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Ruled nodal surfaces of Laplace eigenfunctions and injectivity sets for the spherical mean Radon transform in $\mathbb{R}^3$

Mark L. Agranovsky

Abstract. It is proved that if a Paley–Wiener family of eigenfunctions of the Laplace operator in $\mathbb{R}^3$ vanishes on a real-analytically ruled two-dimensional surface $S \subset \mathbb{R}^3$ then S is a union of cones, each of which is contained in a translate of the zero set of a nonzero harmonic homogeneous polynomial. If S is an immersed C^1 manifold then S is a Coxeter system of planes. Full description of common nodal sets of Laplace spectra of convexly supported distributions is given. In equivalent terms, the result describes ruled injectivity sets for the spherical mean transform and confirms, for the case of ruled surfaces in $\mathbb{R}^3$, a conjecture from [1].

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Contents

1. Introduction

Nodal sets are zeros of the Laplace eigenfunctions. They play an important role in understanding of the wave propagation.

The geometry of a single nodal set can be very complicated and hardly can be well understood. On the other hand, simultaneous vanishing of large families of eigenfunctions on large sets occurs rarely and hence it is natural to expect that common nodal sets in that case should be pretty special and have a simple geometry.

Bourgain and Rudnick [8] obtained a result of such type for the two-dimensional torus T^2. They proved that only geodesics can be common nodal curves for infinitely many Laplace eigenfunctions on T^2. For tori in high dimensions, they proved that Gauss-Kronecker curvature of the common nodal hypersurfaces must be zero. Analogous question for the sphere in the Euclidean space is still open.

In this article, we address similar questions for Euclidean spaces. The case of $\mathbb{R}^2$ was studied in [1], in equivalent terms of injectivity sets for the spherical mean Radon transform. Translated back to the language of nodal sets, the result of [1] says that one-dimensional parts of common nodal sets of large families eigenfunctions (more specifically, of Laplace spectral projections of compactly supported functions) are Coxeter systems of straight lines in the plane.

In the course of that result, it was conjectured in [1] that in higher dimensions, common nodal surfaces for large families of eigenfunctions (injectivity sets of the spherical mean transform) are cones - translates of the zero sets of solid harmonics (harmonic homogeneous polynomials). In this article, we confirm this conjecture for a special case of ruled surfaces in $\mathbb{R}^3$. The proof develops ideas from the article [4] of E. T. Quinto and the author.

Although ruled surfaces (unions of straight lines) are, in a sense, close to cones (union of straight lines with a common point), proving conical structure of ruled nodal surfaces in dimensions higher than two was elusive for a long time.

2. Main results

We will formulate the main results of this article in two equivalent terms: 1) on the language of nodal surfaces and 2) on the language of injectivity sets.

We start with the nodal surfaces version.

2.1. Nodal surfaces version. Let $\varphi_\lambda,\ \lambda > 0$, be a family of eigenfunctions of the Laplace operator Δ in $\mathbb{R}^3$. More precisely, each function φ_λ is a solution (possibly, identically zero) of the Helmholtz equation

$$\Delta\varphi_\lambda = -\lambda^2\varphi_\lambda.$$

Definition 2.1. The family φ_λ is a *Paley–Wiener* family if it can be extended in the complex plane $\lambda \in \mathbb{C}$ as an even nonzero entire function, satisfying the growth condition

$$|\varphi_\lambda(x)| \le C(1+|\lambda|)^N e^{(R+|x|)|\operatorname{Im}\lambda|}.$$

for some positive constants C, R and for some natural N.

By cone in $\mathbb{R}^d$, we understand union of straight lines having a common point-the vertex of the cone. We call a cone C *harmonic cone* if there exists a nonzero harmonic homogeneous polynomial (solid harmonic) h and a vector a such that

$$C \subset a + h^{-1}(0).$$

By curve γ in $\mathbb{R}^d$ we understand the image $\gamma = u(J)$ of a segment $J = [a,b]$ under a nonconstant continuous mapping $u\colon J \mapsto \mathbb{R}^d$ of a segment $J = [a,b]$. The curve γ is closed if $u(a) = u(b)$.

Definition 2.2. Let S be a surface in $\mathbb{R}^3$. We call S an *irreducible real analytically ruled surface* if

(1) there exists a closed continuous curve $\gamma \subset \mathbb{R}^3$ such that S is the union of straight lines, $S = \bigcup_{a\in\gamma} L(a)$, passing through points $a \in \gamma$;

(2) locally, the curve γ is the image of a real analytic mapping $u\colon(-1,1)\mapsto\mathbb{R}^3$ and the surface S is, locally, the image of the (parameterizing) mapping

$$(-1,1)\times\mathbb{R} \ni (t,\lambda) \longmapsto u(t,\lambda) = u(t) + \lambda e(t),$$

where $(-1,1) \ni t \mapsto e(t) \in \mathbb{R}^3$ is a real analytic map with $|e(t)| = 1$.

The curve γ is called the *base curve*, the vector $e(t)$-directional vector, the straight lines $L_t = L(u(t)) = \{u(t) + \lambda e(t),\ \lambda \in \mathbb{R}\}$ are called rulings, or ruling or generating lines. *Real analytically ruled surface* are, by definition, finite unions of irreducible real analytically ruled surfaces.

Remark 2.3. (1) The line foliation (ruling) of the ruled surface S is assumed to be fixed, therefore, formally speaking, a ruled surface is understood as a pair consisting of a surface and a line foliation. For example, the two foliations of the plane $\mathbb{R}^2$: the family of straight lines passing through the origin (the base curve can be taken the unit circle), and a family of parallel lines (the base curve can be taken an orthogonal line) correspond to the two different ruled surfaces. On the other hand, given a foliation, the choice of the base curves is not unique.

The parameterizing mapping $u(t, \lambda)$ does not necessarily define a parametrization of S as a differentiable manifold, since the regularity condition is not required.

(2) Real analytically ruled surfaces are not necessarily everywhere real analytic, and even differentiable. For example, the cone $x^2+y^2-z^2 = 0$ in $\mathbb{R}^3$ is a real analytically ruled surface, parametrized by the mapping $(t, \lambda) \mapsto \lambda(\cos t, \sin t, 1)$, but is not differentiable at its vertex $a = 0$.

Now we are ready to formulate the main results of this article.

Theorem 2.4. *Let S be an irreducible real-analytically ruled surface in which no two generating lines are parallel. Then S is the common nodal set for a Paley–Wiener family if and only if S is a harmonic cone.*

In the reducible case, we have

Theorem 2.5. *Let S be a real-analytically ruled surface in $\mathbb{R}^3$, with no parallel generating lines. If S is the common nodal set for a Paley–Wiener family of eigenfunctions then S is the union of a finite number of harmonic cones, $S = \bigcup_{j=1}^{N} C_j$ such that for any $1 \leq i < j \leq N$ the intersection $C_i \cap C_j \neq \emptyset$ and only the two cases are possible*:

(1) $C_i \cap C_j$ *is the vertex of one of the cones C_i, C_j,*

(2) $C_i \cap C_j$ *is transversal and is an unbounded curve.*

Conjecture from [1] (see section 3 for the details) claims that, in fact, S is a single cone, which means that the cones C_i share their vertices. However, we are not able to prove that at the moment.

Definition 2.6. The union $\Sigma = \bigcup_{j=1}^{N} \Pi_j$ of N hyperplanes in $\mathbb{R}^d$ having a common point is called *Coxeter system* if Σ is invariant with respect to all the reflections around the planes Π_j, $j = 1, \ldots, N$.

Notice that Coxeter systems are harmonic cones, i.e., are, up to translations, zero sets of solid harmonics.

Theorem 2.7. *If in Theorem 2.5 S is an immersed C^1-surface then S is a Coxeter system.*

Recall that an immersed C^1-surface is the image of a two-dimensional C^1-manifold under a C^1-mapping with non-degenerate differential.

Finally, we will formulate one more result about common nodal surfaces for special Paley–Wiener families of eigenfunctions: spectral projections of *convexly supported* distributions:

Theorem 2.8. *Let $f \in D'_{\text{comp}}(\mathbb{R}^3)$ be a nonzero compactly supported distribution or continuous function and*

$$f = \int_0^\infty \varphi_\lambda d\lambda$$

be the Laplace spectral decomposition of f (see [22]*). Assume that the boundary of the unbounded connected component of* $\mathbb{R}^3 \setminus \operatorname{supp} f$ *is a real analytic strictly convex closed surface. If*

$$N = \bigcap_{\lambda>0} \varphi_\lambda^{-1}(0)$$

then $N = S \cup V$ where either $V = \emptyset$ or V is an algebraic variety of $\dim V \leq 1$ *and either $S = \emptyset$ or S is one of the three surfaces*:

(1) *S is a harmonic cone*;

(2) *S is the union of two harmonic cones, $S = C_1 \cup C_2$ such that either $C_1 \cap C_2 = \{b_1\}$ or $C_1 \cap C_2 = \{b_2\}$. where b_1, b_2 are the vertices of the corresponding cones*;

(3) *S is the union of three harmonic cones, $S = C_1 \cup C_2 \cup C_3$, with the vertices b_1, b_2, b_3, correspondingly, such that either*

$$C_1 \cap C_2 = \{b_1\}, \quad C_2 \cap C_3 = \{b_2\}, \quad C_3 \cap C_1 = \{b_3\}$$

or

$$C_1 \cap C_2 = \{b_2\}, \quad C_2 \cap C_3 = \{b_3\}, \quad C_3 \cap C_1 = \{b_1\}.$$

We conjecture that, in fact, $b_1 = b_2 = b_3$ and therefore S itself is a cone, in accordance with Conjecture 3.2.

2.2. Injectivity sets version. The spherical mean Radon transform is defined as the mean value

$$Rf(x,t) = \int_{|\theta|=1} f(x + t\theta) dA(\theta)$$

of f over the sphere $S(x,t)$ centered at $x \in \mathbb{R}^d$ of radius $t > 0$. Here dA is the normalized area measure on the unit sphere $\{|\theta| = 1\}$ in $\mathbb{R}^d$.

The operator R can be extended to distributions $f \in D'(\mathbb{R}^d)$. Namely, for each vector $a \in \mathbb{R}^d$ define the averaging operator

$$R_a \psi(x) := \int_{\mathrm{SO}(d)} \psi(a + \omega(x - a)) d\omega,$$

where $d\omega$ is the normalized Haar measure on the orthogonal group SO(d). The relation between this averaging operator and the operator R is given by

$$(R_a\psi)(x) = R\psi(a, |x - a|).$$

Now, if $f \in D'(\mathbb{R}^d)$ and $a \in \mathbb{R}^d$, then we define the new distribution $R_a f$ by the following action on test-functions ψ:

$$\langle R_a f, \psi \rangle = \langle f, R_a \psi \rangle. \tag{1}$$

It is easy to see that this definition is consistent with the definition of the action of the operator R_a on functions.

Denote R_S the restriction of the transform R on the set $S \times (0, \infty)$:

$$R_S : C_{\text{comp}}(\mathbb{R}^d) \ni f \longmapsto Rf|_{S \times (0,\infty)}.$$

Definition 2.9. We call a set $S \subset \mathbb{R}^d$ an *injectivity set* if given a distribution $f \in D'_{\text{comp}}(\mathbb{R}^d)$ such that $R_a f = 0$ for all $a \in S$ then $f = 0$. Equivalently, S is an injectivity set if the operator R_S is injective, i.e. for every function $f \in C_{\text{comp}}(\mathbb{R}^d)$

$$Rf(x, t) = 0 \text{ for all } x \in S \implies f = 0.$$

Equivalence of definition for functions and distributions can be easily proved by convolving distributions with radial smooth functions.

The spherical mean Radon transform[1] plays an important role in applications, namely, in thermo- and photoacoustic tomography (cf. [17]), which is used in the medical imaging [16]. The mathematical problem behind that is to recover f from the data $Rf(x, t),\ x \in S,\ t > 0$. The uniqueness of the recovery is equivalent to the injectivity of the operator R_S and therefore the first question to be answered is to understand for what *observation surfaces* S the operator R_S is injective, i.e., to understand the injectivity sets. Of course, the case $d = 3$ is most important from the point of view of the applications.

Definition 2.10. Let $\{\varphi_\lambda\}_{\lambda>0,}$ be a measurable family of Laplace eigenfunction: $(\Delta + \lambda^2)\varphi_\lambda = 0$ in $\mathbb{R}^d$. We will call the function

$$f(x) = \int_0^\infty \varphi_\lambda(x) d\lambda \tag{2}$$

[1] We refer to Radon transform because the operator R is defined on complexes of spheres with restricted centers and of arbitrary radii. Such varieties are analogous to varieties of planes with restricted set of normal vectors and arbitrary distances to the origin which are natural in the study of the plane Radon transform.

a *generating function*, assuming that the integral converges (which can be achieved by a proper normalization $\varphi_\lambda \to c(\lambda)\varphi_\lambda, c(\lambda) \neq 0$.) The family φ_λ is called a *Laplace spectral decomposition* of f.

The definition can be extended to distributions $f \in D'(\mathbb{R}^d)$ if we understand the spectral decomposition of f in the distributional sense.

The link between common nodal sets and injectivity sets in the question is very simple: they just coincide (see Proposition 5.1).

Let us briefly explain this relation. It is proved in ([22], Theorem 3.10) that *a family φ_λ of eigenfunctions in $\mathbb{R}^d$ is Paley–Wiener if (and if and only if, when d is odd), after a suitable renormalization $\varphi_\lambda \to c(\lambda)\varphi_\lambda, c(\lambda) \neq 0$, the integral (2) defines a compactly supported distribution $f \in D'(\mathbb{R}^d)$.*

The spectral decomposition $\{\varphi_\lambda\}$ can be recovered from the generating distribution f by means of the convolutions

$$\varphi_\lambda = j^\lambda_{\frac{d-2}{2}} * f \tag{3}$$

of f with the normalized Bessel function

$$j^\lambda_{\frac{d-2}{2}}(x) = (2\pi)^{-\frac{d}{2}} \frac{J_{\frac{d-2}{2}}(|\lambda x|)}{(|\lambda x|)^{\frac{d-2}{2}}}.$$

It follows that $S \subset \bigcap_{\lambda>0} \varphi_\lambda^{-1}(0) = 0$ if and only if $Rf|_{S\times(0,\infty)} = 0$.

Recall that the condition $Rf|_{S\times(0,\infty)} = 0$ for $f \in D'(\mathbb{R}^d)$ means that the average distribution $R_a f$, defined in (1), is the zero distribution: $R_a f = 0$ for all $a \in S$.

Thus, we have

Proposition 2.11. *A set $S \subset \mathbb{R}^d$ serves a common nodal set for a nontrivial Paley–Wiener family $\{\varphi_\lambda\}$ if (and if and only if when d is odd) $Rf|_{S\times(0,\infty)} = 0$ for some nonzero compactly supported distribution (or continuous function) f, i.e., if (and if and only if when d is odd) S fails to be a set of injectivity for the spherical mean Radon transform R.*

Using that equivalence, we can reformulate Theorems 2.4 and 2.5 in the equivalent form:

Theorem 2.12. *Let S be a real-analytically ruled surface in $\mathbb{R}^3$. If S fails to be an injectivity set then S is one of the surfaces listed in Theorem 2.5. If S is irreducible (see Definition 2.2) then S fails to be an injectivity set if and only if S is a harmonic cone.*

The following theorem is a translation, on the injectivity sets language, of Theorem 2.8. Is an equivalent version of Theorem 2.8 and follows from Theorem 2.5 and [1] and [7]. Here the certain restrictions are imposed on the geometric shape of the support of the generating distribution.

Theorem 2.13. *Let $f \in D'_{\text{comp}}(\mathbb{R}^3)$ be nonzero compactly supported distribution or continuous function. Assume that the boundary of the unbounded connected component of $\mathbb{R}^3 \setminus \operatorname{supp} f$ is a real analytic strictly convex closed surface. If $Rf_a = 0$ (see (1)) for all $a \in S$ then S is contained in one of the surfaces listed in Theorem 2.5.*

The proof of Theorems 2.8 and 2.13 is based on Theorem 2.5 and the results of [1] and [7] (Theorem 3.6 from the next section) about ruled structure of observation surfaces for convexly supported functions.

