A weighted estimate for two dimensional Schrödinger, matrix Schrödinger, and wave equations with resonance of the first kind at zero energy

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Abstract. We study the two dimensional Schrödinger operator, $H = -\Delta + V$, in the weighted $L^1(\mathbb{R}^2) \to L^\infty(\mathbb{R}^2)$ setting when there is a resonance of the first kind at zero energy. In particular, we show that if $|V(x)| \lesssim \langle x \rangle^{-4-}$ and there is only s-wave resonance at zero of H, then

$$\|w^{-1}(e^{itH}P_{\mathrm{ac}}f - \frac{1}{\pi i t}Ff)\|_\infty \leq \frac{C}{|t|(\log|t|)^2}\|wf\|_1, \quad |t| > 2,$$

with $w(x) = \log^2(2 + |x|)$. Here $Ff = -\frac{1}{4}\psi\langle\psi, f\rangle$, where ψ is an s-wave resonance function. We also extend this result to wave and matrix Schrödinger equations with potentials under similar conditions.

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Contents

1. Introduction

Recall the propagator of the free Schrödinger equation:

$$e^{-it\Delta} f(x) = \frac{1}{(-4\pi i t)^{n/2}} \int_{\mathbb{R}^n} e^{-i|x-y|^2/4t} f(y) dy \tag{1}$$

which satisfies the dispersive estimate

$$\|e^{-it\Delta} f\|_\infty \lesssim t^{-n/2} \|f\|_1 \tag{2}$$

for any $n \geq 1$. There are many works concerning the validity of such an estimate for the perturbed Schrödinger operator $H = -\Delta + V$, where $V(x)$ is a real-valued and bounded potential with sufficient decay at infinity. See, for example, [33, 43, 23, 21, 24, 52, 19, 9, 12].

Since H may have eigenvalues on $(-\infty, 0]$, the inequality (2) cannot hold in general. Therefore, we consider $e^{itH} P_{ac}(H)$, where $P_{ac}(H)$ is the orthogonal projection onto the absolutely continuous subspace of $L^2(\mathbb{R}^n)$. It was observed that the time decay of the operator $e^{itH} P_{ac}(H)$ is affected by resonances or an eigenvalue at zero energy (see, e.g., [41, 31, 39, 29, 30, 17, 51, 21, 3, 13]).

Recall that, in two dimensions, a distributional solution of $H\psi = 0$ is called an s-wave resonance if $\psi \in L^\infty(\mathbb{R}^2)$, but $\psi \notin L^p(\mathbb{R}^2)$ for any $p < \infty$; and it is called a p-wave resonance if $\psi \in L^p(\mathbb{R}^2)$ for $2 < p \leq \infty$, but $\psi \notin L^2(\mathbb{R}^2)$. We also say there is a resonance of the first kind at zero if there is only an s-wave resonance at zero but there are no p-wave resonances or an eigenvalue at zero. It is important to recall that in this case, there is only one s-wave resonance function up to a multiplicative constant. There are similar definitions for resonance in dimensions $n = 1, 3, 4$, and there are no zero energy resonances in dimensions $n \geq 5$.

We note that by these definitions, the constant function $\psi = 1$ is an s-wave resonance in dimension two for the free Schrödinger operator. In addition, using the formula (1), one can easily prove that

$$\|w^{-1}(e^{-it\Delta} f + \frac{1}{4\pi i t} \psi \langle \psi, f \rangle)\|_{L^\infty(\mathbb{R}^2)} \lesssim \frac{1}{t \log^2 t} \|wf\|_{L^1(\mathbb{R}^2)}. \tag{3}$$

Here we define $w(x) := \log^2(1+|x|)$ and use $w(x)$ with this definition throughout the paper. This suggests that the perturbed Schrödinger evolution should satisfy a similar weighted estimate with an integrable decay rate in the case of an s-wave resonance. Indeed, our main result in this paper is the following.

Theorem 1.1. *Let $|V(x)| \lesssim \langle x \rangle^{-2\beta -}$ for some $\beta > 2$,. If there is a resonance of the first kind at zero for $H = -\Delta + V$, then we have*

$$\|w^{-1}(e^{itH} P_{\mathrm{ac}} f - \frac{1}{\pi i t} Ff)\|_{L^\infty(\mathbb{R}^2)} \leq \frac{C}{|t|(\log |t|)^2} \|wf\|_{L_1(\mathbb{R}^2)}, \quad |t| > 2.$$

Here F is a rank 1 projection onto the one-dimensional space of resonances:

$$Ff = -\frac{1}{4} \psi \langle \psi, f \rangle,$$

where ψ is the canonical s-wave resonance function satisfying $\psi - 1 \in L^p$ for all $p > 2$.

We then, extend this result to the matrix Schrödinger operator and to the low-energy evolution of the solution of the two-dimensional wave equation, [see Theorem 1.3 and 1.4].

The dispersive estimate

$$\|e^{itH} P_{\mathrm{ac}} f\|_{L^\infty(\mathbb{R}^n)} \leq C|t|^{-\frac{n}{2}} \|f\|_{L_1(\mathbb{R}^n)} \tag{4}$$

in dimensions one and two were studied in [23, 45, 22, 34, 38, 13, 14]. In fact, (4) is established by Goldberg and Schlag for $n = 1$ in [23] and Schlag for $n = 2$ in [45], assuming zero is regular, that is, when there is neither a resonance nor an eigenvalue at zero. The result in dimension two is then improved by Erdoğan-Green to a more general case. They showed the same estimate when there is a resonance of the first kind at zero.

The main concern for these estimates is that they are not integrable in time at infinity. An estimate which is integrable at infinity is very useful in the study of nonlinear asymptotic stability of (multi-)solitons in lower dimensions. See [46, 34, 39, 7, 49, 40, 50] for other applications of weighted dispersive estimates to nonlinear PDEs.

The earliest integrable decay rate in dimensions one and two was established by Murata in weighted L^2 spaces. In [39, Theorem 7.6], Murata proved the following statement in polynomially weighted spaces by assuming sufficient decay on V: if zero is a regular point of the spectrum, then for $|t| > 2$,

$$\|w_1^{-1} e^{itH} P_{\mathrm{ac}}(H) f\|_{L^2(\mathbb{R})} \leq C t^{-\frac{3}{2}} \|w_1 f\|_{L^2(\mathbb{R})} \tag{5}$$

and

$$\|w_2^{-1} e^{itH} P_{\mathrm{ac}}(H) f\|_{L^2(\mathbb{R}^2)} \leq \frac{C}{|t|(\log |t|)^2} \|w_2 f\|_{L^2(\mathbb{R}^2)}. \tag{6}$$

In [47], Schlag improved Murata's 1 dimensional result (5) to weighted $L^1 \to L^\infty$ setting and he showed that if zero is regular, then

$$\|\langle x \rangle^{-1} e^{itH} P_{\mathrm{ac}}(H) f\|_{L^\infty(\mathbb{R})} \leq C t^{-\frac{3}{2}} \|\langle x \rangle f\|_{L^1(\mathbb{R}^2)},$$

provided that $|V| \lesssim \langle x \rangle^{-4}$.

The fact that constant functions are resonance in dimension one, together with (3), led Goldberg to ask whether a similar estimate as in Theorem 1.1 can be obtained when zero is not regular. In [22], Goldberg showed that if $(1 + |x|)^4 V \in L^1(\mathbb{R})$, then

$$\|(1 + |x|)^{-2} (e^{itH} P_{\mathrm{ac}} H - (-4\pi i t)^{-\frac{1}{2}}) F) f\|_{L^\infty} \lesssim t^{-\frac{3}{2}} \|(1 + |x|^2) f\|_1, \quad (7)$$

where F is a projection on a bounded function f_0 satisfying $H f_0 = 0$ and $\lim_{x \to \infty} (|f_0(x)| + |f_0(-x)|) = 2$.

Murata's two dimensional result (6) was also improved by Erdoğan and Green. In [14], they showed that if zero is regular, then

$$\|w^{-1} e^{itH} P_{\mathrm{ac}}(H) f\|_{L^1(\mathbb{R}^2)} \leq \frac{C}{|t|(\log|t|)^2} \|w f\|_{L^\infty(\mathbb{R}^2)} \tag{8}$$

provided $|V| \lesssim \langle x \rangle^{-\beta}$, for $\beta > 3$. Theorem 1.1 was motivated by (7) and (8) of Goldberg, and Erdoğan-Green, respectively.

The Schrödinger operator in dimensions $n = 3, 4$ and $n > 4$ has also been studied. For more details about these dimensions one can see [47, 17, 18, 16, 25, 26].

Letting

$$L^{2,\sigma} = \{f \colon \langle x \rangle^\sigma f \in L^2(\mathbb{R}^n)\}$$

and

$$\mathcal{H}_{2,\sigma} = \{f \colon D^\alpha f \in L^{2,\sigma}(\mathbb{R}^n)\},$$

we define the resolvent operator

$$R_V^{\pm}(\lambda^2) \colon L^{2,\sigma} \longrightarrow \mathcal{H}_{2,\sigma}$$

as

$$R_V^{\pm}(\lambda^2) = \lim_{\epsilon \to 0^+} (H - (\lambda^2 \pm i\epsilon))^{-1}.$$

By Agmon's limiting absorption principle [2], $R_V^{\pm}(\lambda^2)$ is well-defined. The proof of Theorem 1.1 relies on expansions of $R_V^{\pm}$ around zero energy and Stone's formula for self-adjoint operators. Let χ be an even smooth cut-off function supported in

$[-\lambda_1, \lambda_1]$ for a fixed, sufficiently small $\lambda_1 > 0$ such that $\chi(\lambda) = 1$ if $|\lambda| \leq \frac{\lambda_1}{2}$. Then, one has

$$e^{itH} P_{\text{ac}}(H)\chi(H)f(x) = \frac{1}{\pi i} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda)[R_V^+(\lambda^2) - R_V^-(\lambda^2)]f(x)d\lambda, \quad (9)$$

for all $f \in \mathcal{S}(\mathbb{R}^2)$. Note that in our analysis, we assume V has enough decay to ensure that H has finitely many eigenvalues of finite multiplicity on $(-\infty, 0]$, with $\sigma_{\text{ac}}(H) = [0, \infty)$. See [42].

We also extend our result to the non-self-adjoint matrix Schrödinger operator. The non-self-adjoint matrix Schrödinger operator is defined as

$$\mathcal{H} = \mathcal{H}_0 + V = \begin{bmatrix} -\Delta + \mu & 0 \\ 0 & \Delta - \mu \end{bmatrix} + \begin{bmatrix} -V_1 & -V_2 \\ V_2 & V_1 \end{bmatrix} \quad (10)$$

on $L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$, where $\mu > 0$ and V_1, V_2 are real valued potentials. Conjugation of $\mathcal{H}$ by the matrix $\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$ gives $\mathcal{H} \simeq \begin{bmatrix} 0 & iL_- \\ iL_+ & 0 \end{bmatrix}$. Hence, by Weyl's criterion and some decay assumptions on V_1 and V_2, $\sigma_{\text{ess}}(\mathcal{H}) = (-\infty, -\mu] \cup [\mu, \infty)$.

We need the following assumptions for the matrix case,

A1) $-\sigma_3 V$ is a positive matrix, where σ_3 is the Pauli spin matrix

$$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix};$$

A2) $L_- = -\Delta + \mu - V_1 + V_2 \geq 0$;

A3) $|V_1| + |V_2| \lesssim \langle x \rangle^{-2\beta-}$ for some $\beta > 2$;

A4) There are no embedded eigenvalues in $(-\infty, -\mu) \cup (\mu, \infty)$.

It is known that the first three assumptions are to hold in the case when the Schrödinger equation is linearized about a positive ground state standing wave $\psi(t, x) = e^{it\mu}\phi(x)$. We need the fourth assumption to be able to define the spectral measure from X_σ to $X_{-\sigma}$, where $X_\sigma = L^{2,\sigma} \times L^{2,\sigma}$. For more details one can see [18] and [15].

Dispersive estimates for the operator (10) are studied in [10, 44, 48, 17, 11, 37, 27]. In the case when thresholds $\pm\mu$ are regular, the following result is obtained in dimension two.

Theorem 1.2 ([15, Theorem 1.1]). *Under the assumptions* A1)–A4), *if* $\pm\mu$ *are regular points of* $\mathcal{H} = \mathcal{H}_0 + V$, *we have*

$$\|e^{itH} P_{\mathrm{ac}} f\|_{L^\infty \times L^\infty} \lesssim \frac{1}{|t|} \|f\|_{L^1 \times L^1} \tag{11}$$

and

$$\|w^{-1} e^{itH} P_{\mathrm{ac}} f\|_{L^\infty \times L^\infty} \lesssim \frac{1}{|t|(\log|t|)^2} \|wf\|_{L^1 \times L^1}, \quad |t| > 2. \tag{12}$$

Our main result for the matrix Schrödinger operator is Theorem 1.3 below. Recall that there is an *s*-wave resonance at μ for $\mathcal{H} = \mathcal{H}_0 + V$ if $\mathcal{H}\psi = \mu\psi$ for some $\psi = (\psi_1, \psi_2) \in L^\infty \times L^\infty$, but $\psi \notin L^p \times L^p$ for any $p < \infty$ (see e.g. [15]) . A distributional solution of $H\psi = \mu\psi$ is called a *p*-wave resonance if $\psi \in L^p \times L^p$ for $2 < p \leq \infty$, but $\psi \notin L^2 \times L^2$. We also say there is a resonance of the first kind at μ if there is only an *s*-wave resonance at μ but there are no *p*-wave resonances or an eigenvalue.

Theorem 1.3. *Under the conditions* A1)–A4), *if there is a resonance of the first kind at the threshold* μ, *then we have*

$$\|e^{itH} P_{\mathrm{ac}} f\|_{L^\infty \times L^\infty} \lesssim \frac{1}{|t|} \|f\|_{L^1 \times L^1},$$

and

$$\left\| w^{-1} \left(e^{itH} P_{\mathrm{ac}} f - \frac{e^{it\mu}}{\pi i t} \mathfrak{F} f \right) \right\|_{L^\infty \times L^\infty} \lesssim \frac{C}{|t|(\log|t|)^2} \|wf\|_{L^1 \times L^1}, \quad |t| > 2.$$

Here $\mathfrak{F}$ *is a rank one operator whose range is the one-dimensional space of resonances*:

$$\mathfrak{F} f(x) = -\frac{1}{4} \psi(x) \langle \sigma_3 \psi, f \rangle,$$

where ψ *is the canonical s-wave resonance function satisfying* $\psi - (1, 0)^T \in L^p \times L^p$ *for all* $p > 2$.

A similar statement holds if there is a resonance of the first kind at $-\mu$.

The resolvent expansions we obtain to prove Theorem 1.1 for the Schrödinger evolution are also applicable to the two-dimensional wave equation with a potential. Recall that the perturbed wave equation is given as

$$u_{tt} - \Delta u + V(x)u = 0, \quad u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \tag{13}$$

with the solution formula

$$u(x, t) = \cos(t\sqrt{H}) f(x) + \frac{\sin(t\sqrt{H})}{\sqrt{H}} g(x)$$

for any $f \in W^{2,1}$ and $g \in W^{1,1}$. By Stone's formula, we have the representations

$$\cos(t\sqrt{H})P_{\mathrm{ac}}f(x) = \frac{1}{\pi i}\int_0^\infty \cos(t\lambda)\lambda[R_V^+(\lambda^2) - R_V^-(\lambda^2)]f(x)d\lambda \qquad (14)$$

and

$$\frac{\sin(t\sqrt{H})}{\sqrt{H}}P_{\mathrm{ac}}g(x) = \frac{1}{\pi i}\int_0^\infty \sin(t\lambda)[R_V^+(\lambda^2) - R_V^-(\lambda^2)]g(x)d\lambda. \qquad (15)$$

For low energy, that is, when $0 < \lambda \ll 1$, this representation leads us to a similar result as in Theorem 1.1. On the other hand, for large energy, i.e., when $\lambda \gtrsim 1$, one needs regularizing powers of $\langle H \rangle^{-\alpha}$ for some $\alpha > 0$ which reflects the loss of derivatives of initial data. See, e.g., [27].

In dimension two, dispersive estimates for the wave equation are studied in [8, 38, 35, 28]. The decay rate $|t|^{-\frac{1}{2}}$ for high energy is first established in [8] between 'regularized' L^p spaces $(\sqrt{H})^{-\frac{\alpha}{2}}$ for $\alpha > 0$. Moulin in [38], improved this result to the 'regularized' $L^1 \to L^\infty$ setting with $H^{-\frac{3}{4}-\epsilon}$ for $0 < \epsilon \ll 1$. Then Kopylova [35] obtained the decay rate $(t\log^2 t)^{-1}$ in the weighted Hilbert space setting for large t when zero is regular. Finally, in [28], Green proved that if there is a resonance of the first kind at zero, then for $t > 0$,

$$\| \cos(t\sqrt{H})\langle H \rangle^{-3/4-}P_{\mathrm{ac}}f(x)\|_{L^\infty} \lesssim |t|^{-\frac{1}{2}}\|f\|_{L^1} \qquad (16\mathrm{a})$$

and

$$\|\frac{\sin(t\sqrt{H})}{\sqrt{H}}\langle H \rangle^{-1/4-}P_{\mathrm{ac}}f(x)\|_{L^\infty} \lesssim |t|^{-\frac{1}{2}}\|f\|_{L^1}. \qquad (16\mathrm{b})$$

Also, if zero is regular, then for $t > 2$,

$$\|\langle x \rangle^{-\frac{1}{2}-}\cos(t\sqrt{H})\langle H \rangle^{-3/4-}P_{\mathrm{ac}}f(x)\|_{L^\infty} \lesssim (t\log^2 t)^{-1}\|\langle x \rangle^{\frac{1}{2}+}f\|_{L^1} \qquad (17\mathrm{a})$$

and

$$\|\langle x \rangle^{-\frac{1}{2}-}\frac{\sin(t\sqrt{H})}{\sqrt{H}}\langle H \rangle^{-1/4-}P_{\mathrm{ac}}f(x)\|_{L^\infty} \lesssim (t\log^2 t)^{-1}\|\langle x \rangle^{\frac{1}{2}+}f\|_{L^1}. \qquad (17\mathrm{b})$$

For more results in other dimensions, one can see [20, 5, 6, 36, 4].

