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Completeness of rank one perturbations of normal operators with lacunary spectrum

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Abstract. Suppose $\mathcal{A}$ is a compact normal operator on a Hilbert space H with certain lacunarity condition on the spectrum (which means, in particular, that its eigenvalues go to zero exponentially fast), and let $\mathcal{L}$ be its rank one perturbation. We show that either infinitely many moment equalities hold or the linear span of root vectors of $\mathcal{L}$, corresponding to non-zero eigenvalues, is of finite codimension in H. In contrast to classical results, we do not assume the perturbation to be weak.

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Contents

1. Introduction and main results

1.1. The main result. Let $\mathcal{L}$ be a compact operator on a separable Hilbert space H. We will say that $\mathcal{L}$ is *complete* if its root vectors, corresponding to non-zero eigenvalues, are complete in H. (Notice that the point 0 of the spectrum of a compact operator plays a special role.) We will say that $\mathcal{L}$ is *nearly complete* if the root vectors of $\mathcal{L}$, corresponding to non-zero eigenvalues, span a subspace of H of finite codimension. One can observe that for any positive integer N, the closed linear span $M(\mathcal{L})$ of root vectors of $\mathcal{L}$, corresponding to non-zero eigenvalues, is contained in $(\ker \mathcal{L}^{*N})^{\perp}$. So, whenever $\ker \mathcal{L}^* \neq 0$, $\mathcal{L}$ cannot be complete and we only can expect the near completeness.

If one wishes to apply the Fourier method to a linear evolution equation associated with a non-normal compact operator $\mathcal{L}$, the first obstacle is that eigenvectors of $\mathcal{L}$ need not form an orthogonal basis. The completeness of $\mathcal{L}$ is probably the weakest substitute of this property. The strongest one in the non-normal case is the Riesz basis property, and there is a whole range of intermediate properties, related with availability of different linear summation methods, such as Cesàro or Abel summability. Most general abstract sufficient conditions for completeness are due to Keldyš [20, 21] and Macaev [24, 25]. A good exposition of these results by Macaev and their generalizations to operator pencils can be found in [26].

Theorem A ([Keldyš, 1951]). *Let A, S be compact Hilbert space operators. Suppose A is normal, belongs to a Schatten ideal $\mathfrak{S}_p$, $0 < p < \infty$, and its spectrum is contained in a finite union of rays* $\arg z = \alpha_k$, $1 \le k \le n$. *Suppose* $\ker A = \ker(I + S) = 0$. *Put $L = A(I + S)$. Then the operators L and L^* are complete.*

The original statement by Keldyš referred only to the case of a selfadjoint operator A; the above formulation appears, for instance, in [29]. A perturbation of a compact operator A of the form $A(I + S)$ or $(I + S)A$, with S compact, is called a *weak perturbation*. Macaev's theorems also concern weak perturbations. In [27], Macaev and Mogul′skii give an explicit condition on the spectrum of A, equivalent to the property that all weak perturbations of A with $\ker(I + S) = 0$ are complete.

Our results deal with the situation which is much more special than in the Keldyš and Macaev theorems: namely, we consider one-dimensional perturbations of normal operators, which are not necessarily weak, and also treat the case of nontrivial kernels. In this case the spectral theory of one-dimensional perturbations of normal (even of selfadjoint) operators becomes rich and complicated (see [3, 2] where a functional model for such perturbations is constructed).

Note that one of the basic methods to prove the near completeness of an operator is to obtain growth estimates of its resolvent and then to apply some appropriate results from the theory of entire functions. This is essentially the method used by Keldyš and Macaev (for a general statement of this type see [12, Theorem XI.9.29]). In the present paper we consider one-dimensional perturbations of normal operators with very sparse (lacunary) spectra. Using the resolvent estimates we will show that in this case a one-dimensional perturbation is nearly complete unless it has strong degeneracy. This phenomenon is related to the lacunarity of the spectrum and does not appear in general.

Let $\{\lambda_n\}$ be a sequence of complex numbers. We will say that this sequence is *lacunary* if there is a positive constant ε such that $|\lambda_n - \lambda_m| \ge \varepsilon \max(|\lambda_m|, |\lambda_n|)$ for all indices $n \ne m$. This is equivalent to the condition that for some $\delta > 0$, the discs $B(\lambda_n, \delta|\lambda_n|)$ are disjoint. Such sequence can accumulate only to 0 and to ∞. If λ_n tend to zero as $n \to +\infty$ and are numbered so that the moduli $|\lambda_n|$ decrease, then they decay exponentially fast, moreover, there exist some $\sigma \in (0,1)$ and some $B \ge 1$ such that $|\lambda_m/\lambda_n| \le B\sigma^{m-n}$ whenever $m \ge n$.

Suppose that $\mathcal{A}$ is a compact normal operator with trivial kernel. By the Spectral Theorem,

$$\mathcal{A} = \sum_{n\in\mathbb{N}} s_n P_n, \tag{1.1}$$

where $\mathbb{N} = \{1, 2, \dots\}$, $s_n \ne 0$, $s_n \to 0$, and P_n are finite dimensional orthogonal projections in H such that $P_n P_m = 0$ for $m \ne n$ and $\sum_n P_n = I$. Our main object of study will be a one-dimensional perturbation of $\mathcal{A}$ of the following form:

$$\mathcal{L}x = \mathcal{A}x + \langle x, b\rangle\, a, \quad a, b \in H. \tag{1.2}$$

To formulate our results, we need to introduce the following sequence of "moment" equations

$$\sum_n s_n^{-1}\langle P_n a, b\rangle = -1, \quad (\mathrm{M}_1) \tag{1.3a}$$

$$\sum_n s_n^{-k}\langle P_n a, b\rangle = 0, \qquad (\mathrm{M}_k) \tag{1.3b}$$

$k = 2, 3, \dots$. Note that, for a general one-dimensional perturbation, the above series need not converge.

Our first main result is Theorem 1.3 below. It might be instructive to precede it with two simpler statements.

Theorem 1.1. *Let $\mathcal{A}$ be a normal operator given by* (1.1) *which belongs to a Schatten ideal $\mathfrak{S}_p$, $0 < p < \infty$, and whose spectrum is contained in a finite union of rays $\arg z = \alpha_\ell$, $1 \le \ell \le n$. Let $\mathcal{L}$ be a one-dimensional perturbation of $\mathcal{A}$, given by* (1.2)*. Assume that, for some $k \in \mathbb{N}$, we have*

$$\sum_n |s_n|^{-k} |\langle P_n a, b\rangle| < \infty,$$

but the equality (M_k) *does not hold. Then $\mathcal{L}$ and $\mathcal{L}^*$ are nearly complete.*

Moreover, for any $\varepsilon > 0$ there is a radius $r > 0$ such that the intersection of the non-zero spectrum $\sigma(\mathcal{L}) \setminus \{0\}$ with the disc $B(0, r)$ is contained in the union of angles $\alpha_\ell - \varepsilon < \arg z < \alpha_\ell + \varepsilon$, $1 \le \ell \le n$.

This assertion can be obtained by an application of standard methods based on resolvent estimates, see Section 2.

Another simple observation is that $\bigcup_n \ker \mathcal{L}^{*n}$ is orthogonal to $M(\mathcal{L})$. So, if the linear manifold $\bigcup_n \ker \mathcal{L}^{*n}$ is infinite dimensional, then, obviously, $\mathcal{L}$ is not nearly complete. It is easy to see that the following fact holds.

Proposition 1.2. *$\bigcup_n \ker \mathcal{L}^{*n}$ is infinite dimensional if and only if $b \in \operatorname{Ran} \mathcal{A}^n$ for any integer $n > 0$ and the equalities* (M_k) *hold for all $k \ge 1$.*

The next theorem shows that a stronger statement than Theorem 1.1 holds for the case of a *lacunary* spectrum (with arbitrary geometry). At the same time, it can be seen as a partial converse of the above sufficient condition for the failure of near completeness.

Theorem 1.3. *Let $\mathcal{L}$ given by* (1.2) *be a one-dimensional perturbation of a compact normal operator $\mathcal{A}$, given by* (1.1)*, whose spectrum is lacunary. If $\mathcal{L}$ is not nearly complete, then the equalities* (M_k) *are valid for all $k \ge 1$.*

Remarks. 1. Since $\mathcal{L}^* x = \mathcal{A}^* x + \langle x, a\rangle b$, it follows that the same criterion holds for near completeness of $\mathcal{L}^*$, where equalities (M_k) are literally the same.

2. In the case when all moment equalities $(M_k), k \ge 1$, are fulfilled, the operator $\mathcal{L}$ may be complete or incomplete. In [3, Theorem 1.3], given any compact selfadjoint operator $\mathcal{A}$ with simple point spectrum and trivial kernel, a bounded rank one perturbation $\mathcal{L}$ of $\mathcal{A}$ with real spectrum was constructed such that $\mathcal{L}$ is complete and $\ker \mathcal{L} = 0$, while $\mathcal{L}^*$ is even not nearly complete. Therefore the near completeness of $\mathcal{L}$ is not equivalent to the near completeness of $\mathcal{L}^*$, even for rank one perturbations of normal operators with lacunary spectrum we are considering here.

It is essential for the construction in [3] that all moment equalities hold. For the lacunary spectra this follows also from our Theorem 1.3.

3. It would be interesting to know whether an analogue of Theorem 1.3 holds true for finite rank perturbations. It also would be interesting to know whether in the conclusion of this theorem, one can replace the completeness with the spectral synthesis property.

1.2. Sharpness of Theorem 1.3. Our second main result says that the lacunarity condition in Theorem 1.3 cannot be weakened much. Namely, if the spectrum of $\mathcal{A}$ is at least slightly more dense than a lacunary one, then there exists a rank one perturbation, which is not nearly complete, but already the first moment does not exist,

$$\sum_n |s_n|^{-1} |\langle P_n a, b\rangle| = \infty. \tag{1.4}$$

To make the conditions on the spectrum clearer it is better to pass to the inverses $t_n = s_n^{-1}$. Note that the lacunarity implies that $n_T(r) = O(\log r)$, $r \to \infty$. Here n_T is the counting function of the sequence $\{t_n\}$: $n_T(r) = \#\{n : |t_n| < r\}$. We show that rank one perturbations satisfying (1.4), which fail to be nearly complete, always exist unless $n_T(r) = O(\log^2 r)$, $r \to \infty$.