3. Background

In dimension $d = 2$, the problem of describing injectivity sets was completely solved in [1]. Let us formulate the result. Denote

$$\Sigma_N = \left(t \cos k\frac{\pi}{N}, t \sin k\frac{\pi}{N}\right), \quad k = 0, 1, \dots, N-1, \ -\infty < t < \infty,$$

the (Coxeter) system of N straight lines passing through the origin and having equal angles between the adjacent lines.

Theorem 3.1. [1] *A set $S \subset \mathbb{R}^2$ is a set of injectivity if and only if S is contained in no set of the form $(a + \omega(\Sigma_N)) \cup V$, where $a \in \mathbb{R}^2$, ω is a rotation in the plane and V is a finite set, invariant under reflections around the lines from the Coxeter system $a + \omega(\Sigma_N)$.*

Observe that the Coxeter system $\omega(\Sigma_N)$ coincides with the zero set of the polynomial $h(x, y) = \operatorname{Im}(e^{i\varphi}(x + iy)^N)$, where ω is the rotation for the angle φ. The polynomial $h(x, y)$ represents the general form of a harmonic homogeneous polynomial in the plane. That observation gives rise to the following conjecture about how injectivity sets look like in arbitrary dimension.

Conjecture 3.2. [1] Suppose $S \subset \mathbb{R}^d$ fails to be an injectivity set Then $S \subset (a + h^{-1}(0)) \cup V$, where h is a harmonic homogeneous polynomial (spatial harmonic) and V is an algebraic variety in $\mathbb{R}^d$ of dimension $\dim V \le d - 2$.

Since in odd dimensions, as it was mentioned in subsection 2.2, non-injectivity sets are precisely common nodal sets of Paley–Wiener families, Conjecture 3.2 can be reformulated as following:

Conjecture 3.3. A set $S \subset \mathbb{R}^d$, d is odd, is a common nodal set for a Paley–Wiener family of Laplace eigenfunctions if and only if $S \subset (a + h^{-1}(0)) \cup V$, where the vector a, the variety V and the polynomial h are as in Conjecture 3.2.

Remark 3.4. A partial case of non-injectivity sets in Conjecture 3.2 are Coxeter systems of hyperplanes. They are arrangements of N hyperplanes with a common point, invariant under reflections around each the hyperplane from the system. The Coxeter systems correspond to the case of completely reducible harmonic homogeneous polynomials h, i.e., those represented as products

$$h = l_1 \dots l_N$$

of $N = \deg h$ linear forms.

Here is some evidences for Conjecture 3.2 (see [5]).

- Any harmonic cone is a non-injectivity set, i.e., if h is a non-zero harmonic homogeneous polynomial, then $S := h^{-1}(0)$ is a non-injectivity set. Namely, define $f(x) := \alpha(|x|)h(x)$ where $\alpha(r)$ is a non-zero smooth even compactly supported function on $\mathbb{R}$. It is an easy exercise to prove that $Rf(x,t) = 0$ for all $x \in S,\ t > 0$.
- If V is an algebraic variety of $\dim V \leq d - 2$ then there exists a nonzero $f \in C_{\text{comp}}(\mathbb{R}^d)$ such that $Rf(x,t) = 0$ for all $(x,t) \in V \times (0,\infty)$ (see [5], Theorem 3.2).

To our knowledge, only partial results towards Conjecture 3.2 are obtained so far ([4], [7], and [2]). Let us mention some of them. It was proved in [2] that among cones only zero sets of spatial harmonics fail to be injectivity sets. Therefore, the main difficulty in proving Conjecture 3.2 is checking that non-injectivity sets are necessarily *cones*. The following two results can be considered as certain steps in that direction.

Theorem 3.5. [3] *Let f be a compactly supported continuous function or distribution in $\mathbb{R}^d$. Assume that* $\operatorname{supp} f$ *is the union of disjoint balls or is finite. If $S \subset \mathbb{R}^d$ and $R_S f = 0$ then $S \subset (a + h^{-1}(0)) \cup V$, where $a \in \mathbb{R}^d$, h is a nonzero harmonic homogeneous polynomial and V is an algebraic variety of* $\dim V \leq d - 2$.

The next result deals with functions with convex compact supports and can be viewed as a motivation for Theorems 2.8 and 2.13.

Theorem 3.6 ([7] and [4]). *Let $f \in C_{\text{comp}}(\mathbb{R}^d)$ be a compactly supported function. Suppose that the outer boundary $\Gamma = \partial(\operatorname{supp} f)$ is a convex closed C^2 surface. If $S \subset \mathbb{R}^d$ is such that $Rf|_{S\times(0,\infty)} = 0$ then S is ruled, i.e., S is the union of straight lines. Moreover, the ruling lines intersect Γ orthogonally at each point where S is differentiable.*

By outer boundary $\partial(\operatorname{supp} f)$ we understand the boundary $\partial(\mathbb{R}^d \setminus \operatorname{supp} f)_\infty$ of the unbounded connected component of the complement.

Remark 3.7. In fact, the ruled structure of S was established in [7] under much milder conditions for Γ for example, under assumption of C^2 smoothness of Γ. However, in the proofs of Theorems 2.8 and 2.13, we will use the weaker version, Theorem 3.6, because some additional properties delivered by the convexity of support will be exploited as well.

4. The strategy of the proof of the main result

The main result of this article is Theorem 2.4. Theorem 2.5 is deduced from Theorem 2.4, Theorems 2.7 and 2.8 follow from Theorems 2.4 and 2.5. All the theorems can be viewed as results towards proving Conjectures 3.2 and 3.3.

The proof of Theorem 2.4 falls apart into several steps.

Step 1. First, we prove that the common nodal surface S for a Paley–Wiener family is algebraic and lies in the zero set of a nontrivial harmonic polynomial. In a different setting, that fact was first observed in [19] (see also [4]).

Step 2. Next, we formulate a local symmetry property, which is based on the results of [1] and [20] about cancelation of analytic wave front sets. The corollary of that property says is that any surface S having a pair of antipodal points-points of smoothness, such that the segment joining them is orthogonal to the surface, fails to be a common nodal surface for a Paley–Wiener family.

Step 3. Assuming that S is not a cone and using a compactness argument we find two generating (ruling) straight lines on S with the maximal distance between them. Then we pick two closest points $a, b \in S$ on those extremal lines. If those extremal points a, b are regular then the previous step implies that S cannot be nodal. Otherwise, one of the extremal points is singular and we encounter the problem of characterization of singularities of algebraic real analytically ruled surfaces in $\mathbb{R}^3$.

Step 4. We obtain the required characterization of the singularities (Theorem 8.1), which is a key ingredient of the proof of the main result.

Step 5. The final arguments are as follows. Theorem 8.1 claims that singular points are either conical or of cuspidal (double tangency) type. However, zero surfaces of nontrivial harmonic polynomials cannot contain cusps (Corollary 8.4) and hence the latter option is ruled out (Step 1) in the irreducible case. Thus, we conclude that S is a cone (in the irreducible case) or a union of cones (in the reducible case). Finally, the proof that the cones are harmonic easily follows by homogenization of harmonic polynomial vanishing on S (obtained on Step 1). This completes the proof.

Remark 4.1. Essentially, Steps 1–3 were presented in [4]. It was proved there that if the extremal points (Step 3) are regular then the surface is an injectivity set (not nodal). The description of singular points obtained in Theorem 8.1 allowed us to further develop the idea of [4] and push forward proving the conical structure of the nodal ruled surfaces, which is the main result of this article.

5. Preliminary observations

In this section, we briefly present auxiliary facts that we will need in the sequel. Most of them are exposed in [1]. It will be convenient to combine those facts in one proposition.

Proposition 5.1. *Let $\Phi = \{\varphi_\lambda,\ \lambda > 0,\}$ be a family of eigenfunctions in $\mathbb{R}^d$ with compactly supported generating distribution $f \in D'(\mathbb{R}^d)$ i.e.,*

$$f = \int_0^\infty \varphi_\lambda d\lambda.$$

Denote

$$N_f = \{a \in \mathbb{R}^d : Rf_a = 0 \text{ for all } a \in S\},$$

where the averaging operator Rf_a is defined in (1), and

$$N(\Phi) = \bigcap_{\lambda>0} \varphi_\lambda^{-1}(0).$$

Then

(1) $N_f = N(\Phi)$;

(2) *the set* $N(\Phi)$ *is algebraic and has the form*

$$N(\Phi) = S \cup V,$$

where $S = \emptyset$ *or* S *is a real algebraic hypersurface*: $S = Q^{-1}(0)$, *where* Q *is a nonzero real polynomial, and* V *is an algebraic variety of* $\dim V \leq d-2$ (*maybe, empty as well*);

(3) *there is a nonzero real harmonic polynomial* H *vanishing on* S, *i.e.* $S \subset H^{-1}(0)$.

Proof. 1. We have

$$f = \int_0^\infty \varphi_\lambda d\lambda,$$

where the equality is understood in the distributional sense.

Suppose $a \in N(\Phi)$, i.e., $\varphi_\lambda(a) = 0$ for all $\lambda > 0$. It follows for the classical Pizzetti formula that Laplace eigenfunctions have the mean value property:

$$R_a \varphi_\lambda(x) = c_{\lambda,x} \varphi_\lambda(a)$$

and hence $R_a \varphi_\lambda$ is identical zero for any $\lambda > 0$. Therefore

$$R_a f = \int_0^\infty R_a \varphi_\lambda d\lambda = 0$$

and hence $a \in N_f$. Thus $N(\Phi) \subset N_f$.

Conversely, let $a \in N_f$. The spectral projection φ_λ is the convolution of the generating distribution f with the Bessel function (3):

$$\varphi_\lambda(a) = (f * j^\lambda_{\frac{d-2}{2}})(a) = \langle f, \psi_a \rangle,$$

where we have denoted

$$\psi_a(x) = j^\lambda_{\frac{d-2}{2}}(|x-a|).$$

Since the function ψ_a depends only on the distance to the point a, it coincides with its spherical average $R_a\psi_a = \psi_a$ and therefore

$$\varphi_\lambda(a) = \langle f, R_a\psi_a\rangle = \langle R_a f, \psi_a\rangle = 0$$

because $a \in N_f$ and therefore $R_a f$ is the zero distribution. We conclude that $\varphi_\lambda(a) = 0$ and $a \in N(\Phi)$. Therefore, $N_f \subset N(\Phi)$ and the sets coincide.

2. Decompose the (even) normalized Bessel function $j_{\frac{d-2}{2}}(\lambda t)$ into power series:

$$j_{\frac{d-2}{2}}(\lambda t) = \sum_{k=0}^{\infty} c_k \lambda^{2k} t^{2k}.$$

Then we have from (3):

$$\varphi_\lambda(x) = \sum_{k=0}^{\infty} c_k \lambda^{2k} |x|^{2k} * f.$$

Define

$$Q_k(x) = c_k|x|^{2k} * f = c_k\langle |x-y|^{2k}, f\rangle,$$

where the right hand side stands for the action of the distribution f with respect to y. It follows that Q_k is a polynomial and $\deg Q_k \le 2k$ and

$$\varphi_\lambda(x) = \sum_{k=0}^{\infty} Q_k(x)\lambda^{2k}. \tag{4}$$

From (4) $\varphi_\lambda(x) = 0$ is equivalent to $Q_k(0) = 0, k = 0, 1, \ldots$ and hence common zeros of φ_λ and Q_k coincide:

$$N(\Phi) = \bigcap_{k=0}^{\infty} Q_k^{-1}(0).$$

Denote Q the greatest common divisor (over $\mathbb{C}$) of Q_k. Then

$$N(\Phi) = (Q^{-1}(0) \cap \mathbb{R}^d) \cup V,$$

where V is the intersection of $\mathbb{R}^d$ with the zero varieties of coprime polynomials and hence $\dim_{\mathbb{R}} V < d-1$.

To complete the proof of Statement 2, we have to show that the polynomial Q has real coefficients. We will do that at the end of the proof.

3. Substituting (4) into the Helmholtz equation

$$\Delta \sum_{k=0}^{\infty} \lambda^{2k} Q_k = -\lambda^2 \sum_{k=0}^{\infty} \lambda^{2k} Q_k$$

yields

$$\Delta Q_k = -Q_{k-1}, \quad k \geq 1.$$

Not all polynomials Q_k are identically zero. Indeed, suppose that $Q_k = c_k |x|^{2k} * f \equiv 0$ for all $k = 0, 1, \ldots$ Since f has compact support and the linear combinations of the polynomials $|y|^{2k}$ approximate, in the C^∞ topology on compact sets, any radial smooth function $\alpha(|y|^2)$, we have $\alpha * f \equiv 0$. Taking the Fourier transform , we obtain $\hat{\alpha} \hat{f} \equiv 0$ which implies $\hat{f} = 0$ due to the arbitrariness of the radial function α. Then $f = 0$ which is not true.

Let $k = k_0$ be the minimal k such that $Q_k \neq 0$ and denote

$$H = Q_{k_0}.$$

Then

$$\Delta H = -Q_{k_0 - 1} = 0$$

and hence H is harmonic. This proves the Statement 3.

It remains to prove that, in fact, Q is a real polynomial, i.e. has the real coefficients. To this end, we first will prove the third statement.

Let

$$H = H_1 \ldots H_q$$

be the decomposition into irreducible, over $\mathbb{C}$, polynomials. Let us prove that all polynomials H_i are real.

Consider the operation of complex conjugation of coefficients:

$$H^*(z) = \overline{H(\bar{z})}, \quad z \in \mathbb{C}^d.$$

Since $H = Q_{k_0}$ has real coefficients, we have

$$H^* = H_1^* \ldots H_q^* = H_1 \ldots H_q.$$

Therefore, each H_i^* coincides with some H_j. If for some $i \neq j$ holds $H_i^* = H_j$ then H is divisible by $H_i H_i^*$ and represents as

$$H = H_i H_i^* R,$$

for some polynomial R. Since in the real space $\mathbb{R}^d$ we have $H_i^* = \overline{H_i}$, we have in $\mathbb{R}^d$

$$H = |H_i|^2 R.$$

However, the Brelot–Choquet theorem [9] states that no non-negative real polynomial can divide a real nonzero harmonic polynomial. Therefore, the only possibility is that $H_i = H_i^*$ for all i. That means that H_i are real polynomials.

The greatest common divisor Q divides H and therefore is a product of some H_i. Since every polynomial H_i has real coefficients, Q does so.

If Q is constant, i.e., all Q_k are coprime, then $S = Q^{-1}(0) = \emptyset$. Otherwise, S is a hypersurface in $\mathbb{R}^d$. Indeed, if $\dim S < n-1$ then $\mathbb{R}^d \setminus Q^{-1}(0)$ is connected and hence everywhere $Q \geq 0$ everywhere or $Q \leq 0$. However, this impossible, since the Brelot–Choquet theorem states that preserving sign polynomials cannot divide harmonic polynomials. This completes the proof of Proposition. □

Remark 5.2. Proposition 5.1 holds for any Paley–Wiener family in odd-dimensional spaces, since according to Theorem 3.10 ([22]) cited in Section 2.2, such families have compactly supported generating distributions. In particular, it is true for $d = 3$, which is our main case.

6. Local symmetry and antipodal points

Definition 6.1. Let $S \subset \mathbb{R}^d$ and let $a, b \in S$, $a \neq b$, be two distinct points in S. We call a and b *antipodal points* if

(1) S is C^1-hypersurface near the points a, b and
(2) $a - b \perp T_a(S)$, $a - b \perp T_b(S)$, where $T_a(S), T_b(S)$ are the tangent spaces to S at a and b correspondingly.

Theorem 6.2 ([1] and [4]). *If $S \subset \mathbb{R}^d$ has a pair of antipodal points a, b and S is real analytic in neighborhoods of those points, then S is an injectivity set.*

Example. The hyperboloid $x_1^2 + x_2^2 - x_3^2 = 1$ in $\mathbb{R}^3$ has antipodal points, for example, $(\pm 1, 0, 0)$ and hence is an injectivity set.