These two results of Green suggest that the techniques we present below to obtain Theorem 1.1 for the Schrödinger evolution can be adapted to the wave evolution. In fact, one obtains the following result.

Theorem 1.4. *Let* $|V(x)| \lesssim \langle x \rangle^{-2\beta-}$ *for some* $\beta > 2$. *If there is a resonance of first kind at zero, then we have*

$$\| \langle x \rangle^{-\frac{1}{2}-} \cos(t\sqrt{H}) \chi(H) P_{ac} f \|_{L^\infty} \leq \frac{C}{|t|(\log|t|)^2} \| \langle x \rangle^{\frac{1}{2}+} f \|_{L^1}$$

and

$$\left\| \langle x \rangle^{-\frac{1}{2}-} \left(\frac{\sin(t\sqrt{H})}{\sqrt{H}} \chi(H) P_{ac} g - \frac{1}{\pi i t} F g \right) \right\|_{L^\infty} \leq \frac{C}{|t|(\log|t|)^2} \| \langle x \rangle^{\frac{1}{2}+} f \|_{L^1},$$

for $|t| > 2$. *Here* $F(x, y) = -\frac{1}{2} \psi \langle \psi_1, f \rangle$ *where* ψ *is the canonical s-wave resonance function satisfying* $\psi - 1 \in L^p$ *for all* $p > 2$.

Theorem 1.4 is valid only for the low energy part, but by including regularizing powers and combining it with the high energy result (16) of Green, we can extend it to all energies.

2. Scalar case

In this section we prove that

Theorem 2.1. *Let* $|V(x)| \lesssim \langle x \rangle^{-3-2\alpha}$. *Then, we have for* $t > 2$

$$\left| \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) [R_V^+(\lambda^2) - R_V^-(\lambda^2)](x, y) d\lambda - \frac{1}{t} F(x, y) \right| \tag{18}$$

$$\lesssim \frac{\sqrt{w(x)w(y)}}{t \log^2(t)} + \frac{\langle x \rangle^{\frac{3}{2}} \langle y \rangle^{\frac{3}{2}}}{t^{1+\alpha}}$$

where $0 < \alpha < \min\left(\frac{1}{4}, \beta - \frac{3}{2}\right)$ *and* $F(x, y) = -\frac{\psi(x)\psi(y)}{4}$ *where* ψ *is the canonical s-wave resonance function satisfying* $\psi - 1 \in L^p$ *for all* $p > 2$.

We combine (18) with the high energy result obtained in [14].

Theorem 2.2. [14, Theorem 5.1] *Let* $|V(x)| \lesssim \langle x \rangle^{-2\beta}$ *for some* $\beta > 3/2$ *and* $\tilde{\chi} := 1 - \chi$. *We have*

$$\sup_{L \geq 1} \left| \int_0^\infty e^{it\lambda^2} \lambda \tilde{\chi}(\lambda) \chi(\lambda/L) [R_V^+(\lambda^2) - R_V^-(\lambda^2)](x, y) d\lambda \right| \lesssim \frac{\langle x \rangle^{\frac{3}{2}} \langle y \rangle^{\frac{3}{2}}}{t^{3/2}}$$

for $|t| > 2$.

This combination together with Stone's formula (9) gives us

$$\left| e^{itH} P_{ac}(H)(x, y) - \frac{1}{\pi i t} F(x, y) \right| \lesssim \frac{\sqrt{w(x)w(y)}}{t \log^2(t)} + \frac{\langle x \rangle^{\frac{3}{2}} \langle y \rangle^{\frac{3}{2}}}{t^{1+\alpha}}.$$

Interpolating this with

$$|e^{itH} P_{ac}(H)(x, y)| \lesssim \frac{1}{t}$$

from [13] which is satisfied when there is a resonance of the first kind at zero and using the inequality, see, e.g., [14]:

$$\min\left(1, \frac{a}{b}\right) \lesssim \frac{\log^2(a)}{\log^2(b)}, \quad a, b > 2,$$

we obtain

$$\left| e^{itH} P_{ac}(H)(x, y) - \frac{1}{\pi i t} F(x, y) \right| \lesssim \frac{w(x)w(y)}{t \log^2(t)}, \quad t > 2.$$

This implies Theorem 1.1.

2.1. The free resolvent and resolvent expansion around zero when there is a resonance of the first kind at zero.

This subsection is devoted to obtain an expansion for the spectral density $[R_V^+(\lambda^2) - R_V^-(\lambda^2)](x, y)$. Recall that in $\mathbb{R}^n$ the integral kernel of the free resolvent is given by Hankel functions, see [32].

For $n = 2$ we have

$$R_0^{\pm}(\lambda^2)(x, y) = \pm \frac{i}{4} H_0^{\pm}(\lambda |x - y|) = \pm \frac{i}{4} [J_0(\lambda |x - y|) \pm i Y_0(\lambda |x - y|)]. \quad (19)$$

Here $J_0(z)$ and $Y_0(z)$ are Bessel functions of the first and second kind of order zero. We use the notation $f = \tilde{O}(g)$ to indicate

$$\frac{d^j}{d\lambda^j} f = O\left(\frac{d^j}{d\lambda^j} g\right), \quad j = 0, 1, 2, \ldots, \quad (20)$$

If (20) is satisfied only for $j = 1, 2, 3, \ldots, k$ we use the notation $f = \tilde{O}_k(g)$.

For $|z| \ll 1$, we have the series expansions for Bessel functions, see, e.g., [1, 13],

$$J_0(z) = 1 - \frac{1}{4} z^2 + \frac{1}{64} z^4 + \tilde{O}_6(z^6), \quad (21)$$

$$Y_0(z) = \frac{2}{\pi} \log(z/2) + \frac{2\gamma}{\pi} + \tilde{O}(z^2 \log(z)). \quad (22)$$

For any $\mathcal{C} \in \{J_0, Y_0\}$ we also have the following representation if $|z| \gtrsim 1$.

$$\mathcal{C}(z) = e^{iz}\omega_+(z) + e^{-iz}\omega_-(z), \qquad \omega_\pm(z) = \widetilde{O}((1+|z|)^{-\frac{1}{2}}). \qquad (23)$$

We prove two lemmas on the behavior of $R_0^\pm(\lambda^2)(x, y)$ for sufficiently small λ.

Lemma 2.3. *Let χ be a smooth cutoff for $[-1, 1]$, and $\widetilde{\chi} = 1 - \chi$. Define $\widetilde{J}_0(z) := \widetilde{\chi}(z)J_0(z)$. Then*

$$|\widetilde{J}_0(\lambda|z|)| \lesssim \lambda^{1/2}|z|^{1/2},$$

$$|\partial_\lambda \widetilde{J}_0(\lambda|z|)| \lesssim \lambda^{-1/2}|z|^{1/2},$$

$$|\partial_\lambda^2 \widetilde{J}_0(\lambda|z|)| \lesssim \lambda^{-1/2}|z|^{3/2}.$$

Similarly, the same bound is satisfied when $\widetilde{J}_0(z)$ is replaced with

$$\widetilde{Y}_0(z) := \widetilde{\chi}(z)Y_0(z) \quad or \quad \widetilde{H}_0(z) := \widetilde{\chi}(z)H_0(z).$$

Proof. Using (23) we have

$$\widetilde{J}_0(\lambda z) = \left|\widetilde{O}\Big(\frac{e^{i\lambda|z|}}{(1+\lambda|z|)^{1/2}}\Big)\right|$$

$$\lesssim |\lambda z|^{0+}$$

$$\lesssim \lambda^{1/2}|z|^{1/2},$$

$$|\partial_\lambda \widetilde{J}_0(\lambda z)| = \widetilde{O}\Big(\frac{ze^{i\lambda z}}{(1+\lambda z)^{1/2}} + \frac{ze^{i\lambda z}}{(1+\lambda z)^{3/2}}\Big)$$

$$\lesssim (\lambda^{-1}|z|)^{\frac{1}{2}}[e^{i\lambda z} + \frac{e^{i\lambda z}}{\lambda z}]$$

$$\lesssim \lambda^{-\frac{1}{2}}|z|^{\frac{1}{2}},$$

$$|\partial_\lambda^2 \widetilde{J}_0(\lambda z)| = \widetilde{O}\Big(\frac{z^2 e^{i\lambda z}}{(1+\lambda z)^{1/2}} + \frac{z^2 e^{i\lambda z}}{(1+\lambda z)^{3/2}} + \frac{z^2 e^{i\lambda z}}{(1+\lambda z)^{5/2}}\Big)$$

$$\lesssim \lambda^{-1/2}|z|^{3/2}. \qquad \square$$

Define

$$G_0 f(x) := (-\Delta)^{-1}(x, y) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \log|x - y|f(y)\,dy \qquad (24)$$

and

$$g^\pm(\lambda) := \|V\|_1\Big(\pm\frac{i}{4} - \frac{1}{2\pi}\log(\lambda/2) - \frac{\gamma}{2\pi}\Big). \qquad (25)$$

The following lemma and its corollary are Lemma 3.1 and Corollary 3.2 in [14].

Lemma 2.4. *The following expansion is valid for the kernel of the free resolvent*

$$R_0^\pm(\lambda^2)(x, y) = \frac{1}{\|V\|_1} g^\pm(\lambda) + G_0(x, y) + E_0^\pm(\lambda)(x, y).$$

$G_0(x, y)$ is the kernel of the operator G_0 in (24)*, and $E_0^\pm$ satisfies the bounds*

$$|E_0^\pm| \lesssim \lambda^{\frac{1}{2}}|x - y|^{\frac{1}{2}}, \qquad |\partial_\lambda E_0^\pm| \lesssim \lambda^{-\frac{1}{2}}|x - y|^{\frac{1}{2}}, \qquad |\partial_\lambda^2 E_0^\pm| \lesssim \lambda^{-\frac{3}{2}}|x - y|^{\frac{3}{2}}.$$

Corollary 2.5. *For $0 < \alpha < 1$ and $b > a > 0$ we have*

$$|\partial_\lambda E_0^\pm(b) - \partial_\lambda E_0^\pm(a)| \lesssim a^{-\frac{1}{2}}|b - a|^\alpha |x - y|^{\frac{1}{2}+\alpha}.$$

Define $U(x)$ as $U(x) = 1$ when $V(x) > 0$ and $U(x) = -1$ when $V(x) \leq 0$, and $v(x) = |V(x)|^{1/2}$. Then using the symmetric resolvent identity for $\Im\lambda > 0$, we have

$$R_V^\pm(\lambda^2) = R_0^\pm(\lambda^2) - R_0^\pm(\lambda^2)vM^\pm(\lambda)^{-1}vR_0^\pm(\lambda^2), \tag{26}$$

where

$$M^\pm(\lambda) = U + vR_0^\pm(\lambda^2)v. \tag{27}$$

Here we derive an expansion for $M^\pm(\lambda)^{-1}$ in a small neighborhood of zero when there is a resonance of the first kind at zero. This derivation is similar to that in [14]. However, we need finer control on the error term.

Let $K: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ with kernel $K(x, y)$. We define the Hilbert–Schmidt norm of K as

$$\|K\|_{\mathrm{HS}} := \sqrt{\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |K(x, y)|^2 dy dx}.$$

Lemma 2.6. *Let $0 < \alpha < 1$. Then*

$$M^\pm(\lambda) = g^\pm(\lambda)P + T + E_1^\pm(\lambda).$$

Here $T = U + vG_0v$ where G_0 is an integral operator defined in (24) *and P is the orthogonal projection onto v. In addition, the error term satisfies the bound*

$$\| \sup_{0<\lambda<\lambda_1} \lambda^{-\frac{1}{2}}|E_1^\pm(\lambda)|\|_{\mathrm{HS}} + \| \sup_{0<\lambda<\lambda_1} \lambda^{\frac{1}{2}}|\partial_\lambda E_1^\pm(\lambda)|\|_{\mathrm{HS}}$$

$$+ \| \sup_{0<\lambda<b<\lambda_1} \lambda^{\frac{1}{2}}(b - \lambda)^{-\alpha}|\partial_\lambda E_1^\pm(b) - \partial_\lambda E_1^\pm(\lambda)|\|_{\mathrm{HS}} \lesssim 1$$

provided that $v(x) \lesssim \langle x \rangle^{-\frac{3}{2}-\alpha-}$.

Proof. Note that

$$E_1^\pm(\lambda) = M^\pm(\lambda) - [g^\pm(\lambda)P + T]$$
$$= vR_0^\pm(\lambda^2)v - g^\pm(\lambda)P - vG_0v$$
$$= vE_0^\pm(\lambda)v.$$

Lemma 2.4 and Corollary 2.5 yield the lemma since $v(x)|x - y|^k v(y)$ is Hilbert–Schmidt on $L^2(\mathbb{R}^2)$ provided that $k > -1$ and $v(x) \lesssim \langle x \rangle^{-k-1-}$. In our case $0 \leq k \leq \frac{1}{2} + \alpha$ and $v(x) \lesssim \langle x \rangle^{-3/2-\alpha-}$. □

The following definitions are from [45] and [32] respectively,

Definition 2.7. We say that an operator $T: L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ with kernel $T(\cdot, \cdot)$ is absolutely bounded if $|T(\cdot, \cdot)|$ is bounded from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$.

Hilbert–Schmidt operators and finite rank operators are absolutely bounded.

Definition 2.8. (1) Let $Q := \mathbf{1} - P$, then zero is defined to be a regular point of the spectrum of $H = -\Delta + V$ if $QTQ = Q(U + vG_0v)Q$ is invertible on $QL^2(\mathbb{R}^2)$.

(2) If zero is not a regular point of spectrum then $QTQ + S_1$ is invertible on $QL^2(\mathbb{R}^2)$ and we define $D_0 = (QTQ + S_1)^{-1}$ as an operator on $QL^2(\mathbb{R}^2)$. Here S_1 is defined as the Riesz projection onto the Kernel of QTQ as an operator on $QL^2(\mathbb{R}^2)$.

(3) We say there is a resonance of the first kind at zero if the operator $T_1 := S_1TPTS_1$ is invertible on $S_1L^2(\mathbb{R}^2)$ and we define D_1 as the inverse of T_1 as an operator on S_1L^2.

Remark 2.9. (1) Throughout this paper we assume that there is a resonance of the first kind at zero. Thus, QTQ is not invertible on QL^2 but $QTQ + S_1$ and $T_1 := S_1TPTS_1$ are invertible on QL^2 and S_1L^2 respectively.

(2) If $|v(x)| \lesssim \langle x \rangle^{-2-}$ then the range of $S_1 - S_2$ (S_2 being the orthogonal projection onto Ker T_1) has dimension at most one, see [32, Theorem 6.2] and [13, Lemma 5.1, Lemma 5.2]. Since in our case $S_2 \equiv 0$, and since zero is not regular, Range S_1 has dimension exactly one. This fact together with the next remark suggests that if there is a resonance of the first kind at zero, then the s-wave resonance is one-dimensional. Also, since Range S_1 has dimension exactly one we write $S_1f = \phi\langle \phi, f \rangle$ for some $\phi \in S_1L^2$ with $\|\phi\|_{L^2} = 1$.

(3) Theorem 6.2 in [32] also states that for $v \lesssim \langle x \rangle^{-1-}$, if $\phi \in S_1 L^2$, then $\phi = \omega \psi$ for an s-wave resonance $\psi \in L^\infty$ such that $H\psi = 0$ in the sense of distributions. Moreover,

$$\psi = c_0 - G_0 v \phi, \tag{28}$$

where

$$c_0 = \frac{1}{\|V\|_{L^1}} \langle v, T\phi \rangle = \frac{1}{\|V\|_1} \int v(x) \, T\phi(x) dx. \tag{29}$$

(4) Denoting $P(x, y)$ the kernel of P we have $P(x, y) = \|V\|_1^{-1} v(x)v(y)$. Hence, in light of second and third remarks above we obtain

$$T_1 = S_1 T P T S_1 = \|V\|_1^{-1} \langle v, T\phi \rangle^2 S_1 = \|V\|_1 c_0^2 S_1$$

and

$$D_1 = T_1^{-1} = \frac{1}{\|V\|_1 c_0^2} S_1.$$

The following lemmas are given without proofs.

