J. Spectr. Theory 7 (2017), 1023–1038
DOI 10.4171/JST/184

Journal of Spectral Theory

Fredholm consistency of upper-triangular operator matrices

Dragana S. Cvetković-Ilić[1]

Abstract. In this paper, for given operators $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, we characterize the set of all $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that the operator matrix $M_C = \left[\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right]$ is Fredholm consistent. We completely describe the sets $\bigcap_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{\mathrm{FC}}(M_C)$ and $\bigcup_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{\mathrm{FC}}(M_C)$. Also, we prove that $\bigcap_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{\mathrm{FC}}(M_C) = \sigma_{\mathrm{FC}}(M_0)$.

Mathematics Subject Classification (2010). 47A05, 47A53.

Keywords. Fredholm operator, Fredholm consistent operator, upper-triangular operator.

1. Introduction

Let $\mathcal{H}, \mathcal{K}$ be infinite dimensional complex separable Hilbert spaces and let $\mathcal{B}(\mathcal{H}, \mathcal{K})$ denote the set of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. For simplicity, we also write $\mathcal{B}(\mathcal{H}, \mathcal{H})$ as $\mathcal{B}(\mathcal{H})$. For a given $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, the symbols $\mathcal{N}(A)$ and $\mathcal{R}(A)$ denote the null space and the range of A, respectively. Let $n(A) = \dim \mathcal{N}(A)$, $\beta(A) = \operatorname{codim} \mathcal{R}(A)$ and $d(A) = \dim \mathcal{R}(A)^{\perp}$.

If $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is such that $\mathcal{R}(A)$ is closed and $n(A) < \infty$, then A is an upper semi-Fredholm operator. If $\beta(A) < \infty$, then A is a lower semi-Fredholm operator. An operator $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is called Fredholm if it is both upper semi-Fredholm and lower semi-Fredholm. The set of all Fredholm operators from the space $\mathcal{B}(\mathcal{H}, \mathcal{K})$ is denoted by $F(\mathcal{H}, \mathcal{K})$. By $F_+(\mathcal{H}, \mathcal{K})$ ($F_-(\mathcal{H}, \mathcal{K})$) we denote the set of all upper (lower) semi-Fredholm operators from $\mathcal{B}(\mathcal{H}, \mathcal{K})$.

Let $\mathcal{S}(\mathcal{H})$ denote a subset of $\mathcal{B}(\mathcal{H})$. An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be $\mathcal{S}(\mathcal{H})$ consistent, or consistent in $\mathcal{S}(\mathcal{H})$, if

$$AB \in \mathcal{S}(\mathcal{H}) \iff BA \in \mathcal{S}(\mathcal{H}),$$

for every $B \in \mathcal{B}(\mathcal{H})$.

[1] The work of the author is supported by Grant No. 174007 of the Ministry of Science, Technology and Development, Republic of Serbia.

A characterization of operators consistent in invertibility (CI operators) and a characterization of CI operators invariant under compact perturbations are given in [7]. It is worth mentioning the very interesting paper of Duggal et al. [6], where the property of being $\mathcal{S}(\mathcal{H})$ consistent is related with the SVEP property for a variety of choices of the subset $\mathcal{S}(\mathcal{H})$ of $\mathcal{B}(\mathcal{H})$.

The work presented in this paper is a continuation of the work of Hai and Chen [8] in which the problem of completions of the upper-triangular operator matrix

$$\begin{bmatrix} A & ? \\ 0 & B \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix}, \tag{1}$$

to consistent invertibility was considered, where $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ are given operators on separable Hilbert spaces.

In this paper we will consider the problem of completion of the upper-triangular operator matrix given by (1) to Fredholm consistent (FC) and answer the following three questions which repeatedly arise.

Question 1. *Is there an operator $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that M_C is FC?*

Question 2. $\bigcap_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{\mathrm{FC}}(M_C) =?$

Question 3. *Is there an operator $C' \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that*

$$\sigma_{\mathrm{FC}}(M_{C'}) = \bigcap_{C \in \mathcal{B}(\mathcal{Y},\mathcal{X})} \sigma_{\mathrm{FC}}(M_C)?$$

Furthermore, for given operators $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, we describe $\bigcup_{C \in \mathcal{B}(\mathcal{Y},\mathcal{X})} \sigma_{\mathrm{FC}}(M_C)$ and characterize the set of all $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that the operator matrix $M_C = \left[\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right]$ is Fredholm consistent.

For $A \in \mathcal{B}(\mathcal{H})$, by $\sigma_{\mathrm{FC}}(A)$ we denote

$$\sigma_{\mathrm{FC}}(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not Fredholm consistent}\}$$

and

$$\rho_{\mathrm{FC}}(A) = \mathbb{C} \setminus \sigma_{\mathrm{FC}}(A).$$

Notice that for given $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, the set of all $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that M_C is Fredholm, will be denoted by $S_F(A, B)$, respectively.

2. Results

We begin by listing the results that will be made use of later in the paper. The next one is a well known useful result.

Lemma 2.1. *Let $S \in \mathcal{B}(\mathcal{H})$, $T \in \mathcal{B}(\mathcal{K},\mathcal{H})$ and $R \in \mathcal{B}(\mathcal{H},\mathcal{K})$ be given operators.*

(i) *If $\mathcal{R}(S)$ is non-closed and $\mathcal{R}\left(\begin{bmatrix} S & T \end{bmatrix}\right)$ is closed, then $n\left(\begin{bmatrix} S & T \end{bmatrix}\right) = \infty$.*

(ii) *If $\mathcal{R}(S)$ is non-closed and $\mathcal{R}\left(\left[\begin{smallmatrix} S \\ R \end{smallmatrix}\right]\right)$ is closed, then $d\left(\left[\begin{smallmatrix} S \\ R \end{smallmatrix}\right]\right) = \infty$.*

In many papers the various problems of completion of the upper-triangular operator matrix

$$M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix}, \tag{2}$$

where $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ are given, were considered. More precisely, for given $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, one is interested in existence of some $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ such that M_C is of a certain given type. Discussions of such completion problems to right (left) invertible, semi-Fredholm, Weyl, Browder or operators with closed range can be found in [2, 3, 9, 10, 11]. Here we will state the result related to the Fredholmness of M_C proved in [4].