The proof of Theorem 6.2 is based on the following theorem about certain symmetry of the support of functions with zero spherical means on a surface:

Theorem 6.3 ([1]). *Let S be a real analytic hypersurface and $a \in S$. Let $f \in C_{\mathrm{comp}}(\mathbb{R}^d)$ be a compactly supported function such that $Rf|_{S\times(0,\infty)} = 0$. Let $x \in \operatorname{supp} f$ be a point of local extremum for the distance function $d(x) := |x - a|$ and denote*

$$x^* = x - 2\langle x - a, \nu_a\rangle \nu_a$$

(ν_a is the unit normal vector of S at a), the point, symmetric to x with respect to the tangent plane $T_a(S)$ (mirror point). Then $x^* \in \operatorname{supp} f$.

The proof of Theorem 6.3 uses microlocal analysis and results about cancelation of analytic wave front sets at mirror points ([1], [14], [13], and [20]).

We are going to exploit Theorem 6.3 for algebraic surfaces $S = Q^{-1}(0)$, where Q is a real nonconstant polynomial. However, Theorem 6.3 cannot be applied directly as S is not necessarily everywhere real analytic and, moreover, even differentiable. Nevertheless, S is real analytic everywhere outside of the critical set

$$\text{crit}\, S := \{x \in S : \nabla Q(x) = 0\},$$

which is a nowhere dense subset of S. It is enough to establish a local symmetry property, though in a slightly weaker form than in Theorem 6.3.

Let us introduce some notations and definitions. Given a point $a \in S$ in a neighborhood of which S is C^1 surface we denote

$$\sigma_a : x \longmapsto x - 2\langle x - a, \nu_a \rangle \nu_a,$$

the reflection of $\mathbb{R}^d$ around the tangent plane $T_a(S)$. Here ν_a, as above, is the unit normal vector to S.

For any $a \in S$ and $r > 0$ denote

$$K_{a,r} := \{x \in \text{supp}\, f : |x - a| = r\},$$

the intersection of supp f with the sphere $S_r(a) = \{|x - a| = r\}$.

Theorem 6.4 (Local symmetry property). *Let $S \subset \mathbb{R}^d$ be a hypersurface, real analytic except for a nowhere dense subset. Let $f \in C_{\text{comp}}(\mathbb{R}^d)$ be such that $Rf|_{S\times(0,\infty)} = 0$. Let $a \in S$ be a C^1 point. Define*

$$r = max\{|x - a| : x \in \text{supp}\, f\}.$$

Then

$$\sigma_a(K_{a,r}) \cap \text{supp}\, f \neq \emptyset.$$

Proof. is based on compactness arguments.

Denote for simplicity $K = K_{a,r}$, $K^* = \sigma_a(K_{a,r})$. If $E \subset S$ is the set where S is not real analytic , the point a is a limit point of $S \setminus E$ and hence we can find a sequence $a_n \in S \setminus E$ such that

$$\lim_{n\to\infty} a_n = a.$$

The surface S is real analytic at any point a_n and the tangent planes

$$T_{a_n}(S) \longrightarrow T_a(S), \quad n \to \infty.$$

Denote

$$r_n = \max\{|a_n - x| : x \in \operatorname{supp} f\}$$

and let $x_n \in \operatorname{supp} f$ be such that

$$|a_n - x_n| = r_n.$$

By the construction, for all $x \in \operatorname{supp} f$ holds

$$|a_n - x| \leq |a_n - x_n| = r_n.$$

By Theorem 6.3, the $T_{a_n}(S)$-symmetric point

$$x_n^* = \sigma_{a_n}(x_n) \in \operatorname{supp} f.$$

Using compactness of supp f, choose a convergent subsequence

$$x_{n_k} \longrightarrow x_0 \in \operatorname{supp} f, \quad k \to \infty.$$

Taking, if necessarily, a subsequence one more time, we can assume that also

$$r_{n_k} \longrightarrow r_0.$$

Then, taking limits $a_n \to a, x_n \to x_0, r_n \to r_0$, we will have

$$|a - x_0| = r_0$$

and for any $x \in \operatorname{supp} f$:

$$|a - x| \leq r_0.$$

Those two inequalities show that

$$r_0 = r,$$

where r is defined in the formulation, and

$$x_0 \in K = K_{a,r}.$$

Now,

$$x_n^* = x_n - 2\langle x_n - a_n, \nu_{a_n}\rangle \nu_{a_n} \longrightarrow x_0 - 2\langle x_0 - a, \nu_a\rangle \nu_a = x_0^*,$$

as $n \to \infty$. Since $x_n^* \in \operatorname{supp} f$ then $x_0^* \in \operatorname{supp} f$. Therefore $K^* \cap \operatorname{supp} f \neq \emptyset$. The theorem is proved. □

Theorems 6.3 and 6.4 can be viewed as non-linear versions of the following global symmetry property, which follows from the uniqueness for Cauchy problem for the wave equation.

Theorem 6.5 ([11], Chapter VI, 8.1). *Let* Π *be a hyperplane in* $\mathbb{R}^d$ *and* $f \in C(\mathbb{R}^d)$. *Then* $Rf|_{\Pi\times(0,\infty)} = 0$ *if and only if* f *is odd with respect to reflections around* Π.

Obviously, supp f in Theorem 6.5 is Π-symmetric. Theorem 6.3 states that if the hyperplane Π is replaced by a hypersurface S then, still, certain symmetry of supp f holds, though in a much weaker (local) sense.

The proof of Theorem 6.2 is geometric and is given in [1]. We present it here to make the text of this article more self-sufficient.

Proof of Theorem 6.2. We will present an analytic exposition of the geometric proof given in [1]. We want to prove that if $f \in C_{\text{comp}}(\mathbb{R}^d)$ and $Rf(x,r) = 0$ for all $x \in S$ and $r > 0$ then $f = 0$ or, equivalently, supp $f = \emptyset$. We assume that $f \neq 0$ and will arrive at a contradiction.

Since the tangent planes at a and b are parallel, the unit normal vectors ν_a and ν_b can be chosen equal

$$\nu_a = \nu_b = \nu = \frac{b-a}{|b-a|}.$$

Denote as above

$$\sigma_a(x) = x - 2\langle x-a, \nu\rangle\nu = x - 2\frac{\langle x-a, b-a\rangle}{|b-a|^2}(b-a)$$

the reflection around the tangent plane $T_a(S)$ and let σ_b be the analogous reflection for the point b.

The idea of the proof in [1] is step by step “eating away” from the support of f, using the local symmetry property. Denote

$$r_1 = \max\{|x-a| : x \in \operatorname{supp} f\}.$$

Consider two cases:

(1) $r_1 < |a-b|$,

(2) $r_1 \geq |a-b|$.

In the first case, supp f lies on one side of $T_b(S)$:

$$\langle x-b, \nu\rangle < 0, \quad x \in \operatorname{supp} f,$$

and therefore the entire $T_b(S)$-symmetric set $\sigma_b(\operatorname{supp} f)$ is disjoint from supp f. This contradicts to Theorem 6.4.

Consider now the case $r_1 \geq |a-b|$ and denote

$$r_2 := \sqrt{r_1^2 - |a-b|^2}.$$

We claim that $\operatorname{supp} f \subset \overline{B(b, r_2)}$, i.e. $|x-b| \leq r_2$ for all $x \in \operatorname{supp} f$. To prove that, consider

$$r = \max\{|x-b| : x \in \operatorname{supp} f\}.$$

Then $\operatorname{supp} f \subset B(b, r)$ and it suffices to prove that $r \leq r_2$.

Suppose that $r > r_2$. Denote

$$K = K_{b,r} = \operatorname{supp} f \cap \{x \in \mathbb{R}^d : |x-b| = r\}.$$

By Theorem 6.4, $K^* = \sigma_b(K)$ meets supp f. That means that there is $x_0 \in K$ such that $\sigma_b(x_0) \in K$, i.e.,

$$x_0 \in \operatorname{supp} f, \quad |x_0 - b| = r, \quad x_0^* = \sigma_b(x) \in \operatorname{supp} f.$$

Since $x_0 \in \operatorname{supp} f$ then by definition of r_1:

$$|x_0 - a| \leq r_1.$$

Therefore,

$$\begin{aligned} r_1^2 &\geq |x_0 - a|^2 \\ &= \langle x_0 - b + (b-a), x_0 - b + (b-a) \rangle \\ &= |x_0 - b|^2 + |b-a|^2 + 2\langle x_0 - b, b-a \rangle. \end{aligned}$$

Taking into account that

$$|x_0 - b| = r, \quad |b-a|^2 = r_1^2 - r_2^2,$$

we obtain the inequality

$$\langle x_0 - b, b-a \rangle \leq \frac{1}{2}(r_2^2 - r^2) < 0.$$

But the same applies to the symmetric point $x_0^* = \sigma_b(x_0)$ because x_0^* meets the same conditions $x_0^* \in \operatorname{supp} f$ and $|x_0^* - b| = |x_0 - b| = r$. Thus, also

$$\langle x_0^* - b, b-a \rangle < 0.$$

Substitution

$$x_0 = \sigma_b(x_0) = x_0 - 2\frac{\langle x-b, b-a\rangle}{|b-a|^2}(b-a)$$

yields

$$-\langle x_0 - b, b-a\rangle < 0.$$

The obtained contradictions shows that $r \le r_2$ and hence

$$\operatorname{supp} f \subset \bar{B}(b,r) \subset \overline{B(b,r_2)}.$$

Then we repeat the argument, replacing a by b and r_1 by r_2, and obtain

$$\operatorname{supp} f \subset \overline{B(a,r_3)},$$

where $r_3 = \sqrt{r_2^2 - |a-b|^2}$.

Proceeding this way, we construct the sequence

$$r_{n+1} = \sqrt{r_n^2 - |a-b|^2},$$

i.e.,

$$r_n = \sqrt{r_1^2 - (n-1)|a-b|^2},$$

such that

$$\operatorname{supp} f \subset \overline{B(a,r_{2k+1})}, \quad \operatorname{supp} f \subset \overline{B(b,r_{2k})}.$$

When $n|a-b|^2 > r_1^2$, we will have $r_n < |a-b|$ which, as explained above, is impossible. Therefore, the only possible conclusion is that $\operatorname{supp} f = \emptyset$ and $f = 0$. Therefore, S is an injectivity set. □

7. Ruled surfaces

Let S be a real analytically ruled surface in $\mathbb{R}^3$ (see Definition 2.2). In accordance with the definition, S consists of straight lines, intersecting the fixed base curve γ.

More precisely, S is locally the image of a map

$$(t,\lambda) \longmapsto u(t,\lambda) = u(t) + \lambda e(t),$$

where

$$u(t)\colon I \longrightarrow \mathbb{R}^3, \quad e(t)\colon I \longrightarrow S^2, \quad I = (-1,1),$$

are real analytic vector-functions.

We denote L_t the straight line

$$L_t = \{u(t) + \lambda e(t),\ \lambda \in \mathbb{R}\}.$$

Lemma 7.1. *The parameterizing mapping $u(t)$ of the base curve γ can be chosen so that the tangent vector to the base curve and the directional vector are orthogonal*:

$$\langle u'(t), e(t)\rangle = 0, \quad t \in (-1, 1). \tag{5}$$

Proof. For any function $\lambda(t)$ we have

$$u(t, \lambda) = u(t) + \lambda(t)e(t) + (\lambda - \lambda(t))e(t).$$

Then $\mu = \lambda - \lambda(t)$ is a new parameter on the line $u(t) + \mathbb{R}e(t)$ and therefore S is the image of the mapping $\hat{u}(t, \mu) = \hat{u}(t) + \mu e(t)$, where $\hat{u}(t) = u(t) + \lambda(t)e(t)$.

The function $\lambda(t)$ is to be found from the condition

$$\begin{aligned}\langle \hat{u}(t)', e(t)\rangle &= \langle u'(t) + \lambda'(t)e(t) + \lambda(t)e'(t), e(t)\rangle \\ &= \langle u'(t), e(t)\rangle + \lambda'(t) \\ &= 0.\end{aligned}$$

We have used here the that $\langle e(t), e(t)\rangle = 1$ and $\langle e'(t), e(t)\rangle = 0$. Therefore $\lambda(t)$ can be taken

$$\lambda(t) = -\int_{t_0}^{t} \langle u'(t), e(t)\rangle dt.$$

The condition of real analyticity preserves for $u(t) + \lambda(t)e(t)$. □

From now on, we assume that the parametrization $u(t, \lambda)$ satisfies the orthogonality condition (5).

7.1. Regularity of the line foliation at smooth points. In this subsection, we will prove that the line foliation of S is regular at the points where the surface S is differentiable.

Notice that, in Definition 2.2, the parameterizing mapping $u(t, \lambda)$ is not assumed necessarily regular, i.e. the condition nondegeneracy of the Jacobi matrix may be not fulfilled.

Recall, that by ruled surface we understand a surface with a fixed line foliation.

Definition 7.2. We call a point $a \in S$ of a ruled surface $S \subset \mathbb{R}^3$ *regular with respect to a parametrization* $I \times I \ni (s, \sigma) \mapsto w(s, \sigma), a = w(0, 0)$, where $I = (-1, 1)$, if

(1) the mappings $\mathbb{R} \ni \sigma \mapsto w(s, \sigma)$ parameterize the original line foliation of S and

(2) the mapping $w(s,\sigma)$ is differentiable and regular at $(0,0)$, i.e., the partial derivatives $\partial_s w(0,0), \partial_\sigma w(0,0)$ are linearly independent and therefore span the tangent space $T_a(S)$.

We will call a just *regular point* of the given line foliation, if a is regular with respect to some parametrization $w(s,\sigma)$.

Lemma 7.3. *Let S_0 be a ruled surface with C^1 open base curve $W \subset S_0$, i.e., $S_0 = \bigcup_{w\in W} L_w$, where L_w is a straight line passing through the point $w \in W$. Suppose that $L_w \perp T_w W, w \in W$. If S is a C^1-near a point $a \in W$ then a is a regular point of the foliation $\{L_w, w \in W\}$.*

Proof. Let Ω_a be the neighborhood of a where S_0 is C^1,

$$I \ni s \longmapsto w(s) \in W,$$

where I is an open interval, be a C^1 parametrization of the base curve W, and $\tau(w(s)) = w'(s)$ the tangent vector to W.

Let $\nu(x), x \in \Omega_a$, be the unit normal C^1 vector field on Ω_a. The surface S_0 is differentiable at a, hence the normal unit vector $\nu(a)$ is well defined, and $\nu(x)$ is C^1 mapping on Ω_a.

Then the cross-product

$$E(w) = \nu(w) \times \tau(w)$$

is both orthogonal to W and tangent to S_0 and hence $E(w)$ is the directional vector of the generating line L_w. The vector field $E(w), w \in W$ is C^1. Let

$$I \ni s \longmapsto w(s) \in W,$$

where I is an open interval, be a C^1 parametrization of the base curve W. Then the mapping

$$I \times I \ni (s,\sigma) \longmapsto w(s,\sigma) = w(s) + \sigma E(s), \quad \sigma \in \mathbb{R}^3,$$

where

$$E(s) := E(w(s)),$$

parameterizes the given line foliation $\{L_w\}$ and satisfies Definition 7.2 of regular point.

Indeed, $w(s,\sigma)$ is differentiable at $(0,0)$, because $w(s)$ and $E(w(s))$ are differentiable. The vectors

$$\partial_s w(0,0) = \tau(w), \quad \partial_\sigma w(0,0) = E(0)$$

are nonzero and orthogonal to each other, hence the point $(0,0)$ is regular with respect to the parametrization $w(s,\sigma)$ of the given foliation, the lemma is proved. □

8. The structure of real analytically ruled algebraic surfaces near singular points. Theorem 8.1

In this section we study singular points of algebraic real-analytically ruled surfaces in $\mathbb{R}^3$. We did not find a relevant result in the literature. The problem is that, to our knowledge, singular points of ruled surfaces (caustics of normal fields), cf. [6], are classified for either generic surfaces or in the case of stable singularities, while in our situation, the surface and a point are given and cannot be perturbed.