Lemma 2.10 ([32, Lemma 2.1]). *Let A be closed operator on a Hilbert space $\mathcal{H}$ and S a projection. Assume $A + S$ has a bounded inverse. Then A has bounded inverse if and only if $B := S - S(A + S)^{-1}S$ has a bounded inverse in $S\mathcal{H}$ and in this case*

$$A^{-1} = (A + S)^{-1} + (A + S)^{-1} S B^{-1} S (A + S)^{-1}.$$

Lemma 2.11 ([13, Lemma 2.5]). *Fix $0 < \alpha < 1$, and assume that $v(x) \lesssim \langle x \rangle^{-3/2-\alpha-}$. Suppose that zero is not a regular point of the spectrum of $-\Delta + V$, and let S_1 be the corresponding Riesz projection. Then for sufficiently small $\lambda_1 > 0$, the operators $M^\pm(\lambda) + S_1$ are invertible for all $0 < \lambda < \lambda_1$ as bounded operators on $L^2(\mathbb{R}^2)$. And one has*

$$(M^\pm(\lambda) + S_1)^{-1} = h_\pm(\lambda)^{-1} S + Q D_0 Q + W_1^\pm(\lambda), \tag{30}$$

Here $h_\pm(\lambda) = g^\pm(\lambda) + c$ where $c \in \mathbb{R}$ and

$$S = \begin{pmatrix} P & -PTQD_0Q \\ -QD_0QTP & QD_0QTPTQD_0Q \end{pmatrix} \tag{31}$$

is a finite-rank operator with real-valued kernel. Furthermore, the error term satisfies the bound

$$\| \sup_{0<\lambda<\lambda_1} \lambda^{-\frac{1}{2}} |W_1 \pm (\lambda)| \|_{\mathrm{HS}} + \| \sup_{0<\lambda<\lambda_1} \lambda^{\frac{1}{2}} |\partial_\lambda W_1 \pm (\lambda)| \|_{\mathrm{HS}}$$

$$+ \| \sup_{0<\lambda<b\lesssim\lambda<\lambda_1} \lambda^{\frac{1}{2}+\alpha} (b-\lambda)^{-\alpha} |\partial_\lambda W_1 \pm (b) - \partial_\lambda W_1 \pm (a)| \|_{\mathrm{HS}} \lesssim 1.$$

Proposition 2.12. *Fix $0 < \alpha < 1$, and assume that $v(x) \lesssim \langle x \rangle^{-3/2-\alpha-}$. If there is a resonance of the first kind at zero, then $B_\pm(\lambda) = S_1 - S_1(M^\pm(\lambda + S_1)^{-1}S_1$ is invertible on $S_1 L^2(\mathbb{R}^2)$. Moreover,*

$$B_\pm^{-1}(\lambda) = -\frac{h_\pm(\lambda)}{c_0^2 \|V\|_1} S_1 + a^\pm(\lambda)S_1,$$

where c_0 is as in Remark 2.9, and

$$\sup_{0<\lambda<\lambda_1} \lambda^{-\frac{1}{2}+} |a\pm(\lambda)| + \sup_{0<\lambda<\lambda_1} \lambda^{\frac{1}{2}-} |a'\pm(\lambda)|$$

$$+ \sup_{0<\lambda<\theta\lesssim\lambda<\lambda_1} \lambda^{\frac{1}{2}+\alpha-} (\theta-\lambda)^{-\alpha} |a'\pm(\theta) - a'\pm(\omega))| \lesssim 1.$$

Proof. We apply Lemma 2.10 to obtain (suppressing '$\pm$' notation)

$$B(\lambda) = S_1 - S_1(h^{-1}(\lambda)S + QD_0Q + W_1(\lambda))S_1$$

$$= -h^{-1}(\lambda)S_1 S S_1 - S_1 W_1(\lambda)S_1$$

$$= h^{-1}(\lambda)S_1 TPT S_1 - S_1 W_1(\lambda)S_1$$

$$= -h^{-1}(\lambda)c_0^2 \|V\|_1 S_1 - S_1 W_1(\lambda)S_1.$$

The second equality follows from the identity

$$QS_1 = S_1 Q = S_1 D_0 = D_0 S_1 = S_1.$$

The third also uses the identity $PS_1 = S_1 P = 0$ and the definition of S. The last equality follows from Remark 2.9 above.

Writing $S_1 W_1(\lambda)S_1 = w(\lambda)S_1$ (where the function w satisfies the error bound of W_1), and noting that by definition of s-wave resonance $c_0 \neq 0$, we obtain $-h^{-1}(\lambda)c_0^2 \|V\|_1 - w(\lambda) \neq 0$ for sufficiently small λ. Therefore

$$B(\lambda)^{-1} = \frac{1}{-h^{-1}(\lambda)c_0^2 \|V\|_1 - w(\lambda)} S_1 = -\frac{h(\lambda)}{c_0^2 \|V\|_1} S_1 + a(\lambda)S_1. \qquad (32)$$

The bounds on $a(\lambda)$ follows from the definition of h and the bounds on w. $\qquad \square$

Using (30) and (32) in Lemma 2.10, we obtain the following expansion for $M^{\pm}(\lambda)^{-1}$.

Corollary 2.13. *Fix $0 < \alpha < 1$, and assume that $v(x) \lesssim \langle x \rangle^{-3/2-\alpha-}$. For all $0 < \lambda < \lambda_1$, we have the following expansion for $M^{\pm}(\lambda)^{-1}$ in case of a resonance of the first kind*

$$M^{\pm}(\lambda)^{-1} = -\frac{h_{\pm}(\lambda)S_1}{c_0^2 \|V\|_1} - \frac{S S_1}{c_0^2 \|V\|_1} - \frac{S_1 S}{c_0^2 \|V\|_1}$$
$$- \frac{S S_1 S}{c_0^2 \|V\|_1 h_{\pm}(\lambda)} + Q D_0 Q + \frac{S}{h_{\pm}(\lambda)} + E(\lambda)(x, y)$$

where $E(\lambda)(x, y)$ is such that

$$\| \sup_{0<\lambda<\lambda_1} \lambda^{-\frac{1}{2}+} |E_{\pm}(\lambda)| \|_{\text{HS}} + \| \sup_{0<\lambda<\lambda_1} \lambda^{\frac{1}{2}+} |\partial_{\lambda} E_{\pm}(\lambda)| \|_{\text{HS}}$$
$$+ \| \sup_{0<\lambda<b<\lambda_1} \lambda^{\frac{1}{2}+\alpha+} (b-\lambda)^{-\alpha} |\partial_{\lambda} E_{\pm}(b) - \partial_{\lambda} E_{\pm}(a)| \|_{\text{HS}} \lesssim 1. \tag{33}$$

Substituting the expansion above for $M^{\pm}(\lambda)^{-1}$ in (26), we obtain the identity

$$R_V^{\pm}(\lambda) = R_0^{\pm}(\lambda^2) + R_0^{\pm}(\lambda^2)v\Big[\frac{h_{\pm}(\lambda)S_1}{c_0^2 \|V\|_1} + \frac{S S_1}{c_0^2 \|V\|_1}$$
$$+ \frac{S_1 S}{c_0^2 \|V\|_1} + \frac{S S_1 S}{c_0^2 \|V\|_1 h_{\pm}(\lambda)}$$
$$- Q D_0 Q - \frac{S}{h_{\pm}(\lambda)} + E^{\pm}(\lambda)\Big] v R_0^{\pm}(\lambda^2). \tag{34}$$

2.2. Proof of the Theorem 2.1. The following proposition takes care of the contribution of the free resolvent in (34) to (9).

Proposition 2.14. [14, Proposition 4.3] *We have*

$$\int_0^{\infty} e^{it\lambda^2} \lambda \chi(\lambda) [R_0^+(\lambda^2) - R_0^-(\lambda^2)](x, y) d\lambda = -\frac{1}{4t} + O\Big(\frac{\langle x \rangle^{\frac{3}{2}} \langle y \rangle^{\frac{3}{2}}}{t^{\frac{5}{4}}}\Big).$$

Below, we obtain similar estimates for each operator included in (34). Simplifying the boundary terms which appear as operators having $\frac{1}{t}$ decay gives us Theorem 2.1.

The following two stationary phase lemmas from [14] will be useful for further calculations.

Lemma 2.15. *For $t > 2$, we have*

$$\left| \int_0^\infty e^{it\lambda^2} \lambda\, \mathcal{E}(\lambda)\, d\lambda - \frac{i\,\mathcal{E}(0)}{2t} \right|$$

$$\lesssim \frac{1}{t} \int_0^{t^{-1/2}} |\mathcal{E}'(\lambda)|\, d\lambda + \left| \frac{\mathcal{E}'(t^{-1/2})}{t^{3/2}} \right| + \frac{1}{t^2} \int_{t^{-1/2}}^\infty \left| \left(\frac{\mathcal{E}'(\lambda)}{\lambda} \right)' \right| d\lambda.$$

Lemma 2.16. *Assume $\mathcal{E}(0) = 0$. For $t > 2$, we have*

$$\left| \int_0^\infty e^{it\lambda^2} \lambda\, \mathcal{E}(\lambda)\, d\lambda \right|$$

$$\lesssim \frac{1}{t} \int_0^\infty \frac{|\mathcal{E}'(\lambda)|}{(1 + \lambda^2 t)}\, d\lambda + \frac{1}{t} \int_{t^{-1/2}}^\infty |\mathcal{E}'(\lambda \sqrt{1 + \pi t^{-1}\lambda^{-2}}) - \mathcal{E}'(\lambda)|\, d\lambda.$$

We start with the contribution of $h_\pm(\lambda) S_1$ from (34) to (9). Recall that

$$h_\pm(\lambda) = g^\pm(\lambda) + c = a_1 \log \lambda + a_2 \pm \frac{\|V\|_1 i}{4},$$

where $c, a_i \in \mathbb{R}$. Using the definition (19) of free resolvent, we write

$$\mathcal{R}_1 := h_+(\lambda) R_0^+(\lambda^2)(x, x_1) R_0^+(\lambda^2)(y_1, y) - h_-(\lambda) R_0^-(\lambda^2)(x, x_1) R_0^-(\lambda^2)(y_1, y)$$

$$= 2i a \log(\lambda)[Y_0(\lambda p) J_0(\lambda q) + J_0(\lambda p) Y_0(\lambda q)]$$

$$+ \frac{\|V\|_1 i}{32}[J_0(\lambda p) J_0(\lambda q) + Y_0(\lambda p) Y_0(\lambda q)],$$

$$(35)$$

where $p = |x - x_1|$ and $q = |y - y_1|$. The following proposition takes care of the contribution of $h_\pm(\lambda) S_1$ in (34) to (9).

Proposition 2.17. *For $t > 2$ and $0 < \alpha < \frac{1}{4}$, if $v(x) \lesssim \langle x \rangle^{-\frac{3}{2} - \alpha -}$, then we have*

$$\left| \int_{\mathbb{R}^4} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \mathcal{R}_1(\lambda, p, q)[v S_1 v](x_1, y_1)\, d\lambda\, dx_1\, dy_1 - \frac{1}{t} F_1(x, y) \right|$$

$$\lesssim \frac{\langle x \rangle^{\frac{1}{2} + \alpha} \langle y \rangle^{\frac{1}{2} + \alpha}}{t^{1 + \alpha}},$$

where

$$F_1(x, y) = -\frac{\|V\|_1}{16\pi^2} \int_{\mathbb{R}^4} \log |x - x_1| v(x_1) S_1(x_1, y_1) v(y_1) \log |y - y_1|\, dx_1\, dy_1$$

$$= -\frac{\|V\|_1}{4} [G_0 v S_1 v G_0](x, y).$$

We prove this proposition in a series of lemmas.

Lemma 2.18. *Let* $0 < \alpha < 1/4$, $v(x) \lesssim \langle x \rangle^{-3/2-\alpha-}$. *For* $t > 2$ *we have*

$$\left| \int_{\mathbb{R}^4} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \log(\lambda) Y_0(\lambda p) [v S_1 v](x_1, y_1) J_0(\lambda q) d\lambda dx_1 dy_1 \right|$$
$$\lesssim \frac{\langle x \rangle^{\frac{1}{2}+\alpha+} \langle y \rangle^{\frac{1}{2}+\alpha+}}{t^{1+\alpha}}, \tag{36}$$

$$\left| \int_{\mathbb{R}^4} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \log(\lambda) J_0(\lambda p) [v S_1 v](x_1, y_1) Y_0(\lambda q) d\lambda dx_1 dy_1 \right|$$
$$\lesssim \frac{\langle x \rangle^{\frac{1}{2}+\alpha+} \langle y \rangle^{\frac{1}{2}+\alpha+}}{t^{1+\alpha}}, \tag{37}$$

$$\left| \int_{\mathbb{R}^4} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) J_0(\lambda p) [v S_1 v](x_1, y_1) J_0(\lambda q) d\lambda dx_1 dy_1 \right|$$
$$\lesssim \frac{\langle x \rangle^{\frac{1}{2}+\alpha+} \langle y \rangle^{\frac{1}{2}+\alpha+}}{t^{1+\alpha}}. \tag{38}$$

To prove Lemma 2.18 we need the following lemma from [45] and [14]. The bounds on G, F, and their first derivatives are in [14, Lemma 3.3]. The claim on the second derivatives follows similarly.

Lemma 2.19. *Let* $p = |x - x_1|$, $r = |x| + 1$, *and*

$$F(\lambda, x, x_1) := \chi(\lambda p) Y_0(\lambda p) - \chi(\lambda r) Y_0(\lambda r),$$

$$G(\lambda, x, x_1) := \chi(\lambda p) J_0(\lambda p) - \chi(\lambda r) J_0(\lambda r).$$

Then for $\lambda \leq \lambda_1$ *and* $0 \leq \tau \leq 1$, *we have*

$$|G(\lambda, x, x_1)| \lesssim \lambda^\tau \langle x_1 \rangle^\tau, \tag{39a}$$

$$|\partial_\lambda G(\lambda, x, x_1)| \lesssim \lambda^{\tau-1} \langle x_1 \rangle^\tau, \tag{39b}$$

$$|\partial_\lambda^2 G(\lambda, x, x_1)| \lesssim \lambda^{-1} \langle x_1 \rangle. \tag{39c}$$

and

$$|F(\lambda, x, x_1)| \lesssim \int_0^{2\lambda_1} |F(0+, x, x_1)| + |\partial_\lambda F(\lambda, x, x_1)| d\lambda \lesssim k(x, x_1), \tag{40a}$$

$$|\partial_\lambda F(\lambda, x, x_1)| \lesssim \lambda^{-1}, \quad |\partial_\lambda^2 F(\lambda, x, x_1)| \lesssim \lambda^{-2+}. \tag{40b}$$

Here $k(x, x_1) := 1 + \log^-(|x - x_1|) + \log^+ |x_1|$, $\log^-(x) = -\log(x)\chi_{(0,1)}(x)$, *and* $\log^+(x) = \log(x)\chi_{(1,\infty)}(x)$.

Proof of Lemma 2.18. We only prove the assertion (36), the second and third assertions are analogous.

Recall that we introduced two expansions to $J_0(\lambda p)$ (or $Y_0(\lambda p)$); they are when $\lambda p \lesssim 1$ and when $\lambda p \gtrsim 1$. In order to use these expansions in the proper context we need to introduce $\chi(\lambda p)$ and $\chi(\lambda q)$ where χ is the same cutoff function defined in the introduction. The phrasing low and high energy has referred only to spectral variable λ until now. However, in the following analysis it refers to λp and λq.

We divide the proof into three cases.

Case 1: $\lambda p \lesssim 1$ and $\lambda q \lesssim 1$.   For low-low energy, we consider

$$\int_{\mathbb{R}^4} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \log(\lambda) Y_0(\lambda p) \chi(\lambda p)[v S_1 v](x_1, y_1) \chi(\lambda q) J_0(\lambda q) d\lambda dx_1 dy_1. \tag{41}$$

Note that by definition $S_1 \le Q$ and $Qv = 0$. Hence, for any $f \in L^2(\mathbb{R}^2)$

$$\int_{\mathbb{R}^4} f(x_1)[S_1](x_1, y_1) v(y_1) dx_1 dy_1 = \int_{\mathbb{R}^4} v(x_1)[S_1](x_1, y_1) f(y_1) dx_1 dy_1 = 0 \tag{42}$$

is satisfied. Using this fact, we can replace $Y_0(\lambda p)\chi(\lambda p)$ with $F(\lambda, x, x_1)$; and $J_0(\lambda q)\chi(\lambda q)$ with $G(\lambda, y, y_1)$ in (41). Thus, we need a bound for

$$\int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \log(\lambda) F(\lambda, x, x_1) G(\lambda, y, y_1) d\lambda. \tag{43}$$

Letting $\mathcal{E}(\lambda) = \chi(\lambda) \log(\lambda) F(\lambda, x, x_1) G(\lambda, y, y_1)$, we see that $\mathcal{E}(0) = 0$. Then taking $\tau = \frac{1}{2}$ in Lemma 2.19, we obtain

$$|\partial_\lambda \mathcal{E}(\lambda)| \lesssim \chi(\lambda) \lambda^{-\frac{1}{2}-} k(x, x_1) \langle y_1 \rangle^{\frac{1}{2}}, \tag{44}$$

$$|\partial_\lambda^2 \mathcal{E}(\lambda)| \lesssim \lambda^{-\frac{3}{2}-} k(x, x_1) \langle y_1 \rangle. \tag{45}$$

Using (45) and the Mean Value Theorem, we have for $a > \lambda$

$$|\partial_\lambda \mathcal{E}(a) - \partial_\lambda \mathcal{E}(\lambda)| \lesssim |a - \lambda| \lambda^{-\frac{3}{2}-} k(x, x_1) \langle y_1 \rangle,$$

whose interpolation with (44) gives us

$$|\partial_\lambda \mathcal{E}(a) - \partial_\lambda \mathcal{E}(\lambda)| \lesssim |a - \lambda|^\alpha \lambda^{-\frac{1}{2}-\alpha-} k(x, x_1) \langle y \rangle^{\frac{1+\alpha}{2}}. \tag{46}$$

Recalling that $\mathcal{E}(0) = 0$, we use Lemma 2.16 and obtain

$$|(43)| \lesssim \frac{1}{t} \int_0^\infty \frac{|\mathcal{E}'(\lambda)|}{1 + \lambda^2 t} d\lambda + \frac{1}{t} \int_{t^{-1/2}}^\infty |\mathcal{E}'(\lambda \sqrt{1 + \pi t^{-1} \lambda^{-2}}) - \mathcal{E}'(\lambda)| d\lambda.$$

Using (44), we can estimate the first integral as

$$\frac{\langle y_1 \rangle^{\frac{1}{2}} k(x, x_1)}{t} \int_0^\infty \frac{\lambda^{-\frac{1}{2}-}}{1 + t\lambda^2} d\lambda \lesssim \frac{\langle y_1 \rangle^{\frac{1}{2}} k(x, x_1)}{t^{\frac{5}{4}-}}.$$

To estimate the second integral we have,

$$\lambda(\sqrt{1 + \pi t^{-1} \lambda^{-2}} - 1) \sim \frac{1}{t\lambda}.$$

And that gives

$$\frac{k(x, x_1) \langle y \rangle^{\frac{1+\alpha}{2}}}{t^{1+\alpha}} \int_{t^{-1/2}}^{\lambda_1} \lambda^{-\frac{1}{2}-2\alpha-} d\lambda \lesssim \frac{k(x, x_1) \langle y_1 \rangle^{\frac{1}{2}+\alpha} \langle y \rangle^{\frac{1}{2}+\alpha}}{t^{1+\alpha}}$$

since $0 < \alpha < \frac{1}{4}$.