Theorem 2.1. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ be given operators. Then M_C is Fredholm for some $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ if and only if one of the following conditions is satisfied.*

(i) *$A \in F(\mathcal{H})$ and $B \in F(\mathcal{K})$ are Fredholm*;

(ii) *$A \in F_+(\mathcal{H})$, $B \in F_-(\mathcal{K})$ and $d(A) = n(B) = \infty$.*

Furthermore, if (i) *is satisfied then $S_F(A,B) = \mathcal{B}(\mathcal{K},\mathcal{H})$, while if* (ii) *holds,*

$$S_F(A,B) = \{C \in \mathcal{B}(\mathcal{K},\mathcal{H}) \colon P_{\mathcal{R}(A)^\perp} C P_{\mathcal{N}(B)} \colon \mathcal{N}(B) \longrightarrow \mathcal{R}(A)^\perp \text{ is Fredholm}\}.$$

In this paper we will consider the problem of completion of the upper-triangular operator matrix M_C given by (2) to Fredholm consistent (FC). First we will state a well-known result which gives us necessary and sufficient conditions for Fredholm consistency of an operator $A \in \mathcal{B}(\mathcal{H})$.

Theorem 2.2. *Let $A \in \mathcal{B}(\mathcal{H})$. Then A is FC if and only if one of the following conditions is satisfied*:

(i) $A \in F(\mathcal{H})$,

(ii) $\mathcal{R}(A)$ *is closed*, $n(A) = d(A) = \infty$,

(iii) $\mathcal{R}(A)$ *is non-closed.*

Throughout the paper for given operators $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, we will suppose that an arbitrary $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ is given by

$$C = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B)^{\perp} \\ \mathcal{N}(B) \end{bmatrix} \longrightarrow \begin{bmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{R}(A)^{\perp} \end{bmatrix}. \tag{3}$$

Now we will begin with giving necessary and sufficient conditions such that M_C given by (2) is FC.

Theorem 2.3. *Let $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{K})$ and $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ be given operators. The operator matrix M_C given by (2) is FC if and only if one of the following conditions is satisfied*:

(i) $A, B \in F(\mathcal{H})$,

(ii) $A \in F_+(\mathcal{H})$, $B \in F_-(\mathcal{K})$, $n(B) = d(A) = \infty$, $C_4 \in F(\mathcal{N}(B), \mathcal{R}(A)^{\perp})$,

(iii) $\mathcal{R}(A)$, $\mathcal{R}(B)$ *are closed and one of the following conditions holds*:

  (a) $\mathcal{R}(C_4)$ *is closed and* $n(A) + n(C_4) = d(B) + d(C_4) = \infty$,

  (b) $\mathcal{R}(C_4)$ *is non-closed,*

(iv) $\mathcal{R}(A)$ *is closed,* $\mathcal{R}(B)$ *is not closed and one of the following conditions holds*:

  (c) $\mathcal{R}(C_4)$ *is closed,* $d(C_4) = \infty$, $\mathcal{R}(B^*) + \mathcal{R}(C_3^* P_{\mathcal{R}(C_4)^{\perp}}) = \overline{\mathcal{R}(B^*)}$, *and* $n(A) + n(C_4) = \infty$,

  (d) $\mathcal{R}(C_4)$ *is closed,* $d(C_4) < \infty$,

  (e) $\mathcal{R}(C_4)$ *is closed,* $\mathcal{R}(B^*) + \mathcal{R}(C_3^* P_{\mathcal{R}(C_4)^{\perp}}) \neq \overline{\mathcal{R}(B^*)}$,

  (f) $\mathcal{R}(C_4)$ *is not closed.*

(v) $\mathcal{R}(A)$ *is non-closed,* $\mathcal{R}(B)$ *is closed and one of the following conditions holds*:

  (a) $\mathcal{R}(C_4)$ *is closed,* $n(C_4) = \infty$, $\mathcal{R}(A) + \mathcal{R}(C_2 P_{\mathcal{N}(C_4)}) = \overline{\mathcal{R}(A)}$, *and* $d(B) + d(C_4) = \infty$,

  (b) $\mathcal{R}(C_4)$ *is closed,* $n(C_4) < \infty$,

  (c) $\mathcal{R}(C_4)$ *is closed,* $\mathcal{R}(A) + \mathcal{R}(C_2 P_{\mathcal{N}(C_4)}) \neq \overline{\mathcal{R}(A)}$,

  (d) $\mathcal{R}(C_4)$ *is not closed.*

(viii) $\mathcal{R}(A)$ *and* $\mathcal{R}(B)$ *are non-closed.*

Proof. We have that M_C is Fredholm consistent if and only if one of conditions (i)–(iii) from Theorem 2.2 holds. So, we will first look for necessary and sufficient conditions which the operators A, B and C have to satisfy in order for (i) from Theorem 2.2 to hold for M_C i.e. so that $M_C \in F(\mathcal{H} \oplus \mathcal{K})$. The answer to this question is completely given in Theorem 2.1. Hence, by this theorem, it follows that $M_C \in F(\mathcal{H} \oplus \mathcal{K})$ if and only if $A \in F(\mathcal{H})$, $B \in F(\mathcal{K})$ or $A \in F_+(\mathcal{H})$, $B \in F_-(\mathcal{K})$, $n(B) = d(A) = \infty$ and C given by (3) is such that $C_4 \in F(\mathcal{N}(B), \mathcal{R}(A)^\perp)$. Now we are looking for necessary and sufficient conditions for the operators A, B and C so that one of the conditions (ii) or (iii) from Theorem 2.2 holds true for M_C. We will consider four cases which depend on the closedness of the ranges of operators A and B.

