with $W^{1,2}(\Omega)$.) It is not in general possible to say that the operators thus defined satisfy boundary conditions in a classical sense, or to guarantee regularity at the boundary. However, in cases where the boundary is sufficiently regular, integration by parts transforms expressions $\langle H\varphi, \varphi\rangle$, where H is a classically defined operator, into a quadratic form of the type (1) for φ in a dense subset of the Sobolev spaces corresponding to Dirichlet or respectively Neumann conditions.

There are certainly significant questions of regularity of the eigenfunctions in the case when Ω is an arbitrary open set, treated for example in [2], but they will play no role in the present article.

The extension of the analysis of [5] from $\mathbb{R}^\nu$ to manifolds is straightforward, because only Hilbert-space structures and locally defined properties of functions and their gradients are used.

2. The averaged variational principle

In this section we recall the averaged variational principle which will be foundational for this article. The following is a restatement of Theorem 3.1 of Harrell-Stubbe [10], along with a characterization of the case of equality.

Theorem 2.1. *Consider a self-adjoint operator H on a Hilbert space $\mathcal{H}$, the spectrum of which is discrete at least in its lower portion, so that $-\infty < \mu_0 \leq \mu_1 \leq \cdots$. The corresponding orthonormalized eigenvectors are denoted $\{\boldsymbol{\psi}^{(\ell)}\}$. The closed quadratic form corresponding to H is denoted $Q(\varphi, \varphi)$ for vectors φ in the quadratic-form domain $\mathcal{Q}(H) \subset \mathcal{H}$. Let $f_\zeta \in \mathcal{Q}(H)$ be a family of vectors indexed by a variable ζ ranging over a measure space $(\mathfrak{M}, \Sigma, \sigma)$. Suppose that $\mathfrak{M}_0$ is a subset of $\mathfrak{M}$. Then for any $z \in \mathbb{R}$,*

$$\sum_j (z - \mu_j)_+ \int_{\mathfrak{M}} |\langle \boldsymbol{\psi}^{(j)}, f_\zeta \rangle|^2 \, d\sigma \geq \int_{\mathfrak{M}_0} (z \|f_\zeta\|^2 - Q(f_\zeta, f_\zeta)) d\sigma, \tag{5}$$

provided that the integrals converge. Moreover, equality holds in (5) for $z \in \mathbb{R}$ if and only if up to sets of measure 0,

$$\{f_\zeta; \zeta \in \mathfrak{M}_0\} \subset E(z) \quad \text{and} \quad \{f_\zeta; \zeta \in \mathfrak{M} \setminus \mathfrak{M}_0\} \perp E_0(z),$$

where $E(z) = \bigoplus_{\mu \leq z} \ker(H - \mu I)$ and $E_0(z) = \bigoplus_{\mu < z} \ker(H - \mu I)$.

Taking $z = \mu_k$ in (5) we obtain

$$\begin{aligned} &\mu_k \Big(\int_{\mathfrak{M}_0} \|f_\zeta\|^2 \, d\sigma - \sum_{j=0}^{k-1} \int_{\mathfrak{M}} |\langle \boldsymbol{\psi}^{(j)}, f_\zeta \rangle|^2 \, d\sigma \Big) \\ &\quad \leq \int_{\mathfrak{M}_0} Q(f_\zeta, f_\zeta) d\sigma - \sum_{j=0}^{k-1} \mu_j \int_{\mathfrak{M}} |\langle \boldsymbol{\psi}^{(j)}, f_\zeta \rangle|^2 \, d\sigma, \end{aligned} \tag{6}$$

Remarks 2.1. 1. The averaged variational principle is an abstract version and sharpening of ideas appearing in various place in the literature, including not only [15], but also work of Lieb and others on coherent states and trace inequalities [20, 22]. In special cases, similar use of averaging and tight frames for the study of eigenvalue sums and related quantities has also been made by Laugesen and Siudeja [18].

2. We point out that the normalization of the test function f_ζ could be incorporated into the measure, so that, for example, Eq. (5) could alternatively be written in terms of integrals of expectation values such as

$$\int \left(\frac{|\langle \boldsymbol{\psi}^{(j)}, f_\zeta \rangle|^2}{\| f_\zeta \|^2} \right) d\sigma, \tag{7}$$

i.e., over norms of projections of the eigenfunctions. Despite the suggestiveness of these alternatives, an advantageous feature of (6)-(5) that we shall later exploit is that useful identities are available for averages of norms of some choices of f_ζ. Still, if the test functions f_ζ and the measure space $\mathfrak{M}$ constitute a *tight frame,* in the sense of satisfying a generalized Parseval identity [13], then alternative forms of the inequalities imply appealing variational principles for sums and Riesz means of eigenvalues, as captured in the next corollary.

Corollary 2.1. *Under the assumptions of the theorem, suppose further that* f_ζ *is a nonvanishing family of test functions with the property that for all* $\phi \in \mathcal{H}$*,*

$$\int_{\mathfrak{M}} \frac{|\langle \boldsymbol{\phi}, f_\zeta \rangle|^2}{\| f_\zeta \|^2} \, d\sigma = A \|\boldsymbol{\phi}\|^2$$

for a fixed constant $A > 0$*. Then for any* $\mathfrak{M}_0 \subset \mathfrak{M}$ *such that* $(|\mathfrak{M}_0| - A\,k)\mu_k \geq 0$*,*

$$\frac{1}{k} \sum_{j=0}^{k-1} \mu_j \leq \frac{1}{|\mathfrak{M}_0|} \int_{\mathfrak{M}_0} \left(\frac{Q(f_\zeta, f_\zeta)}{\| f_\zeta \|^2} \right) d\sigma. \tag{8}$$

For Riesz means,

$$\sum_j (z - \mu_j)_+ \geq \frac{1}{A} \int_{\mathfrak{M}_0} \left(z - \frac{Q(f_\zeta, f_\zeta)}{\| f_\zeta \|^2} \right) d\sigma. \tag{9}$$

The proof of Corollary 2.1 is immediate; see [10] for more in this connection. To make our exposition self-contained, we provide here the proof of inequality (5) in Theorem 2.1 before discussing the case of equality.

Proof of Theorem 2.1. For every integer $l \geq 0$, we denote by P_l the orthogonal projector onto the subspace spanned by $\{\psi^{(j)}\ ,\ j \leq l\}$, i.e.,

$$P_l f = \sum_{j=0}^{l} \langle \psi^{(j)}, f \rangle \psi^{(j)}.$$

Let $z \in \mathbb{R}$, $z > \mu_0$ (inequality (5) being obvious for $z \leq \mu_0$), and let k be the smallest integer such that $z \leq \mu_k$ (that is $z \in (\mu_{k-1}, \mu_k]$). Then

$$z\|f - P_{k-1}f\|^2 \leq \mu_k \|f - P_{k-1}f\|^2 \leq Q(f - P_{k-1}f, f - P_{k-1}f), \quad (10)$$

and, after direct computations,

$$z(\|f\|^2 - \|P_{k-1}f\|^2) \leq Q(f, f) - Q(P_{k-1}f, P_{k-1}f).$$

With

$$\|P_{k-1}f\|^2 = \sum_{j=0}^{k-1} \langle \psi^{(j)}, f \rangle^2$$

and

$$Q(P_{k-1}f, P_{k-1}f) = \sum_{j=0}^{k-1} \mu_j \langle \psi^{(j)}, f \rangle^2,$$

this yields

$$z\|f\|^2 - Q(f, f) \leq \sum_{j=0}^{k-1} (z - \mu_j) \langle \psi^{(j)}, f \rangle^2.$$

Applying this last inequality to f_ζ, $\zeta \in \mathfrak{M}_0$, and integrating over $\mathfrak{M}_0$, we get

$$\begin{aligned} z\int_{\mathfrak{M}_0} \|f_\zeta\|^2 d\sigma - \int_{\mathfrak{M}_0} Q(f_\zeta, f_\zeta) d\sigma &\leq \sum_{j=0}^{k-1} (z - \mu_j) \int_{\mathfrak{M}_0} |\langle \psi^{(j)}, f_\zeta \rangle|^2 d\sigma \\ &= \sum_{j\geq 0} (z - \mu_j)_+ \int_{\mathfrak{M}_0} |\langle \psi^{(j)}, f_\zeta \rangle|^2 d\sigma. \end{aligned} \quad (11)$$

Inequality (5) follows from (11) and the obvious inequality

$$\sum_{j\geq 0} (z - \mu_j)_+ \int_{\mathfrak{M}_0} |\langle \psi^{(j)}, f_\zeta \rangle|^2 d\sigma \leq \sum_{j\geq 0} (z - \mu_j)_+ \int_{\mathfrak{M}} |\langle \psi^{(j)}, f_\zeta \rangle|^2 d\sigma. \quad (12)$$

Assume now that equality holds in (5). This implies that equality holds in (12) and in (10) for $f = f_\zeta$ for almost all $\zeta \in \mathfrak{M}_0$. Equality in (10) holds for f either when $z < \mu_k$ and $f = P_{k-1}f$ or if $z = \mu_k$ and $H(f - P_{k-1}f) = \mu_k(f - P_{k-1}f)$, which implies in both cases that $f \in E(z)$. On the other hand, equality in (12) implies that, for almost all $\zeta \in \mathfrak{M} \setminus \mathfrak{M}_0$ and all $j \in \mathbb{N}$ such that $\mu_j < z$, f_ζ is orthogonal to $\operatorname{span}\{\psi^{(0)}, \ldots, \psi^{(j)}\}$, which means that f_ζ is orthogonal to $E_0(z)$.

Conversely, under the conditions of the statement,

$$
\begin{aligned}
&\sum_j (z-\mu_j)_+ \int_{\mathfrak{M}} |\langle \boldsymbol{\psi}^{(j)}, f_\zeta\rangle|^2\, d\sigma \\
&\quad = \sum_j (z-\mu_j)_+ \int_{\mathfrak{M}_0} |\langle \boldsymbol{\psi}^{(j)}, f_\zeta\rangle|^2\, d\sigma \\
&\quad = \sum_{\mu_j \le z} z \int_{\mathfrak{M}_0} |\langle \boldsymbol{\psi}^{(j)}, f_\zeta\rangle|^2\, d\sigma - \sum_{\mu_j \le z} \mu_j \int_{\mathfrak{M}_0} |\langle \boldsymbol{\psi}^{(j)}, f_\zeta\rangle|^2\, d\sigma \\
&\quad = \sum_{j=0}^{+\infty} z \int_{\mathfrak{M}_0} |\langle \boldsymbol{\psi}^{(j)}, f_\zeta\rangle|^2\, d\sigma - \sum_{j=0}^{+\infty} \mu_j \int_{\mathfrak{M}_0} |\langle \boldsymbol{\psi}^{(j)}, f_\zeta\rangle|^2\, d\sigma \\
&\quad = \int_{\mathfrak{M}_0} (z\|f_\zeta\|^2 - Q(f_\zeta, f_\zeta))\, d\sigma. \qquad \square
\end{aligned}
$$

As the guiding example for this article, when Ω is a bounded subdomain of $\mathbb{R}^\nu$, we may use test functions of the form

$$
f_\zeta(\mathbf{x}) := \frac{1}{(2\pi)^{\nu/2}} e^{i\mathbf{p}\cdot\mathbf{x}},
$$

where ζ has been equated to $\mathbf{p}$, which ranges over $\mathfrak{M} = \mathbb{R}^\nu$ with Lebesgue measure. (The reason for distinguishing ζ logically from $\mathbf{p}$ will be made clear in Theorem 4.1.) Since $\|f_\zeta\|^2 = \frac{|\Omega|}{(2\pi)^\nu}$ for all ζ, where $|\Omega|$ is the Euclidean volume of Ω, Parseval's identity gives

$$
\int_{\mathbb{R}^\nu} |\langle \phi, f_\zeta\rangle|^2 d^\nu p = \|\phi\|^2.
$$

Hence, applying Corollary 2.1 with $\mathfrak{M}_0 \subset \mathfrak{M}$ taken to be the Euclidean ball of radius $2\pi\left(\frac{k}{|\Omega|\omega_\nu}\right)^{\frac{1}{\nu}}$, we recover Kröger's inequality for Neumann eigenvalues of the Euclidean Laplacian (here ω_ν stands for the volume of the ν-dimensional Euclidean unit ball). Indeed, in this case, the Rayleigh quotient of f_ζ is simply given by $\mathcal{R}(f_\xi) = |\mathbf{p}|^2$ and (8) yields

$$\frac{1}{k}\sum_{j=0}^{k-1}\mu_j \le \frac{1}{|\mathfrak{M}_0|}\int_{\mathfrak{M}_0}|\mathbf{p}|^2 d^\nu p = \frac{4\pi^2\nu}{\nu+2}\Big(\frac{k}{|\Omega|\omega_\nu}\Big)^{\frac{2}{\nu}}.$$

This approach can be applied to easily extend Kröger's inequality to Neumann eigenvalues on a bounded subdomain of $\mathbb{R}^\nu$ in the presence of nontrivial potential and weights.

Example 2.1. Let $\mu_0 \le \mu_1 \le \dots$ be the variationally defined Neumann eigenvalues (2) on a bounded open set $\Omega \subset \mathbb{R}^\nu$ endowed with the standard Euclidean metric, where w, ρ, and V satisfy the assumptions stated above. Then

$$\frac{1}{k}\sum_{j=0}^{k-1}\mu_j \le \frac{4\pi^2\nu}{\nu+2}\Big(\frac{k}{|\Omega|\omega_\nu}\Big)^{\frac{2}{\nu}} ⨍_\Omega w(\mathbf{x})d^\nu x + ⨍_\Omega \widetilde{V}(\mathbf{x})w(\mathbf{x})d^\nu x, \tag{13}$$

where $\widetilde{V}(\mathbf{x}) := V(\mathbf{x}) + |\nabla\rho|^2(\mathbf{x})$ and, for every $f \in L^1(\Omega)$,

$$⨍_\Omega f(\mathbf{x})d^\nu x := \frac{1}{|\Omega|}\int_\Omega f(\mathbf{x})d^\nu x$$

is the mean value of f with respect to Lebesgue measure.