Case 2: $\lambda p \lesssim 1$ and $\lambda q \gtrsim 1$. The case $\lambda p \gtrsim 1$ and $\lambda q \lesssim 1$ is similar. Note that Lemma 2.19 is valid for the low energy. Therefore, we can not use (42) to exchange $J_0(\lambda q) \widetilde{\chi}(\lambda q)$ with $G(\lambda, y, y_1)$. Instead, we use the large energy expansion (23) of $J_0(\lambda q)$ and consider the following integral

$$\int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \log(\lambda) F(\lambda, x, x_1) \widetilde{J}_0(\lambda q) d\lambda. \tag{47}$$

Let $\mathcal{E}(\lambda) = \chi(\lambda) \log(\lambda) F(\lambda, x, x_1) \widetilde{J}_0(\lambda q)$. Using the bounds in Lemma 2.19 and Lemma 2.3, we have the estimates

$$|\partial_\lambda \mathcal{E}(\lambda)| \lesssim \chi(\lambda) \lambda^{-\frac{1}{2}-} \langle y \rangle^{\frac{1}{2}} \langle y_1 \rangle^{\frac{1}{2}} k(x, x_1), \tag{48}$$

$$|\partial_\lambda^2 \mathcal{E}(\lambda)| \lesssim \lambda^{-\frac{3}{2}-} k(x, x_1) \langle y_1 \rangle^{\frac{3}{2}} \langle y \rangle^{\frac{3}{2}}. \tag{49}$$

Using the same interpolation argument in Case 1, for $a > \lambda$ we obtain

$$|\partial_\lambda \mathcal{E}(a) - \partial_\lambda \mathcal{E}(\lambda)| \lesssim |a - \lambda|^\alpha \lambda^{-\frac{1}{2}-\alpha-} k(x, x_1) \langle y_1 \rangle^{\frac{1}{2}+\alpha} \langle y \rangle^{\frac{1}{2}+\alpha}. \tag{50}$$

Noting $\mathcal{E}(0) = 0$, one can use (48) and (50) in Lemma 2.16 and obtain

$$|(47)| \lesssim \frac{k(x, x_1) \langle y_1 \rangle^{\frac{1}{2}+\alpha} \langle y \rangle^{\frac{1}{2}+\alpha}}{t^{1+\alpha}}.$$

Case 3: $\lambda p \gtrsim 1$ and $\lambda q \gtrsim 1$. In this case we need to use the large energy expansion for both $Y_0(\lambda p)$ and $J_0(\lambda q)$, see Lemma 2.3. Therefore, we consider the following integral

$$\int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \log(\lambda) \widetilde{Y}_0(\lambda p) \widetilde{J}_0(\lambda q) d\lambda. \tag{51}$$

Note that (51) has slightly faster decay than (47) in terms of λ. Also the largest contribution to the weight function comes when both derivatives act on either $\widetilde{J}_0$ or $\widetilde{Y}_0$ as $\langle \cdot \rangle^{\frac{3}{2}}$. One can reduce this weight to $\langle \cdot \rangle^{\frac{1}{2}+\alpha}$ using the argument that leads to (46) above and obtain

$$|\partial_\lambda \mathcal{E}(a) - \partial_\lambda \mathcal{E}(\lambda)| \lesssim |a - \lambda|^\alpha \lambda^{-\alpha} \log \lambda \langle x_1 \rangle^{\frac{1}{2}+\alpha} \langle y \rangle^{\frac{1}{2}+\alpha} \langle y_1 \rangle^{\frac{1}{2}+\alpha} \tag{52}$$

for $\mathcal{E}(\lambda) = \chi(\lambda) \log(\lambda) \widetilde{Y}_0(\lambda p) \widetilde{J}_0(\lambda q)$. Using (52) in Lemma 2.16, we obtain $|(51)| \lesssim t^{-1-\alpha} \langle y \rangle^{\frac{1}{2}+\alpha} \langle x \rangle^{\frac{1}{2}+\alpha}$.

Hence, combining all four cases we see that

$$\left| \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \log(\lambda) Y_0(\lambda p) J_0(\lambda q) d\lambda \right|$$

$$\lesssim \frac{k(x, x_1) \langle x_1 \rangle^{\frac{1}{2}+\alpha} \langle x \rangle^{\frac{1}{2}+\alpha} \langle y \rangle^{\frac{1}{2}+\alpha} \langle y_1 \rangle^{\frac{1}{2}+\alpha}}{t^{1+\alpha}}$$

for $\alpha \in (0, 1/4)$. That yields

$$|(36)| \lesssim \frac{\langle x \rangle^{\frac{1}{2}+\alpha+} \langle y \rangle^{\frac{1}{2}+\alpha+}}{t^{1+\alpha}}$$

$$\int_{\mathbb{R}^4} k(x, x_1) \langle x_1 \rangle^{\frac{1}{2}+\alpha} v(x_1) [S_1](x_1, y_1) v(y_1) \langle y_1 \rangle dx_1 dy_1$$

$$\lesssim \frac{\langle x \rangle^{\frac{1}{2}+\alpha+} \langle y \rangle^{\frac{1}{2}+\alpha+}}{t^{1+\alpha}}$$

$$\|k(x, x_1) \langle x_1 \rangle^{\frac{1}{2}+\alpha} v(x_1)\|_{L^2_{x_1}} \|\|S_1\|\|_{L^2 \to L^2} \|\langle y_1 \rangle^{\frac{1}{2}+\alpha} v(y_1)\|_{L^2_{y_1}}$$

$$\lesssim \frac{\langle x \rangle^{\frac{1}{2}+\alpha+} \langle y \rangle^{\frac{1}{2}+\alpha+}}{t^{1+\alpha}}.$$

The last inequality follows from the the assumption $v(x) \lesssim \langle x \rangle^{-\frac{3}{2}-\alpha-}$, which implies $\|k(x, x_1) \langle x_1 \rangle^{\frac{1}{2}+\alpha} v(x_1)\|_{L^2_{x_1}} \lesssim 1$. We also used the fact that S_1 is absolutely bounded since it is of finite rank. $\qquad \square$

Lemma 2.20. *Let* $K(\lambda, y, y_1) = \chi(\lambda|y - y_1|) - \chi(\lambda(|y| + 1))$. *Then for any* $0 \le \tau \le 1$, *we have*

$$|K(\lambda, y, y_1)| \lesssim \lambda^\tau \langle y_1 \rangle^\tau,$$

$$|\partial_\lambda K(\lambda, y, y_1)| \lesssim \lambda^{1-\tau} \langle y_1 \rangle^\tau,$$

$$|\partial_\lambda^2 K(\lambda, y, y_1)| \lesssim \lambda^{-1} \langle y_1 \rangle.$$

Proof. Noting that $\chi \in C^\infty$, for the first inequality we use the mean value theorem to conclude

$$
\begin{aligned}
|K(\lambda, y, y_1)| &= |\chi(\lambda q) - \chi(\lambda(|y| + 1))| \\
&\le \lambda \langle y_1 \rangle \max_x |\chi'(x)| \\
&\lesssim \min(1, \lambda \langle y_1 \rangle) \\
&\lesssim \lambda^\tau \langle y_1 \rangle^\tau.
\end{aligned}
$$

For the second inequality note that $\partial_\lambda \chi(\lambda q) = q \chi'(\lambda q)$. For the third equality, we also used that $|K(\lambda, y, y_1)| \lesssim 1$.

Using the fact that $\chi \in C^\infty$, we obtain

$$
\begin{aligned}
|\partial_\lambda K(\lambda, y, y_1)| &= \left| \frac{\lambda q \chi'(\lambda q) - \lambda(|y| + 1)\chi'(\lambda(|y| + 1))}{\lambda} \right| \\
&\lesssim \frac{1}{\lambda} \min(1, \lambda \langle y_1 \rangle) \\
&\lesssim \lambda^{1-\tau} \langle y_1 \rangle^\tau.
\end{aligned}
$$

Finally for the third inequality, note that $\chi''(\lambda q)$ is supported when $\lambda \sim \frac{1}{q}$. Using this and the second derivative of the cut-off functions in terms of λ, we have

$$
\begin{aligned}
|\partial_\lambda^2 K(\lambda, y, y_1)| &\le |q^2 \chi''(\lambda q) - (|y| + 1)^2 \chi''(\lambda(|y| + 1))| \\
&\lesssim \lambda^{-1}|q - (|y| + 1)|. \qquad \square
\end{aligned}
$$

Lemma 2.21. *Under the same conditions of Proposition* 2.17, *we have*

$$
\begin{aligned}
\int_{\mathbb{R}^4} \int_0^\infty & e^{it\lambda^2} \lambda \chi(\lambda) Y_0(\lambda p) v S_1 v Y_0(\lambda q) d\lambda dx_1 dy_1 \\
&= 32i \int_{\mathbb{R}^4} G_0(x, x_1)[v S_1 v](x_1, y_1) G_0(y_1, y) dx_1 dy_1 \\
&\quad + \tilde{O}(t^{-1-\alpha} \langle x \rangle^{\frac{1}{2}+\alpha} \langle y \rangle^{\frac{1}{2}+\alpha}).
\end{aligned}
$$
(53)

Proof. The proof is very similar to the proof of Lemma 2.18 except in the case when $\lambda p, \lambda q \lesssim 1$. This is because the identity (42) leads to an integral with operators $F(\lambda, x, x_1) F(\lambda, y, y_1)$, which doesn't give better decay rate than $1/t$. We have to be more careful obtaining the term behaving like $1/t$ explicitly.

Using the expansion (22) of $Y_0(z)$ for small energy, we obtain

$$Y_0(\lambda p) Y_0(\lambda q)$$
$$= \frac{4}{\pi^2} \log|x - x_1| \log|y - y_1| + A(\lambda, p, q) + E_1(\lambda, p, q) + E_2(\lambda, p, q),$$

where

$$A(\lambda, p, q) := c_1 \log(\lambda)[\log(\lambda p) + \log(\lambda q)] + c_2[\log(\lambda p) + \log(\lambda q)] + c_3,$$

where $c_j \in \mathbb{R} - \{0\}$ and

$$E_1(\lambda, p, q) := \tilde{O}(\log(\lambda p)(\lambda q)^2 \log(\lambda q)),$$

$$E_2(\lambda, p, q) := (\log(\lambda p)(\lambda p)^2 \log(\lambda q)).$$

To handle the terms in the operator $A(\lambda, p, q)$, we need Lemma 2.20. Consider only the first term in $A(\lambda, p, q)$ then, we have

$$\int_{\mathbb{R}^4} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \log \lambda \log(\lambda p) \chi(\lambda p)[v S_1 v](x_1, y_1) \chi(\lambda q) d\lambda dx_1 dy_1. \quad (54)$$

Using (42), we can subtract $\log(\lambda(|x|+1)) \chi(\lambda(|x|+1))$ from the left side of $v(x_1)$ and $\chi(\lambda(|y|+1))$ from the right side of $v(y_1)$. Then, it becomes enough to bound

$$\int_{\mathbb{R}^4} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \log \lambda h(\lambda, x, x_1)[v S_1 v](x_1, y_1) K(\lambda, y, y_1) d\lambda dx_1 dy_1 \quad (55)$$

where $h(\lambda, x, x_1) := \log(\lambda p) \chi(\lambda p) - \log(\lambda(|x| + 1)) \chi(\lambda(|x| + 1))$. However, estimating this integral is equal to estimating the integral (41) from Lemma 2.18. This is because the term $h(\lambda, x, x_1)$ satisfies the bounds in (40) from Lemma 2.19 and the bounds in Lemma 2.20 are similar to the bounds in (39). Hence, $|(54)| \lesssim t^{-1-\alpha} k(x, x_1) \langle y_1 \rangle^{\frac{1}{2}+\alpha} \langle y \rangle^{\frac{1}{2}+\alpha}$.

For the error term $E_1(\lambda, p, q)$, note that using the projection property of S_1 we can subtract $\chi(\lambda(|x| + 1)) \log(\lambda(|x| + 1))$ from the left side of the operator $v S_1 v$ and replace $\log(\lambda p)$ with $h(\lambda, x, x_1)$. Then, using $\lambda q \lesssim 1$, we have

$$|\partial_\lambda[(\lambda q)^2 \log(\lambda q)]| \lesssim q(\lambda q)^{1-} \lesssim \lambda^{-\frac{1}{2}} \langle y \rangle^{\frac{1}{2}} \langle y_1 \rangle^{\frac{1}{2}},$$

$$\left| \partial_\lambda \left(\frac{\partial_\lambda[(\lambda q)^2 \log(\lambda q)]}{\lambda} \right) \right| \lesssim \frac{q^2}{\lambda} \lesssim \lambda^{-\frac{5}{2}-} \langle y \rangle^{\frac{1}{2}} \langle y_1 \rangle^{\frac{1}{2}}.$$

The bound $t^{-\frac{5}{4}}k(x, x_1)\langle y \rangle^{\frac{1}{2}}\langle y_1 \rangle^{\frac{1}{2}}$ follows by Lemma 2.15. Similarly, the error $E_2(\lambda, p, q)$ can be bounded by $t^{-\frac{5}{4}}k(y, y_1)\langle x \rangle^{\frac{1}{2}}\langle x_1 \rangle^{\frac{1}{2}}$.

Finally, we consider the integral

$$\frac{4}{\pi^2} \int_{\mathbb{R}^4} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \log |x - x_1| \qquad\qquad\qquad (56)$$
$$\chi(\lambda p)[vS_1v](x_1, y_1)\chi(\lambda q) \log |y - y_1| d\lambda dx_1 dy_1.$$

Applying integration by parts once, the λ integral of (56) is equal to

$$-\frac{2}{i\pi^2 t} - \frac{1}{\pi^2 t^2} \int_0^\infty e^{it\lambda^2} \frac{d}{d\lambda}(\chi(\lambda)\chi(\lambda p)\chi(\lambda q))d\lambda$$
$$= -\frac{2}{i\pi^2 t} + O(t^{-\frac{5}{4}}(\langle x \rangle\langle x_1 \rangle\langle y \rangle\langle y_1 \rangle)^{\frac{1}{2}}). \qquad (57)$$

For the second inequality note that all the cut-off functions are infinitely differentiable. However two integration by parts would yield too large of a spatial weight. An easy calculation gives $\left|\frac{\partial}{\partial\lambda}(\chi(\lambda)\chi(\lambda p)\chi(\lambda q))\right| \lesssim \lambda^{-\frac{1}{2}}(\langle x \rangle\langle x_1 \rangle\langle y \rangle\langle y_1 \rangle)^{\frac{1}{2}}$. And for $\partial_\lambda\left(\frac{\partial_\lambda(\chi(\lambda)\chi(\lambda p)\chi(\lambda q))}{\lambda}\right)$ the most delicate term comes when all the derivatives fall on either $\chi(\lambda p)$ or $\chi(\lambda q)$. But since $\chi^k(\lambda p)$ for $k \geq 1$ is supported when $p \sim \frac{1}{\lambda}$ we have

$$\left|\frac{\chi(\lambda)\chi''(\lambda p)p^2\chi(\lambda q)}{\lambda}\right| \lesssim \lambda^{-\frac{5}{2}}\langle x \rangle^{\frac{1}{2}}\langle x_1 \rangle^{\frac{1}{2}}$$

and that applying Lemma 2.16 yields (57).

The final result is therefore obtained as

$$(53) = -\frac{2}{\pi^2 it} \int_{\mathbb{R}^4} \log |x - x_1|[vS_1v](x_1, y_1) \log |y - y_1| dy_1 dx_1$$
$$+ O\left(\frac{\langle y \rangle^{\frac{1}{2}+\alpha}\langle x \rangle^{\frac{1}{2}+\alpha}}{t^{1+\alpha}} \int_{\mathbb{R}^4} k(x, x_1)\langle x_1 \rangle^{\frac{1}{2}+\alpha}[vS_1v](x_1, y_1)\right.$$
$$\left. k(y, y_1)\langle y_1 \rangle^{\frac{1}{2}+\alpha} dx_1 dy_1\right),$$

which finishes the proof of Lemma 2.1. $\qquad\qquad\qquad\qquad\qquad\qquad \square$

Multiplying the boundary term with $\frac{\|V\|_1 i}{32}$ gives F_1 in Proposition 2.17. We next consider the contribution of QD_0Q, SS_1, and S_1S, from (34) to (9). Let

$$\mathcal{R}_2(\lambda, p, q) := R_0^+(\lambda^2)(x, x_1)R_0^+(\lambda^2)(y_1, y) - R_0^-(\lambda^2)(x, x_1)R_0^-(\lambda^2)(y_1, y)$$
$$= -\frac{i}{8}[J_0(\lambda p)Y_0(\lambda q) + Y_0(\lambda p)J_0(\lambda q)]. \qquad (58)$$

Note that using this expansion and the projection property of Q the contribution of QD_0Q can be handled as in Proposition 2.17. In fact, since (58) does not contain the term $Y_0(\lambda p)Y_0(\lambda q)$, and since $G(0, x, x_1) = 0$, in the application of integration by parts the boundary term at $\lambda = 0$ is obtained as zero.