Case 1. $\mathcal{R}(A), \mathcal{R}(B)$ are closed. Then M_C has a matrix representation

$$M_C = \begin{bmatrix} A_1 & 0 & C_1 & C_2 \\ 0 & 0 & C_3 & C_4 \\ 0 & 0 & B_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(A)^\perp \\ \mathcal{N}(A) \\ \mathcal{N}(B)^\perp \\ \mathcal{N}(B) \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \\ \mathcal{R}(B) \\ \mathcal{R}(B)^\perp \end{bmatrix},$$

where A_1, B_1 are invertible. We can verify that $\mathcal{R}(M_C)$ is closed if and only if $\mathcal{R}(C_4)$ is closed. Also, we have that

$$n(M_C) = n(A) + n\left(\begin{bmatrix} A_1 & C_2 \\ 0 & C_4 \end{bmatrix}\right) = n(A) + n(C_4), \tag{4}$$

and

$$n(M_C^*) = n(B^*) + n\left(\begin{bmatrix} C_3^* & B_1^* \\ C_4^* & 0 \end{bmatrix}\right) = n(B^*) + n(C_4^*). \tag{5}$$

So, in this case we get that $\mathcal{R}(M_C)$ is non-closed if and only if $\mathcal{R}(C_4)$ is non-closed while $\mathcal{R}(M_C)$ is closed and $n(M_C) = d(M_C) = \infty$ if and only if $\mathcal{R}(C_4)$ is closed and $n(A) + n(C_4) = d(B) + d(C_4) = \infty$.

Case 2. $\mathcal{R}(A)$ is closed, $\mathcal{R}(B)$ is non-closed. In this case, we are looking for necessary and sufficient conditions for M_C to satisfy (ii) or (iii) from Theorem 2.2 (evidently (i) can not be satisfied). M_C has the following matrix representation

$$M_C = \begin{bmatrix} A_1 & 0 & C_1 & C_2 \\ 0 & 0 & C_3 & C_4 \\ 0 & 0 & B_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(A)^\perp \\ \mathcal{N}(A) \\ \mathcal{N}(B)^\perp \\ \mathcal{N}(B) \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \\ \overline{\mathcal{R}(B)} \\ \mathcal{R}(B)^\perp \end{bmatrix},$$

where A_1 is invertible and B_1 is injective with dense range. It can be checked that $\mathcal{R}(M_C)$ is closed if and only if $M_1 = \begin{bmatrix} C_3 & C_4 \\ B_1 & 0 \end{bmatrix}$ has closed range. Using Theorems 2.5 and 2.6 from [5], we have that there exists $C_3 \in \mathcal{B}(\mathcal{N}(B)^{\perp}, R(A)^{\perp})$ such that $\mathcal{R}(M_1)$ is closed if and only if $C_4 \in \mathcal{B}(\mathcal{N}(B), R(A)^{\perp})$ has closed range and $n(C_4^*) = \infty$. In order to describe all C_3 such that $\mathcal{R}(M_1)$ is closed, suppose that C_4 is such that $\mathcal{R}(C_4)$ is closed and $n(C_4^*) = \infty$. In that case M_1 can be represented by the following:

$$M_1 = \begin{bmatrix} C_{31} & C_{41} & 0 \\ C_{32} & 0 & 0 \\ B_1 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B)^{\perp} \\ \mathcal{N}(C_4)^{\perp} \\ \mathcal{N}(C_4) \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{R}(C_4) \\ \mathcal{R}(C_4)^{\perp} \\ R(B)^{\perp} \end{bmatrix}, \tag{6}$$

where C_{41} is invertible. Evidently, $\mathcal{R}(M_1)$ is closed if and only if $\mathcal{R}(B_1^*) + \mathcal{R}(C_{32}^*)$ is closed. Since $C_{32} = P_{\mathcal{R}(C_4)^{\perp}} C_3$ and $\mathcal{R}(B_1^*) = \mathcal{R}(B^*)$, the last condition is equivalent with $\mathcal{R}(B^*) + \mathcal{R}(C_3^* P_{\mathcal{R}(C_4)^{\perp}})$ being closed i.e. with $\mathcal{R}(B^*) + \mathcal{R}(C_3^* P_{\mathcal{R}(C_4)^{\perp}}) = \overline{\mathcal{R}(B^*)}$.

By Lemma 2.1, if $\mathcal{R}(M_C)$ is closed then, since $\mathcal{R}(B)$ is not closed, we get that $d(M_C) = \infty$. Finally, since $n(M_C) = n(A) + n(C_4)$, we can conclude that $\mathcal{R}(M_C)$ is closed and $n(M_C) = d(M_C) = \infty$ if and only if C_4 has closed range, $d(C_4) = \infty$, $\mathcal{R}(B^*) + \mathcal{R}(C_3^* P_{\mathcal{R}(C_4)^{\perp}}) = \overline{\mathcal{R}(B^*)}$ and $n(A) + n(C_4) = \infty$.

Now, we will consider the case when $\mathcal{R}(M_C)$ is not closed. From the discussion above, $\mathcal{R}(M_C)$ is not closed if and only if $\mathcal{R}(M_1)$ is not closed. Using Theorems 2.5 and 2.6 from [5], we conclude that $\mathcal{R}(M_1)$ is not closed precisely in one of the following cases:

1. $\mathcal{R}(C_4)$ is closed, $d(C_4) < \infty$,
2. $\mathcal{R}(C_4)$ is closed, $\mathcal{R}(B^*) + \mathcal{R}(C_3^* P_{\mathcal{R}(C_4)^{\perp}}) \neq \overline{\mathcal{R}(B^*)}$,
3. $\mathcal{R}(C_4)$ is not closed.

Case 3. $\mathcal{R}(A)$ is non- closed, $\mathcal{R}(B)$ is closed. This case is analogous to Case 2.

Case 4. $\mathcal{R}(A)$, $\mathcal{R}(B)$ are non-closed. In this case for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, we have that M_C satisfies conditions from (ii) or (iii) of Theorem 2.2: If $\mathcal{R}(M_C)$ is closed, then by Lemma 2.1 it follows that $n(M_C) = d(M_C) = \infty$, so (ii) from Theorem 2.2 holds while if $\mathcal{R}(M_C)$ is not closed, then (iii) from Theorem 2.2 is satisfied. □

In the following theorem we present certain necessary and sufficient conditions such that for given $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ there exists $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that M_C is Fredholm consistent. The set of all such C, which will be denoted by $S_{\mathrm{FC}}(A, B)$, will be completely described. We will consider four possible cases.