The precise formulation of our second main result is as follows:

Theorem 1.4. *Let $\mathcal{A}_0$ be a compact selfadjoint operator, which has a representation of the form (1.1), where $s_n \in \mathbb{R}$, $s_n \neq 0$. Suppose that* rank $P_n = 1$ *for all n. Put $t_n = s_n^{-1}$. Assume that $\inf_{n\neq k} |t_n - t_k| > 0$ and that, for any $N > 0$, we have*

$$\liminf_{|n|\to\infty} |t_n|^N \prod_{k:\frac{1}{2}\le\frac{t_k}{t_n}\le 2,\, k\neq n} \left|\frac{t_k - t_n}{t_k}\right| = 0. \tag{1.5}$$

Then there exists a rank one perturbation $\mathcal{L}_0 = \mathcal{A}_0 + \langle\cdot, b\rangle a$ of $\mathcal{A}_0$ such that (1.4) holds, but $\mathcal{L}_0$ is not nearly complete. It is true, in particular, if

$$\limsup_{r\to\infty} \frac{n_T(r)}{\log^2 r} = +\infty. \tag{1.6}$$

We will show, in fact, that (1.6) implies (1.5) (see Corollary 6.2). One can express the condition (1.5) in equivalent ways, see the remark in Subsection 6.3.

1.3. Methods of the proof. This work can be considered as a continuation of our papers [3, 4], where the completeness and related properties (e.g., the spectral synthesis) were studied for similar class of perturbations of selfadjoint operators without an assumption on the lacunarity of the spectrum. As in [3, 4], here we also will consider (singular) rank one perturbations of unbounded normal operators with discrete spectrum and obtain parallel completeness results for them.

The spectral analysis of the perturbation (1.2) of operator $\mathcal{A}$ leads to a consideration of the function

$$\beta(z) = 1 + \langle(\mathcal{A} - z^{-1})^{-1}a, b\rangle = 1 - z\langle(I - z\mathcal{A})^{-1}a, b\rangle = 1 + \sum_n c_n\Big(\frac{1}{t_n - z} - \frac{1}{t_n}\Big), \tag{1.7}$$

where $t_n = 1/s_n$ and $c_n = -s_n^{-2}\langle P_n a, b\rangle$. This function is meromorphic in $\mathbb{C}$. It is easy to see that the zero set of β coincides with the set $\{\lambda^{-1}: \lambda \in \sigma(\mathcal{L}), \lambda \neq 0\}$.

We will adopt the following notation. If A is a measurable subset of $[0, +\infty)$, its *linear density* is defined as $\lim_{R\to+\infty} R^{-1}m([0, R] \cap A)$. Given a function f on $\mathbb{C}$, we will write $\lim^*_{z\to\infty} f(z) = w$ if there exists a closed set $A \subset [0, +\infty)$ of linear density one such that

$$\lim_{z\to\infty,\, |z|\in A} f(z) = w.$$

Our main complex variable tool for proving Theorem 1.3 will be the following statement.

Theorem 1.5. *Suppose that a complex sequence $\{t_n\}$ goes to infinity, is lacunary and $t_n \neq 0$ for all n. Let $\varkappa \in \mathbb{C}$ and let c_n be any complex coefficients, not all equal to zero, such that $\sum_n |c_n/t_n^2| < +\infty$. Put*

$$\beta(z) = \varkappa + \sum_n c_n\Big(\frac{1}{t_n - z} - \frac{1}{t_n}\Big). \tag{1.8}$$

If for any $s \in \mathbb{N}$,

$$\lim\nolimits^*_{z\to\infty} z^s\beta(z) = 0, \tag{1.9}$$

then $\sum_n t_n^{-1}c_n = \varkappa$ and $\sum_n t_n^k c_n = 0$ for $k \in \mathbb{Z}$, $k \geq 0$.

It is easy to see that the conditions $\sum_n t_n^{-1}c_n = \varkappa$ and $\sum_n t_n^k c_n = 0$, $k \geq 0$, imply that $\lim z^s\beta(z) = 0$ as $|z| \to \infty$ for any s and $\operatorname{dist}(z, \{t_n\}) \geq 1$. Theorem 1.5 shows that in the lacunary case the converse is true. The proof of this theorem uses a lemma on “peaks” of numerical sequences by Pólya [31], which enables us to obtain some estimates of the function β from below.

Let us mention that in [33], Shkalikov obtained lower estimates for meromorphic functions outside small “exceptional” sets, which he applied in [34] to get a new criterion for eigenvectors of a perturbed selfadjoint operator to form a basis with parenthesis.

The proof of Theorem 1.3 is based on the above Theorem 1.5 and deals directly with resolvent estimates. On the other hand, an important tool in proving Theorem 1.4 is a passage to the functional model of $\mathcal{L}$ in a so-called de Branges space $\mathcal{H}(E)$ of entire functions, associated to an entire function E in the Hermite–Bieler class (see Subsection 6.1 below). The completeness property for $\mathcal{L}^*$ turns out to be equivalent to completeness of a certain system of reproducing kernels in $\mathcal{H}(E)$. This allows us to apply complex analysis tools, such as factorizations of entire functions and partial fraction expansions of their quotients.

De Branges spaces and the related model spaces associated to meromorphic inner functions have numerous and well-known applications to the spectral theory of discrete selfadjoint operators. For instance, Makarov and Poltoratski apply this approach in [28] to get broad extensions of several classical results such as Borg's two spectra theorem and the Hochstadt–Lieberman theorem concerning the unique determination of Schrödinger operators on a finite interval by their spectra. In [35], de Branges spaces were applied to model a subclass of singular Schrödinger operators. In our previous papers [3, 4], we applied the model approach to the completeness of nonselfadjoint rank one perturbations of a discrete selfadjoint operator. These papers also exploit the property of spectral synthesis, which is weaker than the existence of whatever linear summation method for expansions in eigenvectors. Completeness problems for "mixed" systems were treated in recent papers [8, 5, 6, 7]. In particular, in [8], Riesz bases of reproducing kernels in spaces of Cauchy transforms of discrete lacunary measures in $\mathbb{C}$ have been described. It seems that there are some relations between this work and the questions we study here.

We also wish to mention here papers [30] and [36], where de Branges spaces were applied to the spectral theory in a context close to the context of [3, 4].

A result on completeness or near completeness of an abstract compact operator in a Schatten class $\mathfrak{S}_p$ is contained in [12, Theorem XI.9.29], where power estimates of the resolvent in certain angles are assumed. We remark that, although Keldyš and Macaev's theorems can be deduced from this theorem, the condition $\ker(I + S) = 0$ is crucial for getting these power estimates (see the Example 4.4 below).

We refer to [1, 34, 38] for some recent abstract results about completeness and bases of eigenvectors and to the books [16, 17, 12, 29] and the review [32] for an extensive exposition. The papers [19, 15, 14] contain some general results on spectral properties of finite dimensional perturbations of diagonal operators (in particular, on the existence of invariant subspaces). It seems that not much is known in general for weak perturbations with $\ker(I + S) \neq 0$ and for non-weak perturbations.

The paper is organized as follows. In Section 2 we collect some preliminary results relating the near completeness to properties of some entire (meromorphic) functions defined in terms of resolvents. Section 3 contains the proof of our main result, Theorem 1.3. Some additional remarks on the general properties of nearly complete operators are given in Section 4. In Section 5 we introduce (unbounded) singular rank one perturbations and state the counterpart of the main result for this case. Theorem 1.4 about the necessity of some (slightly weaker than lacunarity) sparseness condition is proved in Section 6 by an application of a functional model.

Selfadjoint operators with lacunary spectrum are not so seldom in practice. In Section 4, we give examples of convolution operators on L^2, which are normal, compact and lacunary. In Section 5, we comment on Jacobi matrices, which define unbounded selfadjoint lacunary operators. They arise, in particular, in relation with q-harmonic oscillator and q-orthogonal polynomials.

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2. Preliminary results on near completeness

We start with several simple remarks which will be used in what follows. Note that they are true for general compact operators (not necessarily with lacunary spectra).

Proposition 2.1. *Let $\mathcal{L}$ be a general compact operator on a Hilbert space H. If $\mathcal{L}$ is nearly complete, then the linear set $\bigcup_n \ker \mathcal{L}^{*n}$ is finite dimensional, there is some $k \ge 0$ such that $\ker \mathcal{L}^{*n} = \ker \mathcal{L}^{*k}$ for all $n > k$ and $M(\mathcal{L})^\perp = \bigcup_n \ker \mathcal{L}^{*n} = \ker \mathcal{L}^{*k}$.*

The proof is immediate from the fact that $\mathcal{L}^*|M(\mathcal{L})^\perp$ is always quasinilpotent.

Next, let us make the following observation. Let H^1 be the smallest reducing subspace of $\mathcal{A}$ containing the vectors a, b, and put $H^2 = H \ominus H^1$. Then $\mathcal{L}$

decomposes as $\mathcal{L} = \mathcal{L}^1 \oplus \mathcal{A}^2$, where $\mathcal{L}^1 x = \mathcal{A}x + \langle x, b\rangle a$, $x \in H^1$ and $\mathcal{A}^2 = \mathcal{A}|H^2$. This implies that it suffices to prove Theorem 1.3 for the case when the smallest reducing subspace of $\mathcal{A}$ containing vectors a, b coincides with H. From now on, we will assume that this property is fulfilled. It follows that all spectral projections P_n in (1.1) have at most rank two.

Put $M = M(\mathcal{L})$. Given any $x \in H$ and any $y \in M^\perp$, the function

$$f_{x,y}(\lambda) = \langle (I - \lambda\mathcal{L})^{-1} x, y\rangle$$

is entire.

We will use an easy formula

$$(\mathcal{L} - z)^{-1} = (\mathcal{A} - z)^{-1} - (\mathcal{A} - z)^{-1} a\, \beta^{-1}(z^{-1})\, b^*(\mathcal{A} - z)^{-1}, \tag{2.1}$$

where $\beta(z)$ is defined in (1.7) (here $b^*\colon H \to \mathbb{C}$, $b^*(h) = \langle h, b\rangle$). To check it, one can write down the second resolvent identity

$$(\mathcal{L} - z)^{-1} - (\mathcal{A} - z)^{-1} = -(\mathcal{A} - z)^{-1} a b^* (\mathcal{L} - z)^{-1}.$$

By multiplying it by b^* on the left, one gets $b^*(\mathcal{L} - z)^{-1} = \beta^{-1}(z^{-1}) b^*(\mathcal{A} - z)^{-1}$, which gives (2.1). Then, for any $s \in \mathbb{N}$, $x \in H$ and $y \in M^\perp$ we have

$$\begin{aligned} z^{-s-3} & f_{x,y}(z) \\ &= z^{-s-4} y^*(z^{-1} - \mathcal{L})^{-1} x \\ &= z^{-s-4} y^*(z^{-1} - \mathcal{A})^{-1} x \\ &\qquad + [z^{-2} y^*(\mathcal{A} - z^{-1})^{-1} a] \cdot z^{-s}\beta^{-1}(z) \cdot [z^{-2} b^*(\mathcal{A} - z^{-1})^{-1} x]. \end{aligned} \tag{2.2}$$

Proposition 2.2. *Suppose $N \in \mathbb{Z}$, $N \geq 0$ and $\mathcal{L}$ is a compact operator. Then the following statements are equivalent*:

(i) $M^\perp \subset \ker \mathcal{L}^{*N}$;

(ii) *For any $x \in H$ and any $y \in M^\perp$, $f_{x,y}(\lambda)$ is a polynomial in λ of degree less than N.*

Proof. Since $f_{x,\mathcal{L}^* y}(\lambda) = \lambda^{-1}[f_{x,y}(\lambda) - f_{x,y}(0)]$, it follows that (ii) is equivalent to the condition $f_{x,\mathcal{L}^{*N} y} \equiv 0$ for all $x \in H$, $y \in M^\perp$, that is, to the condition $M^\perp \subset \ker \mathcal{L}^{*N}$. This gives the statement. □

Suppose, in particular, that $\ker \mathcal{L}^{*N}$ is finite dimensional for any $N \in \mathbb{N}$ (it is true for the operator $\mathcal{L}$, given by (1.2)). Then, by the Propositions 2.2 and 2.1, $\mathcal{L}$ is nearly complete if and only if there is an integer $N > 0$ such that for any $x \in H$ and any $y \in M^\perp$, $f_{x,y}(\lambda)$ is a polynomial in λ of degree less than N.