Proposition 2.22. *For $t > 2$ and $0 < \alpha < \frac{1}{4}$ if $v(x) \lesssim \langle x \rangle^{-\frac{3}{2}-\alpha-}$, then we have*

$$\left| \int_{\mathbb{R}^4} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \mathcal{R}_2(\lambda, p, q)[vSS_1v](x_1, y_1) d\lambda dx_1 dy_1 - \frac{1}{t} F_2(x, y) \right|$$

$$\lesssim \frac{\langle x \rangle^{\frac{1}{2}+\alpha} \langle y \rangle^{\frac{1}{2}+\alpha}}{t^{1+\alpha}},$$

$$\left| \int_{\mathbb{R}^4} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \mathcal{R}_2(\lambda, p, q)[vS_1Sv](x_1, y_1) d\lambda dx_1 dy_1 - \frac{1}{t} F_3(x, y) \right|$$

$$\lesssim \frac{\langle x \rangle^{\frac{1}{2}+\alpha} \langle y \rangle^{\frac{1}{2}+\alpha}}{t^{1+\alpha}},$$

where

$$F_2(x, y) = \frac{1}{4} \langle v, SS_1vG_0(\cdot, y) \rangle,$$

$$F_3(x, y) = \frac{1}{4} \langle v, SS_1vG_0(\cdot, x) \rangle.$$

Proof. We consider the first assertion. By (58) we have the following two integrals:

$$\int_{\mathbb{R}^4} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) Y_0(\lambda p) v(x_1)[SS_1](x_1, y_1) v(y_1) J_0(\lambda q) d\lambda dx_1 dy_1, \quad (59)$$

$$\int_{\mathbb{R}^4} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) J_0(\lambda p) v(x_1)[SS_1](x_1, y_1) v(y_1) Y_0(\lambda q) d\lambda dx_1 dy_1. \quad (60)$$

Here the only caveat is that we have S_1 only on the right. This allows us to perform addition and subtraction of $J_0(\lambda(|y| + 1))$ and $Y_0(\lambda(|y| + 1))$ only on the right side of SS_1. Hence, the proofs for high-low and high-high energy are not affected by this caveat. When $\lambda p \lesssim 1, \lambda q \gtrsim 1$ we have the following two integrals for (59) and (60) respectively

$$\int_{\mathbb{R}^4} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda)[1 + \tilde{O}(\log(\lambda p))] \chi(\lambda p) vSS_1v \tilde{J}_0(\lambda q) d\lambda dx_1 dy_1, \quad (61)$$

$$\int_{\mathbb{R}^4} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda)[1 + \tilde{O}((\lambda p)^2)] \chi(\lambda p) vSS_1v \tilde{Y}_0(\lambda q) d\lambda dx_1 dy_1. \quad (62)$$

Letting $\mathcal{E}(\lambda, p, q) = [1 + \tilde{O}(\log(\lambda p))]\chi(\lambda p)\tilde{J}_0(\lambda q)$ we have $\mathcal{E}(0) = 0$. Using Lemma 2.3 and the fact that $(\lambda p) \lesssim 1$, we obtain

$$|\partial_\lambda \mathcal{E}(\lambda, p, q)| \lesssim \lambda^{-\frac{1}{2}-}k(x, x_1)\langle y\rangle^{\frac{1}{2}}\langle y_1\rangle^{\frac{1}{2}},$$

$$|\partial_\lambda^2 \mathcal{E}(\lambda, p, q))| \lesssim \lambda^{-\frac{3}{2}-}k(x, x_1)\langle x_1\rangle^{\frac{3}{2}}\langle y\rangle^{\frac{3}{2}}\langle y_1\rangle^{\frac{3}{2}}.$$

Hence, by interpolation for $b > \lambda$ we have

$$|\partial_\lambda \mathcal{E}(b) - \partial_\lambda \mathcal{E}(\lambda)| \lesssim |b - \lambda|^\alpha \lambda^{-\frac{1}{2}-\alpha-}k(x, x_1)\langle x_1\rangle^{\frac{1}{2}+\alpha}\langle y_1\rangle^{\frac{1}{2}+\alpha}\langle y\rangle^{\frac{1}{2}+\alpha},$$

which gives

$$(61) = O\left(\frac{\langle x\rangle^{\frac{1}{2}+\alpha}\langle y\rangle^{\frac{1}{2}+\alpha}}{t^{1+\alpha-}}\right)$$

using Lemma 2.16 for $\alpha < 1/4$. With a similar argument one can show that (62) satisfies the same decay assumption with the same weight function.

For the low-low case first note that S_1 being only on the right side of the operator allows us to exchange $J_0(\lambda q)$ with $G(\lambda, y, y_1)$ in (59), and $Y_0(\lambda q)$ with $F(\lambda, y, y_1)$ in (60). The decay rate of $G(\lambda, y, y_1)$ cancels out the singularity of $\log \lambda$, which is the dominated term in the expansion (22) of Y_0. Therefore, we don't obtain any boundary term from (59) and can bound it by $\frac{1}{t^{1+\alpha}}$ with the weight $k(x, x_1)\langle y\rangle\langle x\rangle$. However, this is not the case for (60). The following lemma evaluates the contribution of this term. $\qquad\square$

Lemma 2.23. *Under the same conditions of Proposition* 2.22, *for* $\lambda p, \lambda q \lesssim 1$ *we have*

$$\left|(60) - \frac{2}{it}\int_{\mathbb{R}^4} v(x_1)[SS_1](x_1, y_1)v(y_1)G_0(y, y_1)dx_1 dy_1\right|$$
$$\lesssim \frac{\langle x\rangle^{\frac{1}{2}+\alpha+}\langle y\rangle^{\frac{1}{2}+\alpha+}}{t^{!+\alpha}}. \tag{63}$$

Proof. Note that multiplying the boundary term with $-\frac{i}{8}$ gives the the statement of Proposition 2.22.

Using the expansions (21) and (22) for $J_0(\lambda p)$ and $Y_0(\lambda q)$ respectively, we have

$$J_0(\lambda p)Y_0(\lambda q) = [1 + \tilde{O}((\lambda p)^2)][-4G_0(y_1, y) + c(1 + \log \lambda) + \tilde{O}((\lambda q)^{2-}))]$$
$$= -4\,G_0(y_1, y) + c(1 + \log \lambda) + \tilde{O}((\lambda q)^2 \log(\lambda q))$$
$$+ O_2((\lambda p)^2)Y_0(\lambda q).$$

Using this expansion in (60), we obtain

$$\left| (60) + 4 \int_{\mathbb{R}^4} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \chi(\lambda p)) [vSS_1v])(x_1, y_1) \chi(\lambda q) G_0(y_1, y) d\lambda dx_1 dy_1 \right|$$

$$\lesssim \left| \int_{\mathbb{R}^4} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \chi(\lambda p) [vSS_1v](x_1, y_1) \chi(\lambda q) [1 + \log \lambda] d\lambda dx_1 dy_1 \right|$$

$$+ \left| \int_{\mathbb{R}^4} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \chi(\lambda p) [vSS_1v](x_1, y_1) \chi(\lambda q) (\lambda q)^2 \log(\lambda q) d\lambda dx_1 dy_1 \right|$$

$$+ \left| \int_{\mathbb{R}^4} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \chi(\lambda p) (\lambda p)^2 [vSS_1v](x_1, y_1) F(\lambda, y, y_1) d\lambda dx_1 dy_1 \right|.$$

Note that using the property (42), we could exchange $Y_0(\lambda p)$ with $F(\lambda, y, y_1)$ in the last integral.

The first integral is similar to (57). We therefore have

$$\int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \chi(\lambda p) \chi(\lambda q) G_0(y, y_1) d\lambda$$

$$= -\frac{2}{it} G_0(y_1, y) + O(t^{-\frac{5}{4}} (\langle x \rangle \langle x_1 \rangle \langle y \rangle \langle y_1 \rangle)^{\frac{1}{2}} k(y, y_1)).$$

The contribution of the first integral follows as $A(\lambda, p, q)$ in Lemma 2.21 and it can be bounded by $t^{-1-\alpha} \langle x_1 \rangle^{\frac{1}{2}+\alpha} \langle y \rangle \langle y_1 \rangle^{\frac{1}{2}+\alpha}$. Using Lemma 2.15, the other two integrals give the same bound that $E_1(\lambda, p, q)$ in Lemma 2.21 gives. The weights coming from the second derivative of the cut-off functions can be reduced as required using the support of $\chi'(\lambda p)$ and $\chi'(\lambda q)$. Hence, we obtain the inequality (63). □

For the terms arising from $h_\pm(\lambda)^{-1} SS_1 S$ and $h_\pm(\lambda)^{-1} S$, which are the integrals

$$\int_{\mathbb{R}^4} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \mathcal{R}_3(\lambda, p, q) vSS_1 Sv(x_1, y_1) d\lambda dx_1 dy_1 \qquad (64)$$

and

$$\int_{\mathbb{R}^4} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \mathcal{R}_3(\lambda, p, q)] vSv(x_1, y_1) d\lambda dx_1 dy_1, \qquad (65)$$

where

$$\mathcal{R}_3^\pm = \frac{R_0^+(\lambda^2)(x, x_1) R_0^+(\lambda^2)(y, y_1)}{h_+(\lambda)} - \frac{R_0^+(\lambda^2)(x, x_1) R_0^+(\lambda^2)(y, y_1)}{h_-(\lambda)}$$

we have the following Proposition, which is the generalized version of Proposition 4.4 in [14].

Proposition 2.24. *Let* $0 < \alpha < 1/4$, $v(x) \lesssim \langle x \rangle^{-3/2-\alpha-}$. *For any absolutely bounded operator* Γ, *we have*

$$\int_{\mathbb{R}^4} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \mathcal{R}_3(\lambda, p, q) v(x_1) \Gamma(x_1, y_1) v(y_1) d\lambda dx_1 dy_1$$

$$= -\frac{1}{4 \|V\|_1 t} \int_{\mathbb{R}^4} v(x_1) \Gamma(x_1, y_1) v(x_1) dx_1 dy_1$$

$$+ O\left(\frac{\sqrt{w(x)w(y)}}{t \log^2(t)}\right) + O\left(\frac{\langle x \rangle^{\frac{1}{2}+\alpha+} \langle y \rangle^{\frac{1}{2}+\alpha+}}{t^{1+\alpha}}\right).$$

Corollary 2.25. *Under the same conditions, we have*

$$\left|(64) - \frac{1}{t} F_4(x, y)\right| \lesssim O\left(\frac{\sqrt{w(x)w(y)}}{t \log^2(t)}\right) + O\left(\frac{\langle x \rangle^{\frac{1}{2}+\alpha+} \langle y \rangle^{\frac{1}{2}+\alpha+}}{t^{1+\alpha}}\right),$$

$$\left|(65) - \frac{1}{t} F_5(x, y)\right| \lesssim O\left(\frac{\sqrt{w(x)w(y)}}{t \log^2(t)}\right) + O\left(\frac{\langle x \rangle^{\frac{1}{2}+\alpha+} \langle y \rangle^{\frac{1}{2}+\alpha+}}{t^{1+\alpha}}\right),$$

where

$$F_4(x, y) = -\frac{1}{4 \|V\|_1} \int_{\mathbb{R}^2} v(x_1)[SS_1S](x_1, y_1) v(y_1) dx_1 dy_1$$

$$= -\frac{1}{4 \|V\|_1} \langle v, SS_1Sv \rangle,$$

$$F_5(x, y) = -\frac{1}{4 \|V\|_1} \int_{\mathbb{R}^2} v(x_1)[S](x_1, y_1) v(y_1) dx_1 dy_1$$

$$= -\frac{1}{4 \|V\|_1} \langle v, Sv \rangle.$$

Finally, the contribution of the error term $E(\lambda)(x, y)$ can be handled as in Proposition 4.9 in [14]:

Proposition 2.26. *Let* $0 < \alpha < 1/4$, $v(x) \lesssim \langle x \rangle^{-3/2-\alpha}$. *We have the bound*

$$\left| \int_{\mathbb{R}^4} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) [\mathcal{R}_2^+ - \mathcal{R}_2^-] v(x_1) E(\lambda)(x_1, y_1) v(y_1) d\lambda dx_1 dy_1 \right|$$

$$\lesssim \frac{\langle x \rangle^{\frac{1}{2}+\alpha+} \langle y \rangle^{\frac{1}{2}+\alpha+}}{t^{1+\alpha}}.$$

(66)

Using Proposition 2.14, Proposition 2.17, Proposition 2.22, Corollary 2.25, and Proposition 2.26 in the expansion (26) for $R_V^+ - R_V^-$ leads us to (18) with

$$F(x, y) = -\frac{1}{4} + \frac{1}{c_0^2 \|V\|_1} \sum_{i=1}^{4} F_i - F_5. \tag{67}$$

The next proposition calculates $F(x, y)$ explicitly.

Proposition 2.27. *Under the conditions of Theorem* 1.1*,*

$$F(x, y) = -\frac{1}{4}\psi(x)\psi(y) \tag{68}$$

where $\psi - 1 \in L^p$ *for all* $p > 2$.

Proof. Recall that S_1 is a projection operator with the kernel $S_1(x, y) = \phi(x)\phi(y)$ for some $\|\phi\|_{L^2} = 1$. Also, by Remark 2.9 if ψ is an s-wave resonance it has the representation $\psi = c_0 + G_0 v\phi$. Since the operators here are linear we can divide this equality by c_0 to obtain $\tilde{\psi} - 1 = \frac{1}{c_0}G_0 v\phi \in \bigcap_{p>2} L^p$.

Using these and the definition (25) of $G_0 f(x)$, F_1 can be written as

$$F_1(x, y) = -\frac{\|V\|_1}{4}[G_0 v S_1 v G_0](x, y)$$

$$= -\frac{\|V\|_1}{4}[G_0 v\phi](x)[G_0 v\phi](y)$$

$$= -\frac{\|V\|_1}{4}c_0^2(\tilde{\psi}(x) - 1)(\tilde{\psi}(y) - 1).$$

For F_2 and F_3 recall that

$$S = \begin{bmatrix} P & -PTQD_0Q \\ -QD_0QTP & QD_0QTPTQD_0Q \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

Note that multiplying S by v from the left side cancels a_{21} and a_{22}; and by S_1 from the right side cancels a_{11}. Hence, we have

$$F_2(x, y) = \frac{1}{4}\langle v, S S_1 v \, G_0(\cdot, y) \rangle$$

$$= -\frac{1}{4}\langle v, PTQD_0Q S_1 v G_0(\cdot, y) \rangle$$

$$= -\frac{1}{4}\langle Pv, TS_1 v G_0(\cdot, y) \rangle$$

$$= -\frac{1}{4}\langle v, T\phi \rangle [G_0 v\phi](y)$$

For the third equality we used the identities $S_1 D_0 = D_0 S_1 = S_1$ and $QS_1 = S_1$. For the last equality we used $Pv = v$ and the definition of S_1. Hence, recalling the definition of $c_0 = \|V\|_1^{-1} \langle v, T\phi \rangle$ from Remark 2.9 we can write $F_2(x, y)$ as

$$F_2(x, y) = -\frac{\|V\|_1}{4} c_0^2 (\tilde{\psi}(y) - 1)$$

The same calculation shows that

$$F_3(x, y) = -\frac{\|V\|_1}{4} c_0^2 (\tilde{\psi}(x) - 1).$$

Similarly, using $Qv = 0$, $Pv = v$, and $S_1 Q D_0 Q = Q D_0 Q S_1 = S_1$, we calculate

$$F_4(x, y) = -\frac{1}{4\|V\|_1} \langle Sv, S_1 Sv \rangle$$

$$= -\frac{1}{4\|V\|_1} \langle \phi Tv, S_1 Tv \rangle$$

$$= -\frac{1}{4\|V\|_1} \langle v, T\phi \rangle^2$$

$$= -\frac{\|V\|_1}{4} c_0^2.$$

For $F_5(x, y)$, note that we have $v(x)$ both on left and right side of S. Hence, except P everything vanishes and we obtain

$$F_5(x, y) = -\frac{1}{4\|V\|_1} \langle v, Sv \rangle = -\frac{1}{4\|V\|_1} \langle v, Pv \rangle = -\frac{1}{4}.$$

It is easy to see that F_5 cancels out the operator coming from the free resolvent in (67). The other four sum up to $-\frac{c_0^2 \|V\|_1}{4} \tilde{\psi}(x) \tilde{\psi}(y)$ and that establishes the proof. $\qquad\square$

Proposition 2.27 finishes the proof of Theorem 2.1.

We conclude this section by remarking that the bounds that we obtain in this section allows us to reach a similar estimate for the solution of the wave equation with some small modifications. Replacing Proposition 2.14, Proposition 2.22, and Proposition 2.26 with Proposition 5.10, Proposition 5.11, and Proposition 5.15 in [28] respectively, one obtains

$$\left| \int_0^\infty (\sin(t\lambda) + \lambda \cos(t\lambda)\chi(\lambda))[R_V^+(\lambda^2) - R_V^-(\lambda^2)](x, y)d\lambda - \frac{1}{t}\tilde{F}(x, y) \right|$$

$$\lesssim \frac{(1 + \log^+ |x|)(1 + \log^+ |y|)}{t \log^2 t} + \frac{\langle x \rangle^{\frac{1}{2}+\alpha} \langle y \rangle^{\frac{1}{2}+\alpha}}{t^{1+\alpha}}.$$

This estimate gives us Theorem 1.4 with no interpolation. Note that the interpolation with unweighted result (14) does not help us to decrease the weight function to $\log^2(2 + |x|)$ and have the decay $(t \log^2 t)^{-1}$. This is because we need to improve the time decay from $|t|^{-1/2}$ as opposed to Schrödinger time decay $|t|^{-1}$.