Theorem 2.4. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ be the operators with closed ranges. Then there exists $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that M_C is FC if and only if one of the following conditions is satisfied.*

(1.1) $A, B \in F(\mathcal{H})$. *In this case*

$$S_{\mathrm{FC}}(A, B) = \mathcal{B}(\mathcal{K}, \mathcal{H}).$$

(1.2) $A \in F_+(\mathcal{H})$, $B \in F_-(\mathcal{H})$, $d(A) = n(B) = \infty$. *In this case,*

$$\begin{aligned} S_{\mathrm{FC}}(A, B) = \big\{C \in \mathcal{B}(\mathcal{K}, \mathcal{H})\colon & C \textit{ is given by } (3), \\ & C_4 \in F(\mathcal{N}(B), \mathcal{R}(A)^{\perp})\big\} \\ \cup \big\{C \in \mathcal{B}(\mathcal{K}, \mathcal{H})\colon & C \textit{ is given by } (3), \\ & \mathcal{R}(C_4) \textit{ is closed,} \\ & n(C_4) = d(C_4) = \infty\big\} \\ \cup \big\{C \in \mathcal{B}(\mathcal{K}, \mathcal{H})\colon & C \textit{ is given by } (3), \\ & \mathcal{R}(C_4) \textit{ is not closed}\big\}. \end{aligned}$$

(1.3) $n(B) = d(A) = n(A) = d(B) = \infty$. *In this case,*

$$S_{\mathrm{FC}}(A, B) = \mathcal{B}(\mathcal{K}, \mathcal{H}).$$

(1.4) $A \in F_+(\mathcal{H})$, $n(B) = d(A) = d(B) = \infty$. *In this case,*

$$\begin{aligned} S_{\mathrm{FC}}(A, B) = \big\{C \in \mathcal{B}(\mathcal{K}, \mathcal{H})\colon & C \textit{ is given by } (3),\ \mathcal{R}(C_4) \textit{ is closed,} \\ & n(C_4) = \infty\big\} \\ \cup \big\{C \in \mathcal{B}(\mathcal{K}, \mathcal{H})\colon & C \textit{ is given by } (3), \\ & \mathcal{R}(C_4) \textit{ is not closed}\big\}. \end{aligned}$$

(1.5) *$B \in F_-(\mathcal{K})$, $n(A) = n(B) = d(A) = \infty$. In this case,*

$$S_{\mathrm{FC}}(A, B) = \{C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) \colon C \text{ is given by } (3),\ \mathcal{R}(C_4) \text{ is closed}, d(C_4) = \infty\} \cup \{C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) \colon C \text{ is given by } (3), \mathcal{R}(C_4) \text{ is not closed}\}.$$

(1.6) *$A \in F_-(\mathcal{H})$, $B \in F_+(\mathcal{K})$, $n(A) = d(B) = \infty$. In this case,*

$$S_{\mathrm{FC}}(A, B) = \mathcal{B}(\mathcal{K}, \mathcal{H}).$$

(1.7) *$B \in F_+(\mathcal{K})$, $n(A) = d(A) = \infty$. In this case,*

$$S_{\mathrm{FC}}(A, B) = \mathcal{B}(\mathcal{K}, \mathcal{H}).$$

(1.8) *$A \in F_-(\mathcal{H})$, $n(B) = d(B) = \infty$. In this case,*

$$S_{\mathrm{FC}}(A, B) = \mathcal{B}(\mathcal{K}, \mathcal{H}).$$

Proof. Evidently by item (i) of Theorem 2.3, we have that if $A \in F(\mathcal{H})$, $B \in F(\mathcal{K})$, then M_C is FC for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$.

Now, we will first suppose that $n(B) = d(A) = \infty$ and consider four cases depending on whether the values of $n(A)$ and $d(B)$ are finite or not.

Case 1(a). $n(B) = d(A) = \infty$, $n(A) + d(B) < \infty$. Now $A \in F_+(\mathcal{H})$, $B \in F_-(\mathcal{K})$, so by item (ii) of Theorem 2.3, we have that

$$\{C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) \colon C \text{ is given by } (3),\ C_4 \in F(\mathcal{N}(B), \mathcal{R}(A)^{\perp})\} \subseteq S_{\mathrm{FC}}(A, B) \quad (7)$$

and the existence of C such that $C_4 \in F(\mathcal{N}(B), \mathcal{R}(A)^{\perp})$ is evident. Since $\mathcal{N}(B)$ and $\mathcal{R}(A)^{\perp}$ are infinite dimensional spaces, there exist C such that $\mathcal{R}(C_4)$ is not closed and also there exists C such that $\mathcal{R}(C_4)$ is closed and $n(C_4) = d(C_4) = \infty$. The condition that $n(C_4) = d(C_4) = \infty$ is necessary and sufficient for $n(A) + n(C_4) = d(B) + d(C_4) = \infty$ to hold in item (iiia) of Theorem 2.3. Hence,

$$\{C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) \colon C \text{ is given by } (3),\ \mathcal{R}(C_4) \text{ is closed}, n(C_4) = d(C_4) = \infty\} \cup \{C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) \colon C \text{ is given by } (3),\ \mathcal{R}(C_4) \text{ is not closed}\} \subseteq S_{\mathrm{FC}}(A, B). \quad (8)$$

That the converse of (7) and (8) follows from the fact that Theorem 2.3 covers all the cases when M_C is FC.

Case 1(b). $n(B) = d(A) = n(A) = d(B) = \infty$. By item (a) of Theorem 2.3, we have that M_C is FC for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $\mathcal{R}(C_4)$ is closed. Since by item (b) of that theorem, we have that M_C is FC for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $\mathcal{R}(C_4)$ is not closed, we conclude that in this case M_C is FC for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$.