Remark. If the series in (M_1) converges absolutely, then $\mathcal{L}$ is what we called in [3] a *generalized weak perturbation* of $\mathcal{A}$. The case when the hypotheses of Theorem 1.1 hold for $k = 1$ was treated in [3, Theorem 3.3] for rank one perturbations of selfadjoint operators (the assumption that $\mathcal{A} \in \mathfrak{S}_p$ can be dropped). In this case the completeness of a perturbation can be also derived from the results of Macaev as in the proof of [3, Proposition 1.1].

Proof of Theorem 1.1. Assume that $k \in \mathbb{N}$ is the smallest positive integer such that $\sum_n |s_n|^{-k}|\langle P_n a, b\rangle| < \infty$ but the equality (M_k) does not hold. By the above remark, assume that $k \geq 2$. Put $t_n = s_n^{-1}$ and define $\beta(z)$ by (1.7), where $c_n = -t_n^2\langle P_n a, b\rangle$. Then we have $\sum_n |t_n|^k|c_n| < \infty$ and $\sum_n t_n^k c_n = \gamma \neq 0$. By the obvious formula

$$\frac{1}{z-t_n} = \sum_{j=0}^{k} \frac{t_n^j}{z^{j+1}} + \frac{t_n^{k+1}}{z^{k+1}(z-t_n)},$$

we have

$$\beta(z) = -\frac{1}{z^{k+1}}\sum_n t_n^k c_n + \frac{1}{z^{k+1}}\sum_n \frac{t_n^{k+1}c_n}{t_n - z} = \frac{\gamma}{z^{k+1}} + \frac{1}{z^{k+1}}\sum_n \frac{t_n^{k+1}c_n}{t_n - z}.$$

Recall that $\{t_n\}$ is contained in a finite union of rays $\arg z = \alpha_\ell$, $1 \leq \ell \leq n$. Hence, for any $\varepsilon > 0$, we have

$$|\beta(z)| > \frac{c_\varepsilon}{|z|^{k+1}}$$

when $|\arg z - \alpha_\ell| \geq \varepsilon$, $1 \leq \ell \leq n$ and $|z|$ is sufficiently large. Clearly, $\|(\mathcal{A} - z^{-1})^{-1}\|$ also admits a power estimate for such values of z and we conclude, by (2.2), that $|f_{x,y}(z)|$ admits a power estimate for $|\arg z - \alpha_\ell| \geq \varepsilon$. Since $\mathcal{A} \in \mathfrak{S}_p$, we get that $f_{x,y}$ is a function of finite order and, by the Phragmén–Lindelöf principle, we conclude that $f_{x,y}$ is a polynomial of degree less than some fixed N for any $x \in H$, $y \in M^\perp$. □

Now we pass to the analysis of the case when the spectrum is lacunary.

Given an entire function F, we use notations $M_F(r) = \max_{|z|=r}|F(z)|$, $m_F(r) = \min_{|z|=r}|F(z)|$, and put $n_F(r)$ to be the number of zeros of F in the disc $|z| < r$, counted with multiplicities. In what follows, we will say that an entire function F *is of class* Slow if it is of zero order and $\log M_F(r) = \mathcal{O}\big((\log r)^2\big)$ as $r \to \infty$; the last condition can be replaced by the condition $n_F(r) = \mathcal{O}(\log r)$. We will use the following version of [10, Theorem 3.6.1].

Theorem B. *For any entire function F of the class* Slow, *which is not a polynomial, and any $N \in \mathbb{N}$, one has* $\lim^*_{z\to\infty} |z|^{-N}|F(z)| = +\infty$.

Lemma 2.3. *Let $x \in H$ and $y \in M^{\perp}$. Then the entire function $f_{x,y}$ belongs to the class* Slow.

Proof. It is well-known that $\mathcal{L}^*|M^{\perp}$ is quasinilpotent (see, for instance, [29]). Given a linear operator B on a Hilbert space, we denote by $\{\mu_j(B)\}_{j\geq 1}$ the sequence of its singular numbers. The lacunarity of the spectrum of $\mathcal{A}$, the property that rank $P_n \leq 2$ for all n and the well-known estimates for singular numbers of the sum of two operators [12, Corollary XI.9.3] imply that

$$\mu_j(\mathcal{L}^*|M^{\perp}) \leq \mu_j(\mathcal{L}^*) \leq C\sigma^j,$$

where $\sigma < 1$ is a constant. In particular, $\mathcal{L}^*|M^{\perp}$ is a trace class operator. By applying the arguments employed in the proof of [12, Theorem XI.9.26], we get that

$$\begin{aligned}\|(I-\bar{\lambda}\mathcal{L}^*)^{-1}|M^{\perp}\| &= \|\det((I-\bar{\lambda}\mathcal{L}^*)|M^{\perp})\cdot(I-\bar{\lambda}\mathcal{L}^*)^{-1}|M^{\perp}\| \\ &\leq \prod_{j=1}^{\infty}\mu_j((I-\bar{\lambda}\mathcal{L}^*)|M^{\perp}) \\ &\leq \prod_{j=1}^{\infty}(1+|\lambda|\mu_j(\mathcal{L}^*|M^{\perp})) \\ &\leq \exp(C(\log|\lambda|)^2)\end{aligned}$$

(notice that $\det\big((I-\bar{\lambda}\mathcal{L}^*)|M^{\perp}\big) \equiv 1$ for all λ). Since

$$|f_{x,y}(\lambda)| = |\langle x, (I-\bar{\lambda}\mathcal{L}^*)^{-1}y\rangle| \leq \|x\|\cdot\|y\|\cdot\|(I-\bar{\lambda}\mathcal{L}^*)^{-1}|M^{\perp}\|,$$

the assertion of Lemma follows. □

Lemma 2.4. *For any normal compact operator $\mathcal{A}$ with lacunary spectrum and any $\delta > 1$, one has* $\lim^*_{z\to\infty}|z|^{-\delta}\,\|(\mathcal{A}-z^{-1})^{-1}\| = 0$.

Proof. Let $1 < \delta_1 < \delta$, and consider the discs $B_n := B(s_n, |s_n|^{\delta_1})$. Assume that $|z| \geq 1$. If $z^{-1} \notin \bigcup_n B_n$, then $|z^{-1}-s_n| \geq \varepsilon|z|^{-\delta_1}$ for all n, where $\varepsilon > 0$ is some constant (consider the cases $|z^{-1}| \geq 2|s_n|$ and $|z^{-1}| < 2|s_n|$). Therefore $|z|^{-\delta_1}\|(z^{-1}-\mathcal{A})^{-1}\| \leq \varepsilon^{-1}$ for all z such that $z^{-1} \notin \mathbb{C}\setminus\bigcup_n B_n$. One gets from the lacunarity of the spectrum that the set $\big\{|z|^{-1} : z \in \bigcup_n B_n\big\}$ has linear density zero, which implies the statement. □

Lemma 2.5. *If $\mathcal{L}$ is not nearly complete, then for any $s \in \mathbb{N}$,*

$$\lim^*_{z\to\infty} z^s \beta(z) = 0,$$

where $\beta(z)$ is defined in (1.7).

Proof. Suppose $\mathcal{L}$ is not nearly complete, so that $M^\perp$ is infinite dimensional. Fix some $s \in \mathbb{N}$. By Proposition 2.2, Lemma 2.3 and Theorem B, there exist $x \in H$ and $y \in M^\perp$ such that $\lim^*_{z\to\infty} |z|^{-s-3}|f_{x,y}(z)| = +\infty$. By Lemma 2.4,

$$\lim^*_{z\to\infty} z^{-2}u^*(\mathcal{A} - z^{-1})^{-1}v = 0$$

for any pair of vectors $u, v \in H$. Since the limit $\lim^*$ of the modulus of the left hand part in (2.2) equals $+\infty$ and the finite intersection of any subsets of $[0, +\infty)$ of linear density one has linear density one, the assertion of the lemma follows. □

3. Proof of Theorems 1.5 and 1.3

First we show how to deduce Theorem 1.3 from Theorem 1.5.

Proof of Theorem 1.3 assuming Theorem 1.5. Let $\mathcal{L}$ have the form given in the Theorem. Put $t_n = s_n^{-1}$ and define $\beta(z)$ by (1.7), where $c_n = -t_n^2\langle P_n a, b\rangle$. Assume $\mathcal{L}$ is not nearly complete; then by Lemma 2.5, equality (1.9) holds for any positive integer s. Therefore Theorem 1.5 gives us the conclusions of Theorem 1.3. □

The rest of this section is devoted to the proof of Theorem 1.5.

Lemma 3.1. *Let r be a positive integer and let $f \in C^r[a, b]$ be a real function. If $|f^{(r)}| > \varepsilon > 0$ on $[a, b]$, then there exists a subinterval $[c, d]$ of $[a, b]$ of length $\frac{b-a}{3^r}$ such that $|f(x)| \ge \left(\frac{b-a}{6}\right)^r \varepsilon$ for all $x \in [c, d]$.*

Proof. Consider first the case when $r = 1$. Then we can assume without loss of generality that $f' > \varepsilon$ on $[a, b]$. Let $[a, b] = I_1 \cup I_2 \cup I_3$ be the subdivision of $[a, b]$ into three subsequent equal intervals. Then one can take $[c, d] = I_1$ if $f(\frac{a+b}{2}) < 0$ and $[c, d] = I_3$ in the opposite case.

The case of general r now follows by an obvious induction argument. □

We will use the following result by G. Pólya (1923).