Also note that we only need to subtract a finite rank operator from (15). The reason is the following identities (Λ smooth and compactly supported)

$$\int_0^\infty \cos(t\lambda)\lambda\Lambda(\lambda)d\lambda = -\frac{1}{t}\int_0^\infty \sin(t\lambda)(\lambda\Lambda(\lambda))'d\lambda,$$

$$\int_0^\infty \sin(t\lambda)\Lambda(\lambda)d\lambda = -\frac{1}{t}\Lambda(0) + \frac{1}{t}\int_0^\infty \cos(t\lambda)\Lambda'(\lambda)d\lambda.$$

The boundary term in the second identity will result in the finite rank operator, as in the proof of Theorem 1.1.

3. Matrix case

Let $\mathfrak{R}_V^\pm(\lambda) := \lim_{\epsilon\to 0}(\mathcal{H} - (\lambda^2 \pm i\epsilon))^{-1}$ for $\lambda \in (-\infty, \mu] \cup [\mu, \infty)$. Recall that the following representation is valid for $(f, g) \in W^{2,2} \times W^{2,2} \cap X_{1+}$ under the assumptions of A1) - A4), see Section 2 in [17]:

$$\langle e^{it\mathcal{H}} P_{ac} f, g\rangle = \frac{1}{2\pi i}\int_{|\lambda|>\mu} e^{it\lambda}\langle[\mathfrak{R}_V^+(\lambda) - \mathfrak{R}_V^-(\lambda)]f, g\rangle d\lambda. \qquad (69)$$

Using this representation we will prove

Theorem 3.1. *Under the assumptions of A1) - A4), if there is a resonance of the first kind at μ, then we have for any $t > 0$*

$$\sup_{x,y\in\mathbb{R}^2}\left|\int_0^\infty e^{it(\lambda^2+\mu)}\lambda\chi(\lambda/L)[\mathfrak{R}_V^+ - \mathfrak{R}_V^-](\lambda^2 + \mu)(x, y)d\lambda\right| \lesssim \frac{1}{|t|}.$$

Theorem 3.2. *Under the assumptions A1)-A4), if there is a resonance of the first kind at μ, then we have for any $t \geq 2$*

$$\sup_{x,y\in\mathbb{R}^2}\left|\int_0^\infty e^{it\lambda^2}\lambda\chi(\lambda/L)[\mathfrak{R}_V^+ - \mathfrak{R}_V^-](\lambda^2 + \mu)(x, y)d\lambda - \frac{1}{t}\mathfrak{F}(x, y)\right|$$

$$\lesssim \frac{\sqrt{w(x)w(y)}}{t\log^2(t)} + \frac{\langle x\rangle^{3/2}\langle y\rangle^{3/2}}{t^{1+\alpha}}$$

where $0 < \alpha < \min\left(\frac{1}{4}, \frac{\beta-3}{2}\right)$.

The statement of Theorem 3.1 and Theorem 3.2 is established in [15] for $\mathfrak{F}(x, y) = 0$ when μ is regular, see Theorem 1.2. In Section 3.1, we prove the statement of Theorem 3.2 for $\lambda \ll 1$ and combine it with the high energy result of (12) from Theorem 1.2. In Section 3.2, we extend the low energy; when $\lambda \ll 1$, results of (11) from Theorem 1.2 to the case when there is a resonance of the first kind at μ. Then we conclude Theorem 1.3 by interpolation as in the analysis of the scalar case.

Below in Section 3.1, we show that the spectral density $[\mathfrak{R}_V^+(\lambda) - \mathfrak{R}_V^-(\lambda)](x, y)$ has a similar expansion to (34) from the scalar case. Because the same analysis appear in [14] we skip the proofs and refer Section 2 in [14] to the reader.

Note that Theorem 3.1 and Theorem 3.2 are stated only for $\mu > 0$. That is because our analysis below is performed only on the positive branch of the spectrum $[\mu, \infty)$. However, one can perform the same analysis for negative branch taking $\lambda^2 = -\lambda - \mu$ and establish Theorem 1.3 when there is resonance of the first kind at $-\mu$.

3.1. The free resolvent and resolvent expansion around zero in case of s-wave resonance.

The free resolvent $\mathfrak{R}_0(z)$ of matrix Schrödinger equation is given by

$$\mathfrak{R}_0(z) = (\mathcal{H}_0 - z)^{-1} = \begin{bmatrix} R_0(z - \mu) & 0 \\ 0 & -R_0(-z - \mu) \end{bmatrix} \tag{70}$$

for $z \notin (-\infty, -\mu] \cup [\mu, \infty)$. Here $R_0(z)$ is the scalar free resolvent. Writing $z = \mu + \lambda^2$, where $\lambda > 0$ we have

$$\mathfrak{R}_0^\pm(\mu + \lambda^2)(x, y) = \begin{bmatrix} R_0^\pm(\lambda^2)(x, y) & 0 \\ 0 & R_2(\lambda^2)(x, y) \end{bmatrix}$$

where

$$R_2(\lambda^2)(x, y) := -\frac{i}{4} H_0^+(i \sqrt{2\mu + \lambda^2}|x - y|).$$

Note that the bounds

$$|R_2(\lambda^2)(x, y)| \lesssim 1 + \log^- |x - y| \lesssim k(x, y), \tag{71a}$$

$$|\partial_\lambda^k R_2(\lambda^2)(x, y)| \lesssim 1, \quad k = 1, 2, \dots \tag{71b}$$

can be seen directly from the large and small energy expansion of Hankel functions and the fact that μ is strictly greater than zero.

We define the following two matrices

$$M_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and write

$$\mathfrak{R}_0^{\pm}(\mu + \lambda^2)(x, y) = R_0^{\pm}(\lambda^2)(x, y)M_{11} + R_2(\lambda^2)(x, y)M_{22} \qquad (72)$$

for convenience.

Lemma 3.3. *The following expansion is valid for the kernel of the free resolvent*

$$\mathfrak{R}_0^{\pm}(\lambda^2 + \mu)(x, y) = \mathfrak{g}^{\pm}(\lambda)M_{11} + \mathcal{G}_0(x, y) + \mathcal{E}_0^{\pm}(\lambda)(x, y),$$

where

$$\mathfrak{g}^{\pm}(\lambda) = \pm\frac{i}{4} - \frac{1}{2\pi}\log(\lambda/2) - \frac{\gamma}{2\pi},$$

$$\mathcal{G}_0(x, y) = \begin{bmatrix} G_0(x, y) & 0 \\ 0 & -\frac{i}{4}H_0^+(i\sqrt{2\mu}|x - y|) \end{bmatrix},$$

and $\mathcal{E}_0^{\pm}(\lambda)(x, y)$ satisfies the bounds,

$$|\mathcal{E}_0^{\pm}| \lesssim \langle\lambda\rangle^{\frac{1}{2}}\lambda^{\frac{1}{2}}\langle x - y\rangle^{\frac{1}{2}},$$

$$|\partial_\lambda \mathcal{E}_0^{\pm}| \lesssim \langle\lambda\rangle^{\frac{1}{2}}\lambda^{-\frac{1}{2}}\langle x - y\rangle^{\frac{1}{2}},$$

$$|\partial_\lambda^2 \mathcal{E}_0^{\pm}| \lesssim \langle\lambda\rangle^{\frac{1}{2}}\lambda^{-\frac{1}{2}}\langle x - y\rangle^{\frac{3}{2}}.$$

Corollary 3.4. *For $0 < \alpha < 1$ and $d > c > 0$ we have,*

$$|\partial_\lambda \mathcal{E}_0^{\pm}(d) - \partial_\lambda \mathcal{E}_0^{\pm}(c)| \lesssim c^{-\frac{1}{2}}|d - c|^\alpha \langle x - y\rangle^{\frac{1}{2}+\alpha}.$$

We write $V = -\sigma_3 vv := v_1 v_2$ where $v_1 = -\sigma_3 v$, $v_2 = v$, and

$$v = \frac{1}{2}\begin{bmatrix} \sqrt{V_1 + V_2} + \sqrt{V_1 - V_2} & \sqrt{V_1 + V_2} - \sqrt{V_1 - V_2} \\ \sqrt{V_1 + V_2} - \sqrt{V_1 - V_2} & \sqrt{V_1 + V_2} + \sqrt{V_1 - V_2} \end{bmatrix} := \begin{bmatrix} a & b \\ b & a \end{bmatrix}.$$

Using symmetric resolvent identity, we have

$$\mathfrak{R}_V(\mu + \lambda^2) = \mathfrak{R}_0(\mu + \lambda^2) - \mathfrak{R}_0(\mu + \lambda^2)v_1 M^{\pm}(\lambda)^{-1} v_2 \mathfrak{R}_0(\mu + \lambda^2),$$

where

$$M^{\pm}(\lambda) = I + v_2 \mathfrak{R}_0(\mu + \lambda^2)v_1.$$

Employing Lemma 3.3,

$$M^{\pm}(\lambda) = \mathfrak{g}^{\pm}(\lambda)v_2 M_{11}v_1 + T + v_2\mathcal{E}_0^{\pm}v_1$$

where T has kernel $T(x, y) = I + v_2(x)\mathcal{G}_0(x, y)v_1(y)$.

Lemma 3.5. *Let $0 < \alpha < 1$. The following expansion is valid for $\lambda > 0$*

$$M^{\pm}(\lambda) = -\|a^2 + b^2\|_{L_1(\mathbb{R}^2)}\mathfrak{g}^{\pm}(\lambda)P + T + \mathcal{E}_1^{\pm}(\lambda),$$

where P is the orthogonal projection onto the span of the vector $(a, b)^T$ in $L^2 \times L^2$. Further, we have

$$\|\sup_{0<\lambda<\lambda_1} \lambda^{-\frac{1}{2}}|\mathcal{E}_1^{\pm}(\lambda)|\|_{HS} + \|\sup_{0<\lambda<\lambda_1} \lambda^{\frac{1}{2}}|\partial_\lambda\mathcal{E}_1^{\pm}(\lambda)|\|_{HS}$$

$$+ \|\sup_{0<\lambda<b<\lambda_1} \lambda^{\frac{1}{2}}(b-\lambda)^{-\alpha}|\partial_\lambda\mathcal{E}_1^{\pm}(b) - \partial_\lambda\mathcal{E}_1^{\pm}(\lambda)|\|_{HS} \lesssim 1,$$

provided that $a(x), b(x) \lesssim \langle x\rangle^{-3/2-\alpha-}$.

Recall P in the scalar case is defined as projection onto v whereas in matrix case it is defined as projection onto the span of the vector $(a, b)^T$. In light of this difference we give the following modified version of Definition 2.8. Let $Q := \mathbf{1} - P$.

Definition 3.6. (1) μ is defined to be a regular point of the spectrum of $\mathcal{H} = -\Delta + V$ if QTQ is invertible on $Q(L^2 \times L^2)$.

(2) If μ is not a regular point of spectrum then $QTQ + S_1$ is invertible on $Q(L^2 \times L^2)$ and we define $D_0 = (QTQ + S_1)^{-1}$ as an operator on $Q(L^2 \times L^2)$. Here S_1 is defined as Riesz projection onto the Kernel of QTQ as an operator on $Q(L^2 \times L^2)$.

(3) We say there is a resonance of the first kind at zero if the operator $T_1 := S_1TPTS_1$ is invertible on $S_1Q(L^2 \times L^2)$ and we define D_1 as the inverse of T_1.

With the following lemma we can have a representation for the space S_1 as in the scalar case.

Lemma 3.7 ([15, Lemma 4.4]). *If $|a(x)|+|b(x)| \lesssim \langle x\rangle^{-1-}$ and if $\phi \in S_1(L^2 \times L^2)$, then $\phi(x) = v_2(x)\psi_1(x)$ where $\psi_1 \in L^{\infty} \times L^{\infty}$ and $(\mathcal{H}_0 - \mu I)\psi_1 = 0$ in the sense of distribution. Also we have*

$$\psi_1(x) = -\mathcal{G}_0 v_1\phi(y) + (c_0, 0)^T,$$

with $c_0 = \frac{\langle T\phi,(a,b)\rangle}{\|a^2+b^2\|_1}$.

Remark 3.8. (1) $\phi(x) = v_2(x)\psi_1(x)$ implies that $\phi(x) = v_1^T(x)\psi_2(x)$ where

$$\psi_2(x) = \mathcal{G}_0(x, y)v_2\phi - (c_0, 0)^T,$$

since $v_2\psi_1 = v_1^T(-\sigma_3)\psi_1 = v_1^T\psi_2$.

(2) Let S_2 be the orthogonal projection onto Ker T_1. Then the range of $S_1 - S_2$ has dimension at most one. To see this, recall the representation of ψ_1 from Lemma 3.7. Note that first, if $\phi \in S_2(L^2 \times L^2)$ then $c_0 = 0$, [13, Lemma 5.3]. Second, if $|a(x)|, |b(x)| \lesssim \langle x \rangle^{-2-}$ then $\mathcal{G}_0 v_1 \phi(x) \in L^p \times L^p$ for any $p \in (2, \infty]$. Indeed, Lemma 5.1 of [13] suggests that the first entry of $\mathcal{G}_0 v_1 \phi$ is in L^p for any $p \in (2, \infty]$. For the second entry, recall that we analyze the free resolvent of matrix equation on the positive branch. Hence, $\frac{i}{4}H_0^+(i\sqrt{2\mu}|x - y|)$ is well-defined as an operator from L^2 to L^2. Since ϕ is in L^2 and the entries of v_2 are in $L^2 \cap L^\infty$ we can conclude that the second entry of $\mathcal{G}_0 v_1 \phi(x)$ is in L^2. Lemma 3.7 suggests that the second entry is also in L^∞. So by interpolation $\mathcal{G}_0 v_1 \phi(x) \in L^p \times L^p$ for any $p \in (2, \infty]$. Hence, one has that Rank $S_1 \leq$ Rank $S_2 + 1$.

(3) Since in our case $S_2 \equiv 0$, and since zero is not regular, Range S_1 has dimension exactly one. Hence, we take $\|\phi\|_{L^2 \times L^2} = 1$ with $\phi \in S_1(Ł^2 \times Ł^2)$ and $S_1 f = (\phi_1, \phi_2)\langle \phi, f \rangle$ where ϕ is as in the Lemma .3.7.

(4) By Lemma 3.7, we have

$$D_1 = \frac{1}{\|a^2 + b^2\|_1 c_0^2} S_1.$$

Definition 3.6 and Lemma 3.7 give us a similar expansion for $M^\pm(\lambda)^{-1}$ as in the Section 2.1. In the expansion $\|a^2 + b^2\|_1$ exchanges with $\|V\|_1$ due to the definition of $h_\pm(\lambda) = -\|a^2 + b^2\|_1 \mathfrak{g}^\pm(\lambda) + c$ where $c \in \mathbb{R}$. Hence, for $0 < \lambda < \lambda_1$, we have

$$\begin{aligned}
\mathfrak{R}_V^\pm(\lambda) = \mathfrak{R}_0^\pm(\lambda^2) + \mathfrak{R}_0^\pm(\lambda^2)v_1\Big[&\frac{h_\pm}{\|a^2 + b^2\|_1 c_0^2}(\lambda)S_1 \\
&+ \frac{SS_1}{\|a^2 + b^2\|_1 c_0^2} \\
&+ \frac{S_1 S}{\|a^2 + b^2\|_1 c_0^2} \\
&+ \frac{1}{\|a^2 + b^2\|_1 c_0^2}h_\pm^{-1}SS_1S \\
&- h_\pm^{-1}(\lambda)S - QD_0Q - E^\pm(\lambda)\Big]v_2\mathfrak{R}_0^\pm(\lambda^2)
\end{aligned}$$
$$(73)$$

with $E(\lambda)(x, y)$ is such that

$$
\begin{aligned}
\| \sup_{0<\lambda<\lambda_1} \lambda^{-\frac{1}{2}+} &|E^\pm(\lambda)| \|_{\mathrm{HS}} + \| \sup_{0<\lambda<\lambda_1} \lambda^{\frac{1}{2}} |\partial_\lambda E^\pm(\lambda)| \|_{\mathrm{HS}} \\
&+ \| \sup_{0<\lambda<b<\lambda_1} \lambda^{\frac{1}{2}+\alpha} (b-\lambda)^{-\alpha} |\partial_\lambda E^\pm(b) - \partial_\lambda E^\pm(a)| \|_{\mathrm{HS}} \\
&\lesssim 1.
\end{aligned}
$$

Here the matrix S has the same definition (31) as in the scalar case.

3.2. Proof of the Theorem 3.2.

The proof of Theorem 3.2 is similar to the proof of Theorem 2.1. The cancellation property $Qv = 0$ that we used repeatedly is replaced with

$$
M_{11}v_1 S_1 = S_1 v_2 M_{11} = 0, \tag{74}
$$

which allows us to use Lemma 2.19 to gain extra time decay. Furthermore, as in the scalar case, the boundary terms arise only in the low-low energy evolution. For this reason, we present the proof of Theorem 3.2 for the case $\lambda p, \lambda q \lesssim 1$, and omit the cases in which high energy is involved. For high energies one can apply the same methods that we applied in the scalar case using the bound (71) in addition to the bound (23), see [15] for similar arguments.