Case 1(c). $n(B) = d(A) = \infty, n(A) < \infty, d(B) = \infty$. Then $A \in F_+(\mathcal{H})$. In this case, by item (a) of Theorem 2.3, we have that M_C is FC for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $\mathcal{R}(C_4)$ is closed and $n(A)+n(C_4) = \infty$ and by item (b), we have that M_C is FC for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $\mathcal{R}(C_4)$ is not closed. Since in Theorem 2.3 all possible cases are listed when M_C is FC, we conclude that

$$S_{\text{FC}}(A, B) = \left\{C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) \colon C \text{ is given by (3)}, \ \mathcal{R}(C_4) \text{ is closed}, n(C_4) = \infty\right\} \\ \cup \left\{C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) \colon C \text{ is given by (3)}, \ \mathcal{R}(C_4) \text{ is not closed}\right\}. \tag{9}$$

Case 1(d). $n(B) = d(A) = \infty, n(A) = \infty, d(B) < \infty$. Analogously as in the previous case we have that

$$S_{\text{FC}}(A, B) = \left\{C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) \colon C \text{ is given by (3)}, \ \mathcal{R}(C_4) \text{ is closed}, d(C_4) = \infty\right\} \\ \cup \left\{C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) \colon C \text{ is given by (3)}, \ \mathcal{R}(C_4) \text{ is not closed}\right\}. \tag{10}$$

Now, we will suppose that $d(A), n(B) < \infty$. Then $A \in F_-(\mathcal{H})$, $B \in F_+(\mathcal{K})$ and for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ we have that $\mathcal{R}(C_4)$ is closed and $n(C_4), d(C_4) < \infty$. By items (a) and (b) of Theorem 2.3, we have that M_C is FC for some $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ if and only if $n(A) = d(B) = \infty$. In that case M_C is FC for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$.

Let us suppose that $d(A) = \infty$ and $n(B) < \infty$. Then $B \in F_+(\mathcal{K})$ and for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ we have that $\mathcal{R}(C_4)$ is closed, $n(C_4) < d(C_4) = \infty$. By items (a) and (b) of Theorem 2.3, we have that M_C is FC for some $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ if and only if $n(A) = \infty$. Evidently, in this case M_C is FC for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$.

In the case when $d(A) < \infty$ and $n(B) = \infty$ we can analogously conclude that that M_C is FC for some $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ if and only if $d(B) = \infty$. In that case M_C is FC for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. □

Theorem 2.5. *Let $A \in \mathcal{B}(\mathcal{H})$ have closed range and $B \in \mathcal{B}(\mathcal{K})$ have non closed range. Then there exists $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ such that M_C is FC if and only if one of the following conditions is satisfied.*

(2.1) $d(A) = n(B) = \infty$, $n(A) < \infty$. *In this case,*

$$\begin{aligned} S_{\mathrm{FC}}(A,B) = \big\{C \in \mathcal{B}(\mathcal{K},\mathcal{H})\colon & C \text{ is given by } (3), \\ & \mathcal{R}(C_4) \text{ is closed}, \\ & d(C_4) = \infty, \\ & \mathcal{R}(B^*) + \mathcal{R}(C_3^* P_{\mathcal{R}(C_4)^\perp}) = \overline{\mathcal{R}(B^*)}, \\ & n(C_4) = \infty\big\} \\ \cup \big\{C \in \mathcal{B}(\mathcal{K},\mathcal{H})\colon & C \text{ is given by } (3), \\ & \mathcal{R}(C_4) \text{ is closed}, \\ \cup \big\{C \in \mathcal{B}(\mathcal{K},\mathcal{H})\colon & \mathcal{R}(B^*) + \mathcal{R}(C_3^* P_{\mathcal{R}(C_4)^\perp}) \neq \overline{\mathcal{R}(B^*)}\big\} \\ \cup \big\{C \in \mathcal{B}(\mathcal{K},\mathcal{H})\colon & C \text{ is given by } (3), \\ & \mathcal{R}(C_4) \text{ is closed}, d(C_4) < \infty\big\} \\ \cup \big\{C \in \mathcal{B}(\mathcal{K},\mathcal{H})\colon & C \text{ is given by } (3), \\ & \mathcal{R}(C_4) \text{ is not closed}\big\}. \end{aligned} \tag{11}$$

(2.2) $d(A) = \infty$, $n(A) + n(B) < \infty$. *In this case,*

$$\begin{aligned} S_{\mathrm{FC}}(A,B) = \big\{C \in \mathcal{B}(\mathcal{K},\mathcal{H})\colon & C \text{ is given by } (3), \\ & \mathcal{R}(B^*) + \mathcal{R}(C_3^* P_{\mathcal{R}(C_4)^\perp}) \neq \overline{\mathcal{R}(B^*)}\big\}. \end{aligned}$$

(2.3) $d(A) = n(A) = \infty$. *In this case,* $S_{\mathrm{FC}}(A,B) = \mathcal{B}(\mathcal{K},\mathcal{H})$.

(2.4) $d(A) < \infty$. *In this case,* $S_{\mathrm{FC}}(A,B) = \mathcal{B}(\mathcal{K},\mathcal{H})$.

Proof. According to Theorem 2.3 (iv), we will distinguish the following cases.

Case 2(a). $d(A) = n(B) = \infty$, $n(A) < \infty$. Notice that for each of the cases (c)–(f) of item (iv) of Theorem 2.3, there exists C which satisfies the conditions from that case, so $S_{\mathrm{FC}}(A,B)$ is described by (11).

Case 2(b). $d(A) = \infty$, $n(A) + n(B) < \infty$. In this case for any $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ we have that $\mathcal{R}(C_4)$ is closed, $d(C_4) = \infty$ and $n(A) + n(C_4) < \infty$, so items (c), (d), and (f) from Theorem 2.3 can not be satisfied. Hence,

$$\begin{aligned} S_{\mathrm{FC}}(A,B) = \big\{C \in \mathcal{B}(\mathcal{K},\mathcal{H})\colon & C \text{ is given by } (3), \\ & \mathcal{R}(B^*) + \mathcal{R}(C_3^* P_{\mathcal{R}(C_4)^\perp}) \neq \overline{\mathcal{R}(B^*)}\big\}. \end{aligned}$$

Case 2(c). $d(A) = n(A) = \infty$. In this case for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ we have that $n(A) + n(C_4) = \infty$, so by items (c)–(f) from Theorem 2.3, we get that $S_{\mathrm{FC}}(A, B) = \mathcal{B}(\mathcal{K}, \mathcal{H})$.