Lemma C (see Pólya [31], p.170). *Let* $\{p(n)\}$, $\{\alpha(n)\}$ *and* $\{q(n)\}$ $(n \in \mathbb{N})$ *be sequences such that* $p(n) \geq 0$, $q(n) \geq 0$, $\alpha(n) > 0$, *and* $q(n) = \alpha(n)p(n)$ *for all* n. *Suppose that* $\limsup p(n) = +\infty$, $\lim q(n) = 0$, *and that the sequence* $\{\alpha(n)\}$ *decreases and tends to* 0. *Then there exists an increasing index sequence* $\{m_k\}$ *such that*

(1) $p(m_k) = \max\{p(s) \colon 1 \leq s \leq m_k\}$ *for all* k;

(2) $q(m_k) = \max\{q(s) \colon s \geq m_k\}$ *for all* k;

(3) $\lim_k p(m_k) = +\infty$.

The main step in the proof of Theorem 1.5 will be the following statement.

Lemma 3.2. *Suppose that the sequences* $\{t_n\}$, $\{c_n\}$ *and a complex number* $\varkappa$ *meet all the conditions of the above Theorem* 1.5, *but instead of* (1.9), *we only require that*

$$\lim\nolimits^*_{|z|\to+\infty} z\beta(z) = 0. \tag{3.1}$$

Then $\sum_n |t_n^{-1} c_n| < +\infty$ *and* $\sum_n t_n^{-1} c_n = \varkappa$.

Proof. Since $\{t_n\}$ is lacunary, it follows that for some constant $\gamma > 0$, $|t_m - t_n| \geq \gamma \max(|t_m|, |t_n|)$ for all $m \neq n$. Also, there are some constants $g, B > 1$ such that $|t_n/t_m| \leq Bg^{n-m}$ for all $n < m$. We assume that the sequence $\{|t_n|\}$ increases.

First let us prove that $\sum_n |t_n^{-1} c_n| < \infty$. Assume it is not so. Choose some $u \in (1, g)$ close to 1. Put $p(n) = u^n |c_n||t_n|^{-1}$, $\alpha(n) = u^{-n}|t_n|^{-1}$, $q(n) = |t_n|^{-2}|c_n|$. Since the series $\sum_n |t_n^{-1} c_n|$ diverges and the series $\sum_n |t_n^{-2} c_n|$ converges, it follows that all the hypotheses of Pólya's lemma are satisfied. Let $\{m_k\}$ be an index sequence that has properties (1)–(3).

Properties (1) and (2) imply that

$$|c_n| \leq u^{m_k - n} \frac{|t_n|}{|t_{m_k}|} |c_{m_k}| \leq B(g^{-1}u)^{m_k - n} |c_{m_k}| \quad \text{for } n < m_k; \tag{3.2a}$$

$$\frac{|c_n|}{|t_n|^2} \leq \frac{|c_{m_k}|}{|t_{m_k}|^2} \quad \text{for } n > m_k. \tag{3.2b}$$

By (1.8),

$$\frac{\beta''(z)}{2} = \sum_{n=1}^{\infty} \frac{c_n}{(z - t_n)^3}.$$

Let $|z - t_{m_k}| \le \varepsilon |t_{m_k}|$, where ε is a small positive constant, which will be chosen later. If $n < m_k$, then

$$|z - t_n| \ge |t_n - t_{m_k}| - |z - t_{m_k}| \ge (\gamma - \varepsilon)|t_{m_k}|.$$

Similarly, if $n > m_k$, then $|z - t_n| \ge |t_n - t_{m_k}| - |z - t_{m_k}| \ge (\gamma - \varepsilon)|t_n|$. Hence inequalities (3.2) imply the estimates

$$\begin{aligned}
&\left|\frac{\beta''(z)}{2} - \frac{c_{m_k}}{(z - t_{m_k})^3}\right| \\
&\quad \le \sum_{n=1}^{m_k - 1} \frac{|c_n|}{(\gamma - \varepsilon)^3 \, |t_{m_k}|^3} + \sum_{n=m_k+1}^{\infty} \frac{|c_n|}{(\gamma - \varepsilon)^3 \, |t_n|^3} \\
&\quad \le \frac{B|c_{m_k}|}{|t_{m_k}|^3 (\gamma - \varepsilon)^3} \sum_{n=1}^{m_k - 1} \left(\frac{u}{g}\right)^{m_k - n} + \frac{B}{(\gamma - \varepsilon)^3} \sum_{n=m_k+1}^{\infty} \frac{|c_{m_k}|}{|t_{m_k}|^3} \, g^{-n+m_k} \\
&\quad \le K(\varepsilon) \left|\frac{c_{m_k}}{(z - t_{m_k})^3}\right| \quad \text{for } 0 < |z - t_{m_k}| < \varepsilon |t_{m_k}|,
\end{aligned} \tag{3.3}$$

where

$$K(\varepsilon) \stackrel{\text{def}}{=} B\left(\frac{\varepsilon}{\gamma - \varepsilon}\right)^3 \left[\frac{u}{g - u} + \frac{1}{g - 1}\right].$$

Choose a small $\varepsilon \in (0, \gamma)$ such that $K(\varepsilon) < \frac{1}{2}$. Let now $\varepsilon_1 \in (0, \varepsilon)$ be a constant and assume that $z \in [(1 + \varepsilon_1) t_{m_k}, (1 + \varepsilon) t_{m_k})$. Then the inequality $|a^{-3} - b^{-3}| \le 3|a - b| \big(\min(|a|, |b|)\big)^{-4}$ and (3.3), together with the triangle inequality, give

$$\left|\frac{\beta''(z)}{2} - \frac{c_{m_k}}{((1 + \varepsilon_1) t_{m_k} - t_{m_k})^3}\right| \le \frac{2}{3} \left|\frac{c_{m_k}}{((1 + \varepsilon_1) t_{m_k} - t_{m_k})^3}\right| \tag{3.4}$$

if ε_1 is sufficiently close to ε. By property (3) from Lemma C, $|c_{m_k}| = u^{-m_k} t_{m_k} p(m_k) \ge \varepsilon_2 > 0$ for all k. Now it follows from (3.4) that there are constants $\zeta \in \mathbb{C}$, $|\zeta| = 1$ and $\rho > 0$ such that

$$|f''(t)| \ge \rho |t_{m_k}|^{-3} |c_{m_k}| \ge \rho \, \varepsilon_2 \, |t_{m_k}|^{-3}$$

for all k and all $t \in [(1 + \varepsilon_1)|t_{m_k}|, (1 + \varepsilon)|t_{m_k}|)$, where

$$f(t) = \operatorname{Re}(\zeta \beta(t \cdot t_{m_k} / |t_{m_k}|)).$$

So Lemma 3.1 yields that there is a subinterval of $[(1 + \varepsilon_1)|t_{m_k}|, (1 + \varepsilon)|t_{m_k}|)$ of length $(\varepsilon - \varepsilon_1)|t_{m_k}|/9$ on which $|f(t)| \ge \varepsilon_3 t^{-1}$, where $\varepsilon, \varepsilon_1, \varepsilon_3 > 0$ do not depend on k. This contradicts the assumption (3.1).

We conclude that the sum $\sum_n |t_n^{-1} c_n|$ converges. Take some $\tau \in (0, \frac{\gamma}{2})$. Then the discs $B(t_n, \tau\, |t_n|)$ are pairwise disjoint. Let $U = U(\tau)$ be their union. Choose τ so small that the set

$$A = \{r > 0\colon \partial B(0, r) \subset \mathbb{C} \setminus U(\tau)\}$$

satisfies $\limsup_{R\to\infty} R^{-1}|A \cap [0, R]| > 0$. Now one can apply the Lebesgue dominated convergence theorem to the sum $\beta(z) = \varkappa + \sum_n \frac{c_n z}{t_n(t_n - z)}$. Since $|z/(t_n - z)| \le C(\tau) < \infty$ for all $z \notin U$ and all n, one gets that the limit of $\beta(z)$ as $|z| \to \infty$, $|z| \in A$, exists and equals $\varkappa - \sum_n t_n^{-1} c_n$. Hence $\sum_n t_n^{-1}\, c_n = \varkappa$. □

Proof of Theorem 1.5. Given an integer $\ell \ge 1$, consider the statements:

- $(1)_\ell$ $\sum_n |c_n t_n^{\ell-2}| < \infty$;
- $(2)_\ell$ $\sum_n c_n t_n^{\ell-2} = 0$ if $\ell \ge 2$ and $\sum_n c_n t_n^{\ell-2} = \varkappa$ if $\ell = 1$;
- $(3)_\ell$ $\beta(z) = z^{-\ell}\beta_\ell(z)$, where

$$\beta_\ell(z) = \sum_n c_n t_n^\ell \Big(\frac{1}{t_n - z} - \frac{1}{t_n}\Big) = z \sum_n c_n t_n^{\ell-1} \frac{1}{t_n - z}.$$

Lemma 3.2 gives $(1)_1$, $(2)_1$ and $(3)_1$. If for some $\ell \ge 1$, $(1)_\ell$, $(2)_\ell$ and $(3)_\ell$ have been obtained, one applies Lemma 3.2 to $\tilde{c}_n = c_n t_n^\ell$ and to β_ℓ and gets properties $(1)_{\ell+1}$ and $(2)_{\ell+1}$, which imply that $\beta_{\ell+1}(z) = z\beta_\ell(z)$. This gives $(3)_{\ell+1}$.

So, by induction, the equalities $(1)_\ell$, $(2)_\ell$ and $(3)_\ell$ hold for any $\ell \ge 1$. □

4. Some additional remarks

First let us give an example of a natural class of compact normal lacunary operators.

Example 4.1. Let η be an analytic function in an annulus $\mathbb{A} = \{z \in \mathbb{C}\colon r < |z| < R\}$, where $0 < r < 1 < R$. Suppose it has the form $\eta(z) = \eta_0(z) + \eta_1(z)$, where $\eta_0(z) = \tau_1(R - z)^{-a} + \tau_2(r^{-1} - z^{-1})^{-a}$ and η_1 is smoother than η_0 in the sense that $\eta_1 \in H^p(\mathbb{A})$ for some $p > 1/a$ (the powers are defined by using the principal branch of the logarithm). Assume that $a > 1$, $\tau_1, \tau_2 \in \mathbb{C}$ are nonzero and $\tau_1/\tau_2 \notin (0, +\infty)$. Denote by $\mathbb{T}$ the unit circle in $\mathbb{C}$. Then the convolution operator

$$(\mathcal{A}_\eta f)(e^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \eta(e^{i(\theta - x)}) f(e^{ix})\, dx$$

on $L^2(\mathbb{T})$ is compact, normal and has lacunary spectrum. Indeed, in the Fourier representation $\mathcal{A}_\eta$ is just the multiplication operator on $\ell^2(\mathbb{Z})$ by the sequence of Fourier coefficients $\{\hat{\eta}(n)\}$ of the function $\eta|_{\mathbb{T}}$. Notice that $\hat{\eta}_0(n) \sim \tau_1 R^{-a-n} n^{a-1}$ as $n \to +\infty$ and $\hat{\eta}_0(n) \sim \tau_2 r^{a-n} |n|^{a-1}$ as $n \to -\infty$, whereas the assumption on η_1 implies that $\hat{\eta}_1(n)/\hat{\eta}_0(n) \to 0$ as $n \to \pm\infty$, see [13, Theorem 6.4]. This gives our assertion.