Example 2.1 sets the stage for a more general result that we obtain in Section 3 in the context of Riemannian manifolds. We also stress that these estimates will be improved in Section 4, with the aid of a coherent-state analysis relating the upper bounds to phase-space volumes.

Upper bounds for individual Neumann eigenvalues μ_j are also obtainable from the averaged variational principle. In order to somewhat simplify the bound, let us define the shifted Neumann eigenvalues

$$\tilde{\mu}_j := \mu_j - ⨍_\Omega \widetilde{V}(\mathbf{x})w(\mathbf{x})d^\nu x. \tag{14}$$

In terms of these quantities, we will be able to show that (see Corollary 3.1)

$$\tilde{\mu}_k \le 4\pi^2\Big(\frac{k}{|\Omega|\omega_\nu}\Big)^{\frac{2}{\nu}}\Big(1 + 2\sqrt{\frac{1-S_k}{\nu+2}}\Big) ⨍_\Omega w(\mathbf{x})d^\nu x, \tag{15}$$

where

$$S_k := \frac{\frac{1}{k}\sum_{j=0}^{k-1}\tilde{\mu}_j}{\frac{4\pi^2\nu}{\nu+2}\big(\frac{k}{|\Omega|\omega_\nu}\big)^{\frac{2}{\nu}} ⨍_\Omega w(\mathbf{x})d^\nu x} \le 1. \tag{16}$$

3. Bounds for Neumann eigenvalues on domains of Riemannian manifolds

Let (M, g) be a Riemannian manifold of dimension $\nu \geq 2$ and let Ω be a bounded subdomain of M. Of course, when M is a closed manifold, Ω can be equal to the whole of M.

Let $F : (M, g) \to \mathbb{R}^N$, be an isometric embedding (whose existence for sufficiently large N is guaranteed by Nash's embedding Theorem). To any function $u \in L^2(\Omega)$, we associate the function $\hat{u}_F : \mathbb{R}^N \to \mathbb{R}$ defined by

$$\hat{u}_F(\mathbf{p}) = \int_\Omega u(\mathbf{x}) e^{i\mathbf{p}\cdot F(\mathbf{x})} dv_g, \tag{17}$$

where the dot stands for the Euclidean inner product in $\mathbb{R}^N$ (i.e., $\hat{u}_F$ is the Fourier transform of the signed measure $F_*(u dv_g)$ supported by $F(\Omega)$). It is well known, since the works of Hörmander, Agmon–Hörmander, and others (see [1, Theorem 2.1], [14, Theorem 7.1.26], and [25, Corollary 5.2]), that there exists a constant $C_{F(\Omega)}$ such that, for all $u \in L^2(\Omega)$ and $R > 0$,

$$\int_{B_R} |\hat{u}_F(\mathbf{p})|^2 d^N p \leq C_{F(\Omega)} R^{N-\nu} \|u\|^2, \tag{18}$$

where B_R is the Euclidean ball of radius R in $\mathbb{R}^N$ centered at the origin and $\|u\|^2 = \int_\Omega u^2 dv_g$. In other words the Fourier functions appearing in (17) constitute a frame that is not generally tight.

We define the Riemannian constant H_Ω by

$$H_\Omega = \inf_{N \geq \nu} \inf_{F \in I(M, \mathbb{R}^N)} \left(\frac{\nu + 2}{N + 2}\right)^{\frac{\nu}{2}} \frac{1}{\omega_N} C_{F(\Omega)}, \tag{19}$$

where $I(M, \mathbb{R}^N)$ is the set of isometric embeddings from (M, g) to $\mathbb{R}^N$.

When Ω is a domain of $\mathbb{R}^\nu$, we may take for F the identity map so that, for all $u \in L^2(\Omega)$, $\hat{u}_I$ is nothing but the Fourier transform of u extended by zero outside Ω. Using Parseval's identity we get, for all $R > 0$,

$$\int_{B_R} |\hat{u}_F(\mathbf{p})|^2 d^\nu p \leq \int_{\mathbb{R}^\nu} |\hat{u}_F(\mathbf{p})|^2 d^\nu p = (2\pi)^\nu \|u\|^2.$$

Thus $C_{I(\Omega)} = (2\pi)^\nu$ and

$$H_\Omega \leq \frac{(2\pi)^\nu}{\omega_\nu}. \tag{20}$$

In the sequel, the notation $|\Omega|_g$ will designate the Riemannian volume of Ω with respect to g. We also use the notation $⨍_\Omega f \, dv_g$ to represent the mean value of a function $f \in L^1(\Omega)$ with respect to the Riemanian measure dv_g. I.e., $\fint_\Omega f\,dv_g := \frac{1}{|\Omega|_g}\int_\Omega f\,dv_g$.

Theorem 3.1. *Let (M,g) be a Riemannian manifold of dimension $\nu \ge 2$. Let $\mu_l = \mu_l(\Omega, g, \rho, w, V)$, $l \in \mathbb{N}$, be the eigenvalues defined by (2) on a bounded open set $\Omega \subset M$, where w, ρ, and V satisfy the assumptions stated above. Then*

(1) *for all $z \in \mathbb{R}$,*

$$\sum_{j\ge 0}(z-\mu_j)_+ \ge \frac{2\,|\Omega|_g}{(\nu+2)H_\Omega}\left(\fint_\Omega w\,dv_g\right)^{-\frac{\nu}{2}}\left(z-\fint_\Omega \widetilde{V} w\,dv_g\right)_+^{1+\frac{\nu}{2}}, \tag{21}$$

where $\widetilde{V} = V + |\nabla^g \rho|^2$;

(2) *for all $k \in \mathbb{N}$,*

$$\frac{1}{k}\sum_{j=0}^{k-1}\mu_j \le \frac{\nu}{\nu+2}\Big(\frac{H_\Omega}{|\Omega|_g}k\Big)^{\frac{2}{\nu}}\fint_\Omega w\,dv_g + \fint_\Omega \widetilde{V} w\,dv_g; \tag{22}$$

(3) *for all $t > 0$,*

$$\sum_{j\ge 0} e^{-t(\mu_j - \fint_\Omega \widetilde{V} w\,dv_g)} \ge \left(\frac{\pi}{t}\right)^{\frac{\nu}{2}}\frac{|\Omega|_g}{\omega_\nu H_\Omega}\left(\fint_\Omega w\,dv_g\right)^{-\frac{\nu}{2}}. \tag{23}$$

Proof. Let $F\colon (M,g) \to \mathbb{R}^N$ be an isometric embedding. For simplicity, we identify the domain Ω with its image $F(\Omega) \subset R^N$ and any function $u\colon \Omega \to \mathbb{R}$ with $u \circ F^{-1}\colon F(\Omega) \to \mathbb{R}$.

We apply Theorem 2.1 using test functions of the form

$$f_\zeta(\mathbf{x}) := e^{i\mathbf{p}\cdot\mathbf{x}+\rho(\mathbf{x})},$$

where ζ has been equated to $\mathbf{p}$, which ranges over $\mathfrak{M} = B_R \subset \mathbb{R}^N$ endowed with Lebesgue measure, B_R being a Euclidean N-dimensional ball, the radius R of which is to be determined later. Our Hilbert space here is $L^2(\Omega, e^{-2\rho}dv_g)$ (endowed with the norm $\|u\|^2 = \int_\Omega u^2 e^{-2\rho}dv_g$). Hence, for all ζ,

$$\|f_\zeta\|^2 = |\Omega|_g$$

and consequently

$$\int_{\mathfrak{M}} \|f_\zeta\|^2 d^N p = |\Omega|_g \omega_N R^N. \tag{24}$$

On the other hand, in our case the quadratic form is

$$Q(f_\xi, f_\xi) = \int_\Omega (|\nabla^\tau f_\xi|^2 + V|f_\xi|^2) w(\mathbf{x})\, e^{-2\rho(\mathbf{x})} dv_g,$$

where $\nabla^\tau f_\xi$ is the tangential part of the gradient of f_ξ. (More generally, for all $v \in \mathbb{R}^N$, v^τ will designate the tangential vector field induced on Ω by orthogonal projection of v.) Thus, with $|f_\xi|^2 = e^{2\rho}$ and $|\nabla^\tau f_\xi|^2 = |\mathbf{p}^\tau + \nabla^\tau \rho|^2$,

$$Q(f_\xi, f_\xi) = \int_\Omega (|\mathbf{p}^\tau|^2(\mathbf{x}) + 2\mathbf{p}\cdot\nabla^\tau\rho(\mathbf{x})) w(\mathbf{x}) dv_g + \int_\Omega (V + |\nabla^\tau \rho|^2) w(\mathbf{x}) dv_g.$$

Observe that for reasons of symmetry, for all $v \in \mathbb{R}^N \setminus \{0\}$,

$$\int_{B_R} \mathbf{p}\cdot v\, d^N p = 0,$$

and, after elementary calculations,

$$\int_{B_R} (\mathbf{p}\cdot v)^2\, d^N p = \frac{|v|}{N}\int_{B_R} |\mathbf{p}|^2\, d^N p = \frac{|v|}{N+2}\omega_N R^{N+2}.$$

Thus, if $\{v_1, \dots, v_\nu\}$ is an orthonormal basis of the tangent space of Ω at a point $\mathbf{x}$, then

$$\int_\Omega |\mathbf{p}^\tau|^2(\mathbf{x})\, d^N p = \sum_{j\le\nu}\int_{B_R} (\mathbf{p}\cdot v_j)^2\, d^N p = \frac{\nu}{N+2}\omega_N R^{N+2}.$$

This leads to

$$\int_{\mathfrak{M}} Q(f_\xi, f_\xi) d^N p = \frac{\nu}{N+2}\omega_N R^{N+2}\int_\Omega w\, dv_g + \omega_N R^N \int_\Omega \widetilde{V} w\, dv_g. \quad (25)$$

It remains to deal with the integrals $\int_{\mathfrak{M}} |\langle \boldsymbol{\psi}^{(j)}, f_\xi\rangle|^2\, d^N p$, where $\{\psi^{(j)}\}$ is an $L^2(\Omega, e^{-2\rho} dv_g)$-orthonormal basis of eigenfunctions associated to $\{\mu_j\}$. Setting $\phi^{(j)} = e^{-\rho}\psi^{(j)}$,

$$\langle \boldsymbol{\psi}^{(j)}, f_\xi\rangle = \int_\Omega f_\xi \psi^{(j)} e^{-2\rho} dv_g = \int_\Omega e^{i\mathbf{p}\cdot\mathbf{x}}\psi^{(j)} e^{-\rho} dv_g = \hat{\phi}_F^{(j)}(\mathbf{p}).$$

Using (18) we obtain

$$\begin{aligned}\int_{B_R} |\langle \boldsymbol{\psi}^{(j)}, f_\xi\rangle|^2\, d^N p &= \int_{B_R} |\hat{\phi}_F^{(j)}(\mathbf{p})|^2\, d^N p \le C_{F(\Omega)} R^{N-\nu}\int_\Omega |\phi^{(j)}|^2 dv_g \\ &= C_{F(\Omega)} R^{N-\nu}\int_\Omega |\psi^{(j)}|^2 e^{-2\rho} dv_g \\ &= C_{F(\Omega)} R^{N-\nu}. \end{aligned} \quad (26)$$

We put (24), (25), and (26) into (5) after choosing $\mathfrak{M}_0 = \mathfrak{M} = B_R$, and for all $R > 0$ and $z \in \mathbb{R}$, we obtain

$$\sum_{j\geq 0} \left(z-\mu_j\right)_+ C_{F(\Omega)} \geq |\Omega|_g \omega_N R^\nu \left(z - \frac{\nu R^2}{N+2} \fint_\Omega w\, dv_g - \fint_\Omega \widetilde{V} w\, dv_g\right). \tag{27}$$

The right side of this inequality is optimized when $R = 0$ if $z \leq \fint_\Omega \widetilde{V} w\, dv_g$ and when $R^2 = \frac{N+2}{\nu+2}\left(z - \fint_\Omega \widetilde{V} w\, dv_g\right) / \fint_\Omega w\, dv_g$ otherwise. Thus,

$$\begin{aligned}\sum_{j\geq 0} &(z-\mu_j)_+ C_{F(\Omega)} \\ &\geq |\Omega|_g \omega_N \left(\frac{N+2}{\nu+2}\right)^{\frac{\nu}{2}} \frac{2}{\nu+2} \left(\fint_\Omega w\, dv_g\right)^{-\frac{\nu}{2}} \left(z - \fint_\Omega \widetilde{V} w\, dv_g\right)_+^{1+\frac{\nu}{2}}.\end{aligned} \tag{28}$$

Taking the infimum with respect to F and N we get (21).

To prove (22) we first observe that taking $z = \mu_k$ in (27) gives

$$k\mu_k - \sum_{j=0}^{k-1} \mu_j \geq \frac{|\Omega|_g \omega_N R^\nu}{C_{F(\Omega)}} \left(\mu_k - \frac{\nu R^2}{N+2} \fint_\Omega w\, dv_g - \fint_\Omega \widetilde{V} w\, dv_g\right) \tag{29}$$

for all $R > 0$, or

$$\begin{aligned}\sum_{j=0}^{k-1} \mu_j \leq &\left(k - \frac{|\Omega|_g \omega_N R^\nu}{C_{F(\Omega)}}\right)\mu_k \\ &+ \frac{|\Omega|_g \omega_N R^\nu}{C_{F(\Omega)}} \left(\frac{\nu R^2}{N+2} \fint_\Omega w\, dv_g + \fint_\Omega \widetilde{V} w\, dv_g\right).\end{aligned}$$

Choosing R such that $\frac{|\Omega|_g \omega_N R^\nu}{C_{F(\Omega)}} = k$ we get

$$\sum_{j=0}^{k-1} \mu_j \leq k \left(\frac{\nu}{N+2} \left(\frac{C_{F(\Omega)} k}{|\Omega|_g \omega_N}\right)^{\frac{2}{\nu}} \fint_\Omega w\, dv_g + \fint_\Omega \widetilde{V} w\, dv_g\right), \tag{30}$$

which leads to (22) after taking the infimum with respect to F and N.