For convenience we write

$$
\mathfrak{R}_0(\mu + \lambda^2)(x, y) = R_0(\lambda^2)(x, y)M_{11} + R_2(\lambda^2)(x, y)M_{22}. \tag{75}
$$

The following Proposition takes care of the contribution of

$$
\int_{\mathbb{R}^4} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \mathfrak{R}_1(\lambda, p, q)[v_1 S_1 v_2](x_1, y_1) d\lambda dx_1 dy_1 \tag{76}
$$

to (69) where

$$
\begin{aligned}
\mathfrak{R}_1(\lambda, p, q) := \, &h^+(\lambda) \mathfrak{R}_0^+(\mu + \lambda^2)(x, x_1) \mathfrak{R}_0^+(\mu + \lambda^2)(y, y_1) \\
&- h^-(\lambda) \mathfrak{R}_0^-(\mu + \lambda^2)(x, x_1) \mathfrak{R}_0^-(\mu + \lambda^2)(y, y_1).
\end{aligned} \tag{77}
$$

Proposition 3.9. *Let* $0 < \alpha < 1/4$. *If* $|a(x)| + |b(x)| \lesssim \langle x \rangle^{-\frac{3}{2}-\alpha-}$, *then we have*

$$
\left| (76) - \frac{1}{t} \mathfrak{F}_1(x, y) \right| \lesssim \frac{\langle x \rangle^{\frac{1}{2}+\alpha+} \langle y \rangle^{\frac{1}{2}+\alpha+}}{t^{1+\alpha}},
$$

where

$$
\begin{aligned}
\mathfrak{F}_1(x, y) &= \frac{\|a^2 + b^2\|_1}{4} \int_{\mathbb{R}^4} \mathcal{G}_0(x, x_1) v_1(x_1) S_1(x_1, y_1) v_2(y_1) \mathcal{G}_0(y_1, y) dx_1 dy_1 \\
&= \frac{\|a^2 + b^2\|_1}{4} \mathcal{G}_0 v_1 S_1 v_2 \mathcal{G}_0.
\end{aligned}
$$

Proof. Using (75) in (77), $\mathfrak{R}_1(\lambda, p, q)$ can be calculated as

$$h^+(\lambda)R_0^+(\lambda^2)(x, x_1)M_{11}M_{11}R_0^+(\lambda^2)(y_1, y)$$
$$- h^-(\lambda)R_0^-(\lambda^2)(x, x_1)M_{11}M_{11}R_0^-(\lambda^2)(y_1, y)$$
$$+ [h^+(\lambda)R_0^+(\lambda^2)(x, x_1) - h^-(\lambda)R_0^-(\lambda^2)(x, x_1)]M_{11}M_{22}R_2(\lambda^2)(y_1, y)$$
$$+ R_2(\lambda^2)(x, x_1)M_{22}M_{11}[h^+(\lambda)R_0^+(\lambda^2)(y_1, y) - h^-(\lambda)R_0^-(\lambda^2)(y_1, y)]$$
$$+ [h^+(\lambda) - h^-(\lambda)]M_{22}M_{22}R_2(\lambda^2)(x, x_1)R_2(\lambda^2)(y_1, y)$$
$$= A_1(\lambda, p, q) + A_2(\lambda, p, q) + A_3(\lambda, p, q) + A_4(\lambda, p, q).$$

Note that $A_1(\lambda, p, q)$ is similar to (35). Hence, using the projection property (74), its contribution to the integral (76) can be obtained as

$$\frac{\|a^2 + b^2\|_1}{4t} \int_{\mathbb{R}^4} G_0(x, x_1)M_{11}v_1S_1v_2M_{11}G_0(y, y_1)dx_1dy_1$$
$$+ O\left(\frac{\langle x\rangle^{\frac{1}{2}+\alpha}\langle y\rangle^{\frac{1}{2}+\alpha+}}{t^{1+\alpha+}}\right). \tag{78}$$

Next we consider $A_4(\lambda, p, q)$. First note that

$$[h^+(\lambda) - h^-(\lambda)]R_2(\lambda^2)(x, x_1)R_2(\lambda^2)(y_1, y)$$
$$= \frac{\|a^2 + b^2\|_1 i}{32}H_0^+(i\sqrt{2\mu + \lambda^2}\, p)H_0^+(i\sqrt{2\mu + \lambda^2}\, q).$$

Taking

$$\mathcal{E}(\lambda, p, q) = \chi(\lambda)H_0^+(i\sqrt{2\mu + \lambda^2}\, p)H_0^+(i\sqrt{2\mu + \lambda^2}\, q),$$

we see that

$$\mathcal{E}(0) = H_0^+(i\sqrt{2\mu}|x - x_1|)H_0^+(i\sqrt{2\mu}|y - y_1|).$$

Also, the bounds (71) leads us to

$$\left|\frac{\partial}{\partial\lambda}[\chi(\lambda)H_0^+(i\sqrt{2\mu + \lambda^2}|x - x_1|)H_0^+(i\sqrt{2\mu + \lambda^2}|y - y_1|)]\right|$$
$$\lesssim k(x, x_1)k(y, y_1), \tag{79}$$

$$\left|\partial\left(\frac{\frac{\partial}{\partial\lambda}[\chi(\lambda)H_0^+(i\sqrt{2\mu + \lambda^2}|x - x_1|)H_0^+(i\sqrt{2\mu + \lambda^2}|y - y_1|)]}{\lambda}\right)\right|$$
$$\lesssim \lambda^{-2}k(x, x_1)k(y, y_1). \tag{80}$$

Hence, using Lemma 2.15 with the bounds (79) and (80) we obtain the contribution of $A_4(\lambda, p, q)$ to the λ-integral of (76) as

$$-\frac{\|a^2 + b^2\|_1}{64t} H_0^+(i\sqrt{2\mu}\,p)M_{22}M_{22}H_0^+(i\sqrt{2\mu}\,q) + O\left(\frac{k(x,x_1)k(y,y_1)}{t^{3/2}}\right). \tag{81}$$

For $A_2(\lambda, p, q)$, we have

$$[h^+(\lambda)R_0^+(\lambda^2)(x,x_1) - h^-(\lambda)R_0^-(\lambda^2)(x,x_1)]R_2(\lambda^2)(y_1,y)$$
$$= CJ_0(\lambda p)(\log(\lambda) + 1)R_2(\lambda^2)(y_1,y) - i\frac{\|a^2 + b^2\|_1}{8}Y_0(\lambda p)R_2(\lambda^2)(y_1,y) \tag{82}$$

for some $C \in \mathbb{C}$.

Note that we can apply (74) to the left side of this sum and replace $J_0(\lambda p)$ with $G(\lambda, x, x_1)$. Hence, Lemma 2.16 together with the bounds in (71) and in Lemma 2.19 gives us the contribution of the left side to λ-integral of (76) as $t^{-1-\alpha}\langle x\rangle^{\frac{1}{2}+\alpha}\langle x_1\rangle^{\frac{1}{2}+\alpha}k(y,y_1)$.

To find the contribution of the right side of the sum in (82) recall that $Y_0(\lambda|x - x_1|) = \chi(\lambda p)[\frac{2}{\pi}\log(\frac{\lambda p}{2}) + c + \tilde{O}((\lambda p)^2\log(\lambda p))]$. Multiplying this with $R_2(\lambda^2)(y_1,y)$, we have

$$-4G_0(x,x_1)\chi(\lambda p)R_2(\lambda^2)(y_1,y) + [\log\lambda + c]\chi(\lambda p)R_2(\lambda^2)(y_1,y)$$
$$+ \tilde{O}(\lambda p)^2\log(\lambda p)\chi(\lambda p)R_2(\lambda^2)(y_1,y).$$

Using Lemma 2.20 and (74), the contribution of the second term to λ integral in (76) can be obtained as $\langle x_1\rangle^{\frac{1}{2}+\alpha}k(y,y_1)t^{-1-\alpha}$ in a similar way as in $A(\lambda, p, q)$ in Lemma 2.21. And the contribution of the third term follows as $t^{-\frac{5}{4}}k(y,y_1)\langle x\rangle^{\frac{1}{2}}\langle x_1\rangle^{\frac{1}{2}}k(y,y_1)$ with Lemma 2.15.

Finally, for the first term we take $\mathcal{E}(\lambda, p, q) = -4G_0(x,x_1)\chi(\lambda p)R_2(\lambda^2)(y_1,y)$ and see $\mathcal{E}(0, p, q) = iG_0(x,x_1)H_0^+(i\sqrt{2\mu}\,q)$. Using Lemma 2.15 with the bounds of $R_2(\lambda)$ and the support of $\chi(\lambda p)$, the contribution of $A_2(\lambda, p, q)$ is obtained as

$$-\frac{i\|a^2 + b^2\|_1}{16t}\int_{\mathbb{R}^4} G_0(x,x_1)M_{11}[v_1S_1v_2](x_1,y_1)M_{22}H_0^+(i\sqrt{2\mu}\,q)dx_1dy_1$$
$$+ O\left(\frac{\langle x\rangle^{\frac{1}{2}}\langle x_1\rangle^{\frac{1}{2}}k(x,x_1)k(y,y_1)}{t^{\frac{5}{4}}}\right). \tag{83}$$

With a similar argument the contribution of $A_3(\lambda, p, q)$ is

$$- \frac{i \|a^2 + b^2\|_1}{16t} \int_{\mathbb{R}^4} H_0^+(i \sqrt{2\mu} \, p) M_{22}[v_1 S_1 v_2](x_1, y_1) M_{11} G_0(y, y_1) dx_1 dy_1$$

$$+ O\Big(\frac{\langle x \rangle^{\frac{1}{2}} \langle x_1 \rangle^{\frac{1}{2}} k(x, x_1) k(y, y_1)}{t^{\frac{5}{4}}}\Big). \quad (84)$$

Adding up (78), (81), (83), and (84) gives the statement. $\qquad\square$

To find the contribution of the terms SS_1 and S_1S to (69) we define

$$\mathfrak{R}_2(\lambda, p, q) := \mathfrak{R}_0^+(\lambda^2)(x, x_1) \mathfrak{R}_0^+(\lambda^2)(y_1, y) - \mathfrak{R}_0^-(\lambda^2)(x, x_1) \mathfrak{R}_0^-(\lambda^2)(y_1, y).$$

Proposition 3.10. *If* $|a(x)| + |b(x)| \lesssim \langle x \rangle^{-\frac{3}{2}-\alpha-}$ *where* $0 < \alpha < \frac{1}{4}$, *then we have*

$$\Big| \int_{\mathbb{R}^4} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \mathfrak{R}_2(\lambda, p, q)[v_1 S_1 S v_2](x_1, y_1) d\lambda dx_1 dy_1 - \frac{1}{t} \mathfrak{F}_2(x, y) \Big|$$

$$\lesssim \frac{\langle x \rangle^{\frac{1}{2}+\alpha+} \langle x_1 \rangle^{\frac{1}{2}+\alpha+}}{t^{1+\alpha}},$$

$$(85)$$

$$\Big| \int_{\mathbb{R}^4} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \mathfrak{R}_2(\lambda, p, q)[v_1 S S_1 v_2](x_1, y_1) d\lambda dx_1 dy_1 - \frac{1}{t} \mathfrak{F}_3(x, y) \Big|$$

$$\lesssim \frac{\langle x \rangle^{\frac{1}{2}+\alpha+} \langle x_1 \rangle^{\frac{1}{2}+\alpha+}}{t^{1+\alpha}},$$

$$(86)$$

where

$$\mathfrak{F}_2(x, y) = -\frac{1}{4} \mathcal{G}_0 v_1 S_1 S v_2 M_{11}(x, y),$$

$$\mathfrak{F}_3(x, y) = -\frac{1}{4} M_{11} v_1 S S_1 v_2 \mathcal{G}_0(x, y).$$

Proof. We consider only (85). Note that

$$\begin{aligned}
\mathfrak{R}_2^+(\lambda, p, q) &= [R_0^+(\lambda^2)(x, x_1) M_{11} M_{11} R_0^+(\lambda^2)(y_1, y) \\
&\quad - R_0^-(\lambda^2)(x, x_1) M_{11} M_{11} R_0^-(\lambda^2)(y_1, y)] \\
&\quad + [R_0^+(\lambda^2)(x, x_1) - R_0^-(\lambda^2)(x, x_1)] M_{11} M_{22} R_2(\lambda^2)(y_1, y) \\
&\quad + R_2(\lambda^2)(x, x_1) M_{22} M_{11}[R_0^+(\lambda^2)(y_1, y)) - R_0^-(\lambda^2)(y_1, y))] \\
&= B_1(\lambda, p, q) + B_2(\lambda, p, q) + B_3(\lambda, p, q).
\end{aligned}$$

Again a similar kernel to $B_1(\lambda, p, q)$ is examined in the scalar case. It has the contribution

$$-\frac{1}{4t} \int_{\mathbb{R}^4} G_0(x, x_1) M_{11}[v_1 S_1 S v_2](x_1, y_1) M_{11} dx_1 dy_1 + O\Big(\frac{\langle x \rangle^{\frac{1}{2}+\alpha+} \langle x_1 \rangle^{\frac{1}{2}+\alpha+}}{t^{1+\alpha}}\Big) \tag{87}$$

to the integral in (85). For $B_2(\lambda, p, q) = \frac{i}{2} J_0(\lambda p) M_{11} M_{22} R_2(\lambda^2)(y_1, y)$ we can use the orthogonality property (74) on the left side of $S_1 S$ and exchange $J_0(\lambda p)$ with $G(\lambda, x, x_1)$. Then, Lemma 2.15 together with the bounds in Lemma 2.19 and (71) gives us

$$\left| \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) B_2(\lambda, p, q)[v_1 S_1 S v_2](x_1, y_1) d\lambda \right| \lesssim \frac{\langle x_1 \rangle^{\frac{1}{2}} k(y, y_1)}{t^{\frac{5}{4}}}. \tag{88}$$

Lastly, we consider

$$B_3(\lambda, p, q) = \frac{i}{2} R_2(\lambda^2)(x, x_1) J_0(\lambda q) \chi(\lambda q).$$

Applying Lemma 2.15, we have

$$\int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) B_3(\lambda, p, q) d\lambda$$
$$= \frac{i}{16t} H_0^+(i \sqrt{2\mu} p) M_{22} M_{11} + O\Big(\frac{k(x, x_1)\langle y \rangle^{\frac{1}{2}} \langle y_1 \rangle^{\frac{1}{2}}}{t^{\frac{5}{4}}}\Big). \tag{89}$$

Hence, (87), (88), and (89) establishes the proof. $\square$

The following Proposition will take care of the contributions of the following two integrals:

$$\int_{\mathbb{R}^4} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \mathfrak{R}_3(\lambda, p, q) v_1 S S_1 S v_2(x_1, y_1) d\lambda dx_1 dy_1, \tag{90}$$

$$\int_{\mathbb{R}^4} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \mathfrak{R}_3(\lambda, p, q) v_1 S v_2(x_1, y_1) d\lambda dx_1 dy_1, \tag{91}$$

where

$$\mathfrak{R}_3(\lambda, p, q) := \frac{\mathfrak{R}_0^+(\mu + \lambda^2) \mathfrak{R}_0^+(\mu + \lambda^2)}{h_+(\lambda)} - \frac{\mathfrak{R}_0^-(\mu + \lambda^2) \mathfrak{R}_0^-(\mu + \lambda^2)}{h_-(\lambda)}.$$

Proposition 3.11 (Proposition 5.5 in [15]). *Let $0 < \alpha < 1/4$. If $|a(x)| + |b(x)| \lesssim \langle x \rangle^{-3/2-\alpha-}$ then for any absolutely bounded operator Γ we have*

$$\int_{\mathbb{R}^4} \int_0^\infty e^{it\lambda^2} \mathfrak{R}_3(\lambda, p, q) v_1 \Gamma v_2(x_1, y_1) d\lambda dx_1 dy_1$$

$$= \frac{1}{4\|a^2+b^2\|_1} \int_{\mathbb{R}^4} M_{11} v_1 \Gamma v_2 dx_1 dy_1$$

$$+ O\Big(\frac{\sqrt{w(x)w(y)}}{t \log^2(t)}\Big) + O\Big(\frac{\langle x \rangle^{\frac{1}{2}+\alpha+} \langle y \rangle^{\frac{1}{2}+\alpha+}}{t^{1+\alpha}}\Big).$$

Corollary 3.12. *Under the same conditions of Proposition 3.11 we have*

$$\Big|(90) - \frac{1}{t}\mathfrak{F}_4(x, y)\Big| \lesssim O\Big(\frac{\sqrt{w(x)w(y)}}{t \log^2(t)}\Big) + O\Big(\frac{\langle x \rangle^{\frac{1}{2}+\alpha+} \langle y \rangle^{\frac{1}{2}+\alpha+}}{t^{1+\alpha}}\Big),$$

$$\Big|(91) - \frac{1}{t}\mathfrak{F}_5(x, y)\Big| \lesssim O\Big(\frac{\sqrt{w(x)w(y)}}{t \log^2(t)}\Big) + O\Big(\frac{\langle x \rangle^{\frac{1}{2}+\alpha+} \langle y \rangle^{\frac{1}{2}+\alpha+}}{t^{1+\alpha}}\Big),$$

where

$$\mathfrak{F}_4(x, y) = \frac{1}{4\|a^2+b^2\|_1} M_{11} v_1 S S_1 S v_2 M_{11},$$

$$\mathfrak{F}_5(x, y) = \frac{1}{4\|a^2+b^2\|_1} M_{11} v_1 S v_2 M_{11}.$$

The contribution of $E(\lambda)(x, y)$ can be handled as in Proposition 4.9 in [14] and we can obtain the following proposition.