The next proposition gives an explicit form of the space $\bigcup_n \ker \mathcal{L}^{*n}$ whenever this space is finite dimensional.

Proposition 4.2. *Let $\mathcal{L}$ be given by* (1.2) *and not all conditions* (M_n) *are fulfilled. Choose the integer $k \ge 0$ so that* (M_1), ..., (M_k) *are fulfilled and* (M_{k+1}) *either is not fulfilled or has no sense* (*that is, the sum diverges*). *Let $\ell \ge 0$ be the largest integer such that $b \in \operatorname{Ran} \mathcal{A}^{*\ell}$, and put $b = \mathcal{A}^{*s} b_s$, where $b_s \in H$ $(s = 1, \dots, \ell)$. Then*

$$\bigcup\nolimits_n \ker \mathcal{L}^{*n} = \ker \mathcal{L}^{*m} = \operatorname{span}\{b_1, \dots, b_m\},$$

where $m = \min(k, \ell)$.

We omit the proof, which is completely straightforward. The same calculations imply Proposition 1.2.

Let $\mathcal{L}_1, \mathcal{L}_2$ be two bounded operators on Hilbert spaces H_1, H_2, respectively. In [3], we used the following definition: $\mathcal{L}_2$ is said to be *d-subordinate* to $\mathcal{L}_1$ ($\mathcal{L}_1 \overset{d}{\prec} \mathcal{L}_2$) if there exists a bounded linear operator $Y\colon H_1 \to H_2$, which intertwines $\mathcal{L}_1$ with $\mathcal{L}_2$ and has a dense range:

$$Y\mathcal{L}_1 = \mathcal{L}_2 Y; \quad \operatorname{clos} \operatorname{Ran} Y = H_2.$$

As it was mentioned there, if $\mathcal{L}_1 \overset{d}{\prec} \mathcal{L}_2$ and $\mathcal{L}_1$ is complete then $\mathcal{L}_2$ is complete. In connection with the present article, we can also mention the following fact.

Proposition 4.3. *If $\mathcal{L}_1$ and $\mathcal{L}_2$ are compact, $M(\mathcal{L}_1) = \operatorname{clos} \mathcal{L}_1^N H$ and $\mathcal{L}_2$ is d-subordinate to $\mathcal{L}_1$ then $M(\mathcal{L}_2) = \operatorname{clos} \mathcal{L}_2^N H$.*

The proof is straightforward, and we leave it to an interested reader.

As it follows from (2.1) and Theorem 1.5, if some of the infinite sequence of moment equalities (1.3) fail, then there is an estimate

$$\|(\mathcal{L} - z)^{-1}\| \le |z|^{-s}$$

for the resolvent of $\mathcal{L}$ for a set of the form $\{z \colon |z|^{-1} \in A\}$, where $A \subset [0, +\infty)$ is a closed subset of linear density 1 at infinity. This can be compared with the estimates, which are used in the proof of the Keldyš theorem: in the conditions of this theorem, for any $\varepsilon > 0$, an estimate $\|(\mathcal{L} - z)^{-1}\| \le C_\varepsilon |z|^{-1}$ holds for sufficiently small $|z|$ in the complement of the union of the angles $\alpha_k - \varepsilon \le \arg z \le \alpha_k + \varepsilon$ (see [29], Lemma 3.2). In particular, for each $\varepsilon > 0$ there exists $\delta > 0$ such that all non-zero spectrum of $\mathcal{L}$, which is contained in the disc $|z| < \delta$, lies in the union of the above angles.

The next example shows that for weak perturbations $\mathcal{L} = \mathcal{A}(I + S)$ that do not satisfy the requirement $\ker(I + S) = 0$, the last geometric property of the spectrum does not hold in general.

Example 4.4. There is an operator $\mathcal{L} = \mathcal{A}(I+S)$, which has the form (1.2), where $\mathcal{A}$ is a cyclic selfadjoint operator with lacunary spectrum $\{2^{-n} : n \in \mathbb{N}\}$ such that $\sigma(\mathcal{L}) = \{0\} \cup \{i2^{-n} : n \in \mathbb{N}, n \ge 2\}$.

To construct this operator, consider the function

$$\psi(z) = \frac{2}{2-z} \prod_{n=2}^{\infty} \varphi_n(z), \quad \text{where } \varphi_n(z) = \frac{2^n + iz}{2^n - z}.$$

Notice that, given any constant $K \in (0,1)$, there exists $C_K > 0$ such that $|\varphi_n(z) - 1| \le C_K 2^{-n}|z|$ if $2^{-n}|z| < K$ and $|\varphi_n(z) + i| \le C_K 2^n |z|^{-1}$ if $2^n|z|^{-1} < K$. It follows that the above product converges for any $z \ne 2^n, n \in \mathbb{N}$, and defines a meromorphic function on $\mathbb{C}$. The residues $c_n = -\operatorname{Res}_{2^n} \psi$ satisfy $|c_n| \asymp 1$. The above estimates for $\varphi_n(z)$ imply that $\max_{|z|=3\cdot 2^k} |\psi(z)| \le C2^{-k}$. Put $t_n = 2^n$. By writing down the residue theorem for the function $\frac{\psi(\cdot)}{(\cdot)-\zeta}$ on the contours $|z| = 3\cdot 2^k$, where ζ is fixed and letting $k \to \infty$, one gets

$$\psi(\zeta) = \sum_{n\in\mathbb{N}} \frac{c_n}{t_n - \zeta} = 1 + \sum_{n\in\mathbb{N}} c_n \Big(\frac{1}{t_n - \zeta} - \frac{1}{t_n}\Big)$$

(here we use that $\psi(0) = 1$). Take any sequences $a = \{a_n\}$ and $b = \{b_n\}$ such that $\{2^n a_n\}$ and $\{b_n\}$ are in ℓ^2 and $c_n = -t_n^2 a_n b_n$. (For instance, one can put $a_n = t_n^{-3/2}$ and $b_n = -c_n t_n^{-1/2}$.) The operator $\mathcal{A}$ on $H = \ell^2$, defined by $\mathcal{A}\{x_n\} = \{2^{-n}x_n\}$, is cyclic, compact, selfadjoint and has trivial kernel. Since $\{2^n a_n\}$ is in ℓ^2, the operator $\mathcal{L} = \mathcal{A} + ab^*$ on ℓ^2 has the form $\mathcal{L} = \mathcal{A}(I + S)$, where S is a rank one operator, so $\mathcal{L}$ is a weak perturbation of $\mathcal{A}$. Since for this perturbation, $\beta(z^{-1}) = \psi(z)$, we get that the spectrum of $\mathcal{L}$ is $\{0\} \cup \{i2^{-n} : n \in \mathbb{N}, n \ge 2\}$.

Notice that in this example, we get that $\sum a_n b_n s_n^{-1} = -\sum c_n t_n^{-1} = -1$, so that the first moment equation (M_1) holds and one has absolute convergence in its left hand part. The general term of the sum $\sum a_n b_n s_n^{-2}$ in (M_2) does not tend to zero. So the hypotheses of Theorem 1.1 do not hold. This example shows that in this case, although the perturbation is weak, its spectrum is not contained in the union of angles, given by Theorem 1.1. By our Theorem 1.3, $\mathcal{L}$ and $\mathcal{L}^*$ are nearly complete.

In fact, it is easy to get that $|\beta(z)| \geq C|z|^{-1}$ for $|z| \geq 1$, z outside arbitrarily small angles around the two coordinate axes. So in this particular case, it is easy to get that $\mathcal{L}$ and $\mathcal{L}^*$ are nearly complete either by using the argument of the proof of Theorem 1.1 or the criterion of completeness, given in [3], Proposition 3.1.

5. The case of singular perturbations of unbounded normal operators

In this section we consider an analogue of Theorem 1.3 for rank one perturbations of unbounded normal operators. Let $\mathcal{A}_0$ be a compact normal operator and let $\mathcal{L}_0$ be its rank one perturbation such that $\ker \mathcal{A}_0 = \ker \mathcal{L}_0 = 0$. Put $\mathcal{A} = \mathcal{A}_0^{-1}$ and let $\mathcal{L} = \mathcal{L}_0^{-1}$ be the algebraic inverse of $\mathcal{L}_0$ defined on the range of $\mathcal{L}_0$ (In the last two sections, to distinguish between the bounded and unbounded operators, we use the notation $\mathcal{A}_0$, $\mathcal{L}_0$ for compact operators and $\mathcal{A}$, $\mathcal{L}$ for their unbounded inverses.) One should expect that $\mathcal{L}$ is in a certain sense a rank one perturbation of the unbounded normal operator $\mathcal{A}$. However, it is not necessarily a relatively compact (rank one) perturbation $\mathcal{A}$. To formalize this we need to introduce the notion of a singular rank one perturbation.

Now let $\mathcal{A}$ be an unbounded normal operator on a Hilbert space H and let $G(\mathcal{A})$ stand for the graph of $\mathcal{A}$; it is a subspace of $H \oplus H$. We assume that $\mathcal{A}^{-1}$ exists and is bounded. We say that $\mathcal{L}$ is a *singular balanced rank one perturbation of* $\mathcal{A}$ if $G(\mathcal{A}) \cap G(\mathcal{L})$ has codimension one in both spaces $G(\mathcal{A})$ and $G(\mathcal{L})$. Here we follow [3]; in this paper a definition of singular balanced rank n perturbations of a not necessarily normal operator $\mathcal{A}$ was also given. If one takes for $\mathcal{A}$ an ordinary differential operator on an interval and changes its defining boundary conditions without changing the formal differential expression, then one obtains this kind of perturbation of $\mathcal{A}$.

If $\mathcal{L}_0$ is a usual rank one perturbation of $\mathcal{A}_0 = \mathcal{A}^{-1}$, which has zero kernel and $\mathcal{L} = \mathcal{L}_0^{-1}$ is its algebraic inverse, then $\mathcal{L}$ is a singular rank one perturbation of $\mathcal{A}$; moreover, as it will be explained a little bit later, "most" of singular rank one perturbations of $\mathcal{A}$ are obtained in this way. In this Section we reformulate our completeness result, Theorem 1.3, for singular rank one perturbations, and give some examples where singular rank one perturbations of operators with lacunary spectra appear.