Inequality (23) is a consequence of (21) and the following identity relating the heat trace to the Laplace transform of the Riesz mean:

$$\sum_{j\geq 0} e^{-\mu_j t} = t^2 \int_0^\infty e^{-zt} \sum_{j\geq 0} (z-\mu_j)_+ dz. \tag{31}$$

□

Remarks 3.1. 1. In [19, Theorem 1.2], Li and Tang obtained an inequality for the Laplacian (i.e. in the case $V = \rho = 0$, $w = 1$), which is similar to but weaker than (30). Instead of the term $\frac{\nu}{N+2}$ on the right side their inequality has $\frac{N}{N+2}$.

2. It is possible to derive (22) from (21) using the Legendre transform. The Legendre transform of a function f of the form $f(z) = A(z-B)_+^{1+\frac{\nu}{2}}$ with $A > 0$ is given by

$$f^\wedge(p) = \sup_{z\ge 0}(pz - f(z)) = \Big(\frac{2}{A}\Big)^{\frac{2}{\nu}} \frac{\nu}{(\nu+2)^{1+\frac{2}{\nu}}} p^{1+\frac{2}{\nu}} + Bp,$$

while the Legendre transform of $g(z) = \sum_{j\ge 0}(z-\mu_j)_+$ is

$$g^\wedge(p) = \sum_{j=0}^{\lfloor p\rfloor -1} \mu_j + (p - \lfloor p\rfloor)\mu_{\lfloor p\rfloor}.$$

(Indeed, for $z \in [\mu_{k-1}, \mu_k]$, $pz - g(z) = (p-k)z + \sum_{j=0}^{k-1}\mu_j$, which is nondecreasing as soon as $k \le \lfloor p\rfloor$.) Hence, it suffices to apply the Legendre transform to both sides of (21), taking into account that such the transform reverses inequalities.

Corollary 3.1. *Under the assumptions of Theorem 3.1, for any positive integer k,*

$$\tilde\mu_k \le \Big(1 + 2\sqrt{\frac{1-S_k}{\nu+2}}\Big)\Big(\frac{H_\Omega}{|\Omega|_g}k\Big)^{\frac{2}{\nu}} ⨍_\Omega w\, dv_g, \tag{32}$$

where

$$\tilde\mu_k = \mu_k - ⨍_\Omega \tilde V w\, dv_g \quad \textit{and} \quad S_k := \frac{\frac{1}{k}\sum_{j=0}^{k-1}\tilde\mu_j}{\big(\frac{H_\Omega k}{|\Omega|_g}\big)^{\frac{2}{\nu}} ⨍_\Omega w\, dv_g}.$$

Notice that according to Theorem 3.1 (2), $S_k \le 1$.

Proof of Corollary 3.1. We argue as in the proof of Theorem 3.1, rewriting (29) as follows. For every positive R,

$$k\tilde\mu_k - \sum_{j=0}^{k-1}\tilde\mu_j \ge \frac{|\Omega|_g \omega_N R^\nu}{C_{F(\Omega)}}\Big(\tilde\mu_k - \frac{\nu R^2}{N+2} ⨍_\Omega w\, dv_g\Big), \tag{33}$$

which yields

$$\Big(\frac{|\Omega|_g\omega_N R^\nu}{C_{F(\Omega)}} - k\Big)\tilde\mu_k \le \frac{\nu}{N+2}\frac{|\Omega|_g\omega_N R^{\nu+2}}{C_{F(\Omega)}} ⨍_\Omega w\, dv_g - \sum_{j=0}^{k-1}\tilde\mu_j.$$

With the change of variable $\sigma := \frac{|\Omega|_g \omega_N}{C_{F(\Omega)} k} R^\nu$, the last inequality reads

$$(\sigma - 1)\tilde{\mu}_k \le \frac{\nu}{N+2}\Big(\frac{C_{F(\Omega)} k}{|\Omega|_g \omega_N}\Big)^{\frac{2}{\nu}} \fint_\Omega w \, dv_g \; \sigma^{1+\frac{2}{\nu}} - \frac{1}{k}\sum_{j=0}^{k-1} \widetilde{\mu}_j$$

for all $\sigma > 1$. Taking the infimum with respect to F and N and using (19), we get

$$\begin{aligned}(\sigma - 1)\tilde{\mu}_k &\le \Big(\frac{H_\Omega k}{|\Omega|_g}\Big)^{\frac{2}{\nu}} \fint_\Omega w \, dv_g \; \sigma^{1+\frac{2}{\nu}} - \frac{1}{k}\sum_{j=0}^{k-1} \widetilde{\mu}_j \\ &= \Big(\frac{H_\Omega k}{|\Omega|_g}\Big)^{\frac{2}{\nu}} \fint_\Omega w \, dv_g (\sigma^{1+\frac{2}{\nu}} - S_k).\end{aligned}$$

That is,

$$\tilde{\mu}_k \le \Big(\frac{H_\Omega k}{|\Omega|_g}\Big)^{\frac{2}{\nu}} \fint_\Omega w \, dv_g \frac{\sigma^{1+\frac{2}{\nu}} - S_k}{\sigma - 1}. \tag{34}$$

This inequality can be explicitly optimized with respect to $\sigma \in [1, +\infty)$ only when $\nu = 2$, in which case $\sigma_+ = 1 + \sqrt{1 - S_k}$, yielding the desired bound. For $\nu > 2$ we introduce a change of variable, $\sigma = 1 + \alpha z_k$, where $z_k = (1 - S_k)^{\frac{1}{p}}$, $p = \dfrac{\nu+2}{\nu}$, and α is a free positive parameter. Then the bound (34) reads

$$\tilde{\mu_k} \le \Big(\frac{H_\Omega k}{|\Omega|_g}\Big)^{\frac{2}{\nu}} \fint_\Omega w \, dv_g \; \frac{(1 + \alpha z_k)^p - 1 + z_k^p}{\alpha z_k}.$$

Since $1 < p \le 2$ for all $\nu \ge 2$, it follows that

$$\begin{aligned}\frac{1}{p}\,\frac{(1+\alpha z_k)^p - 1 + z_k^p}{\alpha z_k} &= \frac{1}{\alpha z_k}\int_0^{\alpha z_k} (1+s)^{p-1} ds + \frac{z_k^{p-1}}{p\alpha} \\ &\le \frac{1}{\alpha z_k}\int_0^{\alpha z_k} (1 + (p-1)s)\, ds + \frac{z_k^{p-1}}{p\alpha} \\ &= 1 + \frac{(p-1)\alpha z_k}{2} + \frac{z_k^{p-1}}{p\alpha}.\end{aligned}$$

Optimizing with respect to α leads to the choice $\alpha^2 = \frac{2 z_k^{p-2}}{p(p-1)}$. Thus,

$$\frac{1}{p}\,\frac{(1+\alpha z_k)^p - 1 + z_k^p}{\alpha z_k} \le 1 + \sqrt{2}\sqrt{\frac{p-1}{p}} z_k^{\frac{p}{2}} = 1 + 2\frac{\sqrt{1 - S_k}}{\sqrt{\nu + 2}},$$

which implies the the claimed inequality. □

Corollary 3.2. *Under the assumptions of Theorem 3.1, for any integer $k \in \mathbb{N}$ such that $\sum_{j=0}^{k-1} \mu_j \geq 0$,*

$$\mu_k \Big(1 - \frac{\fint_\Omega \widetilde{V} w \, dv_g}{\mu_k}\Big)_+^{1+\frac{2}{\nu}} \leq \Big(\frac{\nu+2}{2}\Big)^{\frac{2}{\nu}} \Big(\frac{H_\Omega}{|\Omega|_g} k\Big)^{\frac{2}{\nu}} \fint_\Omega w \, dv_g. \tag{35}$$

In particular,

$$\mu_k \leq \max \left\{ 2 \fint_\Omega \widetilde{V} w \, dv_g ; 2(\nu+2)^{\frac{2}{\nu}} \Big(\frac{H_\Omega}{|\Omega|_g} k\Big)^{\frac{2}{\nu}} \fint_\Omega w \, dv_g \right\}. \tag{36}$$

Proof. From inequality (29) in the proof of Theorem 3.1, we deduce with

$$\sum_{j=0}^{k-1} \mu_j \geq 0$$

that for all $R \geq 0$,

$$k\mu_k - \frac{|\Omega|_g \omega_N R^\nu}{C_{F(\Omega)}} \left(\mu_k - \frac{\nu R^2}{N+2} \fint_\Omega w \, dv_g - \fint_\Omega \widetilde{V} w \, dv_g \right) \geq 0. \tag{37}$$

The left side achieves its minimum when $R = 0$ if $\mu_k \leq \fint_\Omega \widetilde{V} w \, dv_g$ and otherwise when $R^2 = \frac{N+2}{\nu+2} (\mu_k - \fint_\Omega \widetilde{V} w \, dv_g) / \fint_\Omega w \, dv_g$. Since (35) is obviously satisfied when $\mu_k \leq \fint_\Omega \widetilde{V} w \, dv_g$, we shall assume $\mu_k > \fint_\Omega \widetilde{V} w \, dv_g$ and get

$$k\mu_k - \frac{|\Omega|_g \omega_N}{C_{F(\Omega)}} \Big(\frac{N+2}{\nu+2}\Big)^{\frac{\nu}{2}} \frac{2}{\nu+2} \left(\mu_k - \fint_\Omega \widetilde{V} w \, dv_g \right)^{1+\frac{\nu}{2}} \left(\fint_\Omega w \, dv_g \right)^{-\frac{\nu}{2}} \geq 0,$$

which gives

$$\left(\mu_k - \fint_\Omega \widetilde{V} w \, dv_g \right)^{1+\frac{\nu}{2}} \leq \frac{C_{F(\Omega)}}{|\Omega|_g \omega_N} \Big(\frac{\nu+2}{N+2}\Big)^{\frac{\nu}{2}} \frac{\nu+2}{2} \left(\fint_\Omega w \, dv_g \right)^{\frac{\nu}{2}} k\mu_k.$$

Therefore,

$$\mu_k^{\frac{\nu}{2}} \left(1 - \frac{\fint_\Omega \widetilde{V} w \, dv_g}{\mu_k} \right)^{1+\frac{\nu}{2}} \leq \frac{C_{F(\Omega)}}{|\Omega|_g \omega_N} \Big(\frac{\nu+2}{N+2}\Big)^{\frac{\nu}{2}} \frac{\nu+2}{2} \left(\fint_\Omega w \, dv_g \right)^{\frac{\nu}{2}} k.$$

Raising this to the power $\frac{2}{\nu}$ and taking the infimum with respect to F and N, we obtain (35).

To prove (36) we observe that if $\mu_k > 2 \fint_\Omega \widetilde{V} w \, dv_g$, then $1 - \frac{\fint_\Omega \widetilde{V} w \, dv_g}{\mu_k} > \frac{1}{2}$, so we can deduce from (35) that

$$\Big(\frac{1}{2}\Big)^{1+\frac{2}{\nu}} \mu_k \leq \Big(\frac{\nu+2}{2}\Big)^{\frac{2}{\nu}} \Big(\frac{H_\Omega}{|\Omega|_g} k\Big)^{\frac{2}{\nu}} \fint_\Omega w \, dv_g. \qquad \square$$

Note that when $\rho = V = 0$, inequality (35) of Corollary 3.2 produces

$$\mu_k \le \left(\frac{\nu+2}{2}\right)^{\frac{2}{\nu}} \left(\frac{H_\Omega}{|\Omega|_g} k\right)^{\frac{2}{\nu}} \fint_\Omega w\, dv_g,$$

which coincides with Kröger's estimate [15, Corollary 2] when Ω is a Euclidean domain and $w = 1$ (just replace H_Ω by $\frac{(2\pi)^\nu}{\omega_\nu}$).

Let us highlight some consequences of Theorem 3.1 for Schrödinger operators, Witten Laplacians, and Laplacians associated with conformally Euclidean metrics.

Example 3.1 (Schrödinger operators). From (22) in Theorem 3.1 and (36) in Corollary 3.2 we deduce that for any Schrödinger operator $\Delta_g + V$ on Ω (with $\rho = 0$ and $w = 1$), and any integer $k \ge 0$,

$$\frac{1}{k} \sum_{j=0}^{k-1} \mu_j(\Delta_g + V) \le \frac{\nu}{\nu+2} \left(\frac{H_\Omega}{|\Omega|_g} k\right)^{\frac{2}{\nu}} + \fint_\Omega V\, dv_g. \tag{38}$$

Furthermore, if $\sum_{j=0}^{k-1} \mu_j(\Delta_g + V) \ge 0$, then

$$\mu_k(\Delta_g + V) \le \max\left\{2 \fint_\Omega V\, dv_g ; 2(\nu+2)^{\frac{2}{\nu}} \left(\frac{H_\Omega}{|\Omega|_g} k\right)^{\frac{2}{\nu}}\right\}. \tag{39}$$

These estimates are to be compared with [6, Theorem 2.2 and Corollary 2.8], [7, Theorem 2.1], and the results by Grigor'yan, Netrusov and Yau [9, Theorem 5.15 and (1.14)] by which, under the assumption that $\mu_0(\Delta_g + V) \ge 0$,

$$\mu_k(\Delta_g + V) \le C(\Omega) k + \frac{1}{\varepsilon(\Omega)} \fint_\Omega V\, dv_g \tag{40}$$

where $C(\Omega) > 0$ and $\varepsilon(\Omega) \in (0, 1)$ are two Riemannian constants that do not depend on V or k. They asked whether such an estimate holds true with $\varepsilon(\Omega) = 1$. Inequality (38) answers this question for the eigenvalue sums $\sum_{j=0}^{k-1} \mu_j$ in the affirmative, without any positivity condition.