Case 2(d). $d(A) < \infty$. For any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ we have that $\mathcal{R}(C_4)$ is closed and $d(C_4) < \infty$. Hence by item (d) of Theorem 2.3 it follows that $S_{\mathrm{FC}}(A, B) = \mathcal{B}(\mathcal{K}, \mathcal{H})$ in this case. □

Theorem 2.6. *Let $A \in \mathcal{B}(\mathcal{H})$ have non closed range and $B \in \mathcal{B}(\mathcal{K})$ have closed range. Then there exists $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that M_C is FC if and only if one of the following conditions is satisfied.*

(3.1) $d(A) = n(B) = \infty$, $d(B) < \infty$. *In this case,*

$$\begin{aligned} S_{\mathrm{FC}}(A, B) = \big\{C \in \mathcal{B}(\mathcal{K}, \mathcal{H})\colon & C \text{ is given by } (3), \\ & \mathcal{R}(C_4) \text{ is closed}, \\ & n(C_4) = \infty, \\ & \mathcal{R}(A) + \mathcal{R}(C_2 P_{\mathcal{N}(C_4)}) = \overline{\mathcal{R}(A)}, \\ & d(C_4) = \infty\big\} \\ \cup \big\{C \in \mathcal{B}(\mathcal{K}, \mathcal{H})\colon & C \text{ is given by } (3), \\ & \mathcal{R}(C_4) \text{ is closed}, \\ & \mathcal{R}(A) + \mathcal{R}(C_2 P_{\mathcal{N}(C_4)}) \neq \overline{\mathcal{R}(A)}\big\} \\ \cup \big\{C \in \mathcal{B}(\mathcal{K}, \mathcal{H})\colon & C \text{ is given by } (3), \\ & \mathcal{R}(C_4) \text{ is closed}, \\ & n(C_4) < \infty\big\} \\ \cup \big\{C \in \mathcal{B}(\mathcal{K}, \mathcal{H})\colon & C \text{ is given by } (3), \\ & \mathcal{R}(C_4) \text{ is not closed},\big\}. \end{aligned}$$

(3.2) $n(B) = \infty$, $d(A) + d(B) < \infty$. *In this case,*

$$\begin{aligned} S_{\mathrm{FC}}(A, B) = \big\{C \in \mathcal{B}(\mathcal{K}, \mathcal{H})\colon & C \text{ is given by } (3), \\ & \mathcal{R}(A) + \mathcal{R}(C_2 P_{\mathcal{N}(C_4)}) \neq \overline{\mathcal{R}(A)}\big\}. \end{aligned}$$

(3.3) $n(B) = d(B) = \infty$. *In this case,* $S_{\mathrm{FC}}(A, B) = \mathcal{B}(\mathcal{K}, \mathcal{H})$.

(3.4) $n(B) < \infty$. *In this case,* $S_{\mathrm{FC}}(A, B) = \mathcal{B}(\mathcal{K}, \mathcal{H})$.

Proof. The proof can be given analogously to that of Theorem 2.5. □

Theorem 2.7. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ be the operators with non-closed ranges. Then*

$$S_{\mathrm{FC}}(A, B) = \mathcal{B}(\mathcal{K}, \mathcal{H}).$$

Proof. By item (vi) of Theorem 2.3, we have that if $\mathcal{R}(A)$, $\mathcal{R}(B)$ are non-closed, then M_C is FC for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. □

As a corollary of the previous result we get the following result.

Theorem 2.8. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ be given operators. Then there exists $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that M_C is FC if and only if one of the following conditions is satisfied:*

(a) *$A \in F(\mathcal{H})$ and $B \in F(\mathcal{K})$,*

(b) *$n(A) + n(B) = d(A) + d(B) = \infty$,*

(c) *$\mathcal{R}(A)$ is non-closed,*

(d) *$\mathcal{R}(B)$ is non-closed.*

Now, using Theorems 2.2 and 2.4–2.7 we can compute $\bigcap_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{\mathrm{FC}}(M_C)$.

Corollary 2.1. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. Then*

$$\begin{aligned}\bigcap_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{\mathrm{FC}}(M_C) = & \ (\rho_F(A) \cap \sigma_F(B) \cap \rho_{F_+}(B)) \\ & \cup (\rho_F(B) \cap \sigma_F(A) \cap \rho_{F_-}(A)) \\ & \cup (\rho_{F_+}(A) \cap \sigma_F(A) \cap \rho_{F_+}(B)) \\ & \cup (\rho_{F_-}(A) \cap \sigma_F(B) \cap \rho_{F_-}(B)).\end{aligned}$$

As in the case of ordinary spectrum (see [1]) and consistent invertibility spectrum (see [8]) we have the following result.

Corollary 2.2. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. Then for every $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, we have that*

$$\sigma_{\mathrm{FC}}(M_C) \subseteq \sigma_{\mathrm{FC}}(A) \cup \sigma_{\mathrm{FC}}(B). \tag{12}$$

Proof. Suppose that $0 \notin \sigma_{\mathrm{FC}}(A) \cup \sigma_{\mathrm{FC}}(A)$ (instead of 0 we can take any $\lambda \in \mathbb{C}$). Then A and B are FC operators. So by Theorem 2.2 we have 9 possible cases. We can check using Theorems 2.4-2.7 that in each of these cases we have that $S_{\mathrm{FC}}(A, B) = \mathcal{B}(\mathcal{K}, \mathcal{H})$. Hence $0 \notin \sigma_{\mathrm{FC}}(M_C)$ for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. □

Remark. The reverse inclusion of (12) does not hold in general. For $A \in F_-(\mathcal{H}) \setminus F(\mathcal{H})$ and $B \in B(\mathcal{K})$ such that $\mathcal{R}(B)$ is closed and $n(B) = d(B) = \infty$, by Theorem 2.4 (1.8), we have that $S_{\mathrm{FC}}(A, B) = \mathcal{B}(\mathcal{K}, \mathcal{H})$. So A is not FC and $0 \in \sigma_{\mathrm{FC}}(A) \cup \sigma_{\mathrm{FC}}(B)$ while $0 \notin \sigma_{\mathrm{FC}}(M_C)$ for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. Using Theorems 2.4–2.7 we can easy find necessary and sufficient conditions for the operators A and B to satisfy so that

$$\sigma_{\mathrm{FC}}(M_C) = \sigma_{\mathrm{FC}}(A) \cup \sigma_{\mathrm{FC}}(B), \tag{13}$$

for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$.