To give a description of all singular rank one perturbations, we need to introduce some new notions. We define *the extrapolation Hilbert space* $\mathcal{A}H$ as the set of formal expressions $\mathcal{A}x$, where x ranges over *the whole space* H. Put $\|\mathcal{A}x\|_{\mathcal{A}H} = \|x\|_H$ for all $x \in H$. The formula $x = \mathcal{A}(\mathcal{A}^{-1}x)$ allows one to interpret H as a linear submanifold of $\mathcal{A}H$. We consider the scale of spaces

$$\mathcal{D}(\mathcal{A}) \subseteq H \subseteq \mathcal{A}H.$$

Notice that $\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A}^*)$. The pairing $\langle x, y\rangle \overset{\text{def}}{=} \langle \mathcal{A}x, \mathcal{A}^{*-1}y\rangle$, $x \in \mathcal{D}(\mathcal{A})$, $y \in \mathcal{A}H$, gives rise to a natural identification $\mathcal{D}(\mathcal{A}) = (\mathcal{A}H)^*$.

The set of rank 1 singular balanced perturbations of $\mathcal{A}$ can be parametrized by what we will call 1-data for $\mathcal{A}$. By 1-*data* we mean a triple $(a, b, \varkappa)$, where $a, b \in \mathcal{A}H$ are non-zero, $\varkappa \in \mathbb{C}$ and the following condition is fulfilled:

(A) If $a \in H$, then $\varkappa \neq \langle \mathcal{A}^{-1}a, b\rangle$.

Given 1-data $(a, b, \varkappa)$, the corresponding rank 1 singular balanced perturbation $\mathcal{L} = \mathcal{L}(a, b, \varkappa)$ of $\mathcal{A}$ is defined as follows:

$$\begin{aligned} \mathcal{D}(\mathcal{L}) \overset{\text{def}}{=} \big\{y = y_0 + c\mathcal{A}^{-1}a :& \\ & c \in \mathbb{C},\ y_0 \in \mathcal{D}(\mathcal{A}),\ \varkappa c + b^* y_0 = 0\big\}; \end{aligned} \tag{5.1a}$$

$$\mathcal{L}\, y \overset{\text{def}}{=} \mathcal{A}y_0, \quad y \in \mathcal{D}(\mathcal{L}). \tag{5.1b}$$

Condition (A) is equivalent to the uniqueness of the decomposition $y = y_0 + c\mathcal{A}^{-1}a$ for $y \in \mathcal{D}(\mathcal{L})$ and hence to the correctness of the definition of $\mathcal{L}$.

As it is shown in [3], any singular balanced rank one perturbation of $\mathcal{A}$ has a form $\mathcal{L} = \mathcal{L}(a, b, \varkappa)$ for certain 1-data $(a, b, \varkappa)$. Moreover, if $\varkappa \neq 0$, then, by [3, Proposition 2.4], $\mathcal{L} = \mathcal{L}_0^{-1}$ (the algebraic inverse), where $\mathcal{L}_0$ is a rank one perturbation of $\mathcal{A}_0 = \mathcal{A}^{-1}$:

$$\mathcal{L}_0 = \mathcal{A}_0 - \varkappa^{-1}\mathcal{A}_0 a(\mathcal{A}_0^* b)^*. \tag{5.2}$$

One can write

$$\mathcal{A}x = \sum_{n\in\mathbb{N}} t_n P_n x, \tag{5.3}$$

where the finite dimensional orthogonal projections P_n are as above: $P_n P_m = 0$ for $m \neq n$, $\sum_n P_n = I$, but now $|t_n| \to \infty$ (and $t_n \neq 0$ for all n). The domain of $\mathcal{A}$ is the set of vectors $x \in H$, for which the above sum converges.

We will say that the singular perturbation $\mathcal{L}(a,b,\varkappa)$ is *degenerate* if $\langle P_n a, b\rangle = 0$ for all n and at the same time $\varkappa = 0$ (it is consistent with the condition (A) if $a \notin H$). We will say that $\mathcal{L}(a, b, \varkappa)$ is *non-degenerate* in all other cases.

It is easy to check that the spectrum of $\mathcal{L}$ coincides with its point spectrum and equals to the zero set of the meromorphic function

$$\beta_{\mathcal{L}}(\lambda) = \varkappa + \lambda b^*(\mathcal{A} - \lambda)^{-1}\mathcal{A}^{-1}a = \varkappa + \lambda \sum_k \frac{\langle P_n \mathcal{A}^{-1}a, b\rangle}{t_n - \lambda}$$

(see [3]). So, if the operator $\mathcal{L}(a, b, \varkappa)$ is degenerate, then each point $\lambda \in \mathbb{C}$ is its eigenvalue. If $\mathcal{L}(a, b, \varkappa)$ is non-degenerate, then its spectrum is discrete.

Whereas the point 0 was a special point of the spectrum for compact operators, in the present context of unbounded operators, this role passes to the point ∞. Analogously to the case of bounded operators, given an operator $\mathcal{L}$ on H with compact resolvent, we say that $\mathcal{L}$ is *complete* if its root vectors span H and we say that $\mathcal{L}$ is *nearly complete* if its root vectors span a subspace of H of finite codimension.

Theorem 5.1. *Let $\mathcal{A}$ be a normal operator with compact resolvent, given by (5.3), which has lacunary spectrum $\{t_n\}_{n\in\mathbb{N}}$. Suppose that $0 \notin \sigma(\mathcal{A})$. Let $(a, b, \varkappa)$ be 1-data for $\mathcal{A}$, and let $\mathcal{L} = \mathcal{L}(a, b, \varkappa)$ be the corresponding singular perturbation of $\mathcal{A}$, which is non-degenerate. If $\mathcal{L}$ is not nearly complete, then the following infinite sequence of "moment" equations holds for all $k \in \mathbb{Z}$, $k \geq -1$:*

$$(\mathrm{S}_k) \qquad \sum_n t_n^k \langle P_n a, b\rangle = \begin{cases} \varkappa, & k = -1, \\ 0, & k \geq 0. \end{cases}$$

Proof. We reduce this assertion to the case of a compact operator. First we observe that it is suffices to consider the case when $\varkappa \neq 0$. Indeed, by [3, Proposition 2.5], for any $\lambda \in \rho(\mathcal{A})$, one has

$$\mathcal{L}(\mathcal{A}, a, b, \varkappa) - \lambda I = \mathcal{L}(\mathcal{A} - \lambda I, a, b, \beta_{\mathcal{L}}(\lambda)). \tag{5.4}$$

Notice that the function $\beta_{\mathcal{L}}(\lambda)$ has poles exactly at the points $\{t_n\}$ and its residue at t_n equals $-\langle P_n a, b\rangle$. If all these residues are zero for all n, then $\varkappa \neq 0$, by the non-degeneracy assumption.

If not all numbers $\langle P_n a, b\rangle$ are zero, then $\beta_{\mathcal{L}}$ is non-constant, and therefore $\beta_{\mathcal{L}}(\lambda) \neq 0$ for some λ. Operator $\mathcal{L} - \lambda I$ is nearly complete if and only if $\mathcal{L}$ is. A direct calculation shows that moment equations (S_k) hold for $\mathcal{L}$ if and only if they hold for $\mathcal{L} - \lambda I$. So we may assume that $\varkappa \neq 0$, just by replacing $\mathcal{L}$ with $\mathcal{L} - \lambda I$, where $\lambda \in \mathbb{C} \setminus \sigma(\mathcal{A})$ is any number such that $\beta_{\mathcal{L}}(\lambda) \neq 0$.

If $\varkappa \neq 0$, then $\mathcal{L} = \mathcal{L}_0^{-1}$ where $\mathcal{L}_0$ is a rank one perturbation of $\mathcal{A}_0 = \mathcal{A}^{-1}$ given by formula (5.2). Notice that $\mathcal{A}^{-1}$ is a compact normal operator with lacunary spectrum and that $\mathcal{L}$ is nearly complete if and only if $\mathcal{L}_0$ is. Now the statement follows by applying Theorem 1.3 to $\mathcal{L}_0$. ☐

Next we give examples of selfadjoint operators with compact resolvents and their singular perturbations, which are motivated by applications.

Example 5.2. Consider an infinite symmetric Jacobi matrix with diagonal entries b_n and off-diagonal entries a_n, $n = 0, 1, \dots$, where at least one of the sequences $\{a_n\}$, $\{b_n\}$ grows exponentially. In many cases, these matrices give rise to unbounded selfadjoint operators with lacunary spectrum. As a particular example, take the position operator of the Biedenharn–Macfarlane q-oscillator, studied by Klimyk in [22]. It corresponds to the values $b_n = 0$, $a_n = ((q^n - 1)/(q - 1))^{1/2}$, where $q > 1$ is a fixed number. This Jacobi matrix defines a symmetric operator $\mathcal{A}_0$ with indices $(1, 1)$. In [22, Theorem 1], a parametrization of all self-adjoint extensions of $\mathcal{A}_0$ is given and their spectra are calculated. Any self-adjoint extension of $\mathcal{A}_0$ is lacunary. Therefore Theorem 5.1 applies to any operator $\mathcal{L}$ such that $\mathcal{A}_0 \subset \mathcal{L}$ and $\dim \mathcal{D}(\mathcal{L})/\mathcal{D}(\mathcal{A}_0) = 1$. There are many other Jacobi matrices, which give rise to different classes of q-orthogonal polynomials; see [23]. By inspecting the orthogonality relations one sees that there are other cases when the corresponding unbounded selfadjoint operators are lacunary.

An easier case is obtained if the off-diagonal entries of the Jacobi matrix are subordinate to the diagonal ones. Namely, assume that $b_n > 0$, $a_n \in \mathbb{R}$,

$$\frac{b_{n+1}}{b_n} \longrightarrow Q > 1 \quad \text{and} \quad \frac{a_n}{b_n} \longrightarrow 0 \quad \text{as } n \to +\infty.$$

Then our Jacobi matrix defines an unbounded selfadjoint operator $\mathcal{A}$ on $\ell^2(\mathbb{N})$, which is semi-bounded from below and has lacunary spectrum. To see it, consider the diagonal matrix D with the sequence $\{b_n\}$ on the diagonal. Then D defines a positive unbounded selfadjoint operator on $\ell^2(\mathbb{Z}_+)$. Define $\mathcal{A}$ by $\mathcal{A}x = Dx + Fx$, $x \in \mathcal{D}(\mathcal{A}) \stackrel{\text{def}}{=} \mathcal{D}(D)$, where F is the operator given by the Jacobi matrix with the same off-diagonal entries a_n and zero entries on the diagonal. It is easy to see that $\mathcal{A}$ is a relatively compact perturbation of D and therefore $\mathcal{A}$ is selfadjoint and has discrete spectrum, see [37, Theorem 9.9]. Consider the circles $|z - b_n| = \sigma b_n$ around eigenvalues of D. If $\sigma > 0$ is small, each of them contains exactly one eigenvalue of D. Moreover, it is easy to see that $\|(D - z)^{-1}F\| \le 1/2$ if z is outside all these circles and is sufficiently large. It follows that for large n, the eigenvalues of the operator $D + tF$ (which has discrete spectrum) do not cross the circle $|z - b_n| = \sigma b_n$ when the parameter t traces the interval $[0, 1]$. By [29, Lemma 8.1], there is exactly one eigenvalue of $\mathcal{A} = D + F$ inside each such circle. Therefore in this case, $\mathcal{A}$ is lacunary.