On the other hand, unlike the upper bound in (40), our estimates (38) and (39) are consistent with the Weyl law regarding the power of k. Notice that (40) has been recently improved by A. Hassannezhad [11] who obtained

$$\mu_k(\Delta_g + V) \le C(\Omega) + A_\nu \fint_\Omega V\, dv_g + B_\nu \left(\frac{k}{|\Omega|_g}\right)^{\frac{2}{\nu}}$$

under the same assumption of positivity of $\mu_0(\Delta_g + V)$, where $A_\nu > 1$ and B_ν are two constants that only depend on the dimension ν, and $C(\Omega)$ is a Riemannian constant that does not depend on V or k. Our estimates are valid, however, under weaker assumptions and, moreover, the coefficient in front of $\fint_\Omega V\, dv_g$ in (39) is equal to 1 while the other coefficient is explicitly computable at least in the elementary case where Ω is conformally Euclidean.

Example 3.2 (Witten Laplacians). Let Ω be a bounded domain of a Riemannian manifold (M, g) and let Δ_ρ be the Witten Laplacian associated with the density $e^{-2\rho}$, that is,

$$\Delta_\rho \varphi = \Delta_g \varphi + 2(\nabla^g \rho, \nabla^g \varphi).$$

The Neumann eigenvalues $\{\mu_l\}$ of Δ_ρ in Ω satisfy the following estimates.

(1) For all $z \in \mathbb{R}$,

$$\sum_{j\geq 0}(z-\mu_j)_+ \geq \frac{2\,|\Omega|_g}{(\nu+2)H_\Omega}\left(z - \fint_\Omega |\nabla^g \rho|^2\, dv_g\right)_+^{1+\frac{\nu}{2}}. \tag{41}$$

(2) For all $k \in \mathbb{N}^*$,

$$\frac{1}{k}\sum_{j=0}^{k-1}\mu_j \leq \frac{\nu}{\nu+2}\Big(\frac{H_\Omega}{|\Omega|_g}k\Big)^{\frac{2}{\nu}} + \fint_\Omega |\nabla^g \rho|^2\, dv_g. \tag{42}$$

(3) For all $k \in \mathbb{N}^*$,

$$\mu_k\left(1 - \frac{\fint_\Omega |\nabla^g \rho|^2\, dv_g}{\mu_k}\right)_+^{1+\frac{2}{\nu}} \leq \Big(\frac{\nu+2}{2}\Big)^{\frac{2}{\nu}}\Big(\frac{H_\Omega}{|\Omega|_g}k\Big)^{\frac{2}{\nu}}. \tag{43}$$

In particular,

$$\mu_k \leq \max\left\{2\fint_\Omega |\nabla^g \rho|^2\, dv_g ; 2(\nu+2)^{\frac{2}{\nu}}\Big(\frac{H_\Omega}{|\Omega|_g}k\Big)^{\frac{2}{\nu}}\right\}. \tag{44}$$

This last inequality is to be compared with the estimates obtained in [4].

For example, when Ω is a bounded domain of $\mathbb{R}^\nu$ endowed with the Gaussian density $e^{-|x|^2/2}$, the consequence for the corresponding Witten Laplacian is

$$\frac{1}{k}\sum_{j=0}^{k-1}\mu_j \leq \frac{\nu}{\nu+2}\Big(4\pi^2\Big(\frac{k}{|\Omega|\omega_\nu}\Big)^{\frac{2}{\nu}} + \frac{\omega_\nu}{|\Omega|}R^{\nu+2}\Big),$$

where R is chosen so that Ω is contained in the Euclidean ball B_R.

Example 3.3 (Laplacian associated with a conformally Euclidean metric). Let Ω be a bounded domain of $\mathbb{R}^\nu$ and let $g = \alpha^{-2} g_E$ be a Riemannian metric that is conformal to the Euclidean metric g_E. The Neumann eigenvalues $\{\mu_l\}$ of the Laplacian Δ_g in Ω satisfy the following estimates, in which $|\Omega|$ denotes the Euclidean volume of Ω.

(1) For all $z \in \mathbb{R}$,

$$\sum_{j\geq 0}(z-\mu_j)_+ \geq \frac{2\omega_\nu|\Omega|}{(\nu+2)(2\pi)^\nu}\left(\fint_\Omega \alpha^2\, d^\nu x\right)^{\frac{\nu}{2}}\left(z-\frac{\nu^2}{4}\fint_\Omega |\nabla\alpha|^2\, d^\nu x\right)_+^{1+\frac{\nu}{2}}. \tag{45}$$

(2) For all $k \in \mathbb{N}$,

$$\frac{1}{k}\sum_{j=0}^{k-1}\mu_j \leq \frac{4\pi^2\nu}{\nu+2}\Big(\frac{k}{\omega_\nu|\Omega|}\Big)^{\frac{2}{\nu}}\fint_\Omega \alpha^2\, d^\nu x + \frac{\nu^2}{4}\fint_\Omega |\nabla\alpha|^2\, d^\nu x. \tag{46}$$

(3) For all $k \in \mathbb{N}$,

$$\mu_k\left(1-\frac{\nu^2}{4}\frac{\fint_\Omega |\nabla\alpha|^2\, d^\nu x}{\mu_k}\right)_+^{1+\frac{2}{\nu}} \leq 4\pi^2\Big(\frac{\nu+2}{2}\Big)^{\frac{2}{\nu}}\Big(\frac{k}{\omega_\nu|\Omega|}\Big)^{\frac{2}{\nu}}\fint_\Omega \alpha^2\, d^\nu x. \tag{47}$$

In particular,

$$\mu_k \leq \max\left\{\frac{\nu^2}{2}\fint_\Omega |\nabla\alpha|^2\, d^\nu x; 8\pi^2(\nu+2)^{\frac{2}{\nu}}\Big(\frac{k}{\omega_\nu|\Omega|}\Big)^{\frac{2}{\nu}}\fint_\Omega \alpha^2\, d^\nu x\right\}. \tag{48}$$

Note that a domain of the hyperbolic space $\mathbf{H}$ can be identified with a domain of the Euclidean unit ball endowed with the metric $g = \big(\frac{2}{1-|x|^2}\big)^2 g_E$. For any such domain, with $\alpha = \frac{1-|x|^2}{2}$, $\fint_\Omega \alpha^2\, d^\nu x \leq \frac{1}{4}$, and $\fint_\Omega |\nabla\alpha|^2\, d^\nu x = \fint_\Omega |x|^2\, d^\nu x$, we get

$$\frac{1}{k}\sum_{j=0}^{k-1}\mu_j \leq \frac{\pi^2\nu}{\nu+2}\Big(\frac{k}{\omega_\nu|\Omega|}\Big)^{\frac{2}{\nu}} + \frac{\nu^2}{4}\fint_\Omega |x|^2\, d^\nu x, \tag{49}$$

$$\mu_k\left(1-\frac{\nu^2}{4}\frac{\fint_\Omega |x|^2\, d^\nu x}{\mu_k}\right)_+^{1+\frac{2}{\nu}} \leq \pi^2\Big(\frac{\nu+2}{2}\Big)^{\frac{2}{\nu}}\Big(\frac{k}{\omega_\nu|\Omega|}\Big)^{\frac{2}{\nu}}, \tag{50}$$

and

$$\mu_k \leq \max\left\{\frac{\nu^2}{2}\fint_\Omega |x|^2\, d^\nu x; 2\pi^2(\nu+2)^{\frac{2}{\nu}}\Big(\frac{k}{\omega_\nu|\Omega|}\Big)^{\frac{2}{\nu}}\right\}. \tag{51}$$

4. Sums of Neumann eigenvalues on domains conformal to Euclidean sets, and phase-space volumes

A phase-space analysis can considerably sharpen the upper bounds on sums of eigenvalues from the previous sections so that they become sharp in the semiclassical regime. Following physical tradition, it is shown in [21] how this may be achieved in some circumstances with the aid of coherent states. We carry out such an analysis in this section for (1) when $(M, g) = (\mathbb{R}^\nu, d^\nu x)$. We must first introduce a few quantities that will be helpful to relate spectral estimates to phase-space volumes. To avoid complications we assume that the potential energy V is Lipschitz continuous and bounded from below. We do not assume that Ω is necessarily bounded, but if it is not, we require V to be *confining* in the sense that there is a radial function $V_{\mathrm{rad}}(r)$ tending to $+\infty$ as $r \to \infty$ with $V(\mathbf{x}) \geq V_{\mathrm{rad}}(|\mathbf{x}|)$ for all $\mathbf{x} \notin \Omega$. This condition is sufficient to ensure that the eigenvalues form a discrete sequence tending to $+\infty$.

Definition 4.1. The *effective potential* incorporating a correction for the conformal transformation will be denoted $\widetilde{V}(\mathbf{x}) := V(\mathbf{x}) + |\nabla\rho|^2(\mathbf{x})$, and the maximal Lipschitz constant of $\widetilde{V}(\mathbf{x})$ on the region $\Omega \cap \{\mathbf{x}: \widetilde{V}(\mathbf{x}) \leq \Lambda\}$ will be denoted $\mathrm{Lip}(\Lambda)$.

The L^2-normalized ground-state Dirichlet eigenfunction for the ball of geodesic radius r in M will be denoted h_r and $\mathcal{K}(h_r) := \int_{B_r} |\nabla h_r(\mathbf{x})|^2 d^\nu x$. I.e., in this section where $M = \mathbb{R}^\nu$, h is a scaled Bessel function and

$$\mathcal{K}(h_r) = \frac{j^2_{\frac{\nu}{2}-1,1}}{r^2}.$$

Remark 4.1. The function h_r will ensure that some coherent-state functions to be defined below are localized in configuration space. Its specific form is but one of many plausible choices.

We next recall some quantities that arise in phase-space analysis.

Definition 4.2. The *Euclidean phase-space volume* for energy Λ is defined as

$$\Phi_1(\Lambda) := \frac{1}{(2\pi)^\nu}|(\mathbf{x}, \mathbf{p}): |\mathbf{p}|^2 + \widetilde{V}(\mathbf{x}) \leq \Lambda| = \frac{\omega_\nu}{(2\pi)^\nu}\int_\Omega (\Lambda - \widetilde{V}(\mathbf{x}))_+^{\frac{\nu}{2}} d^\nu x,$$

according to a standard calculation to be found, for example, in [21]. If the weight in (1) is not constant, we make use of a *weighted phase-space volume,*

$$\Phi_w(\Lambda) = \frac{\omega_\nu}{(2\pi)^\nu}\int_\Omega (\Lambda - \widetilde{V}(\mathbf{x}))_+^{\frac{\nu}{2}} w(\mathbf{x}) d^\nu x.$$

The total energy associated with this quantity is correspondingly

$$\begin{aligned} E_w(\Lambda) &:= \frac{1}{(2\pi)^\nu} \int_{\{(\mathbf{x},\mathbf{p}):\mathbf{x}\in\Omega,|\mathbf{p}|^2+\widetilde{V}(\mathbf{x})\le\Lambda\}} (|\mathbf{p}|^2 + \widetilde{V}(\mathbf{x}))w(\mathbf{x})d^\nu x d^\nu p \\ &= \frac{\nu}{\nu+2}\frac{\omega_\nu}{(2\pi)^\nu}\int_\Omega (\Lambda - \widetilde{V}(\mathbf{x}))_+^{1+\frac{\nu}{2}} w(\mathbf{x}) d^\nu x. \end{aligned} \tag{52}$$

We note that according to (52),

$$\frac{dE_w}{d\Lambda}(\Lambda) = \Phi_w(\Lambda), \tag{53}$$

and that Φ_w increases strictly monotonically in Λ, implying that E_w is strictly convex.

Theorem 4.1. *Let $\mu_0 \le \mu_1 \le \cdots$ be the variationally defined Neumann eigenvalues (2) on an open set $\Omega \in \mathbb{R}^\nu$, where w, ρ, and V satisfy the assumptions stated above, and define $\Lambda(k)$ as the minimal value of Λ for which $\Phi_1(\Lambda) \ge (2\pi)^\nu k$. Then*

$$\sum_{j=0}^{k-1}\mu_j \le E_w(\Lambda(k)) + 3(2j_{\nu-1,1}^2\operatorname{Lip}(\Lambda(k)))^{\frac{1}{3}}\Phi_w(\Lambda(k) + (2j_{\nu-1,1}^2\operatorname{Lip}(\Lambda(k))^{\frac{1}{3}})). \tag{54}$$

The Riesz-mean form of the inequality reads

$$\begin{aligned} \sum_{j=0}^{k-1}(z-\mu_j)_+ \ge{}& \frac{2}{\nu+2}\frac{\omega_\nu}{(2\pi)^\nu}\int (z - V(y))_+^{1+\frac{\nu}{2}} dy \\ &- \left(\frac{\omega_\nu}{(2\pi)^\nu}\int (z-V(y))_+^{\frac{\nu}{2}} dy\right)\left(\|\nabla h_r\|^2 + \int |\mathbf{x}|h_r^2\right). \end{aligned} \tag{55}$$

Remarks 4.1. 1. We call attention to the fact that the condition in this theorem defining Λ uses the Euclidean phase space, whereas weighted phase-space quantities appear in (54).