Using Theorems 2.4-2.7 we can compute $\bigcup_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{\mathrm{FC}}(M_C)$.

Corollary 2.3. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. Then*

$$\begin{aligned} \bigcup_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{\mathrm{FC}}(M_C) = {} & (\rho_{F_+}(A) \cap \sigma_F(A)) \\ & \cup (\rho_{F_-}(B) \cap \sigma_F(B)) \\ & \cup (\rho_F(A) \cap \sigma_F(B) \cap \rho_{F_+}(B)) \\ & \cup (\rho_F(B) \cap \sigma_F(A) \cap \rho_{F_-}(A)). \end{aligned} \tag{14}$$

Proof. Notice that the set $\bigcup_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{\mathrm{FC}}(M_C)$ is exactly the union of two disjoint sets: $\bigcap_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{\mathrm{FC}}(M_C)$ and

$$T = \bigcup_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{\mathrm{FC}}(M_C) \setminus \Big(\bigcap_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{\mathrm{FC}}(M_C) \Big).$$

Since the first one is described in Corollary 2.1, we will describe the other one. Notice that $0 \in T$ if and only A and B satisfy conditions from one of the items given in Theorems 2.4-2.7 in which we have that $S_{\mathrm{FC}}(A, B) \neq \mathcal{B}(\mathcal{K}, \mathcal{H})$ (instead of 0 we can consider any $\lambda \in \mathbb{C}$). Hence, by Theorems 2.4-2.7, we have that $S_{\mathrm{FC}}(A, B) \neq \mathcal{B}(\mathcal{K}, \mathcal{H})$ if and only if one of the following conditions holds:

- $A \in F_+(\mathcal{H})$, $B \in F_-(\mathcal{H})$, $d(A) = n(B) = \infty$,
- $A \in F_+(\mathcal{H})$, $n(B) = d(A) = d(B) = \infty$,
- $B \in F_-(\mathcal{K})$, $n(A) = n(B) = d(A) = \infty$,
- $\mathcal{R}(A)$ is closed, $\mathcal{R}(B)$ is non-closed, $d(A) = n(B) = \infty$, $n(A) < \infty$,
- $\mathcal{R}(A)$ is closed, $\mathcal{R}(B)$ is non-closed, $d(A) = \infty$, $n(A) + n(B) < \infty$,
- $\mathcal{R}(A)$ is non-closed, $\mathcal{R}(B)$ is closed and $d(A) = n(B) = \infty$, $d(B) < \infty$,
- $\mathcal{R}(A)$ is non-closed, $\mathcal{R}(B)$ is closed and $n(B) = \infty$, $d(A) + d(B) < \infty$.

Hence,

$$\begin{aligned}
T = \{\lambda \in \mathbb{C} \colon & A-\lambda \in F_+(\mathcal{H}) \setminus F(\mathcal{H}),\\
& B-\lambda \in F_-(\mathcal{K}) \setminus F(\mathcal{K})\}\\
\cup \{\lambda \in \mathbb{C} \colon & A-\lambda \in F_+(\mathcal{H}) \setminus F(\mathcal{H}),\\
& \mathcal{R}(B-\lambda) \text{ is closed},\\
& n(B-\lambda) = d(B-\lambda) = \infty\}\\
\cup \{\lambda \in \mathbb{C} \colon & B-\lambda \in F_-(\mathcal{K}) \setminus F(\mathcal{K}),\\
& \mathcal{R}(A-\lambda) \text{ is closed},\\
& n(A-\lambda) = d(A-\lambda) = \infty\}\\
\cup \{\lambda \in \mathbb{C} \colon & A-\lambda \in F_+(\mathcal{H}) \setminus F(\mathcal{H}),\\
& \mathcal{R}(B-\lambda) \text{ is non-closed}\}\\
\cup \{\lambda \in \mathbb{C} \colon & B-\lambda \in F_-(\mathcal{H}) \setminus F(\mathcal{H}),\\
& \mathcal{R}(A-\lambda) \text{ is non-closed}\}.
\end{aligned}$$

Using Theorem 2.2 we get that the union of the second and the forth set appearing above and the third set appearing in the expression for

$$\bigcap_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{\mathrm{FC}}(M_C)$$

(Corollary 2.1) is equal to

$$\begin{aligned}
\{\lambda \in \mathbb{C} \colon & A-\lambda \in F_+(\mathcal{H}) \setminus F(\mathcal{H}),\\
& B-\lambda \text{ is FC}\}\\
\cup \{\lambda \in \mathbb{C} \colon & A-\lambda \in F_+(\mathcal{H}) \setminus F(\mathcal{H}),\\
& B-\lambda \in F_+(\mathcal{K}) \setminus F(\mathcal{K})\}.
\end{aligned}$$

Also, the union of the third and the fifth set appearing in the expression for T and the forth set appearing in the expression for

$$\bigcap_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{\mathrm{FC}}(M_C)$$

(Corollary 2.1) is equal to

$$\begin{aligned}
\{\lambda \in \mathbb{C} \colon & B-\lambda \in F_-(\mathcal{H}) \setminus F(\mathcal{H}),\\
& A-\lambda \text{ is FC}\}\\
\cup \{\lambda \in \mathbb{C} \colon & B-\lambda \in F_-(\mathcal{H}) \setminus F(\mathcal{H}),\\
& A-\lambda \in F_-(\mathcal{K}) \setminus F(\mathcal{K})\}.
\end{aligned}$$