We remark that in general, to find out whether a selfadjoint Jacobi matrix with exponentially growing entries is selfadjoint or has defect indices $(1, 1)$, one can apply [11], Theorem 2 and its Corollary.

6. Proof of Theorem 1.4

6.1. Functional model for rank one perturbations. The proof of the main part of Theorem 1.4 uses a functional model for singular rank one perturbations of unbounded selfadjoint operators with discrete spectrum, which are essentially the algebraic inverses to rank one perturbations of compact selfadjoint operators. This model was introduced in [3]. Let us briefly recall its statement (in the generality we need here). For the details see [3] or [2, Section 4].

We consider the following objects.

- $\{t_n\}$ is, as above, a sequence of real points such that $|t_n| \to \infty$ as $|n| \to \infty$, and $t_n \neq 0$. We can assume without loss of generality that $\{t_n\}$ is an increasing sequence enumerated by $\mathbb{Z}$, $\mathbb{N}$ or $-\mathbb{N}$.
- A is an entire function which is real on $\mathbb{R}$ and has simple zeros exactly at the points t_n.
- Two sequences $\{a_n\}$ and $\{b_n\}$, $b_n \neq 0$ for any n, and a complex number $\varkappa \neq 0$ satisfy
  (1) $\sum_n \frac{|a_n|^2+|b_n|^2}{t_n^2} < \infty$;
  (2) $\sum_n \frac{a_n \bar{b}_n}{t_n} \neq \varkappa$ in the case when $\sum_n |a_n|^2 < \infty$.
- The entire function E is given by $E = A - iB$, where

$$\frac{B(z)}{A(z)} = \delta + \sum_n \Big(\frac{1}{t_n - z} - \frac{1}{t_n}\Big)|b_n|^2, \tag{6.1}$$

  and δ is an arbitrary real constant. It is a Hermite–Biehler function, which means that $|E(z)| > |E(\bar{z})|$ if $\operatorname{Im} z > 0$.
- The de Branges space $\mathcal{H}(E)$, associated with the function E. It can be defined as the space of exactly those entire functions F which have the representation

$$\frac{F(z)}{A(z)} = \sum_n \frac{u_n |b_n|}{t_n - z}$$

  for some sequence $\{u_n\} \in \ell^2$. It is a Hilbert space with the norm given by $\|F\|_{\mathcal{H}(E)} = \|\{u_n\}\|_{\ell^2}$, so that $\mathcal{H}(E)$ is essentially the space of the discrete Cauchy transforms. We refer to [9] (or [3, 2]) for an alternative (more standard) definition of $\mathcal{H}(E)$ and more background.

- Entire function G is given by

$$\frac{G(z)}{A(z)} = \varkappa + \sum_n \Big(\frac{1}{t_n - z} - \frac{1}{t_n}\Big) a_n \bar{b}_n. \tag{6.2}$$

- The model operator $\mathcal{T}$, defined by the formulas

$$\mathcal{D}(\mathcal{T}) := \{F \in \mathcal{H}(E) \colon \text{there exists } c = c(F) \in \mathbb{C} \text{ such that } zF - cG \in \mathcal{H}(E)\},$$

$$\mathcal{T}F := zF - cG, \qquad F \in \mathcal{D}(\mathcal{T}).$$

Now the functional model from [3, Theorem 4.4] combined with [3, Proposition 2.4] (see also [2, Section 4]) can be stated as follows.

Theorem D. *Any singular rank one perturbation $\mathcal{L} = \mathcal{L}(\mathcal{A}, a, b, \varkappa)$ of the selfadjoint operator $\mathcal{A}$ (with simple spectrum and trivial kernel) is unitary equivalent to the model operator $\mathcal{T}$ whose parameters $\{t_n\} = \{s_n^{-1}\}$, A and G are related to $a, b, \varkappa$ as above. Conversely, any function G as above appears in the model of some rank one perturbation of $\mathcal{A}$.*

The reproducing kernels of the de Branges space $\mathcal{H}(E)$ are given by

$$K_w(z) = \frac{\overline{E(w)}E(z) - E(\bar{w})\overline{E(\bar{z})}}{2\pi i(\bar{w} - z)} = \frac{\overline{A(w)}B(z) - \overline{B(w)}A(z)}{\pi(z - \bar{w})}.$$

One has $K_w \in \mathcal{H}(E)$ and $\langle F, K_w\rangle = F(w)$ for any $w \in \mathbb{C}$.

If $\sum_n |b_n|^2 = \infty$ or $\sum_n |b_n|^2 < \infty$ and $\sum_n t_n^{-1} a_n \bar{b}_n \neq \varkappa$, then the adjoint operator $\mathcal{L}^*$ is well-defined and also is a singular rank one perturbation of $\mathcal{A}$ (see [3, Proposition 2.2]). Moreover, in this case the eigenvectors of $\mathcal{L}^*$ are mapped by the same unitary equivalence as in the above theorem onto the reproducing kernels $\{K_\lambda\}_{\lambda \in Z_G}$, see [3, Lemma 5.4]. By Z_f we denote the zero set of an entire function f. To avoid unessential technicalities, we assume that all zeros of G are simple.

6.2. Strategy of the proof of Theorem 1.4. In view of relation (5.2) between bounded rank one perturbations of compact normal operators and singular rank one perturbations, we can solve an equivalent problem for singular rank one perturbations. Let $\mathcal{A}_0$ be a compact selfadjoint operator with simple spectrum $\{s_n\}$, $s_n \neq 0$ (we can identify $\mathcal{A}_0$ with a diagonal operator on ℓ^2) such that $t_n = s_n^{-1}$ satisfy condition (1.5). Consider the unbounded operator $\mathcal{A} = \mathcal{A}_0^{-1}$ and assume that we were able to construct a singular rank one perturbation $\mathcal{L} = \mathcal{L}(\mathcal{A}, a, b, \varkappa)$ such that $\varkappa \neq 0$, $a = \{a_n\}$, $b = \{b_n\}$ satisfy $\sum_n \frac{|a_n|^2+|b_n|^2}{t_n^2} < \infty$ and

$$\sum_n |t_n^{-1} a_n b_n| = \infty, \tag{6.3}$$

so that $\mathcal{L}$ is incomplete with infinite defect (that is, the linear span of root vectors of $\mathcal{L}$ has infinite codimension in ℓ^2). Consider the vectors $a^0 = \mathcal{A}^{-1}a = \{t_n^{-1}a_n\}$ and $b^0 = \mathcal{A}^{-1}b = \{t_n^{-1}b_n\}$ from ℓ^2. Then, by (5.2)

$$\mathcal{L}_0 = \mathcal{L}^{-1} = \mathcal{A}_0 - \varkappa^{-1} a^0 (b^0)^*$$

is a bounded rank one perturbation of $\mathcal{A}_0$ whose set of root vectors has infinite codimension. Note that equality (6.3) coincides with $\sum_n |s_n^{-1} a_n^0 b_n^0| = \infty$, i.e., nonexistence of the first moment for $\mathcal{L}_0$.

Thus, the statement of Theorem 1.4 is reduced to an equivalent problem for singular rank one perturbations. In view of the above functional model, this problem is equivalent to a completeness problem for a system of reproducing kernels in de Branges spaces. Namely, to prove Theorem 1.4 we will need to construct an entire function G such that

$$\frac{G(z)}{A(z)} = \varkappa + \sum_n c_n \Big(\frac{1}{t_n - z} - \frac{1}{t_n}\Big), \quad \sum_n \frac{|c_n|}{t_n^2} < \infty, \quad \sum_n \frac{|c_n|}{|t_n|} = \infty, \tag{6.4}$$

and $\varkappa \neq 0$, but the system $\{K_\lambda\}_{\lambda\in Z_G}$ has an infinite defect in $\mathcal{H}(E)$. Then we can define a singular rank one perturbation $\mathcal{L}$ of $\mathcal{A}$ such that $\mathcal{L}^*$ is also a well-defined singular rank one perturbation, but $\mathcal{L}^*$ will not be complete.

We will construct G of the form A/S where S is a canonical product of order less than one, whose zeros form a subset of the set $\{t_n\}$. Then, necessarily, $\varkappa = 1/S(0)$, $c_n = -1/S'(t_n)$ and the equation (6.4) rewrites as

$$\frac{G(z)}{A(z)} = \frac{1}{S(z)} = \frac{1}{S(0)} - \sum_{t_n\in Z_S} \frac{1}{S'(t_n)} \Big(\frac{1}{t_n - z} - \frac{1}{t_n}\Big). \tag{6.5}$$

In Subsection 6.3 the relations between conditions (1.5) and (1.6) will be discussed as well as some equivalent forms of condition (1.5). The function S from (6.5) will be constructed in Subsection 6.4, while in Subsection 6.5 the proof of Theorem 1.4 will be completed.

6.3. Discussion of condition (1.5)**.** We begin with the proof of the fact that any separated sequence $T = \{t_n\}$, for which (1.5) does not hold, satisfies $n_T(r) = O(\log^2 r)$ (and thus is sufficiently sparse).

Lemma 6.1. *Let $R > 0$ and let the interval $[R, 2R]$ contain at least $2M$ points t_k. Then there exists $t_n \in [R, 2R]$ such that*

$$\prod_{k: R \le t_k \le 2R, k \ne n} \left| \frac{t_k - t_n}{t_k} \right| \le 2^{-M+1}.$$

Proof. Clearly, we can choose M points $t_{n_1}, \dots t_{n_M} \in \{t_k\}$ such that $|t_{n_1} - t_{n_j}| \le R/2$, $j = 2, \dots, M$. Hence, we have

$$\prod_{k: R \le t_k \le 2R, k \ne n_1} \left| \frac{t_k - t_{n_1}}{t_k} \right| \le \left(\frac{R}{2}\right)^{M-1} \left(\frac{1}{R}\right)^{M-1} = 2^{-M+1}$$

(we dropped the factors for which $|t_k - t_{n_1}| > R/2$ since they are anyway smaller than 1). ☐

Corollary 6.2. *Condition* (1.6) *implies* (1.5).

Proof. Assume that (1.5) is not satisfied. Given $R > 0$, put $M = M(R) = [M'(R)/2]$, where $M'(R)$ is the number of points t_k in the interval $[R, 2R]$. Then, by Lemma 6.1, $R^N 2^{-M(R)} \gtrsim 1$ for some N which is independent on R. Hence $M(R) = O(\log R)$. Thus, we conclude that, for any $m \in \mathbb{N}$, there is always no more than $O(m)$ points t_n between 2^m and 2^{m+1}, and so $n_T(r) = O(\log^2 r)$, which contradicts (1.6). ☐

Remark. One can rewrite (1.5) in equivalent ways. Recall that *the Krein class* consists of entire functions F which are real on $\mathbb{R}$, have only simple and real zeros and for some positive integer k and some polynomial P of degree at most k satisfy the following absolutely convergent expansion:

$$\frac{1}{F(z)} = P(z) + \sum_n \frac{1}{F'(t_n)} \left(\frac{1}{z - t_n} + \frac{1}{t_n} + \dots + \frac{z^{k-1}}{t_n^k} \right),$$

where s_n are zeros of F.