Proposition 3.13. *Let $0 < \alpha < 1/4$. If $|a(x)| + |b(x)| \lesssim \langle x \rangle^{-3/2-\alpha-}$, then we have*

$$F \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) [\mathcal{R}_0^+(\mu+\lambda^2) v_1 E v_2 \mathcal{R}_0^+(\mu+\lambda^2)$$

$$- \mathcal{R}_0^-(\mu+\lambda^2) v_1 E v_2 \mathcal{R}_0^-(\mu+\lambda^2)](x, y) d\lambda$$

$$= O\Big(\frac{\langle x \rangle^{\frac{1}{2}+\alpha} \langle y \rangle^{\frac{1}{2}+\alpha}}{t^{1+\alpha}}\Big).$$

We found the boundary terms $\mathfrak{F}_i(x, y)$, $i = 1, \ldots, 5$ that has $\frac{1}{t}$ decay for every term appearing in the expansion (73). Also we note that the contribution of free resolvent is calculated in [15] as

$$\int_0^\infty e^{it\lambda^2} \lambda\chi(\lambda)[\mathcal{R}_0^+(\mu + \lambda^2) - \mathcal{R}_0^-(\mu + \lambda^2)](x, y)d\lambda$$

$$= -\frac{1}{4t}M_{11} + O\left(\frac{\langle x\rangle^{\frac{3}{2}}\langle y\rangle^{\frac{3}{2}}}{t^{\frac{5}{4}}}\right). \tag{92}$$

Considering this and the expansion (73) we see that the assertion of Theorem 3.2 is satisfied for

$$\mathfrak{F}(x, y) = \mathfrak{F}_0(x, y) + \frac{1}{\|a^2 + b^2\|_1 c_0^2} \sum_{i=1}^4 \mathfrak{F}_i(x, y) - \mathfrak{F}_5(x, y).$$

The following proposition concludes the explicit representation of $\mathfrak{F}(x, y)$ in Theorem 3.2.

Proposition 3.14. *Under the conditions of Theorem* 1.3 *we have*

$$\mathfrak{F}(x, y) = -\frac{1}{4}\psi(x)[\sigma_3\psi(y)]^T$$

where $(\mathcal{H}_0 - \mu I)\psi = 0$ *in the sense of distribution and* $\psi - (1, 0)^T \in \bigcap_{p>2} L^p \times \bigcap_{p>2} L^p$.

Proof. First note that $\mathcal{G}_0^T(x, y) = \mathcal{G}_0(x, y)$ and $v_2^T = v_2$. Then, recalling the integral kernel of S_1, which is $S_1(x, y) = \phi(x)\phi^T(y)$, we can write the operator obtained as $\mathfrak{F}_1$ as

$$\mathfrak{F}_1(x, y) = \frac{\|a^2 + b^2\|_1}{4}\mathcal{G}_0 v_1 S_1 v_2 \mathcal{G}_0(x, y)$$

$$= \frac{\|a^2 + b^2\|_1}{4}[\mathcal{G}_0 v_1\phi](x)[\mathcal{G}_0 v_2\phi]^T(y) \tag{93}$$

$$= \frac{\|a^2 + b^2\|_1}{4}((c_0, 0)^T - \psi_1(x))(-(c_0, 0) - \psi_2^T(y)).$$

For the last equality we used the representation of ϕ from Lemma 3.7 and Remark 3.8-(1). Note that here $((c_0, 0)^T - \psi_1(x))$ is a column vector and $(-(c_0, 0) - \psi_2^T(y))$ is a row vector. Hence, their vector product gives an operator which is represented by a 4×4 matrix.

For $\mathfrak{F}_2(x, y)$, the definition of S_1 gives us

$$\mathfrak{F}_2(x, y) = -\frac{1}{4}\mathcal{G}_0 v_1 S_1 S v_2 M_{11}(x, y)$$

$$= \mathcal{G}_0 v_1 S_1 T P v_2 M_{11}(x, y)$$

$$= -\frac{1}{4}[\mathcal{G}_0 v_1 \phi](x)\langle\phi, T P v_2 M_{11}(\cdot, y)\rangle$$

$$= -\frac{1}{4}[\mathcal{G}_0 v_1 \phi](x)\langle\phi, T P(a, b)^T\rangle(1, 0)$$

$$= -\frac{1}{4}[\mathcal{G}_0 v_1 \phi](x)\|a^2 + b^2\|_1(c_0, 0).$$

For the second equality, we used the definition of S from Lemma 2.11 together with identities $Q v_2 M_{11} = 0$, $S_1 D_0 = D_0 S_1 = S_1$, and $Q S_1 = S_1$. For the fourth equality, we used the fact that T is symmetric and $v_2 M_{11} = (a, b)^T (1, 0)$. Hence, we have

$$\mathfrak{F}_2 = \frac{\|a^2 + b^2\|_1}{4}((c_0, 0)^T - \psi_1(x))(c_0, 0). \tag{94}$$

Consequently, since $M_{11} v_1 = (-1, 0)^T (a, b)$ one obtains

$$\mathfrak{F}_3(x, y) = \frac{\|a^2 + b^2\|_1}{4}(-c_0, 0)(-(c_0, 0) - \psi_2^T(y)). \tag{95}$$

Using the orthogonality property (74) and the definition of S, one has

$$\mathfrak{F}_5(x, y) = \frac{1}{4\|a^2 + b^2\|_1} M_{11} v_1 S v_2 M_{11}$$

$$= \frac{1}{4\|a^2 + b^2\|_1} M_{11} v_1 P v_2 M_{11}$$

$$= \frac{-\|a^2 + b^2\|_1}{4\|a^2 + b^2\|_1} M_{11}$$

$$= -\frac{1}{4} M_{11}. \tag{96}$$

With a same argument as in the Proposition 2.27 using the orthogonality

property (74) in the definition of S, we have

$$
\begin{aligned}
\mathfrak{F}_4(x, y) &= \frac{1}{4\|a^2 + b^2\|_1} M_{11} v_1 S S_1 S v_2 M_{11} \\
&= \frac{1}{4\|a^2 + b^2\|_1} M_{11} v_1 P T S_1 T P v_2 M_{11} \\
&= \frac{c_0^2}{4\|a^2 + b^2\|_1} M_{11} v_1 P v_2 M_{11} \\
&= -\frac{\|a^2 + b^2\|_1}{4} c_0^2.
\end{aligned}
\tag{97}
$$

For the third equality we used the definition of $PT\phi$ from Lemma 3.7 and the fact that T is symmetric.

Multiplying (93), (94), (95), (97), and (96) with required constants and summing up together with the boundary term (92) from the free resolvent for matrix Schrödinger operator, we obtain

$$
\sum_{i=0}^{5} \mathfrak{F}_i(x, y) = \frac{1}{4c_0^2} \psi_1(x) \psi_2^T(y) = -\frac{1}{4c_0^2} \psi_1(x) [\sigma_3 \psi_1(y)]^T.
$$

As in Proposition 2.27, dividing ψ_1 by c_0 establishes the proof.   □

3.3. Proof of the Theorem 3.1. The $\frac{1}{t}$ bound for the free resolvent, for a similar error term to E, and for the term $h_{\pm}(\lambda)^{-1}S$ were examined in [15] in Proposition 5.4, Proposition 7.5, and Proposition 7.2 respectively. Since the proof of Proposition 7.2 requires the operator S only to be absolutely bounded it can be extended to the term $h_{\pm}(\lambda)^{-1} S S_1 S$.

For the operators $Q D_0 Q$, $S S_1$, and $S_1 S$ recall the expansion from Proposition 3.10:

$$
\begin{aligned}
&\mathfrak{R}_0^+(\lambda^2)(x, x_1) \mathfrak{R}_0^+(\lambda^2)(y_1, y) - \mathfrak{R}_0^-(\lambda^2)(x, x_1) \mathfrak{R}_0^-(\lambda^2)(y_1, y) \\
&= B_1(\lambda, p, q) + B_2(\lambda, p, q) + B_3(\lambda, p, q).
\end{aligned}
$$

The $\frac{1}{t}$ bound for a similar kernel to $B_1(\lambda, p, q)$ is established in Proposition 3.11 in [13] for the operator $Q D_0 Q$, $S S_1$, and $S_1 S$. Furthermore, Proposition 7.2 in [14] shows that $B_2(\lambda, p, q)$ and $B_3(\lambda, p, q)$ can also be estimated by $\frac{1}{t}$ for the operator $Q D_0 Q$. Since the proof of Proposition 7.2 requires the operator $Q D_0 Q$ only to be absolutely bounded it can be adopted to $S S_1$ and $S_1 S$.

Hence, it is enough to establish the $\frac{1}{t}$ bound for the operator $h_{\pm}(\lambda) S_1$. The following Proposition will conclude Theorem 3.1

Proposition 3.15. *If $|a(x)| + |b(x)| \lesssim \langle x \rangle^{-\frac{3}{2}-}$, then we have*

$$\int_{\mathbb{R}^4} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \mathfrak{R}_1^+(\lambda, p, q)[v_1 S_1 v_2](x_1, y_1) d\lambda dx_1 dy_1 = O\left(\frac{1}{t}\right).$$

Recall the calculation from Proposition 3.9

$$\mathfrak{R}_1(\lambda, p, q) = A_1(\lambda, p, q) + A_2(\lambda, p, q) + A_3(\lambda, p, q) + A_4(\lambda, p, q).$$

Not that Theorem 3.1 in [13] establishes the $1/t$ bound for a similar operator to $A_1(\lambda, p, q)$. Using (74) one can adopt the same proof to $A_1(\lambda, p, q)$.

Using the bounds (71), the contribution of

$$A_4(\lambda, p, q) = \frac{i}{2} R_2(x, x_1) M_{22} M_{22} R_2(y_1, y)$$

can be handled as

$$\int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \frac{i}{2} R_2(\lambda^2)(x, x_1) R_2(\lambda^2)(y_1, y) d\lambda$$

$$\lesssim \frac{1}{t} \int_0^{2\lambda_1} |\partial_\lambda [R_2(\lambda^2)(x, x_1) R_2(\lambda^2)(y_1, y)]| d\lambda$$

$$\lesssim k(x, x_1) k(y, y_1) O\left(\frac{1}{t}\right).$$

The assertion for $A_4(\lambda, p, q)$ follows with $\|v_1(x_1)k(x, x_1)\|_{L^2_{x_1}} \lesssim 1$.

To prove the contribution of the operators $A_2(\lambda, p, q)$ and $A_3(\lambda, p, q)$ we need the following lemma.

Lemma 3.16. *If $|a(x)| + |b(x)| \lesssim \langle x \rangle^{-\frac{3}{2}-}$, then we have*

$$\int_{\mathbb{R}^4} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) A_2(\lambda, p, q) v_1(x_1)[S_1](x_1, y_1) v_2(y_1) d\lambda dx_1 dy_1 = O\left(\frac{1}{t}\right).$$

The same bound is valid if $A_2(\lambda, p, q)$ is exchanged with $A_3(\lambda, p, q)$.

Proof. We have to consider the large and the small energy contribution separately.

Case 1: $\lambda |x - x_1| \lesssim 1$. Recall that

$$A_2(\lambda, p, q) = C J_0(\lambda p)(\log(\lambda) + 1) R_2(\lambda^2)(y_1, y) + z Y_0(\lambda p) R_2(\lambda^2)(y_1, y)$$

for some $C \in \mathbb{R}$ and $z \in \mathbb{C}$. Taking this expansion and the projection property (74)

of S_1 into account it is enough to consider the contribution of the following two integrals:

$$\int_0^\infty e^{it\lambda^2}\lambda\chi(\lambda)F(\lambda, x, x_1)R_2(\lambda^2)(y_1, y)d\lambda, \tag{98}$$

$$\int_0^\infty e^{it\lambda^2}\lambda\chi(\lambda)\log(\lambda)G(\lambda, x, x_1)R_2(\lambda^2)(y_1, y)d\lambda. \tag{99}$$

By integration by part once, we have

$$\begin{aligned}
|98| &\lesssim \frac{1}{t}\int_0^\infty e^{it\lambda^2}\chi'(\lambda)F(\lambda, x, x_1)R_2(\lambda^2)(y_1, y)d\lambda \\
&\quad + \frac{1}{t}\int_0^{2\lambda_1}|\partial_\lambda F(\lambda, x, x_1)R_2(\lambda^2)(y_1, y)|d\lambda \\
&\quad + \frac{1}{t}\int_0^{2\lambda_1}|F(\lambda, x, x_1)\partial_\lambda R_2(\lambda^2)(y_1, y)|d\lambda \\
&\lesssim \frac{k(y, y_1)k(x, x_1)}{t} \\
&\quad + \frac{k(y, y_1)}{t}\int_0^{2\lambda_1}|F(\lambda, x, x_1)|d\lambda \\
&\quad + \frac{1}{t}\int_0^{2\lambda_1}|\partial_\lambda F(\lambda, x, x_1)|d\lambda \\
&\lesssim \frac{k(y, y_1)k(x, x_1)}{t}.
\end{aligned}$$

For the last inequality observe that by Lemma 2.19 we have $|F(\lambda, x, x_1)| \lesssim k(x, x_1)$ and

$$\begin{aligned}
\int_0^{2\lambda_1}|\partial_\lambda F(\lambda, x, x_1)|d\lambda &\lesssim \int_0^{2\lambda_1}[p|\chi'(\lambda p)\log(\lambda p)| + q|\chi'(\lambda q)\log(\lambda q)|d\lambda \\
&\lesssim k(x, x_1)]d\lambda + \int_0^{2\lambda_1}\frac{1}{\lambda}|\chi(\lambda p) - \chi(\lambda q)|d\lambda \\
&\lesssim k(x, x_1).
\end{aligned}$$

To see the last equality note that first integral is bounded by a constant because of the support of χ'. The second integral is bounded by $\log\left(\frac{x-x_1}{|x|+1}\right)$ since the support of $\chi(\lambda p) - \chi(\lambda q)$ is $\left[\frac{\lambda_1}{2p}, \frac{2\lambda_1}{q}\right]$.

With a similar argument one can conclude that $|99| \lesssim \frac{\langle x_1\rangle^{1/2}k(y, y_1)}{t}$.

Case 2: $\lambda|x - x_1| \gtrsim 1$. Note that using (74) the λ-integral of

$$\int_{\mathbb{R}^4} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \log(\lambda) \tilde{J}_0(\lambda p)$$
$$M_{11}[v_1 S v_2] M_{22}(x_1, y_1) R_2(\lambda^2)(y_1, y) d\lambda dx_1 dy_1$$

can be written as

$$\int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \log(\lambda) [\tilde{J}_0(\lambda p) - \tilde{J}_0(\lambda(1 + |x|))] R_2(\lambda^2)(y_1, y) d\lambda. \quad (100)$$

Let $s = \max(|x - x_1|, 1 + |x|)$ and $r = \min(|x - x_1|, 1 + |x|)$. Using the large energy representation (23) of Bessel functions and pulling the slower oscillation $e^{\pm i\lambda r}$ out, (100) can be rewritten as the sum of

$$\int_0^\infty e^{it(\lambda^2 \pm \lambda r t^{-1})} \lambda \chi(\lambda) \log(\lambda) \tilde{G}_\pm(\lambda, s, r) R_2(\lambda^2)(y_1, y) d\lambda, \quad (101)$$

where

$$\tilde{G}^\pm(\lambda, s, r) := \tilde{\chi}(\lambda s) \omega_\pm(\lambda s) - e^{\pm i\lambda(s-r)} \tilde{\chi}(\lambda r) \omega_\pm(\lambda r).$$

By Lemma 3.7 in [13], for $0 < \tau < 1$ and $\lambda \le 2\lambda_1$, $\tilde{G}^\pm(\lambda, s, r)$ satisfies

$$|\tilde{G}^\pm(\lambda, s, r)| \lesssim (\lambda|r - s|)^\tau \left(\frac{\tilde{\chi}(\lambda r)}{|\lambda r|^{\frac{1}{2}}} + \frac{\tilde{\chi}(\lambda s)}{|\lambda s|^{\frac{1}{2}}} \right), \quad (102a)$$

$$|\partial_\lambda \tilde{G}^\pm(\lambda, s, r)| \lesssim |s - r| \left(\frac{\tilde{\chi}(\lambda r)}{|\lambda r|^{\frac{1}{2}}} + \frac{\tilde{\chi}(\lambda s)}{|\lambda s|^{\frac{1}{2}}} \right). \quad (102b)$$

In Lemma 3.8 of [13], it is proven that

$$\int_0^\infty e^{it(\lambda^2 \pm \lambda r t^{-1})} a(\lambda) d\lambda = O\left(\frac{1}{t^2} \right)$$

provided

$$|a(\lambda)| \lesssim k(x, x_1) \langle y_1 \rangle^{0+} \lambda \chi(\lambda) \left(\frac{\tilde{\chi}(\lambda r)}{|\lambda r|^{\frac{1}{2}}} + \frac{\tilde{\chi}(\lambda s)}{|\lambda s|^{\frac{1}{2}}} \right), \quad (103)$$

$$|a'(\lambda)| \lesssim k(x, x_1) \langle y_1 \rangle \lambda \chi(\lambda) \left(\frac{\tilde{\chi}(\lambda r)}{|\lambda r|^{\frac{1}{2}}} + \frac{\tilde{\chi}(\lambda s)}{|\lambda s|^{\frac{1}{2}}} \right). \quad (104)$$

Hence, it is enough to show that $a(\lambda) = \lambda \chi(\lambda) \log(\lambda) \tilde{G}_\pm(\lambda, s, r) R_2(\lambda^2)(y_1, y)$ satisfies (103) and (104), which follows immediately from the inequalities (71), (102), and Lemma 2.19.

$\square$

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