Theorem 8.1. *Let $I \ni t \mapsto u(t) \in \mathbb{R}^3$ and $I \ni t \mapsto e(t) \in S^2$ be two real analytic vector mappings of the interval $I = (-1,1)$, $u(t) \neq \text{const}$. Denote S the ruled surface $S := \{u(t) + \lambda e(t), t \in (-1,1), \lambda \in \mathbb{R}\}$ and assume that S is algebraic. Then the following five cases are possible.*

(1) *Every point $a \in S$ is a **C^1-point** in the following sense: for any $(t_0, \lambda_0) \in I \times \mathbb{R}$, such that $a = u(t_0, \lambda_0)$, there is an open neighborhood $A \subset I \times \mathbb{R}$ such that $u(A)$ is a C^1 manifold. The line foliation $\{L_t\}$ is regular at a.*

(2) *S is a plane.*

(3) *S is a cone, i.e. all the lines L_t have a common point (vertex).*

(4) *S has a **cuspidal (double tangency) point** $a \in S$, which means the following: if H is a polynomial vanishing on S and*

$$H(x+a) = H_k(x) + H_{k+1}(x) + \cdots + H_N(x),$$

where H_j are homogeneous polynomials of degree j and $H_k \neq 0$, then the minor homogeneous term H_k is divisible by a nonzero degenerate quadratic form $Q(x) = (A_1x_1 + A_2x_2 + A_3x_3)^2$.

Remark 8.2. In Case 1, S can be C^1 surface, possibly, with self intersections. In Case 2, S is a smooth manifold (a plane), however the given line foliation can be singular (have caustics). For example, all the lines L_t can pass through the same point, so that S belongs to Case 3, or there can be caustics of more complicated forms. On the other hand, planes can be viewed also as a regular ruled surface (foliated into parallel lines) although this foliation can be not the same as the initial one.

Example 8.3. (1) It was proved in [15] that a generic ruled surface in $\mathbb{R}^3$ is equivalent, near its singular point, to the Whitney umbrella, the image S of the mapping

$$(t, \lambda) \longmapsto (t^2, \lambda, \lambda t).$$

The Whitney umbrella is an algebraic surface, defined by the algebraic equation

$$z^2 - yx^2 = 0.$$

The origin $a = (0, 0, 0)$ is the only singular point. Whitney umbrella is a typical ruled surface with cuspidal singular point, as defined in Case 4 of Theorem 8.1. Indeed, any polynomial H vanishing on S is divisible by $x_3^2 - x_1x_2^2$. Then the minor homogeneous term H_k of H is divisible by x_3^2, i.e., property 3 holds with $Q(x_1, x_2, x_3) = x_3^2$.

(2) Another example of cuspidal surface is the swallow tail ruled surface in $\mathbb{R}^3$– the zero variety of the discriminant of the quartic polynomial

$$t \longmapsto t^4 + x_1t^2 + x_2t + x_3,$$

i.e.,

$$16x_1^4x_3 - 4x_1^3x_2^2 - 128x_1^2x_3^2 + 144x_1x_2^2x_3 - 27x_2^4 + 256x_3^3 = 0$$

(cf. [10]). The minor homogeneous term at $a = 0$ in this case is $H_3(x_1, x_2, x_3) = 256x_3^3$, the quadratic form Q is $Q(x_1, x_2, x_3) = x_3^2$. Therefore, the origin $a = 0$ is a cuspidal singular.

An important corollary of Theorem 8.1 is the following result.

Corollary 8.4. *Let S be as in Theorem 8.1. Suppose that $S \subset H^{-1}(0)$, where H is a nonzero harmonic polynomial. Then S is the surface of one of the first three cases in Theorem 8.1.*

Proof. Suppose that S is a surface of the fourth type, i.e, S has a cuspidal point $a \in S$. Let H be a harmonic polynomial such that the restriction $H|S = 0$. Then the minor term H_k in the homogeneous decomposition

$$H(x+a) = H_k(x) + \cdots + H_N(x)$$

is divisible by a nonzero quadratic polynomial $A^2(x)$ where

$$A(x) = A_1 x_1 + A_2 x_2 + A_3 x_3$$

is a nonzero linear form. Then

$$H_k(x) = 0,\ \nabla H_k(x) = 0, \quad \text{whenever } A(x) = 0.$$

Thus, H_k satisfies on the plane $\Pi = \{A(x) = 0\}$ both the zero Dirichlet and Neumann conditions. Since H_k is harmonic, this implies $H_k = 0$ identically. Therefore, the homogeneous decomposition of H begins with H_{k+1}. The same argument yields $H_{k+1} = 0$. Proceeding this way, we obtain $H = 0$. This contradiction shows that Case 4 is impossible. □

8.1. Outline of the proof of Theorem 8.1. First of all, we will show that if a is not a conical point of S then by a suitable changing parameters t (reparametrization) and λ (rescaling), we can pass to a parametrization (12) of S of the form

$$u(s,\sigma) = s^m v_m + \sigma s^m e_0 + D(s,\sigma)\tau,$$

where v_m, e_0, and τ are nonzero pairwise orthogonal vectors and $D(s,\sigma)$ is a nonzero (if S is not a plane) real analytic function.

Then we show that if m is odd then S is C^1-differentiable at a and, even more, a is a regular point of the line foliation on S (Lemmas 8.10 and 7.3).

In the case of even m we reduce the situation, by consequent descending the power m, to the case of even m and D not even function of s (we assume that $D \neq 0$ identically since otherwise S is a plane).

Then we prove in Lemma 8.9 that in this case the point a is of cuspidal type, i.e., the fourth case of Theorem 8.1 takes place.

Thus, we conclude that if S contains no cuspidal points then either S is a plane or a cone, or the power m associated with any point $a \in S$ is odd and therefore S is everywhere C^1 differentiable and the line foliation is everywhere regular.

8.2. Preliminary constructions. Let a be a singular point of the real analytically ruled surface S.

As it is showed in Lemma 7.1, we can choose the parametrization $u(t,\lambda) = u(t) + \lambda e(t)$ near a so that $\langle u'(t), e(t)\rangle = 0$. Using translation we can always move a to the origin and assume that $a = 0$. We can also assume that the value of the parameter corresponding to the point a is $t = 0$.

Lemma 8.5. *Let $a = u(0) + \lambda_0 e(0) = 0$ be a singular point of the ruled surface S. Then the parameterizing mapping $u(t,\lambda) = u(t) + \lambda t$ can be rewritten as $u(t,\mu) = v(t) + \mu e(t)$, where*

$$v(t) = u(t) + \lambda_0 e(t), \quad \mu = \lambda - \lambda_0, \tag{6}$$

and

(1) $v'(0) = 0$,

(2) *if $v(t) = 0$ identically then S is a cone with the vertex* 0. *Otherwise, $v(t)$ decomposes in a neighborhood of $t = 0$ into power series*:

$$v(t) = v_m t^m + v_{m+1} t^{m+1} + \cdots, \quad v_m \neq 0,$$

where $m \geq 2$, v_j are vectors in $\mathbb{R}^3$,

(3) $\langle v_m, e(0)\rangle = 0$.

Proof. Since a is singular, the vectors

$$\frac{\partial u}{\partial t}(0,\lambda_0) = u'(0) + \lambda_0 e'(0) \quad \text{and} \quad \frac{\partial u}{\partial \lambda}(0,\lambda_0) = e(0)$$

are linearly dependent at $0, \lambda_0$:

$$c_1(u'(0) + \lambda_0 e'(0)) + c_2 e(0) = 0,$$

for some $c_1, c_2 \in \mathbb{R}$, $c_1^2 + c_2^2 \neq 0$.

The unit vector $e(0)$ is orthogonal both to $u'(0)$ and $e'(0)$, therefore $c_2 = 0$ and

$$u'(0) + \lambda_0 e'(0) = 0.$$

Now rewrite $u(t,\lambda)$ as

$$u(t,\lambda) = u(t) + \lambda_0 e(t) + (\lambda - \lambda_0)e(t),$$

and denote $\lambda - \lambda_0 = \mu$. Then we get the parametrization

$$u(t,\mu) = v(t) + \mu e(t), \quad u(0,0) = a,$$

where

$$v(t) = u(t) + \lambda_0 e(t),$$

Then

$$v(0) = u(0) + \lambda_0 e(0) = 0, \quad v'(0) = 0.$$

Two cases are possible.

1) $v(t) \equiv 0$.

   Then $u(t, \lambda_0) = u(t) + \lambda_0 e(t) = v(t) = 0$, i.e., all the lines L_t pass through the origin and therefore S is a cone with the vertex 0.

2) $v(t)$ is not identical zero.

   Then, by real analyticity,

$$u(t, \mu) = v_m t^m + \cdots + \mu(e_0 + e_1 t + \cdots), \tag{7}$$

   where $v_m \neq 0$. Since $v'(0) = 0$ then $m \geq 2$.

Also we have

$$\langle v'(t), e(t) \rangle = \langle u'(t) + \lambda_0 e'(t), e(t) \rangle = 0.$$

Thus,

$$\langle m v_m t^{m-1} + \cdots, e_0 + e_1 t + \cdots \rangle = 0$$

and dividing by t^{m-1} and letting $t \to 0$ yields

$$\langle v_m, e_0 \rangle = 0.$$

The lemma is proved. □

On the next step, we will replace the parameters μ, t by new parameters σ, s which are more convenient for further investigation. We start with re-scaling the parameter μ on the ruling lines.

8.3. Re-scaling: changing the linear parameter μ. Thus, by Lemma 8.5, the surface S is parameterized, near $a = 0$, by the mapping $u(t, \mu) = v(t) + \mu e(t)$, where

$$v(t) = \sum_{j=m}^{\infty} v_j t^j, \; e(t) = \sum_{j=0}^{\infty} e_j t^j.$$

We assume that S is not a cone, $v(t)$ is not identical zero and $\langle v_m, e(0) \rangle = 0$ in accordance with Lemma 8.5.

Let τ be a unit vector orthogonal both to v_m and e_0. Then the triple

$$v_m, e_0, \tau$$

constitutes a basis in $\mathbb{R}^3$.

Decompose the vector-coefficients $v_m, v_{m+1}, \ldots$ and $e_0, e_1, \ldots,$ into linear combinations of the basis vectors:

$$v_j = A_j v_m + B_j e_0 + C_j \tau, \quad j \geq m,$$

$$e_j = \widehat{A}_j v_m + \widehat{B}_j e_0 + \widehat{C}_j \tau, \quad j \geq 0,$$

and since v_m, e_0, τ constitute the basis, one has

$$A_m = 1, \quad B_m = 0, \quad C_m = 0, \quad \widehat{A}_0 = 0, \quad \widehat{B}_0 = 1, \quad \widehat{C}_0 = 0.$$

Substitution the expressions for v_j, e_j into the power series for $v(t)$ and $e(t)$ leads to

$$v(t) = A(t)v_m + B(t)e_0 + C(t)\tau,$$

$$e(t) = \widehat{A}(t)v_m + \widehat{B}(t)e_0 + \widehat{C}(t)\tau,$$

where we have denoted

$$A(t) = \sum_{j=m}^{\infty} A_j t^j, \quad B(t) = \sum_{j=m+1}^{\infty} B_j t^j, \quad C(t) = \sum_{j=m+1}^{\infty} C_j t^j \tag{8}$$

and

$$\widehat{A}(t) = \sum_{j=1}^{\infty} \widehat{A}_j t^j, \quad \widehat{B}(t) = \sum_{j=0}^{\infty} \widehat{B}_j t^j, \quad \widehat{C}(t) = \sum_{j=1}^{\infty} \widehat{C}_j t^j. \tag{9}$$

Correspondingly, the parameterizing function $u(t,\mu) = v(t) + \mu e(t)$ takes the form

$$u(t,\mu) = (A(t) + \mu\widehat{A}(t))v_m + (B(t) + \mu\widehat{B}(t))e_0 + (C(t) + \mu\widehat{C}(t))\tau. \tag{10}$$

Let us fix a real number $\sigma \in \mathbb{R}$ and write the functional equation

$$B(t) + \mu\widehat{B}(t) = \sigma(A(t) + \mu\widehat{A}(t)). \tag{11}$$

This equation defines the parameter μ as a function of σ and t:

$$\mu = \mu(\sigma, t) = \frac{\sigma A - B}{\widehat{B} - \sigma\widehat{A}}.$$

Since from (8)

$$B(t) = B_{m+1}t^{m+1} + \cdots, \qquad \widehat{B}(t) = 1 + B_1 t + \cdots,$$

and

$$A(t) = t^m + A_{m+1}t^{m+1} + \cdots, \quad \widehat{A}(t) = A_1 t + \cdots,$$

and $m > 1$, we obtain

$$\mu = \frac{\sigma A - B}{\widehat{B} - \sigma \widehat{A}} = \frac{\sigma t^m + \cdots - B_{m+1}t^{m+1} + \cdots}{(1 + \widehat{B}_1 t + \cdots) - \sigma(\widehat{A}_1 t + \cdots)},$$

and hence

$$\mu = \mu(t) = \sigma t^m + o(t^m).$$

Then the coefficient $A(t) + \mu(t,\sigma)\widehat{A}(t)$ in front of v_m in (10) is

$$\begin{aligned} A(t) + \mu(t,\sigma)\widehat{A}(t) &= t^m + A_{m+1}t^{m+1} + \cdots + (\sigma t^m + \cdots)(A_1 t + \cdots) \\ &= t^m + o(t^m), \quad t \to 0. \end{aligned}$$

Remark 8.6. The base curve $\{t \to v(t)\}$ of the foliation is given by the condition $\mu = 0$ which corresponds, due to (11), to

$$\sigma = \frac{B(t)}{A(t)} = B_{m+1}t + o(t).$$

8.4. Re-parametrization: changing the parameter t of the base curve. Now introduce the new parameter s by the relation

$$s^m = A(t) + \mu \widehat{A}(t) = t^m + o(t^m), \quad t \to 0.$$

If m is odd, then the real parameter $s = s(t)$ is well defined near $t = 0$. If m is even then $s = s(t)$ near $t = 0$ is the real branch of $(A(t) + \mu \widehat{A}(t))^{\frac{1}{m}}$ for which

$$s = s(t) = t + o(t).$$

Thus, that asymptotic holds for both odd and even m.

From (10) and (11), one can rewrite, in a neighborhood of $s = 0$, the function $u(t,\mu)$ as a function of the new parameters s, σ:

$$u(s,\sigma) = s^m v_m + \sigma s^m e_0 + D(s,\sigma)\tau, \tag{12}$$

where we have denoted

$$D(s,\sigma) := C(t) + \mu \widehat{C}(t).$$

Since $s = t + o(t)$, we have from (8) and (9):

$$C(t) = C_{m+1}t^{m+1} + o(t^{m+1}) = C_{m+1}s^{m+1} + o(s^{m+1}),$$

$$\widehat{C}(t) = \widehat{C}_1 t + o(t) = \widehat{C}_1 s + o(s),$$

$$\mu = \sigma s^m + o(s^m).$$

Then we have

$$D(s,\sigma) = C(t) + \mu\widehat{C}(t) = (C_{m+1} + \sigma\widehat{C}_1)s^{m+1} + o(s^{m+1}). \tag{13}$$

Lemma 8.7. *If H is a polynomial vanishing on S and $H = H_k + H_{k+1} + \cdots$ is its decomposition into homogeneous polynomials, then $H_k(x) = 0$ for all vectors $x \in \operatorname{span}\{v_m, e_0\}$.*

Proof. We have $H(u(s,\sigma)) = 0$ for all $\sigma \in \mathbb{R}$ and s close to 0. From (13), $D(s,\sigma) = o(s^m)$ and then formula (12) implies

$$\begin{aligned} &H(u(s,\sigma)) \\ &\quad = H_k(s^m v_m + \sigma s^m e_0 + o(s^m)) + H_{k+1}(s^m v_m + \sigma s^m e_0 + o(s^m)) + \cdots \\ &\quad = 0. \end{aligned}$$

Since H_j are homogeneous of degree j, dividing by s^{km} and letting $s \to 0$ yields:

$$H_k(v_m + \sigma e_0) = 0.$$

Then

$$H_k(\alpha v_m + \alpha\sigma e_0) = \alpha^k H_k(v_m + \sigma e_0) = 0$$

for any $\alpha \in \mathbb{R}$. Since σ is arbitrary, the real numbers $\alpha, \alpha\sigma$ are arbitrary as well, and hence H vanishes on any linear combination of the vectors v_m and e_0. The lemma is proved.