It is not difficult to see that condition (1.5) is equivalent to any of the following properties (i) and (ii):

(i) There exists an entire function Q of order less than one, whose zeros are simple and lie in the set $\{t_n\}$, such that for any N we have

$$\liminf_{t_n \to \infty, t_n \in Z_Q} |t_n^N Q'(t_n)| = 0$$

(the lower limit is taken here over those t_n which are zeros of Q).

(ii) There exists a subsequence of $\{t_n\}$, which is not the zero set of a function in the Krein class.

For instance, to prove that (1.5) implies (i), it suffices to put

$$Q(z) = \prod_k \prod_{m: t_{n_k}/2 \le t_m \le t_{n_k}} \Big(1 - \frac{z}{t_m}\Big),$$

where the sequence $\{n_k\}$ grows fast enough (by the above proof of Lemma 6.3, this product defines a zero order entire function). We leave the details to the reader.

6.4. A key lemma. To construct a function S satisfying (6.5) we will need the following lemma, which is the main technical step in the proof of Theorem 1.4.

Lemma 6.3. *Under the hypothesis of Theorem 1.4, there exists a canonical product S of order less than one with zeros in the set $\{t_n\}$ such that*

$$\sum_{t_n \in Z_S} \frac{1}{t_n^2 |S'(t_n)|} < \infty, \qquad \sum_{t_n \in Z_S} \frac{1}{|t_n S'(t_n)|} = \infty.$$

Proof. Without loss of generality we assume that $t_n > 0$ and that (1.5) holds. Put $S = \prod_k S_{T_k}$, where

$$S_{T_k}(z) = \prod_{t_n \in T_k} \Big(1 - \frac{z}{t_n}\Big), \tag{6.6}$$

and $T_k \subset \{t_n : n \ne 1,\ t_{n_k}/2 \le t_n \le 2t_{n_k}\}$, where n_k go rapidly to infinity so that, for any $N > 0$,

$$t_{n_k}^N \cdot \prod_{l \ne n_k : t_{n_k}/2 \le t_l \le 2t_{n_k}} \left| \frac{t_{n_k} - t_l}{t_l} \right| \longrightarrow 0, \quad k \to \infty. \tag{6.7}$$

We will show that for an appropriate choice of $\{T_k\}$ either $S(z)$ or $(z - t_1)S(z)$ is the desired function.

The sets T_k will be chosen inductively. Suppose that the sets $T_1, \dots T_{k-1}$ have been already chosen and put $U_{k-1} = \prod_{j=1}^{k-1} S_{T_j}$. Then, clearly, $|U_{k-1}(z)| \sim q_k |z|^{N_k}$, $|z| \to \infty$, for some constants $q_k > 0$ and $N_k \in \mathbb{N}$.

Let us first consider the case when $T_k = \{t_n : t_{n_k}/2 \le t_n \le 2t_{n_k}\}$, assuming that n_k is sufficiently large. Then, by (6.7), the corresponding function S_{T_k} satisfies

$$t_{n_k}^2 |U_{k-1}(t_{n_k}) S'_{T_k}(t_{n_k})| = t_{n_k} |U_{k-1}(t_{n_k})| \prod_{l \ne n_k : t_{n_k}/2 \le t_l \le 2t_{n_k}} \left| \frac{t_{n_k} - t_l}{t_l} \right| < 1.$$

In particular,

$$\sum_{t_n \in T_k} \frac{1}{|t_n U_{k-1}(t_n) S'_{T_k}(t_n)|} \geq t_{n_k} \gg 1.$$

Now consider another extreme case where T_k consists only of the point t_{n_k}, that is, $T_k = \{t_{n_k}\}$. Then $S_{T_k}(z) = 1 - z/t_{n_k}$, and we have

$$\sum_{t_n \in T_k} \frac{1}{|t_n U_{k-1}(t_n) S'_{T_k}(t_n)|} = \frac{1}{|U_{k-1}(t_{n_k})|} \ll 1.$$

Hence, there exists a (not necessarily unique) set $T_k \subset \{t_n : t_{n_k}/2 \leq t_n \leq 2t_{n_k}\}$ such that S_{T_k} satisfies

$$\sum_{t_n \in T_k} \frac{1}{|t_n U_{k-1}(t_n) S'_{T_k}(t_n)|} > 1 \tag{6.8}$$

and T_k is minimal in the sense that the estimate (6.8) no longer holds if one removes any point from T_k. This will be our choice of T_k.

Now let t_j be any point in T_k and let $\tilde{S}_j = S_{T_k \setminus \{t_j\}}$. Then, by the above property of minimality,

$$\begin{aligned} 1 &\geq \sum_{t_n \in T_k, t_n \neq t_j} \frac{1}{|t_n U_{k-1}(t_n) \tilde{S}'_j(t_n)|} \\ &= \sum_{t_n \in T_k, t_n \neq t_j} \frac{1}{|t_n U_{k-1}(t_n) S'_{T_k}(t_n)|} \cdot \frac{|t_j - t_n|}{|t_j|} \\ &\gtrsim \sum_{t_n \in T_k, t_n \neq t_j} \frac{1}{t_n^2 |U_{k-1}(t_n) S'_{T_k}(t_n)|}, \end{aligned}$$

where the last inequality follows from the hypothesis $\inf_{n \neq j} |t_n - t_j| > 0$. Since $t_j \in T_k$ was arbitrary, we conclude that, uniformly with respect to k,

$$\sum_{t_n \in T_k} \frac{1}{t_n^2 |U_{k-1}(t_n) S'_{T_k}(t_n)|} \lesssim 1. \tag{6.9}$$

Obviously, by choosing t_{n_k} to grow sufficiently fast, we may achieve that, for the function $S = \prod_k S_{T_k}$ the factors S_{T_j} with $j > k$ almost do not influence the product at the points $t_n \in T_k$ so that $\frac{1}{2} \leq \prod_{j=k+1}^{\infty} |S_{T_j}(t_n)| \leq 2$ for $t_n \in [t_{n_k}/2, 2t_{n_k}]$. Then

$$\frac{1}{2} \sum_{t_n \in T_k} \frac{1}{|t_n S'(t_n)|} \leq \sum_{t_n \in T_k} \frac{1}{|t_n U_{k-1}(t_n) S'_{T_k}(t_n)|} \leq 2 \sum_{t_n \in T_k} \frac{1}{|t_n S'(t_n)|}, \tag{6.10}$$

$$\frac{1}{2}\sum_{t_n\in T_k}\frac{1}{t_n^2|S'(t_n)|}\le\sum_{t_n\in T_k}\frac{1}{t_n^2|U_{k-1}(t_n)S'_{T_k}(t_n)|}\le 2\sum_{t_n\in T_k}\frac{1}{t_n^2|S'(t_n)|}. \qquad (6.11)$$

Also, it follows from Lemma 6.1 and from (6.9) that $\#T_k$, the number of elements in T_k, satisfies $\#T_k \lesssim N_k \ln t_{n_k} + \ln q_k$. Since N_k and q_k do not depend on the choice of t_{n_k}, the function S will be of zero order if t_{n_k} grow sufficiently fast.

By the construction of S_{T_k} (namely, by (6.8) and (6.10)) we clearly have

$$\sum_{t_n\in Z_S}\frac{1}{|t_nS'(t_n)|}=\infty.$$

If, at the same time,

$$\sum_{t_n\in Z_S}\frac{1}{t_n^2|S'(t_n)|}<\infty,$$

then our construction is completed. If the latter sum is also infinite, then put $\tilde S=(z-t_1)S=(z-t_1)\prod_k S_{T_k}$. Then, clearly, $|\tilde S'(t_n)|\asymp|t_nS'(t_n)|$, $t_n\in Z_S$, and so, by (6.11) and (6.9), we have

$$\sum_{t_n\in Z_S}\frac{1}{t_n^2|\tilde S'(t_n)|}\lesssim\sum_k\frac{1}{t_{n_k}}\sum_{t_n\in T_k}\frac{1}{t_n^2|U_{k-1}(t_n)S'_{T_k}(t_n)|}\lesssim\sum_k\frac{1}{t_{n_k}}<\infty.$$

Thus, $\tilde S$ has the required properties. □

6.5. End of the proof of Theorem 1.4. Let S be the entire function constructed in Lemma 6.3. Then the function $G=A/S$ is entire. The proof of (6.5) follows by the standard interpolation series argument. Indeed, the series in the right-hand side of (6.5) converges absolutely by the conditions on S. Note that

$$H(z)=\frac{1}{S(z)}-\frac{1}{S(0)}-\sum_{t_n\in Z_S}\frac{1}{S'(t_n)}\Big(\frac{1}{z-t_n}+\frac{1}{t_n}\Big)$$

is an entire function (the poles disappear). Since S is of order less than one with real zeros, we conclude that $1/S$ is of Smirnov class in the upper and in the lower half-planes, as well as the regularized Cauchy transform in the right-hand side of (6.5). Hence, by the classical theorem of M.G. Krein (see, e.g., [18, Part II, Chapter 1]), H is an entire function of zero exponential type. Note also that $|H(iy)|=o(|y|)$, $|y|\to\infty$, whence H is a constant. Since $H(0)=0$, we finally get that $H\equiv 0$.

Thus, G satisfies (6.4). Put $a_n=|c_n|^{1/2}$, $b_n=c_n/|c_n|^{1/2}$. Since we have $\sum_n|a_nb_nt_n^{-1}|=\infty$, we conclude that $\sum_n|a_n|^2=\sum_n|b_n|^2=\infty$. By Theorem D, the function G corresponds to the rank one perturbation $\mathcal{L}$ of $\mathcal{A}$, generated by $\{a_n\}$, $\{b_n\}$ and $\varkappa = 1/S(0)$. Moreover, $\mathcal{L}^*$ also is a well-defined singular rank one perturbation of $\mathcal{A}$ and the system of its eigenvectors is unitary equivalent to the system of reproducing kernels $\{K_\lambda\}_{\lambda\in Z_G}$ in $\mathcal{H}(E)$. It remains to see that the latter system is not complete in $\mathcal{H}(E)$. However, it is a basic fact of the de Branges theory that $\{K_\lambda\}_{\lambda\in Z_A}$ is an orthogonal basis of $\mathcal{H}(E)$ (see [9, Theorem 22]). Hence, $\{K_\lambda\}_{\lambda\in Z_A\setminus Z_S}$ is incomplete with infinite defect. Theorem 1.4 is proved. □

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