2. The dominant term in the semiclassical regime can be identified by introducing a small parameter α as a coefficient of $|\nabla\varphi|^2$ in (1), i.e.,

$$\mathcal{R}_\alpha(\varphi) := \frac{\int_\Omega (\alpha|\nabla\varphi(\mathbf{x})|^2 + V(\mathbf{x})|\varphi(\mathbf{x})|^2)w(\mathbf{x})e^{-2\rho(\mathbf{x})}dv_g}{\int_\Omega |\varphi(\mathbf{x})|^2 e^{-2\rho(\mathbf{x})}dv_g}. \tag{56}$$

The result, in the Riesz-mean form after choosing a convenient relationship between r and α, is

$$\sum_{j=0}^{k-1} (z-\mu_j)_+ \geq \alpha^{-\frac{\nu}{2}} \frac{2}{\nu+2} \frac{\omega_\nu}{(2\pi)^\nu} \int (z-V(y))_+^{1+\frac{\nu}{2}}\, dy - \left(\alpha^{\frac{1}{3}-\frac{\nu}{2}} \frac{\omega_\nu}{(2\pi)^\nu} \int (z-V(y))_+^{\frac{\nu}{2}}\, dy\right) \left(\|\nabla h_1\|^2 + \int |\mathbf{x}| h_1^2\right), \tag{57}$$

in which the leading term is precisely the expected semiclassical expression, in contrast to results of the previous section such as Example 3.3.

3. Inequalities of the type (54) imply estimates of quantities including the trace of the heat kernel (= the partition function in quantum physics) and the spectral zeta function by simple transforms. For instance (54) implies for the Riesz mean $R_1(z) := \sum_j (z-\mu_j)_+$ that

$$\begin{aligned} R_1(z) \geq \frac{1}{(2\pi)^\nu} (z\Phi_w(\Lambda(z)) &- E_w(\Lambda(k)) \\ &+ 3(2j_{\nu-1,1}^2 \operatorname{Lip}(\Lambda(k)))^{\frac{1}{3}} \Phi_w(\Lambda(k) \\ &+ (2j_{\nu-1,1}^2 \operatorname{Lip}(\Lambda(k))^{\frac{1}{3}})). \end{aligned} \tag{58}$$

The Riesz mean is in turn related to the heat trace by the Laplace transform (31).

Proof of Theorem 4.1. We apply Theorem 2.1 to the Neumann eigenvalues of (1) as defined by (2), using for test functions "coherent states" [21, 27] of the form

$$f_\zeta(\mathbf{x}) := \frac{1}{(2\pi)^{\nu/2}} e^{i\mathbf{p}\cdot(\mathbf{x})+\rho(\mathbf{x})} h_r(\mathbf{x}-\mathbf{y}).$$

In this formula, $\zeta = (\mathbf{p}, \mathbf{y})$ ranges over the phase space $\mathfrak{M} = \mathbb{R}^{2\nu}$ with Lebesgue measure. The radius r will be chosen below.

We note that the inner product that appears is a Fourier transform with respect to the variable $\mathbf{x}$, *viz.*,

$$\langle \phi, f_\zeta \rangle = \mathfrak{F}[h_r(\mathbf{x}-\mathbf{y}) e^{-\rho(\mathbf{x})} \phi(\mathbf{x})],$$

where if Ω is a strict subset of $\mathbb{R}^\nu$, then ϕ is extended by 0 outside Ω. Thus, with the Parseval identity,

$$\begin{aligned} \int_{\mathbb{R}^{2\nu}} |\langle \phi, f_\zeta \rangle|^2 d^\nu p\, d^\nu y &= \int_{\mathbb{R}^\nu} \int_{\mathbb{R}^\nu} h_r(\mathbf{x}-\mathbf{y})^2 |\phi|^2 e^{-2\rho} d^\nu y\, d^\nu x \\ &= \int_{\mathbb{R}^\nu} |\phi|^2 e^{-2\rho} d^\nu x \\ &= \|\phi\|^2. \end{aligned} \tag{59}$$

The set $\mathfrak{M}_0$ in Theorem 2.1 must be taken large enough so that

$$\begin{aligned} k &\le \int_{\mathfrak{M}_0} \|f_\xi\|^2 d\sigma \\ &= \frac{1}{(2\pi)^\nu} \int_{\mathfrak{M}_0} \int_\Omega h_r^2(\mathbf{x}-\mathbf{y}) e^{2\rho(\mathbf{x})-2\rho(\mathbf{x})} d^\nu x d^\nu p\, d^\nu y \\ &\le \frac{1}{(2\pi)^\nu} |\mathfrak{M}_0|, \end{aligned} \tag{60}$$

in which case

$$\begin{aligned} \sum_{j=0}^{k-1} \mu_j &\le \int_{\mathfrak{M}_0} \mathcal{R}(f_\xi) d^\nu p\, d^\nu y \\ &= \frac{1}{(2\pi)^\nu} \int_{\mathfrak{M}_0 \times \Omega} ((|\mathbf{p}|^2 + V(\mathbf{x})) h_r^2(\mathbf{x}-\mathbf{y}) \\ &\qquad + |\nabla h_r(\mathbf{x}-\mathbf{y}) + h_r(\mathbf{x}-\mathbf{y}) \nabla\rho(\mathbf{x})|^2) w(\mathbf{x}) d^\nu x\, d^\nu p\, d^\nu y. \end{aligned} \tag{61}$$

We now make the ansatz that $\mathfrak{M}_0 = \mathfrak{M}_0(\Lambda) := \{(\mathbf{x}, \mathbf{p}) : \mathbf{x} \in \Omega, |\mathbf{p}|^2 + \widetilde{V}(\mathbf{x}) \le \Lambda\}$, where $\Lambda \ge \Lambda(k)$, defined as the minimum value of Λ for (60) to be valid. Thus the upper bound in (60) becomes $\frac{1}{(2\pi)^\nu} \Phi_1(\Lambda)$, whence the condition in the theorem.

Since the support of h is restricted to a ball of radius r, the x-integral may be restricted to the set

$$\{\mathbf{x} \in \Omega : \text{there exists } \mathbf{y} \text{ such that } |\mathbf{y}-\mathbf{x}| \le r, |\mathbf{p}|^2 + \widetilde{V}(\mathbf{x}) \le \Lambda\} \subset \mathfrak{M}_0(\Lambda + \mathrm{Lip}(\Lambda) r).$$

Thus, integrating first in $\mathbf{y}$, the right side of (61) is bounded above by

$$\begin{aligned} &\frac{1}{(2\pi)^\nu} \int_{\{\mathbf{p} : |\mathbf{p}|^2 \le \Lambda\}} \\ &\qquad \int_{\{\mathbf{x} : \widetilde{V}(\mathbf{x}) \le \Lambda + \mathrm{Lip}(\Lambda) r - |\mathbf{p}|^2\}} \\ &\qquad\qquad \int_{\mathbb{R}^\nu} ((|\mathbf{p}|^2 + V(\mathbf{x})) h_r^2(\mathbf{x}-\mathbf{y}) \\ &\qquad\qquad\quad + |\nabla h_r(\mathbf{x}-\mathbf{y}) + h_r(\mathbf{x}-\mathbf{y}) \nabla\rho(\mathbf{x})|^2) w(\mathbf{x}) d^\nu y\, d^\nu x\, d^\nu p \\ &= \frac{1}{(2\pi)^\nu} \int_{\{\mathbf{p} : |\mathbf{p}|^2 \le \Lambda\}} \\ &\qquad \int_{\{\mathbf{x} : \widetilde{V}(\mathbf{x}) \le \Lambda + \mathrm{Lip}(\Lambda) r - |\mathbf{p}|^2\}} \\ &\qquad\qquad \int_{\mathbb{R}^\nu} ((|\mathbf{p}|^2 + \widetilde{V}(\mathbf{x})) h_r^2(\mathbf{x}-\mathbf{y}) + |\nabla h_r(\mathbf{x}-\mathbf{y})|^2 \\ &\qquad\qquad\quad + \nabla\rho(\mathbf{x}) \cdot \nabla h_r^2(\mathbf{x}-\mathbf{y})) w(\mathbf{x}) d^\nu y\, d^\nu x\, d^\nu p. \end{aligned}$$

The last contribution vanishes because

$$\int_{\mathbb{R}^\nu} \nabla\rho(\mathbf{x}) \cdot \nabla h_r^2(\mathbf{x}-\mathbf{y}) d^\nu y = \int_{\mathbb{R}^\nu} \nabla\rho(\mathbf{x}) \cdot \nabla 1 d^\nu y = 0, \tag{62}$$

leaving

$$\sum_{j=0}^{k-1} \mu_j \leq \frac{1}{(2\pi)^\nu} \int_{\mathfrak{M}_0(\Lambda+\mathrm{Lip}(\Lambda)r)} (|\mathbf{p}|^2 + \widetilde{V}(\mathbf{x}) + \mathcal{K}(h_r)) w(\mathbf{x}) d^\nu x \, d^\nu p \tag{63}$$

for all values of $r > 0$. The upper bound (63) is of the form

$$\frac{1}{(2\pi)^\nu} (E_w(\Lambda + \mathrm{Lip}(\Lambda)r) + \Phi_w(\Lambda + \mathrm{Lip}(\Lambda)r)\mathcal{K}(h_r)) \tag{64}$$

$$\leq \frac{1}{(2\pi)^\nu} \Big(E_w(\Lambda) + \Big(\frac{j_{\nu-1,1}^2}{r^2} + \mathrm{Lip}(\Lambda) r \Big) \Phi_w(\Lambda + \mathrm{Lip}(\Lambda)r) \Big), \tag{65}$$

where we have made use of (53) and the monotonicity of Φ_w in a first-order expansion of E_w. Choosing the optimal value $r = \big(\frac{2j_{\nu-1,1}^2}{L}\big)^{\frac{1}{3}}$, we get the claim (54).

The derivation of (55) proceeds similarly. □

Examples 4.1. *We note the following special cases of particular interest.*

1. *Laplace operators with Neumann conditions on a compact Euclidean domain* ($V = \rho = 0$, $w = 1$). *In this case* $\mathrm{Lip}(\Lambda) = 0$,

$$\Lambda^{\frac{\nu}{2}} := \frac{(2\pi)^\nu}{B_\nu} \frac{k}{|\Omega|},$$

and we recover the inequality of Kröger, that

$$\sum_{j=0}^{k-1} \mu_j \leq \frac{\nu}{\nu+2} \frac{\omega_\nu}{(2\pi)^\nu} |\Omega| \Lambda^{1+\frac{\nu}{2}} = \frac{\nu}{\nu+2} (2\pi)^2 \omega_\nu^{-\frac{2}{\nu}} \frac{k^{\frac{\nu+2}{\nu}}}{|\Omega|^{\frac{2}{\nu}}}.$$

We observe that without the potential V the introduction of the function h_r is not needed for the proof.

2. *Nonhomogeneous problems with $\rho = V = 0$, but w is variable, under Neumann conditions*:

$$\sum_{j=0}^{k-1} \mu_j \leq \frac{\nu}{\nu+2} \frac{\omega_\nu}{(2\pi)^\nu} \left(\int_\Omega w(\mathbf{x}) d^\nu x \right) \Lambda^{1+\frac{\nu}{2}}.$$

The eigenvalue bounds of Corollary 3.1 are sharp as k tends to infinity. Since in this case $\widetilde{\mu}_k = \mu_k$, *we get*

$$\mu_k \le 4\pi^2 \fint_\Omega w(\mathbf{x}) d^\nu x \Big(\frac{k}{|\Omega|\omega_\nu}\Big)^{\frac{2}{\nu}} \Big(1 + 2\sqrt{\frac{1 - S_k}{\nu + 2}}\Big)$$

with S_k *as given in Corollary 3.1*:

$$S_k = \frac{\frac{\nu+2}{\nu}\frac{1}{k}\sum_{j=0}^{k-1}\mu_j}{4\pi^2 \fint_\Omega w(\mathbf{x}) d^\nu x \big(\frac{k}{|\Omega|\omega_\nu}\big)^{\frac{2}{\nu}}} \le 1.$$

5. Bounds for Neumann eigenvalues on subdomains of compact homogeneous spaces

In this section, we deal with the case where the ambient space is a compact homogeneous Riemannian manifold (M, g) with isomorphism group denoted G. In particular, we shall recover Strichartz's result [26] with a more efficient proof and extend it to a wider class of operators. We begin with bounds in the spirit of Theorem 3.1 and then derive a phase-space bound analogous to Theorem 4.1.

Let us denote by

$$\mathrm{spec}(M) = \{0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots\}$$

the spectrum of the Laplace-Beltrami operator Δ_g on M (each eigenvalue is repeated according to its multiplicity). Although 0 is a simple eigenvalue, all the other eigenvalues are degenerate owing to the transitive action of the isometry group G (recall that the eigenspaces are invariant under the action of G).

Given a regular domain $\Omega \subset M$ endowed with densities $e^{-2\rho}$ and $e^{-2\theta} = we^{-2\rho}$, and a potential V, we consider the eigenvalues $\mu_l(\Omega, g, \rho, w, V)$, $l \in \mathbb{N}$, defined by (1) and (2) and seek for relationships between the μ_l's and the λ_l's . As before, we use the notation $\widetilde{V} = V + |\nabla^g \rho|^2$. We also require the subspaces introduced in Theorem 2.1:

$$E_0(R) = \bigoplus_{\mu < R} \ker(H - \mu I) \quad \text{and} \quad E(R) = \bigoplus_{\mu \le R} \ker(H - \mu I),$$

where $H = H(\Omega, g, \rho, w, V)$ is the operator defined by (4). The corresponding subspaces associated with the Laplacian Δ_g on M will be denoted

$$F_0(R) = \bigoplus_{\lambda < R} \ker(\Delta_g - \lambda I) \quad \text{and} \quad F(R) = \bigoplus_{\lambda \le R} \ker(\Delta_g - \lambda I).$$

Theorem 5.1. *Let (M, g) be a compact homogeneous Riemannian manifold. Let $\mu_l = \mu_l(\Omega, g, \rho, w, V)$, $l \in \mathbb{N}$, be the eigenvalues defined by (2) on a bounded open set $\Omega \subset M$. Then, for all $z \in \mathbb{R}$,*

$$\sum_{j\geq 0} (z-\mu_j)_+ \geq \frac{|\Omega|_g}{|M|_g} \sum_{j\geq 0} (z-\tilde{\lambda}_j)_+, \tag{66}$$

where $\tilde{\lambda}_j := \lambda_j \fint_\Omega w\, dv_g + \fint_\Omega \widetilde{V} w\, dv_g$. Equality holds in (66) for some $z \in \mathbb{R}$ if and only if

$$E_0(z) \subset e^\rho F(\tilde{z}) \subset E(z),$$

with $\tilde{z} := \frac{1}{\fint_\Omega w\, dv_g}\big(z - \fint_\Omega \widetilde{V} w\, dv_g\big)$.