Notice that if $D(s,\sigma) = 0$ identically then S locally is a plane (Case 1 of Theorem 8.1). Indeed, if $D(s,\sigma) \equiv 0$ then we have from (12) $u(s,\sigma) = s^m v_m + \sigma e_0$ and hence the image of u is contained in the plane spanned by the vectors v_m and e_0. □

Lemma 8.8. *Suppose that $D(s,\sigma)$ is not identically zero. Then a suitable change of the parameter s leads to one of the following cases*:

(1) *the integer m in (12) is odd*;

(2) *m is even but $D(s,\sigma)$ is not an even function with respect to s.*

Proof. We will consequently descend the power m until we reach one of the above cases.

If m is odd then we are done. Suppose that m is even, $m = 2m'$. If $D(s, \sigma))$ is not an even function with respect to s, then we are done.

If $D(s, \sigma)$ is still even in s then $D(s, \sigma) = D'(s^2, \sigma)$, where $D'(s', \sigma)$ is a new function, real analytic in s' near 0.

Then introduce new parameter

$$s' = s^2$$

and pass to the new parameter s' and the new parameterizing function

$$u(s', \sigma) = (s')^{m'} v_m + \sigma (s')^{m'} e_0 + D'(s', \sigma)\tau,$$

which extends as a real analytic function to negative values of s'.

If, again, m' is even and $D'(s', \sigma)$ is an even function of s', then we introduce the new parameter

$$s'' = (s')^2.$$

Proceeding that way, we finally end up either with odd m or with even m but not even (with respect to s) function $D(s, \sigma)$. The lemma is proved. □

8.5. The case of even m. The following lemma shows that the case of even power m leads to the Case 4 in Theorem 8.1, of double tangency at the singular point a (which here is assumed to be $a = 0$).

Lemma 8.9. *Let m be even and let $D(s, \sigma)$ be not identically zero function (i.e. due to (12) the surface S is not a plane). Then a is a cuspidal point as defined in Case 4 of Theorem 8.1.*

Proof. We will divide the proof in several steps.

8.5.1. Extracting the even part of $D(s, \sigma)$. As usual, we assume that $a = 0$. By Lemma 8.8 we can make, using a suitable reparametrization, the function $D(s, \sigma)$ not even with respect to the variable s.

Now fix an arbitrary σ such that $D(s, \sigma)$ is not even in s when σ is near 0. Let us split $D(s, \sigma)$ into a sum

$$D(s, \sigma) = D_1(s, \sigma) + D_2(s, \sigma)$$

of even and odd functions with respect to s:

$$D_1(-s, \sigma) = D(s, \sigma), \quad D_2(-s, \sigma) = -D_2(s, \sigma),$$

and D_2 is not identical zero.

By the construction, the power series for D contains no powers of s less than $m+1$:

$$D(s,\sigma)=\sum_{j=m+1}^{\infty} D_j(\sigma)s^j.$$

Then

$$D_2(s,\sigma)=\sum_{j=j_0}^{\infty} D_{2,j}(\sigma)s^j,$$

where $j_0 \geq m+1$ and $D_{2,j_0}(\sigma) \neq 0$ for σ near $\sigma=0$. Further, we will use only the fact that

$$D_2(s,\sigma)=D_{2,j_0}(\sigma)s^{j_0}+o(s^{j_0}), \quad s\to 0. \tag{14}$$

Substituting the above representations for $D(s,\sigma)$ formula (12) for $u(s,\sigma)$ we obtain

$$u(s,\sigma)=s^m v_m+\sigma s^m e_0+(D_1(s,\sigma)+D_2(s,\sigma))\tau. \tag{15}$$

8.5.2. Taylor series for $H(u(s,\sigma))$. Now, let H be a polynomial vanishing on S:

$$H(x)=0, \quad \text{for all } x\in S.$$

We want to prove that S has a double tangency at $a=0$, more precisely, that the property 4 of Theorem 8.1 is satisfied for the polynomial H.

From the representation (15), we have

$$H(u(s,\sigma))=H(s^m v_m+\sigma s^m e_0+(D_1(s,\sigma)+D_2(s,\sigma))\tau)=0.$$

Now, let us write Taylor formula for the polynomial H, at the point

$$s^m v_m+\sigma s^m e_0+D_1(s,\sigma)\tau,$$

evaluated on the vector

$$D_2(s,\sigma)\tau.$$

It yields

$$H(u(s,\sigma))=\sum_{r=0}^{\deg H} d^r H(s^m v_m+\sigma s^m e_0+D_1(s,\sigma)\tau; D_2(s,\sigma)\tau)=0, \tag{16}$$

where $d^r H(a;h)$ stands for the $r-$th differential of H at a point a, evaluated on a vector h.

Replacing s by $-s$, we also have, taking into account that $D_1(-s,\sigma) = D(s,\sigma)$:

$$H(u(-s,\sigma)) = \sum_r d^r H(s^m v_m + \sigma s^m e_0 + D_1(s,\sigma)\tau; D_2(-s,\sigma)\tau) = 0. \quad (17)$$

Now, if we subtract (17) from (16) , then the term corresponding to $r = 0$ cancels and we will have

$$H(u(s,\sigma)) - H(u(-s,\sigma)) = \sum_{r=1}^{\deg H} T_r = 0 \quad (18)$$

where we have denoted

$$\begin{aligned} T_r &= d^r H(s^m v_m + \sigma s^m e_0 + D_1(s,\sigma)\tau; D_2(s,\sigma)\tau) \\ &\quad - d^r H(s^m v_m + \sigma s^m e_0 + D_1(s,\sigma)\tau; D_2(-s,\sigma)\tau). \end{aligned} \quad (19)$$

8.5.3. Contribution of the first differential. Now let us look at the first term T_1 in the expression (18) and (19), corresponding to the first differential of H (which we will write in the gradient form):

$$T_1 = \langle \nabla H(s^m v_m + \sigma s^m e_0 + D_1(s,\sigma)\tau), (D_2(s,\sigma) - D_2(-s,\sigma))\tau \rangle. \quad (20)$$

The formula (14) implies

$$D_2(s,\sigma) - D_2(-s,\sigma) = 2D_{2,j_0}(\sigma)s^{j_0} + o(s^{j_0}), \quad (21)$$

since j_0 is odd.

Let also $m + \alpha$ be the order of zero of $D_1(s,\sigma)$ at $s = 0$:

$$D_1(s,\sigma) = D_{1,m+\alpha}(\sigma)s^{m+\alpha} + o(s^{m+\alpha}), \quad s \to 0, \quad (22)$$

for some $\alpha > 0$.

Now decompose H

$$H = H_k + \cdots + H_{\deg H}$$

into sum of homogeneous polynomials, $\deg H_j = j$, and substitute the decomposition into (20):

$$\begin{aligned} T_1 := [dH_k(\dots) - dH_k(\dots)] + [dH_{k+1}(\dots) - dH_{k+1}(\dots)] \\ + \text{higher order differentials}. \end{aligned}$$

Here all the differentials dH_k are evaluated at the point

$$s^m v_m + \sigma s^m e_0 + D_1(s,\sigma)\tau$$

and on the vector

$$D_2(\pm s,\sigma)\tau,$$

depending whether $+$ or $-$ stands in front of dH_k in (20).

Now using (21) and homogeneity of the polynomials H_k we obtain

$$\begin{aligned} T_1 = s^{(k-1)m+j_0}\langle \nabla H_k(v_m + \sigma e_0 + (D_{1,m+\alpha}(\sigma)s^\alpha + o(s^\alpha))\tau, \\ (2D_{2,j_0}(\sigma) + o(s))\tau\rangle \\ + s^{km+j_0}\langle \nabla H_{k+1}(\ldots),\ldots\rangle + \cdots . \end{aligned}$$

and at last

$$T_1 = 2D_{2,j_0}(\sigma)s^{(k-1)m+j_0}\langle \nabla H_k(v_m + \sigma e_0), \tau)\rangle + o(s^{(k-1)m+j_0}). \tag{23}$$

Similarly, substituting the above asymptotic (21),(22) of D_1 and D_2 into the next homogeneous terms $H_{k+1}, H_{k+2}, \ldots$ leads to the expressions similar to (23) were k is replaced by $k+1$, $k+2$ and so on. Therefore, the least power that comes from $H_{k+1}, H_{k+2}, \ldots$ is s^{km+j_0}.

8.5.4. Contribution of the higher differentials. Let us turn now to the higher differentials and consider the contribution of the terms corresponding to d^2H_k, $d^3H_k \ldots$ in the asymptotic near $s = 0$.

Analogously to T_1, consider the term T_2 in (18), corresponding to the second differential d^2H:

$$\begin{aligned} d^2H(s^m v_m + \sigma s^m e_0 + (D_{1,m+\alpha}(\sigma)s^{m+\alpha} + o(s^{m+\alpha}))\tau, \\ (2D_{2,j_0}(\sigma)s^{j_0} + o(s^{j_0}))\tau). \end{aligned}$$

The asymptotic of (18) near $s = 0$ is determined again by the minimal degree homogeneous polynomial H_k, more precisely, by the difference

$$\begin{aligned} d^2H_k(s^m v_m + \sigma s^m e_0 + (D_{1,m+\alpha}(\sigma)s^{m+\alpha} + o(s^{m+\alpha}))\tau, \\ (2D_{2,j_0}(\sigma)s^{j_0} + o(s^{j_0}))\tau), \end{aligned}$$

which comes from the minor homogeneous term H_k in H.

By the homogeneity, it equals to

$$4D_{2j_0}(\sigma)^2 s^{(k-2)m+2j_0} d^2 H_k(v_m + \sigma e_0 + o(s), \tau)) + o(s^{(k-2)m+2j_0}).$$

However,

$$(k-2)m + 2j_0 \geq (k-1)m + j_0,$$

because $j_0 \geq m+1$.

Moreover, for the next terms, coming from the higher differentials d^r, we will have the following order of the asymptotic

$$(k-r)m + rj_0 = (k-1)m + j_0 - (r-1)m + (r-1)j_0 > (k-1)m + j_0.$$

Thus, we see that only the first differential dH_k of the minor homogeneous term H_k contributes the term $s^{(k-1)m+j_0}$ of the minimal power to the asymptotic of $H(u(s,\sigma))$ near $s=0$.

Therefore, the main term of the asymptotic, which is determined by the minimal power of s , equals to

$$H(u(s,\sigma)) - H(u(-s,\sigma)) = 2D_{2,j_0}(\sigma)s^{(k-1)m+j_0}\langle \nabla H_k(v_m + \sigma e_0), \tau\rangle + \cdots . \tag{24}$$

8.5.5. Double tangency property. Since the left hand side in (24) is identically zero

$$H(u(s,\sigma)) - H(u(-s,\sigma)) = 0,$$

the main term of the decomposition in the left hand side is zero as well. It follows then from $D_{2,j_0}(\sigma) \neq 0$ that

$$\langle \nabla H_k(v_m + \sigma e_0), \tau\rangle = 0.$$

Now recall that σ is an arbitrary real number. Since the polynomial H_k is homogeneous, we have

$$\langle \nabla H_k(h), \tau\rangle = 0,$$

for all $h \in \Pi := \operatorname{span}\{v_m, e_0\}$. Since the vector τ is orthogonal to the plane Π, the normal derivative

$$\frac{\partial H_k}{\partial \tau} = 0$$

on Π.

Also, we know from Lemma 8.7 that $H_k = 0$ on Π. Thus H_k vanishes on Π at least to the second order and therefore if to define linear form

$$A(x) = \langle x, \tau \rangle,$$

then H is divisible by the degenerate quadratic form$Q = A^2$:

$$H = A^2 R,$$

where R is a polynomial. The lemma is proved. □

8.6. The case of odd *m*. We say that $S \subset \mathbb{R}^3$ is a *surface, differentiable at a point* $a = (a_1, a_2, a_3) \in S$, if S is representable near a as the graph $S = \{z = z(x, y)\}$ of a function $z(x, y)$, differentiable at the point (a_1, a_2).

Lemma 8.10. *If m is odd then the surface S is differentiable at $a = 0$. If, moreover, S is differentiable in a neighborhood of the point a then S is a C^1-manifold near a.*

Proof. By Lemma 8.5, the surface S is the image of the function

$$u(t, \mu) = v(t) + \mu e(t),$$

where

$$v(t) = v_m t^m + \cdots, \quad e(t) = e_0 + e_1 t + \cdots, \quad m = 2s + 1.$$

Since m is odd, the mapping $t \to t^m$ is one-to-one and hence the base curve $\mu = 0$ parameterized by

$$u(t, 0) = v(t), \quad t \in I = (-\varepsilon, \varepsilon),$$

can be re-parametrized by the change of the parameter $t^m = s$:

$$v(s) = v_m s + v_{m+1} s^{\frac{m+1}{m}} + \cdots = v_m s + o(s).$$

The mapping $s \to v(s)$ is differentiable near s and $v'(s) \neq 0$. Therefore, it defines a differentiable curve near $v(0) = 0$.

We also have from definition (6) of $v(t)$ and Lemma 7.1:

$$\langle v'(t), e(t) \rangle = \langle u'(t) + \lambda_0 e'(t), e(t) \rangle = 0.$$

Therefore, the image of the function $u(t, \mu)$ describes a ruled surface consisting of straight lines orthogonal to the differentiable curve $v \colon I \to \mathbb{R}^3$.

Apply an orthogonal transformation so that the triple v_m, e_0, τ becomes the axis. Denote x_1, x_2, x_3 the coordinates of points in the basic v_m, e_0, τ.

Then, according to (12), the mapping $u(s,\sigma)$ has the following representation in the new coordinates:

$$u(s,\sigma) = (x_1, x_2, x_3) = (s^m, \sigma s^m, D(s,\sigma)).$$

We have

$$x_1 = s^m, \quad x_2 = \sigma s^m, \quad x_3 = D(s,\sigma),$$

and therefore

$$s = x_1^{\frac{1}{m}}, \quad \sigma = \frac{x_2}{x_1}.$$

The function $D(s,\sigma)$ is real analytic at $s=0, \sigma=0$:

$$D(s,\sigma) = \sum_{\alpha,\beta \in Z_+} c_{\alpha,\beta} s^\alpha \sigma^\beta,$$

in a neighborhood of $s=0, \sigma=0$.

Moreover, according to (13),

$$D(s,\sigma) = (C_{m+1} + \sigma \widehat{C}_1) s^{m+1} + o(s^{m+1}), \quad s \to 0$$

and hence the summation index α in the above Taylor series satisfies

$$\alpha \geq m+1. \tag{25}$$

Substituting the expressions for s, σ through x_1, x_2 yields the representation of the function

$$x_3 = z(x_1,x_2) = D\left(x_1^{\frac{1}{m}}, \frac{x_2}{x_1}\right)$$

as a Newton–Puiseux fractional power series:

$$z(x_1,x_2) = \sum_{\substack{\alpha=m+1,\\ \beta=0}}^{\infty} c_{\alpha,\beta} x_1^{\frac{\alpha}{m}-\beta} x_2^\beta. \tag{26}$$

8.6.1. Differentiability of $z(x_1,x_2)$ at $(0,0)$. We know that the line $L_0 = \{\lambda e_0, \lambda \in \mathbb{R}\}$ is one of the generating lines and belongs to S. In the coordinates x_1, x_2, x_3, the line L_0 has the equation $x_1 = x_3 = 0$. Since $x_3 = z(x_1,x_2)$ is the equation of S, we conclude that

$$\lim_{x_1 \to 0} z(x_1,x_2) = 0$$

for any fixed x_2. This implies that the series (26) contains only positive powers of x_1.

Therefore, the series (26) can be rewritten as

$$z(x,y)=\sum_{\substack{\nu>0,\\ \beta\geq 0}} b_{\nu,\beta}x_1^{\nu}x_2^{\beta}, \tag{27}$$

where we have introduced the new coefficients

$$b_{\nu,\beta}=c_{\alpha,\beta}, \quad \nu=\frac{\alpha}{m}-\beta.$$

In our case ν is strictly positive because $z(0,0,0)=0$.

Notice, that since m is odd, the fractional power x_1^{ν} is well defined for $x_1<0$ as well, so the decomposition (27) holds in a full neighborhood of $(0,0)$.