Now, we have

$$\begin{aligned}\bigcup_{C\in\mathcal{B}(\mathcal{K},\mathcal{H})}\sigma_{\mathrm{FC}}(M_C) = \{\lambda\in\mathbb{C}\colon A-\lambda\in F_+(\mathcal{H})\setminus F(\mathcal{H}),\\ B-\lambda\in F_-(\mathcal{K})\setminus F(\mathcal{K})\}\\ \cup\{\lambda\in\mathbb{C}\colon A-\lambda\in F_+(\mathcal{H})\setminus F(\mathcal{H}),\\ B-\lambda \text{ is FC}\}\\ \cup\{\lambda\in\mathbb{C}\colon A-\lambda\in F_+(\mathcal{H})\setminus F(\mathcal{H}),\\ B-\lambda\in F_+(\mathcal{K})\setminus F(\mathcal{K})\}\\ \cup\{\lambda\in\mathbb{C}\colon B-\lambda\in F_-(\mathcal{H})\setminus F(\mathcal{H}),\\ A-\lambda \text{ is FC}\}\\ \cup\{\lambda\in\mathbb{C}\colon B-\lambda\in F_-(\mathcal{H})\setminus F(\mathcal{H}),\\ A-\lambda\in F_-(\mathcal{K})\setminus F(\mathcal{K})\}\\ \cup(\rho_F(A)\cap\sigma_F(B)\cap\rho_{F_+}(B))\\ \cup(\rho_F(B)\cap\sigma_F(A)\cap\rho_{F_-}(A)).\end{aligned}$$

Using that B is not FC if and only if $B\in F_-(\mathcal{H})\setminus F(\mathcal{H})$ or $B\in F_+(\mathcal{H})\setminus F(\mathcal{H})$, we get that the union of the first three sets on the hand right side of the last equality is equal to

$$\{\lambda\in\mathbb{C}\colon A-\lambda\in F_+(\mathcal{H})\setminus F(\mathcal{H})\}$$

while the union of the first, fourth and fifth is equal to

$$\{\lambda\in\mathbb{C}\colon B-\lambda\in F_-(\mathcal{H})\setminus F(\mathcal{H})\}$$

Hence (14) holds. □

In the following result we give an answer to Question 3.

Corollary 2.4. *Let* $A\in\mathcal{B}(\mathcal{H})$ *and* $B\in\mathcal{B}(\mathcal{K})$. *Then*

$$\bigcap_{C\in\mathcal{B}(\mathcal{K},\mathcal{H})}\sigma_{\mathrm{FC}}(M_C)=\sigma_{\mathrm{FC}}(M_0).$$

Proof. Since one inclusion is trivial, we need only show that

$$\sigma_{\mathrm{FC}}(M_0)\subseteq\bigcap_{C\in\mathcal{B}(\mathcal{K},\mathcal{H})}\sigma_{\mathrm{FC}}(M_C).$$

So, let $\lambda\notin\bigcap_{C\in\mathcal{B}(\mathcal{K},\mathcal{H})}\sigma_{\mathrm{FC}}(M_C)$ be arbitrary. Without any loss of generality we can suppose that $\lambda=0$. So, we have that there exists $C_0\in\mathcal{B}(\mathcal{K},\mathcal{H})$ such that M_{C_0} is FC. Hence, we can conclude that for A and B the conditions from one of the items given in Theorems 2.4-2.7 must hold. It can be checked that in all of these cases, we have $0\in S_{\mathrm{FC}}(A,B)$ i.e. $\lambda\notin\sigma_{\mathrm{FC}}(M_0)$. □

References

[1] C. Benhida, E. H. Zerouali, and H. Zguitti, Spectra of upper-triangular operator matrices. *Proc. Amer. Math. Soc.* **133** (2005), no. 10, 3013–3020. MR 2159780 Zbl 1067.47005

[2] X. H. Cao, M. Z. Guo, and B. Meng, Semi-Fredholm spectrum and Weyl's theorem for operator matrices. *Acta Math. Sin.* (*Engl. Ser.*) **22** (2006), no. 1, 169–178. MR 2200773 Zbl 1129.47014

[3] X. H. Cao and B. Meng, Essential approximate point spectra and Weyl's theorem for operator matrices. *J. Math. Anal. Appl.* **304** (2005), no. 2, 759–771. MR 2127605 Zbl 1083.47006

[4] D. S. Cvetković -Ilić, G. Hai, and A. Chen, Some results on Fredholmness and boundedness below of an upper triangular operator matrix. *J. Math. Anal. Appl.* **425** (2015), no. 2, 1071–1082. MR 3303908 Zbl 1304.47017

[5] Y. N. Dou, G. C. Du, C. F. Shao, and H. K. Du, Closedness of ranges of upper-triangular operators. *J. Math. Anal. Appl.* **356** (2009), no. 1, 13–20. MR 2524211 Zbl 1206.47001

[6] B. P. Duggal, D. S. Djordjević, and R. Harte, Consistent in invertibility operators and SVEP. *Filomat* **27** (2013), no. 5, 767–776. MR 3186097 Zbl 1324.47040

[7] W. Gong and D. Han, Spectrum of the products of operators and compact perturbations. *Proc. Amer. Math. Soc.* **120** (1994), no. 3, 755–760. MR 1197538 Zbl 0842.47003

[8] G. Hai and A. Chen, Consistent invertibility of upper-triangular operator. *Linear Algebra Appl.* **455** (2014), 22–31. MR 3217398 Zbl 1303.47006

[9] J. K. Han, H.Y. Lee, and W.Y. Lee, Invertible completions of 2×2 upper triangular operator matrices. *Proc. Amer. Math. Soc.* **128** (2000), no. 1, 119–123. MR 1618686 Zbl 0944.47004

[10] I. S. Hwang, W. Y. Lee, The boundedness below of 2×2 upper triangular operator matrix. *Integral Equations Operator Theory* **39** (2001), no. 3, 267–276. MR 1818061 Zbl 0986.47004

[11] K. Takahashi, Invertible completions of operator matrices. *Integral Equations Operator Theory* **21** (1995), no. 3, 355–361. MR 1316548 Zbl 0824.47002

Received July 10, 2015; revised October 4, 2015

Dragana Cvetković-Ilić, Department of Mathematics,
Faculty of Science and Mathematics, University of Nis, 18000 Nis, Serbia

e-mail: dragana@pmf.ni.ac.rs