Proof. Let $\{y_\lambda : \lambda \in \operatorname{spec}(M)\}$ be an orthonormal basis of $L^2(M, g)$ with $\Delta_g y_\lambda = \lambda y_\lambda$. The proof relies on Theorem 2.1, in which we take $\mathfrak{M} = \operatorname{spec}(M)$ endowed with the uniform discrete measure and use test functions of the form

$$f_\lambda = y_\lambda e^\rho.$$

For any function $\psi \in L^2(\Omega, e^{-2\rho} dv_g)$, endowed with the norm $\|\psi\|^2 = \int_\Omega \psi^2 e^{-2\rho} dv_g$,

$$\begin{aligned}
\int_{\mathfrak{M}} \langle f_\lambda, \psi\rangle^2 d\lambda &= \sum_{\lambda\in\operatorname{spec}(M)} \Big(\int_\Omega f_\lambda \psi e^{-2\rho} dv_g\Big)^2 \\
&= \sum_{\lambda\in\operatorname{spec}(M)} \Big(\int_\Omega y_\lambda \psi e^{-\rho} dv_g\Big)^2 \\
&= \sum_{\lambda\in\operatorname{spec}(M)} \Big(\int_M y_\lambda \psi e^{-\rho} dv_g\Big)^2 \\
&= \int_M \psi^2 e^{-2\rho} dv_g \\
&= \|\psi\|^2,
\end{aligned}$$

where we used the same notation ψ to designate the extension of ψ by zero outside Ω.

Let $R > 0$ and let $\mathfrak{M}_0 = \{\lambda \in \mathfrak{M} : \lambda \leq R\}$. Due to the transitive action of the isometry group G on M, for every eigenvalue Λ of Δ_g, with multiplicity m_Λ, the basis $\{y_\lambda : \lambda = \Lambda\}$ of the corresponding eigenspace is such that $\sum_{\lambda=\Lambda} y_\lambda^2$ is constant on M. Integrating over M, we get

$$\sum_{\lambda=\Lambda} y_\lambda^2 = \frac{m_\Lambda}{|M|_g}. \tag{67}$$

Moreover, $0 = \frac{1}{2}\Delta_g(\sum_{\lambda=\Lambda} y_\lambda^2) = \sum_{\lambda=\Lambda}(\Lambda y_\lambda^2 - |\nabla^g y_\lambda|^2)$, that is,

$$\sum_{\lambda=\Lambda} |\nabla^g y_\lambda|^2 = \frac{m_\Lambda}{|M|_g}\Lambda. \tag{68}$$

Therefore,

$$\int_{\mathfrak{M}_0} \|f_\lambda\|^2 d\lambda = \sum_{\lambda\le R}\int_\Omega y_\lambda^2 dv_g = \sum_{\Lambda\le R}\frac{|\Omega|_g}{|M|_g} m_\Lambda = \frac{|\Omega|_g}{|M|_g}N(R),$$

where $N(R)$ is the number of eigenvalues of Δ_g on M that are less than or equal to R (counted with multiplicity). On the other hand, using (67) and (68), for every Λ we get

$$\begin{aligned}\sum_{\lambda=\Lambda} |\nabla^g f_\lambda|^2 &= e^{2\rho}\sum_{\lambda=\Lambda}(|\nabla^g y_\lambda|^2 + y_\lambda^2|\nabla^g\rho|^2 + g(\nabla^g\rho, \nabla^g y_\lambda^2))\\ &= \frac{m_\Lambda}{|M|_g}e^{2\rho}(\Lambda + |\nabla^g\rho|^2).\end{aligned}$$

Thus

$$\begin{aligned}\int_{\mathfrak{M}_0} Q(f_\lambda, f_\lambda)d\lambda &= \sum_{\lambda\le R}\int_\Omega (|\nabla^g f_\lambda|^2 + V f_\lambda^2) w e^{-2\rho} dv_g\\ &= \sum_{\Lambda\le R}\int_\Omega \frac{m_\Lambda}{|M|_g}(\Lambda + |\nabla^g\rho|^2 + V) w dv_g \frac{\int_\Omega w\, dv_g}{|M|_g}\\ &\quad \sum_{\lambda\le R}\lambda + \frac{\int_\Omega \widetilde{V} w\, dv_g}{|M|_g}N(R).\end{aligned}$$

Inserting this into (5), we get for every $z \in \mathbb{R}$ and $R > 0$,

$$\begin{aligned}\sum_{j\ge 0}(z-\mu_j)_+ &\ge z\frac{|\Omega|_g}{|M|_g}N(R) - \frac{\int_\Omega w\, dv_g}{|M|_g}\sum_{\lambda\le R}\lambda - \frac{\int_\Omega \widetilde{V} w\, dv_g}{|M|_g}N(R)\\ &= \frac{|\Omega|_g}{|M|_g}\sum_{\lambda\le R}\left(z - \lambda \fint_\Omega w\, dv_g - \fint_\Omega \widetilde{V} w\, dv_g\right).\end{aligned} \tag{69}$$

Notice that the right side is negative if $z \le \fint_\Omega \widetilde{V} w\, dv_g$. When $z > \fint_\Omega \widetilde{V} w\, dv_g$, we can choose

$$R = \tilde{z} = \frac{z - \fint_\Omega \widetilde{V} w\, dv_g}{\fint_\Omega w\, dv_g}$$

so that the last sum is taken over all eigenvalues λ for which the contributions are nonnegative, thus

$$\sum_{j\geq 0}(z-\mu_j)_+ \geq \frac{|\Omega|_g}{|M|_g}\sum_{j\geq 0}\left(z-\lambda_j \fint_\Omega w\,dv_g - \fint_\Omega \tilde{V} w\,dv_g\right)_+ .$$

Regarding the case of equality, it follows from Theorem 2.1 that equality holds in (66) if and only if $f_\lambda \in E(z)$ for $\lambda \leq \tilde{z}$ and f_λ is orthogonal to $E_0(z)$ for $\lambda > \tilde{z}$. Equivalently, $e^\rho F(\tilde{z}) \subset E(z)$ and, since $\mathrm{Span}\{f_\lambda : \lambda > \tilde{z}\}$ is the orthogonal complement of $e^\rho F(\tilde{z})$, $E_0(z) \subset e^\rho F(\tilde{z})$. □

As we have seen in the previous sections, our technique allows bounds on eigenvalue sums to be obtained. In order to simplify the statement of these bounds, we intoduce the following notation. Given a sequence $(a) = (a_k)_{k\geq 0}$ of real numbers and $p \in [1, +\infty)$, we set

$$\mathfrak{S}_{(a)}(p) = \sum_{j=0}^{\lfloor p\rfloor -1} a_j + (p - \lfloor p\rfloor) a_{\lfloor p\rfloor},$$

so that when p is an integer, $\mathfrak{S}_{(a)}(p)$ is nothing other than the sum of the first p terms $a_0, \cdots, a_{p-1}$ of the sequence (a).

Theorem 5.2. *Let (M, g) be a compact homogeneous Riemannian manifold. Let $(\mu) = (\mu_l)_{l\geq 0}$ be the sequence of eigenvalues defined by (2) on an open set $\Omega \subset M$. Then, for every $p \in [1, +\infty)$,*

$$\mathfrak{S}_{(\mu)}(p) \leq \frac{|\Omega|_g}{|M|_g}\, \mathfrak{S}_{(\tilde{\lambda})}\Big(\frac{|M|_g}{|\Omega|_g}\, p\Big), \tag{70}$$

where $(\tilde{\lambda}) = (\tilde{\lambda}_l)_{l\geq 0}$ is the sequence defined by

$$\tilde{\lambda}_l = \lambda_l \fint_\Omega w\,dv_g + \fint_\Omega \tilde{V} w\,dv_g .$$

Moreover, equality holds in (70) for some $p = k \in \mathbb{N}^*$ if and only if

$$E_0(\mu_k) \subset e^\rho F_0(\lambda_{\check{k}}) \quad \textit{and} \quad e^\rho F(\lambda_{\hat{k}-1}) \subset E(\mu_k). \tag{71}$$

with $\check{k} = \left\lfloor \frac{|M|_g}{|\Omega|_g} k \right\rfloor$ and $\hat{k} = \left\lceil \frac{|M|_g}{|\Omega|_g} k \right\rceil$, where $\lfloor\ \rfloor$ and $\lceil\ \rceil$ denote the floor and the ceiling functions, respectively.

Observe that we have $F_0(\lambda_{\check{k}}) \subset F(\lambda_{\hat{k}-1})$ with equality if and only if $\lambda_{\hat{k}-1} < \lambda_{\check{k}}$.

Proof of Theorem 5.2. As mentioned in Remark 3.1, the Legendre transform enables us to obtain (70) from (66). Alternatively, we can prove (70) using the averaged variational principle, which has the advantage of allowing us to characterize the case of equality. Taking $z = \mu_k$ in (69), we immediately get

$$\sum_{j=0}^{k-1} \mu_j \le \frac{|\Omega|_g}{|M|_g} \sum_{j=0}^{N(R)-1} \tilde{\lambda}_j + \Big(k - \frac{|\Omega|_g}{|M|_g} N(R)\Big)\mu_k \tag{72}$$

for all $R > 0$. Denote by $1 = N_0 < N_1 < N_2 < \cdots < N_j < \cdots$ the values taken by the function $N(R)$, $R \in \mathbb{R}$, that is, $N_j = m_0 + m_1 + \cdots + m_j$. The sequence of eigenvalues of Δ_g on M is then numbered as follows:

$$0 = \lambda_0 < \lambda_1 = \lambda_2 = \cdots = \lambda_{N_1 - 1} < \lambda_{N_1} = \cdots = \lambda_{N_2 - 1} < \lambda_{N_2} = \cdots$$
$$= \lambda_{N_j - 1} < \lambda_{N_j} = \cdots = \lambda_{N_{j+1} - 1} < \lambda_{N_{j+1}} = \cdots .$$

Let $q \in \mathbb{N}$ be chosen so that

$$N_q \le \frac{|M|_g}{|\Omega|_g} k < N_{q+1}.$$

We consider the inequality (72) with first $N(R) = N_q$ and, then, $N(R) = N_{q+1}$. We multiply the first inequality by $\alpha = \big(N_{q+1} - \frac{|M|_g}{|\Omega|_g} k\big)/m_{q+1}$ and add the second inequality multiplied by $1 - \alpha = \big(\frac{|M|_g}{|\Omega|_g} k - N_q\big)/m_{q+1}$ to get

$$\begin{aligned}\sum_{j=0}^{k-1} \mu_j &\le \frac{|\Omega|_g}{|M|_g} \sum_{j=0}^{N_q - 1} \tilde{\lambda}_j + (1-\alpha)\frac{|\Omega|_g}{|M|_g} m_{q+1} \tilde{\lambda}_{N_q} \\ &\quad + k\mu_k - \frac{|\Omega|_g}{|M|_g}(\alpha N_q + (1-\alpha)N_{q+1})\mu_k \\ &= \frac{|\Omega|_g}{|M|_g} \sum_{j=0}^{N_q - 1} \tilde{\lambda}_j + \Big(k - \frac{|\Omega|_g}{|M|_g} N_q\Big)\tilde{\lambda}_{N_q} \\ &= \frac{|\Omega|_g}{|M|_g}\Big(\sum_{j=0}^{N_q - 1} \tilde{\lambda}_j + \Big(\frac{|M|_g}{|\Omega|_g} k - N_q\Big)\tilde{\lambda}_{N_q}\Big),\end{aligned}$$

since α is chosen such that $\alpha N_q + (1-\alpha) N_{q+1} = \frac{|M|_g}{|\Omega|_g} k$. Now, from the definition of N_q we have $\lambda_{N_q} = \lambda_{N_q + 1} = \cdots = \lambda_{\lfloor \frac{|M|_g}{|\Omega|_g} k \rfloor}$ and, consequently,

$$\sum_{j=0}^{N_q - 1} \tilde{\lambda}_j + \Big(\frac{|M|_g}{|\Omega|_g} k - N_q\Big)\tilde{\lambda}_{N_q} = \mathfrak{S}_{(\tilde{\lambda})}\Big(\frac{|M|_g}{|\Omega|_g} k\Big),$$

which yields

$$\sum_{j=0}^{k-1} \mu_j \le \frac{|\Omega|_g}{|M|_g} \mathfrak{S}_{(\tilde\lambda)}\Big(\frac{|M|_g}{|\Omega|_g}\, k\Big).$$

This means that (70) holds for all $p \in \mathbb{N}$. Since the functions $\mathfrak{S}_{(\mu)}$ and $\mathfrak{S}_{(\tilde\lambda)}$ are piecewise-affine in p, the extension of (70) to all positive p is immediate.

Let k be a positive integer. Equality is achieved in (70) for $p = k$ if and only if one of the following holds:

- $\frac{|M|_g}{|\Omega|_g} k = N_q$ and equality holds in (72) for R such that $N(R) = N_q$, i.e. for $R = \lambda_{N_q - 1}$;
- $\frac{|M|_g}{|\Omega|_g} k > N_q$ and equality holds in (72) for the values of R such that $N(R) = N_q$ and $N(R) = N_{q+1}$, i.e. for both $R = \lambda_{N_q-1}$ and $R = \lambda_{N_{q+1}-1}$.