The general term in the Newton–Puiseux series (27) is of homogeneity degree

$$\nu+\beta=\left(\frac{\alpha}{m}-\beta\right)+\beta=\frac{\alpha}{m}\geq 1+\frac{1}{m}.$$

The latter inequality is due to (25).

For the further analysis, it will be convenient to write the series (27) in the polar coordinates

$$x_1=r\cos\theta, \quad x_2=r\sin\theta$$

as

$$z(x_1,x_2)=\sum_{\substack{nu>0,\\ \beta\geq 0}} b_{\nu,\beta}r^{\nu+\beta}(\cos\theta)^{\nu}(\sin\theta)^{\beta}.$$

Since the exponents $\nu,\beta\geq 0$ then $|\cos\theta|^{\nu},|\sin\theta|^{\beta}\leq 1$, the inequality

$$\nu+\beta>1+\frac{1}{m}$$

implies

$$z(x_1,x_2)=o(r), \quad r\to 0.$$

Therefore the function $z(x_1,x_2)$ is differentiable at $(0,0)$ with $dz(0,0)=0$. The lemma is proved. □

8.6.2. C^1 differentiabilty of S in a neighborhood of a. Now we want to prove that if we know that the surface S under consideration is differentiable at any point near $a=0$ then it is continuously differentiable. It was established earlier that the surface S is the graph $S=\{z=z(x_1,x_2)\}$ and the assumption means the function $z(x_1,x_2)$ is differentiable at any point in a neighborhood U of $(0,0)$. Due to (27)

$$\frac{\partial z}{\partial x_1}(x_1,x_2)=\sum_{\substack{\nu>0,\\ \beta\geq 0}} b_{\nu,\beta}\nu x_1^{\nu-1}x_2^{\beta}. \tag{28}$$

Since $\nu > 0$ is fractional, the number $\nu - 1$, in principle, can be negative. However, this is not the case, because if series (28) contains negative powers of x_1 then for small $x_2 \neq 0$ we have $\lim_{x_1 \to 0} \frac{\partial z}{\partial x_1}(x_1, x_2) = \infty$ which contradicts to the differentiability of $z(x_1, x_2)$ at the points $(0, x_2)$ with small x_2. Then the series

$$\frac{\partial z}{\partial x_1}(x_1^m, x_2) = \sum_{\substack{\nu > 0, \\ \beta \geq 0}} b_{\nu,\beta} \nu x_1^{m(\nu-1)} x_2^{\beta}$$

is a power series, since $m(\nu - 1) = \alpha - m\beta - m$ is integer and nonnegative. Power series are continuous in their domains of convergence, therefore $\frac{\partial z}{\partial x_1}(x_1^m, x_2)$ is continuous in a neighborhood of $(0, 0)$. Since m is odd, the mapping $x_1 \mapsto x_1^m$ is a homeomorphisms and hence the continuity of $\frac{\partial z}{\partial x_1}(x_1^m, x_2)$ follows.

Same argument implies the continuity of the $\frac{\partial z}{\partial x_2}$ because the series (27) is just a usual power series with respect to integer powers of x_2. The proof of the lemma is completed.

8.7. End of the proof of Theorem 8.1. Now we are ready to finish the proof of Theorem 8.1.

We start with assumption that S is neither a plane nor a cone, i.e., we exclude Cases 2 and 3 in the formulation of Theorem 8.1. Then we have to prove that either every point $a \in S$ is C^1 and the line foliation is regular there (Case 1) or S has a cuspidal point (Case 4).

Lemma 8.9 says that cuspidal singular points $a \in S$ correspond to the even powers m associated with the decomposition (12). Therefore, if S is free of cuspidal points (Case 4 is not realized) , then for any singular, with respect to the initial parametrization of our line foliation, point the associated power m is odd.

But Lemma 8.10 implies that then any singular point $a \in S$ with the associated odd power m is a differentiable point. Surely, S is also differentiable at any regular point. Therefore every point $a \in S$ is differentiable. But then the second assertion of Lemma 8.10 yields that any point $a \in S$ is C^1 and the line foliation of S is regular (with respect to some parametrization of the foliation).

Thus, we have proven that one of Cases 1–4, enlisted in Theorem 8.1, holds. The theorem is proved.

9. Irreducible case. Proof of Theorem 2.4

9.1. Extremal ruling lines and antipodal points. For any two ruling straight lines $L_t, L_s \subset S$ define the distance function

$$d(t,s) := \text{dist}(L_t, L_s) = \min\{|u-v|: u \in L_t, v \in L_s\}.$$

Lemma 9.1. *If $d(s,t) = 0$ for all t, s then the surface S is a cone.*

Proof. The condition implies that any two ruling lines meet. Fix two non-parallel ruling lines L_t, L_s. They intersect at some point $b \in L_t \cap L_s$.

Due to real analyticity of the one-dimensional connected family $\{L_t\}$ of the ruling lines, the two cases are possible:

1) all the lines L_t pass through the point b, and then S is a cone with the vertex b;

2) at most finite number of lines $L_{t_1}, \dots, L_{t_N}$ contain b.

Suppose that Case 2 takes place. Take any third ruling line L_r for $r \neq t_1, \dots, t_N$. Since, by the assumption, any two ruling lines have a common point, the line L_r must intersect both lines L_t, L_s and none of the points of intersections is a.

Then the line L_r belongs to the two-dimensional plane Π spanned by the lines L_t, L_s. Thus, we have checked that all but at most finite number of ruling lines in S belong to the two-dimensional plane Π. Since the surface S is algebraic $S = Q^{-1}(0)$ (Proposition 5.1), it coincides with the entire plane: $S = \Pi$.

Then, of course, S, as a surface, is a cone, which means that it can be foliated by straight lines with a common point (though the original foliation of the surface S may be not conical, i.e., the ruling lines L_t can have no common point). The lemma is proved. □

Now we are interested in the case when $d(s,t) = \text{dist}(L_t, L_s)$ is not identically zero function.

Lemma 9.2. *If S is not a cone then there are two maximally distant ruling lines L_{t_0}, L_{s_0}, i.e., the distance function $d(t,s)$ attains its maximum*:

$$d(t_0, s_0) = \max_{t,s} d(t,s) > 0.$$

at some values t_0, s_0 of the parameters.

Proof. The function $d(s,t)$ is defined on the compact set $[-1,1]\times[-1,1]$. It is upper semi-continuous, i.e., the upper limit

$$\limsup_{(t,s)\to(t_0,s_0)} d(t,s) \le d(t_0,s_0).$$

Indeed, let $a = u(t_0) + \lambda_0 e(t_0) \in L_{t_0}$, $b = u(s_0) + \mu_0 e(s_0) \in L_{s_0}$, be the points on the straight lines L_{t_0}, L_{s_0} such that

$$|a-b| = \operatorname{dist}(L_{t_0}, L_{s_0}).$$

If $(t_n, s_n) \to (t_0, s_0)$ then

$$a_n = u(t_n) + \lambda_0 e(t_n) \longrightarrow a, \quad b_n = u(s_n) + \mu_0 e(s_n) \longrightarrow b.$$

Then we have

$$d(s_n, t_n) \le |a_n - b_n|$$

and hence

$$\lim_{n\to\infty} d(t_n, s_n) \le \lim_{n\to\infty} |a_n - b_n| = |a-b| = d(t_0, s_0).$$

Due to the arbitrariness of the sequence $(t_n, s_n) \to (t_0, s_0)$, the function $d(t,s)$ is upper semi-continuous. By Weierstrass theorem it attains its maximal value $d(t_0, s_0)$. Since we have assumed that $d(t,s)$ is not identically zero, we have $|a-b| = d(t_0,s_0) > 0$. We will call a, b *extremal points.* □

So far, there was no need in assumption that the foliation $\{L_t\}$ contains no parallel lines. If we assume that , then the closest points of two ruling lines and, in particular, the extremal points a and b, are uniquely determined.

Lemma 9.3. *Suppose that the line foliation of S contains no parallel lines. Suppose that the surface S is differentiable at the extremal points a and b and the foliation $S = \bigcup_t L_t$ is regular at both extremal points a and b. Then a and b are antipodal points* (*see Definition* 6.1).

According to Definition 7.2, regularity means that near the points a and b, the surface S can be parametrized by the mappings

$$w_a(t,\lambda) = w_a(t) + \lambda E_a(t), \quad w_b(s,\mu) = w_b(s) + \mu E_b(s),$$

correspondingly, which define the original foliation of S and are differentiable and regular at the points $(t_0,\lambda_0), (s_0,\mu_0)$. Here $a = w_a(t_0,\lambda_0)$, $b = w_b(s_0,\mu_0)$.

We denote the straight lines

$$L_t = \{w_a(t) + \lambda E_a(t), \lambda \in \mathbb{R}\}, \quad L_s = \{w_b(s) + \mu E_b(s), \mu \in \mathbb{R}\}.$$

The tangent spaces at a and b are spanned by the corresponding partial derivatives, which are linearly independent due to regularity:

$$T_a(S) = \text{span}\{\partial_t w_a(t_0, \lambda_0), E_a(t_0)\},$$

$$T_b(S) = \text{span}\{\partial_t w + b(s_0, \mu_0), E_b(s_0)\}.$$

We know that the function

$$(\lambda, \mu) \longmapsto |w_a(t_0, \lambda) - w_b(s_0, \mu)|^2$$

attains minimum at $\lambda = \lambda_0, \mu = \mu_0$. Therefore, the partial derivatives vanish at (t_0, λ_0).

Differentiation in λ at $t = t_0, \lambda = \lambda_0$ yields

$$\langle E_a(t_0), w_a(t_0, \lambda_0) - w_b(s_0, \mu_0)\rangle = \langle E_a(t_0), a - b\rangle = 0. \tag{29}$$

Analogously, differentiation in μ gives

$$\langle E_b(s_0), a - b\rangle = 0. \tag{30}$$

For any pair L_t, L_s of the ruling lines in S, denote $a(t,s), b(t,s)$ the points

$$a(t,s) = w_a(t) + \lambda(t,s)E_a(t), \quad b(t,s) = w_b(t) + \mu(t,s)E_b(s),$$

belonging to the lines L_t, L_s correspondingly, at which the distance between the lines is attained:

$$d(t,s) = \text{dist}(L_t, L_s) = |a(t,s) - b(t,s)|.$$

The coefficients $\lambda(t,s), \mu(t,s)$ can be found from the orthogonality conditions

$$\langle a(t,s) - b(t,s), E_a(t)\rangle = 0, \quad \langle a(t,s) - b(t,s), E_b(s)\rangle = 0.$$

The solutions of the corresponding linear system are

$$\lambda(t,s) = \frac{-\langle E_a(t), E_b(s)\rangle\langle w_a(t) - w_b(s), E_b(s)\rangle + \langle w_a(t) - w_b(s), E_b(s)\rangle}{1 - \langle E_a(t), E_b(s)\rangle^2},$$

$$\mu(t,s) = \frac{\langle E_a(t), E_b(s)\rangle\langle w_a(t) - w_b(s), E_b(s)\rangle - \langle w_a(t) - w_b(s), E_a(t)\rangle}{1 - \langle E_a(t), E_b(s)\rangle^2}.$$

The denominator is different from zero as the lines L_t, L_s are not parallel by the condition and hence $1 - \langle E_a(t), E_b(s)\rangle \neq 0$.

The above formulas show that the functions $\lambda(t,s), \mu(t,s)$ are differentiable at the point (t_0, s_0).

Since the distance function $d(t,s)$ attains its maximum at t_0, s_0 we have

$$\partial_t d(t_0, s_0) = \langle a_t'(t_0, s_0) - b_t'(t_0, s_0), a - b\rangle = 0,$$

$$\partial_s d(t_0, s_0) = \langle a_s'(t_0, s_0) - b_s'(t_0, s_0), a - b\rangle = 0,$$

or

$$\langle w_a'(t_0) + \lambda'(t_0, s_0) e E_a(t_0) + \lambda_0 E_a'(t_0), a - b\rangle = 0,$$

$$\langle w_b'(s_0) + \mu'(t_0, s_0) E_b(s_0) + \mu_0 E_b'(s_0), a - b\rangle = 0.$$

Since $a - b$ is orthogonal to $E_a(t_0)$ and $E_b(s_0)$, we obtain

$$\langle w_a'(t_0) + \lambda_0 E_a'(t_0), a - b\rangle = 0,$$

$$\langle w_b'(s_0) + \mu_0 E_b'(s_0), a - b\rangle = 0.$$

Thus, the vector $a-b$ is orthogonal to the vectors $(\partial_t w_a)(t_0, \lambda_0)$, $\partial_s w_b(s_0, \mu_0)$ and also to the vectors $\partial_t w_a(t_0, \lambda_0) = E_a(t_0)$, $\partial_s w_b(s_0, \mu_0) = E_b(s_0)$, due to (29) and (30). The partial derivatives of w_a and w_b at the points $(t_0, \lambda_0), (s_0, \mu_0)$ respectively span the corresponding tangent planes $T_a(S), T_b(S)$, therefore

$$a - b \perp T_a(S), \quad a - b \perp T_b(S).$$

The two orthogonality relations show that the points a and b are antipodal. This completes the proof.

9.2. End of the proof of Theorem 2.4

9.2.1. The "if" part. Notice, that the "if" statement holds in any dimension d. Suppose that S is a harmonic cone with a vertex a. This means that there exists a nonzero harmonic homogeneous polynomial (solid harmonic) h such that

$$h(a + x) = 0, \quad \text{for all } x \in S.$$

By shifting, we can assume $a = 0$.

Define

$$\varphi_\lambda(x) = \int\limits_{|\omega|=1} e^{i\lambda\langle x,\omega\rangle} h(\omega) dA(\omega).$$

Then

$$\Delta\varphi_\lambda = -\lambda^2\varphi_\lambda.$$

Now fix $x_0 \in \mathbb{R}^d \setminus 0$ such that $h(x_0) = 0$. Denote $\mathrm{SO}_{x_0}(d)$ the group of orthogonal transformations $\rho \in \mathrm{SO}(d)$ of $\mathbb{R}^d$ such that $\rho(x_0) = x_0$. Then

$$\begin{aligned}\varphi_\lambda(x_0) &= \varphi_\lambda(\rho(x_0)) \\ &= \int\limits_{|\omega|=1} e^{i\lambda\langle\rho(x_0),\omega\rangle} h(\omega) dA(\omega) \\ &= \int\limits_{|\omega|=1} e^{i\lambda\langle x_0,\rho^{-1}(\omega)\rangle} h(\omega) dA(\omega).\end{aligned}$$

Change of variables $\omega' = \rho^{-1}(\omega)$ leads to

$$\varphi_\lambda(x_0) = \int\limits_{|\omega'|=1} e^{i\lambda\langle x_0,\omega'\rangle} h(\rho\omega') dA(\omega').$$

Integrating the equality in ω' against the normalized Haar measure $d\rho$ on $\mathrm{SO}(d)$ yields

$$\varphi_\lambda(x_0) = \int\limits_{|\omega'|=1} e^{i\lambda\langle x_0,\omega'\rangle} \tilde{h}(\omega') d\omega', \tag{31}$$

where $\tilde{h}(\omega')$ is the average

$$\tilde{h}(\omega') = \int\limits_{\rho\in \mathrm{SO}_{x_0}(d)} h(\rho\, \omega') d\rho.$$

The function $\tilde{h}(\omega')$ is a spherical harmonic, invariant under rotations $\rho \in \mathrm{SO}(d)$, preserving x_0, and therefore it is proportional to the zonal harmonic Z_{x_0} (see [21]) with the pole $\frac{x_0}{|x_0|}$, of the same degree as $\tilde{h}$:

$$\tilde{h} = c Z_{x_0}. \tag{32}$$

However,

$$\frac{1}{|x_0|^{\deg h}}\tilde{h}(x_0) = \tilde{h}\Big(\frac{x_0}{|x_0|}\Big) = h\Big(\frac{x_0}{|x_0|}\Big) = 0,$$

because $\rho\, x_0 = x_0$, $h(x_0) = 0$ and h is homogeneous. On the other hand the value of the zonal harmonic at its pole is

$$Z_{x_0}\Big(\frac{x_0}{|x_0|}\Big) = \alpha\Omega_{d-1}^{-1},$$

where α is the dimension of the space of spherical harmonics of degree $\deg h$ and Ω_{d-1} is the area of the unit sphere in $\mathbb{R}^d$. ([21], Corollary 2.9), Therefore, we have form (32):

$$c\alpha\Omega_{d-1}^{-1} = 0$$

and $c = 0$. Then (32) implies $\tilde{h} \equiv 0$ and then $\varphi_\lambda(x_0) = 0$ because of (31).