The first case corresponds to the case of equality in Theorem 2.1 with $z = \mu_k$, $\mathfrak{M} = \operatorname{spec}(M)$, $\mathfrak{M}_0 = \{\lambda \in \operatorname{spec}(M); \lambda \le \lambda_{N_q-1}\}$. As in the proof of Theorem 5.1, this situation occurs if and only if $E_0(\mu_k) \subset e^\rho F(\lambda_{N_q-1}) \subset E(\mu_k)$, with $N_q = \frac{|M|_g}{|\Omega|_g} k$. Since $\lambda_{N_q-1} < \lambda_{N_q}$, $F(\lambda_{N_q-1}) = F_0(\lambda_{N_q})$, and the last conditions can be written as

$$E_0(\mu_k) \subset e^\rho F_0(\lambda_{\frac{|M|_g}{|\Omega|_g} k}) \quad \text{and} \quad F(\lambda_{\frac{|M|_g}{|\Omega|_g} k - 1}) \subset E(\mu_k), \tag{73}$$

which is equivalent to (71).

In the second case, similar considerations show that equality holds if and only if

$$E_0(\mu_k) \subset e^\rho F(\lambda_{N_q-1}) \subset e^\rho F(\lambda_{N_{q+1}-1}) \subset E(\mu_k). \tag{74}$$

Since $N_q < \frac{|M|_g}{|\Omega|_g} k < N_{q+1}$, it is clear that

$$N_q \le \left\lfloor \frac{|M|_g}{|\Omega|_g} k \right\rfloor \le N_{q+1} - 1$$

and

$$N_q \le \left\lceil \frac{|M|_g}{|\Omega|_g} k \right\rceil - 1 \le N_{q+1} - 1.$$

Thus,

$$\lambda_{N_q} = \lambda_{N_{q+1}-1} = \lambda_{\lfloor \frac{|M|_g}{|\Omega|_g} k \rfloor} = \lambda_{\lceil \frac{|M|_g}{|\Omega|_g} k \rceil - 1}.$$

Consequently,

$$F(\lambda_{N_{q+1}-1}) = F\Big(\lambda_{\lceil \frac{|M|_g}{|\Omega|_g} k \rceil - 1}\Big),$$

and, since $\lambda_{N_q-1} < \lambda_{N_q}$,

$$F(\lambda_{N_q-1}) = F_0(\lambda_{N_q}) = F_0\Big(\lambda_{\lfloor \frac{|M|_g}{|\Omega|_g} k \rfloor}\Big).$$

Therefore, (74) is equivalent to (71). □

Remarks 5.1. 1. The particular case of (70) in which $w = 1$ and $\rho = V = 0$ corresponds to the inequality obtained by Strichartz [26, Theorem 2.2].

In the same paper [26], following Gallot [8, Proposition 2.9], Strichartz also proved that for Dirichlet eigenvalues μ_l^D of the Laplacian on a domain Ω of a compact homogeneous Riemannian manifold (M, g), the reverse inequality

$$\mathfrak{S}_{(\mu^D)}(p) \geq \frac{|\Omega|_g}{|M|_g}\, \mathfrak{S}_{(\lambda)}\Big(\frac{|M|_g}{|\Omega|_g} p\Big)$$

holds. Contrary to what was found for Neumann eigenvalues in Theorem 5.2, a straightforward extension of the latter inequality to Dirichlet eigenvalues of a Laplacian with potential cannot hold in general. Indeed, such an extension would imply for $p = 1$ that $\mu_0^D(\Delta_g + V) \geq \frac{1}{|M|_g} \int_\Omega V\, dv_g$, which is not always true. For example, if Ω is a spherical cap of radius r and if u is a positive first eigenfunction of the Dirichlet Laplacian on Ω, we can take the family of continuous potentials V_ε with $V_\varepsilon = \frac{1}{u^2}$ on the spherical cap of radius $(1-\varepsilon)r$, and let V_ε be constant on the complement. Then, by using u as a test function, it is easy to see that $\mu_0^D(\Delta_g + V_\varepsilon) \leq \mu_0^D(\Delta_g) + |\Omega|$, while $\int_\Omega V_\varepsilon\, dv_g$ tends to infinity as $\varepsilon \to 0$.

2. Assuming that $|\Omega| \geq \frac{2}{3}|M|$, an immediate consequence of Theorem 5.2 and the fact that the first positive eigenvalue λ_1 of the Laplacian on a homogeneous manifold (M, g) has multiplicity at least 2, is the inequality

$$\mu_0 + \mu_1 \leq \frac{|\Omega|_g}{|M|_g}\Big(2\lambda_1 \fint_\Omega w\, dv_g + 3 \fint_\Omega \widetilde{V} w\, dv_g\Big),$$

which yields

$$\mu_1 \leq 2\frac{|\Omega|_g}{|M|_g}\lambda_1$$

for the Neumann Laplacian (with $\rho = V = 0$ and $w = 1$).

In the case where Ω is equal to the whole of M, Theorem 5.2 leads to the following corollary.

Corollary 5.1. *Let* (M, g) *be a compact homogeneous Riemannian manifold. Let* μ_l, $l \in \mathbb{N}$, *be the eigenvalues defined by* (2) *on* M. *Then, for every* $k \in \mathbb{N}^*$,

$$\sum_{j=0}^{k-1} \mu_j \leq \sum_{j=0}^{k-1} \tilde{\lambda}_j, \tag{75}$$

where equality holds if and only if

$$E_0(\mu_k) \subset e^\rho F_0(\lambda_k) \quad \textit{and} \quad e^\rho F(\lambda_{k-1}) \subset E(\mu_k).$$

In particular, if m_1 *is the multiplicity of* λ_1, *then equality holds in* (75) *for* $k \leq m_1$ *if and only if* $(V + |\nabla^g \rho|^2)w - div_g(w\nabla\rho)$ *is constant on* M *and* $\mu_j = \tilde{\lambda}_j$ *for* $j = 0, 1, \cdots, k-1$.

Proof of Corollary 5.1. Assume that equality holds in (75) for $k \leq m_1$. Then $E_0(\mu_k) \subset e^\rho F_0(\lambda_k)$. Since $k \leq m_1$, $F_0(\lambda_k) = F(\lambda_0) = \text{span}\{1\}$, it follows that $E_0(\mu_k)$ has dimension 1, that is, $E_0(\mu_k) = E(\mu_0) = \text{span}\{e^\rho\}$. Consequently, $\mu_1 = \mu_2 = \cdots = \mu_k$, and e^ρ is an eigenfunction of H associated with μ_0. Thus

$$\begin{aligned} He^\rho &= e^{2\rho}\text{div}_g(we^{-2\rho}\nabla^g e^\rho) + Vwe^\rho = \cdots \\ &= ((V + |\nabla^g \rho|^2)w - \text{div}_g(w\nabla\rho))e^\rho = \mu_0 e^\rho, \end{aligned}$$

which implies that $(V + |\nabla^g \rho|^2)w - \text{div}_g(w\nabla\rho) = \mu_0$. Integrating, we get $\mu_0 = \fint_\Omega \tilde{V} w \, dv_g = \tilde{\lambda}_0$. Now,

$$\mu_0 + (k-1)\mu_1 = \sum_{j=0}^{k-1} \mu_j = \sum_{j=0}^{k-1} \tilde{\lambda}_j = \tilde{\lambda}_0 + (k-1)\tilde{\lambda}_1$$

and, consequently, $\mu_1 = \tilde{\lambda}_1$. □

Remarks 5.2. 1. An immediate consequence of Corollary 5.1 is that for any potential V on a compact homogeneous (M, g) and every positive k,

$$\frac{1}{k}\sum_{j=0}^{k-1} \mu_j(\Delta_g + V) \leq \frac{1}{k}\sum_{j=0}^{k-1} \mu_j(\Delta_g) + \fint_M V \, dv_g,$$

to be compared with the results of [6].

2. We know that for any $k \geq 2$, either $\lambda_{k-1} = \lambda_k$ or else $\lambda_{k-1} = \lambda_{k-2}$. Notice that if $\lambda_{k-1} = \lambda_k$ and if equality holds in (75) for k, then, necessarily, $\tilde{\lambda}_{k-1} = \mu_{k-1} = \mu_k = \tilde{\lambda}_k$. This follows directly from the combination of $\sum_{j=0}^{k-1} \mu_j = \sum_{j=0}^{k-1} \tilde{\lambda}_j$ with $\sum_{j=0}^{k} \mu_j \leq \sum_{j=0}^{k} \tilde{\lambda}_j$ and $\sum_{j=0}^{k-2} \mu_j \leq \sum_{j=0}^{k-2} \tilde{\lambda}_j$. Consequently, equality also holds in (75) for $k-1$ and $k+1$.

Moreover, if $\mu_k > \mu_{k-1}$, then equality holds in (75) for k if and only if $\lambda_k > \lambda_{k-1}$ and $E(\mu_{k-1}) = e^{\rho} F(\lambda_{k-1})$. Indeed, in this case, $\dim E_0(\mu_k) = \dim E(\mu_{k-1}) = k$ and $\dim F_0(\lambda_k) = \dim F_0(\lambda_{k-1}) = k$.

Applying the Laplace transform to both sides of (66), we obtain the following comparison of heat traces (see (31)):

Corollary 5.2. *Let (M, g) be a compact homogeneous Riemannian manifold. Let μ_l, $l \in \mathbb{N}$, be the eigenvalues defined by (2) on a bounded open set $\Omega \subset M$. Then, for all $t > 0$,*

$$\sum_{j\geq 0} e^{-\mu_j t} \geq \frac{|\Omega|_g}{|M|_g} \sum_{j\geq 0} e^{-\tilde{\lambda}_j t}, \tag{76}$$

where $\tilde{\lambda}_j := \lambda_j ⨍_\Omega w \, dv_g + ⨍_\Omega \tilde{V} w \, dv_g$.

We define the theta function via

$$\Theta(t) = \frac{1}{4\pi t} \sum_{(p,q)\in\mathbb{Z}^2} e^{-\frac{p^2+q^2+pq}{4t}}.$$

Corollary 5.3. *Let $\Gamma = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \subset \mathbb{R}^2$ be a lattice, where $\{e_1, e_2\}$ is a basis of $\mathbb{R}^2$. Let ρ, $w > 0$, V be Γ-periodic functions on $\mathbb{R}^2$ and denote by $\mu_l = \mu_l(\rho, w, V)$, $l \in \mathbb{N}$, the eigenvalues of the operator $H(\rho, w, V)$ defined by (4), acting on Γ-periodic functions on $\mathbb{R}^2$. Then, for all $t > 0$,*

$$\sum_{j\geq 0} e^{-\mu_j t} \geq \Theta\left(\frac{⨍_\Omega w \, dx}{|\Omega|} t\right) e^{-t ⨍_\Omega \tilde{V} w \, dx}, \tag{77}$$

where Ω is a fundamental domain for the action of Γ on $\mathbb{R}^2$.

Proof. This result is a direct consequence of (76) combined with Poisson's formula and Montgomery's Theorem [23]. □

We turn now to the phase-space analysis taking into account the form of the potential V and allowing conformal transformations and nontrivial weights.

Let us denote by

$$0 = \Lambda_0 < \Lambda_1 < \Lambda_2 < \cdots < \Lambda_l < \cdots$$

the increasing sequence of eigenvalues of the Laplacian of the compact homogeneous space (M, g). The multiplicity of Λ_l is denoted m_l, and we designate by $\{y_{l,1}, y_{l,2}, \cdots, y_{l,m_l}\}$ an L^2-orthonormal basis of the eigenspace associated with Λ_l.

In the case of a domain Ω in a manifold $X \simeq (M, e^{-2\rho} g)$ that is conformally equivalent to (M, g), we shall use coherent-state test functions of the form:

$$f_\zeta(\mathbf{x}) := y_{\ell m}(\mathbf{x}) e^{\rho(\mathbf{x})} h_{\mathbf{y}}(\mathbf{x}). \tag{78}$$

In this formula, $h(\mathbf{x})$ is a nonnegative H^1 function supported in the geodesic ball of radius r in the canonical metric on M, with $\int_{B_r} h_r^2(\mathbf{x}) d^\nu x = 1$, and $\mathbf{y}$ ranges over the isometry group G. As before we choose it specifically as the ground-state Dirichlet eigenfunction on the geodesic ball of radius r and set $\mathcal{K}(h_r) := \int_{B_r} |\nabla h(\mathbf{x})|^2 d^\nu x$, which is thus the fundamental Dirichlet eigenvalue for the Laplacian on the geodesic disk of radius r. Denoting by $T_{\mathbf{y}}(\mathbf{x})$ the action by the group element $\mathbf{y}$ on the point $\mathbf{x}$, we let

$$h_{\mathbf{y}}(\mathbf{x}) := h(T_{\mathbf{y}}(\mathbf{x})).$$

Recall that one can designate an arbitrary point of M as 0 and cover M with translates $T_{\mathbf{y}}(0)$. We normalize the uniform measure $d\gamma$ on G so that for any $f \in L^\infty(M)$, $\int_G f(T_{\mathbf{y}}(\mathbf{x})) d\gamma(\mathbf{y}) = \int_M f(\mathbf{x}) dv_g$. The index $\zeta = (\ell, m, \mathbf{y})$ ranges over $\mathfrak{M} = \mathcal{J} \times G$, where $\mathcal{J}$ is the set of all pairs of integer indices for the normalized eigenfunctions $y_{\ell m}(\mathbf{x})$, and the associated measure $d\sigma$ is the product of the counting measure on $\mathcal{J}$ with $d\gamma$.

As in Section 2, we find it helpful to define some auxiliary quantities:

Definition 5.1. As before,

$$\widetilde{V}(\mathbf{x}) := V(\mathbf{x}) + |\nabla \rho|^2.$$

The *weighted phase-space volume* is

$$\begin{aligned}\Phi_w^h(\Lambda) &:= |\{\ell, m, \mathbf{y}\}: m \le m_\ell, T_{\mathbf{y}}(0) \in \Omega, \Lambda_\ell + \widetilde{V}(T_{\mathbf{y}}(0)) \le \Lambda| \\ &= \int_{\{\mathbf{y}: T_{\mathbf{y}}(0) \in \Omega, \widetilde{V}(T_{\mathbf{y}}(0)) \le \Lambda\}} \Big(\sum_{\{\ell: \Lambda_\ell + \widetilde{V}(T_{\mathbf{y}}(0)) \le \Lambda\}} m_\ell \Big) d\gamma(\mathbf{y}).\end{aligned}$$

The *total energy* associated with this phase-space volume is correspondingly

$$E_w^h(\Lambda) := \int_{\{\mathbf{y}: T_{\mathbf{y}}(0) \in \Omega, \widetilde{V}(T_{\mathbf{y}}(0)) \le \Lambda\}} \Big(\sum_{\{\ell: \Lambda_\ell + \widetilde{V}(T_{\mathbf{y}}(0)) \le \Lambda\}} m_\ell (\Lambda_\ell + \widetilde{V}(T_{\mathbf{y}}(0))) \Big) d\gamma(\mathbf{y}).$$

Theorem 5.3. *Let $\mu_0 \le \mu_1 \le \cdots$ be the variationally defined Neumann eigenvalues (2) on a bounded open set $\Omega \subset M$, where w, ρ, and V satisfy the assumptions stated in Section 1. Then for all $r > 0$,*

$$\sum_{j=0}^{k-1} \mu_j \le E_w^h(\Lambda + \mathrm{Lip}(\Lambda) r) + \mathcal{K}(h_r) \Phi_w^h(\Lambda + \mathrm{Lip}(\Lambda) r). \tag{79}$$

Proof. Note that

$$\langle \phi, f_\zeta \rangle_\Omega = \langle e^{-\rho(\mathbf{x})} h_{\mathbf{y}}(\mathbf{x})\phi(\mathbf{x}), y_{\ell m}\rangle_M,$$

where, if Ω is a strict subset of M, then ϕ is extended by 0 outside Ω. By the Fourier completeness relation,

$$\begin{aligned}\int_G \Big(\sum_{\ell,m} |\langle \phi, f_\zeta\rangle_\Omega|^2\Big) d\gamma(\mathbf{y}) &= \int_G \|e^{-\rho(\mathbf{x})} h_{\mathbf{y}}(\mathbf{x})\phi(\mathbf{x})\|^2_{L^2(M,dv_g)} d\gamma(\mathbf{y}) \\ &= \int_\Omega |\phi|^2 e^{-2\rho} \Big(\int_G h_{\mathbf{y}}(\mathbf{x})^2 d\gamma(\mathbf{y})\Big) dv_g \\ &= \int_\Omega |\phi|^2 e^{-2\rho} dv_g \\ &= \|\phi\|^2.\end{aligned} \tag{80}$$

To apply the theorem, choose $\mathfrak{M}_0$ of the form $\{(\ell, m, \mathbf{y}) : m \le m_\ell, T_{\mathbf{y}}(0) \in \Omega, \Lambda_\ell + \widetilde{V}(T_{\mathbf{y}}(0)) \le \Lambda\}$ for a finite value of Λ large enough so that

$$k \le \int_{\mathfrak{M}_0} \|f_\zeta\|^2_{L^2(\Omega)} d\sigma = \int_\Omega \int_{\mathfrak{M}_0} h^2_{\mathbf{y}}(\mathbf{x}) e^{2\rho(\mathbf{x}) - 2\rho(\mathbf{x})} d\sigma dv_g = |\Omega| \Phi^h_w(\Lambda). \tag{81}$$

We define $\Lambda(k)$ as the minimal value of Λ for which (81) is valid and henceforth choose $\mathfrak{M}_0 = \{(\ell, m, \mathbf{y}) : m \le m_\ell, T_{\mathbf{y}}(0) \in \Omega, \Lambda_\ell + \widetilde{V}(\mathbf{y}) \le \Lambda(k)\}$. Then

$$\begin{aligned}\sum_{j=0}^{k-1} \mu_j &\le \int_{\mathfrak{M}_0} \mathcal{R}(f_\zeta) d\sigma(\zeta) \\ &= \int_\Omega \int_{\mathfrak{M}_0} w(\mathbf{x}) (y^2_{\ell m} (h^2_{\mathbf{y}}(\mathbf{x}) \widetilde{V}(\mathbf{x}) + |\nabla h_{\mathbf{y}}(\mathbf{x})|^2 + \nabla \rho(\mathbf{x}) \cdot \nabla h^2_{\mathbf{y}}(\mathbf{x})) \\ &\qquad + h^2_{\mathbf{y}}(\mathbf{x}) |\nabla y_{\ell m}|^2 + 2 h_{\mathbf{y}}(\mathbf{x}) \nabla h_{\mathbf{y}}(\mathbf{x}) \cdot y_{\ell m} \nabla y_{\ell m}) d\sigma dv_g \\ &\le \int_\Omega \int_{\{\mathbf{y}:\widetilde{V}(T_{\mathbf{y}}(0)) \le \Lambda\}} w(\mathbf{x}) \\ &\qquad \Big(\sum_{\{\ell:\Lambda_\ell + \widetilde{V}(T_{\mathbf{y}}(0)) \le \Lambda\}} \Big(\frac{m_\ell}{|M|}\Big) (h^2_{\mathbf{y}}(\mathbf{x})(\Lambda_\ell + \widetilde{V}(\mathbf{x})) + |\nabla h_{\mathbf{y}}(\mathbf{x})|^2)\Big) d\gamma(\mathbf{y}),\end{aligned}$$

by dint of (67) and (68). (The final cross term dropped out because it was proportional to the gradient of a constant function (67), in analogy with (62).) Because h is supported in a ball of radius r, we restrict the x-integration to $\mathbf{x}$: $\mathrm{dist}(\mathbf{x}, \mathbf{y}) \le r$ with $(\ell, m, \mathbf{y}) \in \mathfrak{M}_0$ and estimate the integral in analogy with (63), obtaining

$$\sum_{j=0}^{k-1} \mu_j \le \Big(\frac{1}{|M|}\Big) \int_{(\ell,m,\mathbf{x}) \in \mathfrak{M}_0(\Lambda + \mathrm{Lip}(\Lambda) r)} w(\mathbf{x}) (\Lambda_\ell + \widetilde{V}(\mathbf{x}) + \mathcal{K}(h_r)) d\sigma, \tag{82}$$

which yields the statement in the theorem. □

Acknowledgments. E.H. is grateful to the Université F. Rabelais and to École Polytechnique Fédérale de Lausanne for hospitality that supported this collaboration.

References

[1] S. Agmon and L. Hörmander, Asymptotic properties of solutions of differential equations with simple characteristics. *J. Analyse Math.* **30** (1976), 1–38. MR 0466902 Zbl 0335.35013

[2] S. Agmon, *Lectures on elliptic boundary value problems.* Prepared for publication by B. Frank Jones, Jr., with the assistance of G. W. Batten, Jr. Van Nostrand Mathematical Studies, 2. D. Van Nostrand Co., Princeton, N.J., etc., 1965. MR 0178246 Zbl 0142.37401

[3] E. F. Beckenbach and R. Bellman, *Inequalities.* Second revised printing. Ergebnisse der Mathematik und ihrer Grenzgebiete. Neue Folge, Band 30. Springer-Verlag, Berlin etc., 1965. MR 0192009 Zbl 0126.28002

[4] B. Colbois, A. El Soufi, and A. Savo, Eigenvalues of the Laplacian on a compact manifold with density. *Comm. Anal. Geom.* **23** (2015), no. 3, 639–670. MR 3310527 Zbl 1317.58031

[5] D. E. Edmunds and W. D. Evans, *Spectral theory and differential operators.* Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1987. MR 0929030 Zbl 0628.47017

[6] A. El Soufi and S. Ilias, Majoration de la seconde valeur propre d'un opérateur de Schrödinger sur une variété compacte et applications. *J. Funct. Anal.* **103** (1992), no. 2, 294–316. MR 1151550 Zbl 0766.58055

[7] A. El Soufi and S. Ilias, Second eigenvalue of Schrödinger operators and mean curvature. *Comm. Math. Phys.* **208** (2000), no. 3, 761–770. MR 1736334 Zbl 0955.58025

[8] S. Gallot, Inégalités isopérimétriques et analytiques sur les variétés riemanniennes. In *On the geometry of differentiable manifolds.* Papers from the workshop held at the Università degli Studi di Roma "La Sapienza", Rome, June 23–27, 1986. Astérisque 163-164 (1988). Société Mathématique de France, Paris, 1989, 31–91. MR 0999971 Zbl 0674.53001

[9] A. Grigor′yan, Yu. Netrusov, and S.-T. Yau, Eigenvalues of elliptic operators and geometric applications. In A. Grigor′yan, and S.-T. Yau (eds.), *Surveys in differential geometry.* Vol. IX. Surveys in Differential Geometry, 9. International Press, Somerville, MA, 2004, 147–217. MR 2195408 Zbl 1061.58027

[10] E. M. Harrell II and J. Stubbe, On sums of graph eigenvalues. *Linear Algebra Appl.* **455** (2014), 168–186. MR 3217405 Zbl 1305.05127

[11] A. Hassannezhad, Eigenvalues of perturbed Laplace operators on compact manifolds. *Pacific J. Math.* **264** (2013), no. 2, 333–354. MR 3089400 Zbl 1291.35146

[12] R. Hempel, L. A. Seco, and B. Simon, The essential spectrum of Neumann Laplacians on some bounded singular domains. *J. Funct. Anal.* **102** (1991), no. 2, 448–483. MR 1140635 Zbl 0741.35043

[13] C. E. Heil and D. F. Walnut, Continuous and discrete wavelet transforms. *SIAM Rev.* **31** (1989), no. 4, 628–666. MR 1025485 Zbl 0683.42031

[14] L. Hörmander, *The analysis of linear partial differential operators.* I. Distribution theory and Fourier analysis. Grundlehren der Mathematischen Wissenschaften, 256. Springer-Verlag, Berlin, 1983. MR 0717035 Zbl 0521.35001

[15] P. Kröger, Upper bounds for the Neumann eigenvalues on a bounded domain in Euclidean space. *J. Funct. Anal.* **106** (1992), no. 2, 353–357. MR 1165859 Zbl 0777.35044

[16] A. Laptev, Dirichlet and Neumann eigenvalue problems on domains in Euclidean spaces. *J. Funct. Anal.* **151** (1997), no. 2, 531–545. MR 1491551 Zbl 0892.35115

[17] A. Laptev, Recent improvements of Berezin–Li & Yau type inequalities. BIRS lecture, 2013. http://www.birs.ca/workshops/2013/13w5059/files/Laptev.pdf

[18] R. S. Laugesen and B. A. Siudeja, Sums of Laplace eigenvalues–rotationally symmetric maximizers in the plane. *J. Funct. Anal.* **260** (2011), no. 6, 1795–1823. MR 2754893 Zbl 1215.35112

[19] L. Li and L. Tan, Some upper bounds for sums of eigenvalues of the Neumann Laplacian. *Proc. Amer. Math. Soc.* **134** (2006), no. 11, 3301–3307. MR 2231915 Zbl 1111.35021

[20] E. H. Lieb, Coherent states as a tool for obtaining rigorous bounds. In: D. H. Feng and J. Klauder (eds.), *Coherent states: past, present, and future.* Proceedings of the conference in Oak Ridge, TN, 1993. World Scientific, Singapore, 1994, 267-278. Reprinted in E. H. Lieb, *Inequalities.* Selecta of E. H. Lieb. Edited, with a preface and commentaries by M. Loss and M. B. Ruskai. Springer, Berlin, 2002, 377-388.

[21] E. H. Lieb and M. Loss, *Analysis.* Second edition. Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, R.I., 2001. MR 1817225 Zbl 0966.26002

[22] E. H. Lieb and J. P. Solovej, Quantum coherent operators: a generalization of coherent states. *Lett. Math. Phys.* **22** (1991), no. 2, 145–154. Reprinted in E. H. Lieb, *Inequalities.* Selecta of E. H. Lieb. Edited, with a preface and commentaries by M. Loss and M. B. Ruskai. Springer, Berlin, 2002, 145–154. MR 1122052 Zbl 0757.46064

[23] H. L. Montgomery, Minimal theta functions. *Glasgow Math. J.* **30** (1988), no. 1, 75–85. MR 0925561 Zbl 0639.10017

[24] M. Reed and B. Simon, *Methods of modern mathematical physics.* IV. Analysis of operators. Academic Press, New York and London, 1978. MR 0493421 Zbl 0401.47001

[25] R. S. Strichartz, Fourier asymptotics of fractal measures. *J. Funct. Anal.* **89** (1990), no. 1, 154–187. MR 1040961 Zbl 0693.28005

[26] R. S. Strichartz, Estimates for sums of eigenvalues for domains in homogeneous spaces. *J. Funct. Anal.* **137** (1996), no. 1, 152–190. MR 1383015 Zbl 0848.58050

[27] W. Thirring, *Quantum mathematical physics.* Atoms, molecules and large systems. Second edition. Translated from the 1979 and 1980 German originals by E. M. Harrell II. Springer-Verlag, Berlin, 2002. MR 2133871 Zbl 1060.81025

[28] J. Weidmann, Linear operators in Hilbert spaces. Translated from the German by J. Szücs. Graduate Texts in Mathematics, 68. Springer-Verlag, Berlin etc., 1980. MR 0566954 Zbl 0434.47001

Received July 9, 2015

Evans M. Harrell II, School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA

e-mail: harrell@math.gatech.edu

Saïd Ilias, Laboratoire de Mathématiques et Physique Théorique, Université de Tours, UMR-CNRS 6083, Parc de Grandmont, 37200 Tours, France

e-mail: ilias@univ-tours.fr

Joachim Stubbe, MATH-GEOM, EPFL, Station 8, CH-1015 Lausanne, Switzerland

e-mail: Joachim.Stubbe@epfl.ch