Thus, we have proven that $\varphi_\lambda(x_0) = 0$ whenever $h(x_0) = 0$ and hence the harmonic cone $h^{-1}(0)$ is a common nodal set for a nontrivial Paley–Wiener family of eigenfunctions.

9.2.2. The "only if" part. We assume that an irreducible real analytically ruled hypersurface $S \subset \mathbb{R}^3$, without parallel generating lines, is contained in the common zero set of a Paley–Wiener family of eigenfunctions. We need to prove that S is a cone.

We start with the case when every point $a \in S$ is at least C^1 point and the foliation $\{L_t\}$ of S is everywhere regular. In particular, it is regular at the extremal points a, b at which the distance function $d(t,s)$ attains its maximum. Then the points a and b are antipodal by Lemma 9.3, and then Theorem 6.2 implies that S is an injectivity set. By Proposition 5.1, this contradicts to the assumption that S is the common nodal set for Paley–Wiener family of eigenfunctions.

Therefore the line foliation of S has at least one singular point, say, a. By Corollary 8.4 of Theorem 8.1 a is a conical point. This means that a belongs to an open family of lines $\{L_t\}$. Since S is irreducible, the base curve γ that parameterizes the family L_t is real analytic and connected. Therefore, all lines L_t pass through a and therefore S is a cone with the vertex a.

Moreover, S is a harmonic cone. Indeed, we know from Proposition 5.1 that there exists a nonzero harmonic polynomial H such that $S \subset H^{-1}(0)$. Since S is a cone with the vertex a we have

$$H(a + \lambda(x - a)) = 0$$

for all $x \in S$ and $\lambda \in \mathbb{R}$. Therefore, if $H(a+u) = \sum_{j=0}^{N} H_j(u)$ is the homogeneous decomposition, then $H_j(x-a) = 0$, $j = 0, \ldots, N$ and it remains to note that all H_j are harmonic and homogeneous. Then $a + S \subset h^{-1}(0)$, where h can be taken any nonzero polynomial H_j. Theorem 2.4 is proved.

10. Reducible case. Proof of Theorem 2.5

Now we turn to the proof of more general Theorem 2.5 where we do not assume that the base curve γ of the ruled surface S is connected.

In general situation, S decomposes into irreducible components:

$$S = \bigcup_{j=1}^{M} S_j,$$

where each S_j is a real analytically ruled surface with a real analytic closed connected base curve γ_j. So, the ruled surface S is parameterized by the base curve

$$\gamma = \gamma_1 \cup \cdots \cup \gamma_M.$$

Each surface S_j satisfies all the conditions of Theorem 2.4 and therefore is a harmonic cone with a vertex $a_j \in S_j$. All we need now is to prove the additional properties of the decomposition of S into union of cones claimed in Theorem 2.5.

We will start with proving that the cones pairwise meet.

Lemma 10.1. *If there are i, j such that $S_i \cap S_j = \emptyset$ then S is an injectivity set.*

Proof. Assume that S fails to be an injectivity set. Since S_i and S_j do not meet, any two generating lines L_a, $a \in \gamma_i$ and L_b, $b \in \gamma_j$, are disjoint and $\mathrm{dist}(L_a, L_b) > 0$.

Since there are no parallel generating lines, the function $(a, b) \mapsto \mathrm{dist}(L_a, L_b)$ is continuous and attains its *minimum*. Let $a_0 \in S_i, b_0 \in S_j$ are the points where the minimal distance between the generating lines is realized:

$$|a_0 - b_0| = \min_{\substack{a \in S_i, \\ b \in S_j}} \mathrm{dist}(L_a, L_b) > 0.$$

The two cases are possible:

(1) a_0 and b_0 are regular points of the foliation $S = \bigcup_{a \in S} L_a$.

(2) One of the points a_0, b_0 is a singular point.

Let $a_0 = u(t_0, \lambda_0),\ b_0 = u(s_0, \mu_0)$.

In Case 1, the same computations as in the proof of Lemma 9.3 show that the vector $a_0 - b_0$ is orthogonal to the both tangent spaces $T_{a_0}(S),\ T_{b_0}(S)$ which by the definition means that the points a_0 and b_0 are antipodal. Theorem 6.2 implies that S is an injectivity set. This is a contradiction.

Consider now Case 2, i.e., assume that one of the extremal points, say, a_0 is singular. Since S is not an injectivity set, a_0 is a conical point, due to Theorem 8.1. The ruled surface S_i has the real analytic connected base curve γ_i hence S_i is a cone with the vertex a_0.

Now, the straight lines $L_{t_0} \subset S_i$ and $L_{s_0} \subset S_j$ are the closest generating lines belonging to S_i and S_j correspondingly. Since $a_0 \in L_{t_0}, b_0 \in L_{s_0}$ are the closest points, the the segment joining them is perpendicular to the both lines:

$$[a_0, b_0] \perp L_{t_0}, L_{s_0}.$$

However, since S_i is the cone with the vertex a_0, all the straight lines L_t generating S_i all pass through a_0. If L_t is not orthogonal to $[a_0, b_0]$ then

$$\mathrm{dist}(L_t, L_{s_0}) < |a_0 - b_0| = \mathrm{dist}(L_{t_0}, L_{s_0})$$

which is impossible.

Therefore, for all generating lines $L_t \subset S_i$ we have

$$L_t \perp [a_0, b_0]$$

and hence $L_t \subset \Pi$, where Π is the plane passing through a_0 and orthogonal to $[a_0, b_0]$. Then S_i coincides with the plane Π and $S_i = \Pi$ can be viewed as a line foliation, regular at a_0. If the second extremal point b_0 is regular for the given foliation $\{L_t\}$ then both points a_0, b_0 are regular antipodal points and S is an injectivity set. If b_0 is a conical point, then the same argument with closest generating lines shows that S_j is a plane. Then again a_0, b_0 are regular antipodal points and S is an injectivity sets. The lemma is proved. □

Now we will prove that the cones intersect transversally.

Lemma 10.2. *If some S_i and S_j are tangent at a point a which is not a vertex of any cone S_i, S_j then S is an injectivity (not nodal) set.*

Proof. We saw in the proof Theorem 8.1 that if a is not a vertex of the cone S_i then it is either a point of real analyticity or a point of differentiability , but is a singular point of the line foliation corresponding to the case of odd m in the parametrization (12). The same is true for the second cone S_j.

After a suitable translation and rotation, we can make $a = 0$ and

$$T_a(S_i) = T_a(S_j) = \{x_3 = 0\}.$$

The representation (27) shows that the surfaces S_i, S_j are defined near $a = 0$ as the graphs:

$$S_i \colon x_3 = z_i(x_1, x_2),$$

$$S_j\colon x_3 = z_j(x_2, x_2),$$

where

$$z_i(x_1, x_2) = o(r), \quad z_j(x_1, x_2) = o(r), \quad r = \sqrt{x_1^2 + x_2^2} \to 0.$$

Moreover, by the construction, these functions are algebraic and for some odd integers m, n the functions

$$z_i(x_1^m, x_2), \quad z_j(x_1^n, x_2)$$

are real analytic.

If S is not an injectivity set, then due to Proposition 5.1, there exists the nonzero harmonic polynomial H vanishing on S (Proposition 5.1). Since $H = 0$ on $S_i = \{x_3 - z_i(x_1, x_2) = 0,\}$ the polynomial

$$H(x_1^{mn}, x_2, x_3) = 0 \quad \text{whenever } \rho_i(x) := x_3 - z_i(x_1^{mn}, x_2) = 0.$$

The function ρ_i is real analytic and $\nabla \rho_i \neq 0$ on S_i hence the polynomial H is divisible by ρ_i which means that

$$H(x_1^{mn}, x_2, x_3) = (x_3 - z_i(x_1^{mn}, x_2))R(x_1, x_2, x_3),$$

where R is real analytic near 0.

Since S_i and S_j can coincide only on a nowhere dense subset, and $H = 0$ on S_j, the function R must vanish on the surface $\rho_j(x) := x_3 - z_j(x_1^{mn}, x_2) = 0$. Further, since both functions H and ρ_j are real analytic and $\nabla \rho_j \neq 0$ on S_j, the function R is divisible by ρ_j, meaning that

$$R = \rho_j G,$$

where the function G is real analytic near 0.

Finally, returning to x_1 instead of x_1^{mn} we have

$$H(x) = (x_3 - z_i(x_2, x_3))(x_3 - z_j(x_1, x_2))G(x^{\frac{1}{mn}}, x_2, x_3).$$

Decompose

$$G(x^{\frac{1}{mn}}, x_2, x_3) = \sum_{\alpha,\beta,\gamma \geq 0} g_{\alpha,\beta,\gamma} x_1^{\frac{\alpha}{mn}} x_2^{\beta} x_3^{\gamma}$$

and let G_0 be the sum of the terms with the minimal homogeneity degree

$$\frac{\alpha_0}{mn} + \beta_0 + \gamma_0.$$

If

$$H = H_k + H_{k+1} + \cdots + H_N, H_k \neq 0,$$

is the homogeneous decomposition for H, then since $z_i, z_j = o(r), r \to 0$ we have for the minimal degree homogeneous term:

$$H_k(x) = x_3^2 G_0(x).$$

Thus,

$$H(x_1, x_2, 0) = 0.$$

Notice that G_0 is a polynomial with respect to $x_1^{\frac{1}{mn}}, x_2, x_3$. Therefore, differentiation in x_3 yields

$$\partial_{x_3} H(x) = 2x_3 G_0(x) + x_3^2 \partial_{x_3} G_0(x)$$

and hence

$$\partial_{x_3} H(x_1, x_2, 0) = 0.$$

However, H_k is harmonic and satisfies the overdetermined Dirichlet–Neumann conditions on the plane $x_3 = 0$. This implies $H_k \equiv 0$. This contradiction completes the proof. □

10.1. End of the proof of Theorem 2.5. First of all, according to Theorem 2.4, each irreducible ruled component of S is a harmonic cone and therefore, S is the union of harmonic cones, $S = \bigcup_{j=1}^N S_j$.

Moreover, the vertices are the only singular points of the cones S_i. The cones S_i are real analytic everywhere except, maybe, for the vertex. If S_i is differentiable also at the vertex then S_i is a plane and then, of course, is real analytic everywhere.

Further, Lemma 10.1 implies that $S_i \cap S_j \neq \emptyset$ for any $i \neq j$, since otherwise S is an injectivity set. In turn, Lemma 10.2 says that $S_i \neq S_j$ is transversal. The intersection $S_i \cap S_j$ is either 0-dimensional (discrete) or 1-dimensional.

Consider the first case. If $S_i \cap S_j$ is discrete, then since S_i, S_j are two-dimensional, any point $a \in S_i \cap S_j$, at which S_i and S_j are differentiable, must be a tangency point, which is not the case. Therefore, a must be singular for either cone S_i, S_j and hence is a vertex of one of them. This is exactly Case 1 of Theorem 2.5.

Now consider the second case. In this case the transversal intersection $S_i \cap S_j$ is a curve. Moreover, this curve must be unbounded. Indeed, if $S_i \cap S_j$ is a bounded curve, say, G, then G bounds bounded domains $D_i \subset S_i$, $D_j \subset S_j$ on the cones S_i, S_j. According to Proposition 5.1 there exists a nontrivial harmonic polynomial H vanishing on S and, in particular, on G, D_i and D_j. This contradicts the maximum modulus principle, because the union $G \cup D_i \cup D_j$ bounds a bounded domain $D \subset \mathbb{R}^3$. Thus, the case of 1-dimensional intersection corresponds to Case 2 of Theorem 2.5. Since the two cases are the only possible, the proof is complete.

Theorem 2.5 is proved.

11. Coxeter systems of planes. Proof of Theorem 2.7

Theorem 2.4 asserts that S is a cone. The only cone which has no differentiable singularities is a plane. Therefore, if S in Theorem 2.4 is a differentiable surface then S is a plane.

Then Theorem 2.7 follows from the following lemma.

Lemma 11.1. *Any finite union S of hyperplanes in $\mathbb{R}^d$ is an injectivity set unless S can be completed to a Coxeter system.*

Proof. We will give the proof for the case $d = 3$ which is under consideration in this article.

Let

$$S = \bigcup_{i=1}^{N} \Pi_i$$

where Π_i are the hyperplanes. Suppose that S fails to be an injectivity set. Then there exists a nonzero function $f \in C_{\text{comp}}(\mathbb{R}^3)$ such that $Rf(x,t) = 0,\ t > 0,$ for all $x \in S$. It is known [11], v.II, that then f is odd with respect to reflections around each plane Π_i.

Denote $W_{\Pi_1,\ldots,\Pi_N}$ the group generated by the reflections around the planes $\Pi_1, \ldots, \Pi_N$.

Now we are going to use the additional information about existence of nonzero harmonic polynomial vanishing on S (Proposition 5.1), which rules out, due to Maximal Modulus Principle, the possibility for the action of the group $W_{\Pi_1,\ldots,\Pi_N}$ to have compact fundamental domain.

If $N = 2$ then the angle between Π_1 and Π_2 must be a rational multiple of π since otherwise

$$\bigcup_{w \in W_{\Pi_1,\Pi_2}} w(\Pi_1) \cup \bigcup_{w \in W_{\Pi_1,\Pi_2}} w(\Pi_2)$$

is dense in $\mathbb{R}^3$ and then $f = 0$ identically because f vanishes on each Π_1, Π_2. Therefore S is a subsystem of the Coxeter system generated by the planes Π_1, Π_2.

Let $N \geq 3$. The following cases are possible:

(1) all the planes $\Pi_i, i = 1, \ldots, N$, have a common point,

(2) there are two parallel planes Π_{i_1}, Π_{i_2},

(3) there are three planes $\Pi_{i_1}, \Pi_{i_2}, \Pi_{i_3}$ that bound a right triangular prism,

(4) $N \geq 4$ and there are four planes $\Pi_{i_1}, \Pi_{i_2}, \Pi_{i_3}, \Pi_{i_4}$ that bound a bounded simplex.

In the first case, the reflection group W generated by the planes Π_i must be finite, since otherwise $\bigcup_{w \in W_{\Pi_1,\ldots,\Pi_N}} w(S)$ is dense in $\mathbb{R}^3$ and then $f = 0$. Therefore, in the first case S can be included in a Coxeter system of planes.

The second case is impossible, since supp f, being symmetric both with respect to Π_{i_1} and Π_{i_2}, must be unbounded, which is not the case.

In the third case, the normal vectors ν_1, ν_2, ν_3 of the corresponding planes are linearly dependent and span a plane P orthogonal to all $P_{i_j}, j = 1, 2, 3$. For any $b \in \mathbb{R}^3$ the intersection $(P + b) \cap (\Pi_{i_1} \cup \Pi_{i_2} \cup \Pi_{i_3})$ is three lines L_1, L_2, L_3 in the 2-plane $P + b$, bounding a triangle.

The restriction $f|_{P+b}$ can be regarded as a compactly supported function defined in $\mathbb{R}^2$, and this function is odd-symmetric with respect to the lines L_1, L_2, L_3.

In particular, it has zero spherical means on the lines. As it was proven in Proposition 5.1, if f is not identically zero on $P + b$ then there is a nonzero harmonic polynomial vanishing on $L_1 \cup L_2 \cup L_3$ which is impossible due to Maximum Modulus Principle since the union contains a bounded contour. Therefore, $f = 0$ on $P + b$ and then $f = 0$ everywhere as b is arbitrary. Thus, the third case is ruled out as well.

Also, the fourth case is impossible, since if f is not zero then we again have contradiction with existence of a nonzero harmonic polynomial vanishing on S, as in the previous case. The lemma is proved. □

Proof of Theorem 2.7. Since any two-dimensional cone in $\mathbb{R}^3$, which is a differentiable surface, is a two-dimensional plane, Theorem 2.5 implies that the surface S in Theorem 2.7 is a finite union of 2-planes and hence is a Coxeter system of planes, due to Lemma 11.1. □

12. Proof of Theorem 2.8 (the case of convexly supported generating function)

12.0.1. Some lemmas. We are given a nonzero function $f \in C_{\text{comp}}(\mathbb{R}^3)$ such that the outer boundary Γ of $\operatorname{supp} f$ is a strictly convex real analytic closed hypersurface.

Consider the set

$$N_f = \{x \in \mathbb{R}^3 \colon Rf(x,t) = 0 \text{ for all } t > 0\}.$$

By Proposition 5.1, the set N_f represents as

$$N_f = S \cup V,$$

where S is either empty or an algebraic hypersurface

$$S = Q^{-1}(0),$$

where Q is a polynomial, dividing a nonzero harmonic polynomial H. We assume $S \neq \emptyset$.

Now, Theorem 3.6 yields that the observation surface S is foliated into straight lines, each of which intersects orthogonally, at two points, the strictly convex surface Γ.

The surfaces Γ and S intersect orthogonally. The intersection

$$\gamma \colon \Gamma \cap S$$

is a curve, smooth at all points $a \in \gamma$ at which S is smooth.

Lemma 12.1. *The surface S is a real analytically ruled surface.*

Proof. Denote

$$\gamma = \Gamma \cap S.$$

Pick a point $a \in \gamma$. Let $T_a(\Gamma)$ be the tangent plane. Applying translation and rotation, one can assume that $a = 0$ and

$$T_a(\Gamma) = \{x_3 = 0\}.$$

The projection

$$\pi \colon T_a(\Gamma) \longrightarrow \Gamma$$

along the normals to Γ is well defined in a neighborhood

$$U \subset T_a(\Gamma)$$

of a.

Since Γ is real analytic, the normal field to Γ is real analytic as well and hence π is real analytic diffeomorphism near $a = 0$. Also, $\pi(U \cap S)$ is an open neighborhood of a in γ.

It is easy to understand that the polynomial Q is not identically zero on $T_a(\Gamma)$ since its zero variety $S = Q^{-1}(0)$ is transversal to $T_a(\Gamma)$ near a. Therefore, the intersection

$$C := T_a(\Gamma) \cap S$$

is an open algebraic curve in the plane $T_a(\Gamma) = \{z = 0\}$, defined by the equation $C = \{Q(x, y, 0) = 0\}$.

Then we use Puiseux theorem ([18], Chapter II, 9.6; [23], Theorem 2.1.1; [12], Chapter 2, p. 3–11) which claims that each branch C_i of C is parameterized either by

$$I \ni t \longmapsto (0, t, 0)$$

or by

$$I \ni t \longmapsto (t^m, \alpha_i(t), 0),$$

where I is an open interval (which can be taken $I = (-1, 1)$), m is natural and $\alpha_i(t)$ is a real analytic function.

Then γ decomposes, near a, into the union of the curves $\gamma_i = \pi(C_i)$ and each γ_i is the image $\gamma_i = u(I)$ where the mapping

$$I \ni t \longmapsto u_i(t) = \pi(t^m, B_i(t), 0)$$

is real analytic, because π is so. By Corollary 8.4 of Theorem 8.1, the ruled surface

$$S_i = \{u_i(t) + \lambda\nu(u_i(t)), t \in I, \lambda \in \mathbb{R}\},$$

where ν is unit normal vector to Γ, is real analytically ruled surface. □

Lemma 12.2. *Let a be the vertex of the cone C_i. Let γ_i be a connected closed subarc of $C_i \cap \Gamma$ where Γ is the outer boundary of* supp f. *Then the distance $|x - a|$ from a to an arbitrary point $x \in \gamma_i$ is constant.*

Proof. Consider the parametrization $u(t, \lambda) = u(t) + \lambda e(t), t \in I$, of the cone C_i. The mapping $t \mapsto u(t)$ parameterizes the curve $\gamma_i = C_i \cap \Gamma$. Consider the distance function

$$d(t) = |a - u(t)|^2.$$

Then

$$d'(t) = (a - u(t), u'(t)).$$

Since a is the vertex of C_i, it belongs to any line L_t. Therefore $a = u(t) + \lambda(t)e(t)$ and hence

$$d'(t) = (a - u(t), u'(t)) = \lambda(t)(e(t), u'(t)) = 0,$$

because $u'(t)$ is tangent to Γ, $e(t)$ is the directional vector of the Line L_t and L_t is orthogonal to Γ, as stated in Theorem 3.6. □

Lemma 12.3. *If two cones C_i, C_j meet outside of* supp f *then they have a common vertex and hence the union $C_i \cup C_j$ is itself a cone.*

Proof. The cones C_i, C_j consist of straight lines orthogonal to the outer boundary Γ of supp f. Also, Γ is a real analytic strictly convex surface. If C_i meet C_j in the exterior of Γ then C_i and C_j share a ruling straight line L passing through a common point of the two cones and orthogonal to Γ. The vertices of both cones C_i and C_j belong to L. The common line L meets the convex surface Γ at two points b^+, b^-:

$$\{b^+, b^-\} = L \cap \Gamma.$$

Let γ_i and γ_j be the connected closed subarcs of the smooth curves $C_i \cap \Gamma$ and $C_j \cap \Gamma$, correspondingly, containing the point b^+.

Then γ_i, γ_j are smooth closed curves on Γ, sharing the common point $b^+ \in \gamma_i \cap \gamma_j$.

Suppose that γ_i and γ_j are tangent at b^+ and let τ be the common tangent vector at b^+. Since the tangent planes of the cones C_i and C_j coincide:

$$T_{b^+}(C_i) = T_{b^+}(C_j) = \text{span}\{L, e\},$$

the two cones are tangent. However, this is impossible due to Lemma 10.2.

Thus, the two closed curves γ_i and γ_j intersect at b^+ transversally. Then they must intersect in at least one more point, $c \in \Gamma$. Then both cones $C_i, C - j$ contain the straight line L_c intersecting Γ orthogonally at the point c. The two cases are possible:

(1) $c \neq b^-$,

(2) $c = b^-$.

In Case 1, the straight lines L and L_c are different. Both of them belong to the cones C_i and C_j and hence the intersection of the two lines $L \cap L_c$ is just a single point which is the vertex of both C_i and C_j. Thus, C_i and C_j share the vertex and the lemma is proved in this case.

In Case 2 the two straight lines coincide, $L = L_c$, as they both pass through the points b^+ and $b^- = c$. Let a_i, a_j be the vertices of the cones C_i, C_j correspondingly. By Lemma 12.2, the distance $|x - a_i|$ is constant on γ_i. Since $b^+, b^- \in \gamma_i$, we have

$$|b^+ - a_i| = |b^- - a_i|.$$

The three points a, b^+, b^- belong to the same line L and therefore, a is the midpoint:

$$a_i = \frac{1}{2}(b^+ + b^-).$$

The same can be repeated for γ_j and then we obtain

$$a_j = \frac{1}{2}(b^+ + b^-).$$

Thus, $a_i = a_j$ and the statement of the lemma is true in Case 2 as well. □

Lemma 12.4. *Suppose that $S_i \cap S_j$ is 0-dimensional. Then*

(1) $S_i \cap S_j \subset \{c_i, c_j\}$, *where c_i, c_j are the vertices of the cones S_i, S_j correspondingly*;

(2) *if $S_i \cap S_j = \{c_i, c_j\}$ then $c_i = c_j$.*

Proof. We know that S_i and S_j are differentiable everywhere except maybe at the vertices. If $a \in S_i \cap S_j$ and $a \neq c_i, a \neq c_j$, then a is the point of smoothness for both S_i and S_j and hence the cones S_i, S_j cannot intersect at a transversally since in this case the intersection $S_i \cap S_j$ must be one-dimensional. Therefore, S_i and S_j are tangent at a. This possibility is ruled out by Lemma 10.2. This proves the Statement 1.

If $S_i \cap S_j = \{c_i, c_j\}$ and $c_i \neq c_j$ then both cones S_i and S_j contain the straight line passing through the vertices c_i and c_j. This contradicts to the assumption that the intersection is 0-dimensional. □

Lemma 12.5. *If $S_i \cap S_j$ is one-dimensional then the cones S_i and S_j share the vertex so that $S_i \cup S_j$ is a cone.*

Proof. Let $\gamma = S_i \cap S_j$. If the curve γ is unbounded, then S_i and S_j intersect outside of Γ and by Lemma 12.3 S_i and S_j have a common vertex. Otherwise, γ is a bounded curve. It is also closed as it is algebraic. Then γ bounds two-dimensional domains D_i and D_j on the surfaces S_i, S_j correspondingly. Therefore, $S_i \cap S_j$ contain a cycle $D_i \cup D_j$. However, it is impossible due to Maximum Modulus Principle, since there exists a nonzero harmonic polynomial H vanishing on $S_i \cup S_j$. □

Corollary 12.6. *If S_i and S_j have different vertices, $c_i \neq c_j$, then $S_i \cap S_j$ consists of a single point, which is either c_i or c_j.*

Proof. The intersection $S_i \cap S_j$ is discrete (0-dimensional) since otherwise the cones S_i, S_j have equal vertices, by Lemma 12.5. Then Lemma 12.4 says the intersection coincides with one of the vertices. □

12.0.2. End of the proof of Theorem 2.8. Let us group all the cones S_i whose vertices coincide. The union of such cones is again a cone and hence the union S can be regrouped in the union

$$S = C_1 \cup \cdots \cup C_P$$

of cones C_i with pairwise different vertices b_i.. Each C_i is the union of the cones S_j with equal vertices. Due to Lemma 12.5, the pairwise intersections $C_i \cap C_j$, $i \neq j$, are 0-dimensional.

First of all , each cone C_j is harmonic, i.e. belongs to the zero set of a nontrivial harmonic homogeneous polymnomial. Indeed, we know that there is a nonzero harmonic polynomial H vanishing on S. By translation, we can assume that the vertex b_i of the cone C_i is $b_i = 0$. Since C_i is a cone, we have

$$H(\lambda x) = 0$$

for all $x \in C_i$ and all $\lambda \in \mathbb{R}$. If $H = H_0 + \cdots + H_N$ is the homogeneous decomposition, then $H_0(x)+\lambda H_1(x)+\cdots+\lambda^N H_N(x) = 0$ and hence $H_k(x) = 0$ for all k. Denoting h any nonzero homogeneous term of H we will have $h(x) = 0$ for all $x \in C_i$ and hence C_i ia a harmonic cone because.the polynomial h is homogeneous and harmonic.

Further, we know that for any $i \neq j$ the intersection $S_i \cap S_j$ is either c_i or c_j. It follows that for the cones C_i, which are unions of groups of S_j, holds $C_i \cap C_j \subset \{b_i, b_j\}$. If $C_i \cap C_j = \{b_i, b_j\}$ then both cones C_i and C_j contain the points $b_j \neq b_j$ and hence, the straight line through these points, which is not the case.

Thus, $C_i \cap C_j$ is a single point,which is a vertex of C_i or C_j:

$$C_i \cap C_j = \{b_i\} \quad \text{or} \quad C_i \cap C_j = \{b_j\}. \tag{33}$$

Lemma 12.7. $P \leq 3$.

Proof. Suppose that $P \geq 4$. Consider the cones C_1, C_2, C_3, C_4. We have

$$C_1 \cap C_2 = \{b_1\} \text{ or } \{b_2\}.$$

Without loss of generality, we can assume that

$$C_1 \cap C_2 = \{b_1\}.$$

Then we claim that

$$C_1 \cap C_3 = \{b_3\}.$$

Indeed, C_1 and C_3 meet either at b_1 or at b_3. However, if they meet at b_1 then b_1 belongs both to the cones C_3 and C_2 and therefore b_1 coincides with one of their vertices b_2, b_3, which is impossible because b_1, b_2, b_3 are all different. Therefore, the remaining option is that C_1 and C_3 meet at b_3. For the same reason,

$$C_1 \cap C_4 = \{b_4\}.$$

Analogously,

$$C_2 \cap C_3 = \{b_2\},$$

because otherwise $C_2 \cap C_3 = \{b_3\}$ and then $b_3 \in C_2, b_3 \in C_1$ and therefore $b_3 \in \{b_1, b_2\}$ which is not the case.

At last, consider the intersection of C_2 and C_4:

$$C_2 \cap C_4 = \{b_2\} \quad \text{or} \quad C_2 \cap C_4 = \{b_4\}.$$

If $C_2 \cap C_4 = \{b_2\}$ then we have

$$b_2 \in C_4, \quad b_2 \in C_3$$

and therefore

$$b_2 \in \{b_3, b_4\}$$

which is not the case. If, alternatively, $C_2 \cap C_4 = \{b_4\}$, then we have $b_4 \in C_2$, $b_4 \in C_1$ and therefore

$$b_4 \in \{b_1, b_2\},$$

which is not the case. Thus, neither option is possible. Thus, $P \leq 3$. The lemma is proved. □

Let us continue the proof of Theorem 2.8.

If $P = 1$ then $S = C_1$ is a cone and, moreover, a harmonic cone. This is Case 1) in Theorem 2.8.

Suppose $P = 2$ so that $S = C_1 \cup C_2$. Formula (33) says that S is a chain of two cones corresponding to Case 2) of Theorem 2.7.

Finally, suppose that $P = 3$ and therefore

$$S = C_1 \cup C_2 \cup C_3.$$

Lemma 12.8. *No two cones of C_1, C_2, C_3 can have vertices belonging to the third one.*

Proof. Suppose, for example, that

$$b_1, b_2 \in C_3.$$

We know that $C_1 \cap C_2$ is either b_1 or b_2. Suppose that $C_1 \cap C_2 = \{b_1\}$. Then $b_1 \in C_2$ and also $b_1 \in C_3$. Hence

$$b_1 \in C_2 \cap C_3.$$

This implies that either $b_1 = b_2$ or $b_1 = b_3$. Neither is possible as all the vertices are different.

In the second case we have $b_2 \in C_1$ and also $b_2 \in C_3$. Then $b_2 \in C_1 \cap C_3$, which is either b_1 or b_3 and we have the same kind of contradiction. The lemma is proved. □

Now we can finish the proof of Theorem 2.8 in the case $S = C_1 \cup C_2 \cup C_3$.

We have $C_1 \cap C_2$ is either b_1 or b_2. If

$$C_1 \cap C_2 = \{b_1\},$$

then $C_2 \cap C_3$ can be only b_2 since otherwise $b_1, b_3 \in C_2$ which is ruled out by Lemma 12.8. Analogously, $C_3 \cap C_1$ cannot be equal to b_1 since then $b_1 \in C_2 \cap C_3$ and hence b_1 is either b_2 or b_3 which is not the case.

The case $C_1 \cap C_2 = \{b_2\}$ is treated in a similar way. Thus, finally we conclude that in the case $P = 3$ the configuration of the cones is exactly as it is pointed out in the Case 3 of Theorem 2.8. The theorem is proved.

13. Concluding remarks

Proving Conjecture 3.2 for ruled surfaces requires verification that the configurations of cones in Theorem 2.5 is itself a cone, i.e., the vertices of all the cones C_i coincide. At the moment, we do not know how to do that.

To fully prove Conjecture 3.2 about conical structure of the common nodal sets of Paley–Wiener families of eigenfunctions, it would be sufficient to prove that the common nodal sets are ruled surfaces. Then one could apply Theorems 2.4 and 2.5 which deliver a bridge from ruled surfaces to cones. In turn, as it is mentioned in Section 2, the ruled structure of common nodal sets is confirmed in several partial cases, namely, in the two-dimensional case (Theorem 3.1), in the case of generating distributions with finite (Theorem 3.5) or convex (Theorem 2.8) supports. Also, the conjecture on ruled structure is consistent with the result of [8] for the periodic case. This result states that the common nodal sets for large families of eigenfunctions on the torus T^d have the zero Gaussian curvature. In view of those results, the hypothesis about ruled structure of common nodal sets in Euclidean spaces seems plausible.

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Mark L. Agranovsky, Department of Mathematics, Bar-Ilan University, Ramat-Gan, 52900, Israel

e-mail: agranovs@math.biu.ac.il