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Differential equations for discrete Jacobi–Sobolev orthogonal polynomials

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Abstract. The aim of this paper is to study differential properties of orthogonal polynomials with respect to a discrete Jacobi–Sobolev bilinear form with mass point at -1 and/or $+1$. In particular, we construct the orthogonal polynomials using certain Casorati determinants. Using this construction, we prove that when the Jacobi parameters α and β are nonnegative integers the Jacobi–Sobolev orthogonal polynomials are eigenfunctions of a differential operator of finite order (which will be explicitly constructed). Moreover, the order of this differential operator is explicitly computed in terms of the matrices which define the discrete Jacobi–Sobolev bilinear form.

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Contents

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1. Introduction and main results

Classical polynomials are orthogonal polynomials (with respect to a positive measure) which are in addition eigenfunctions of a second-order differential operator. As a consequence of S. Bochner classification theorem of 1929 (see [3]), it follows that there are only three families of classical orthogonal polynomials: Hermite, Laguerre and Jacobi (and Bessel polynomials if signed measures are considered). Although such result actually goes back to E. J. Routh in 1884 (see [28]).

H. L. Krall raised in 1939 (see [23, 24]) the problem of finding orthogonal polynomials which are also common eigenfunctions of a higher-order differential operator with polynomial coefficients. He obtained a complete classification for the case of a differential operator of order four (see [24]). Besides the classical families of Hermite, Laguerre and Jacobi, he found three other families of orthogonal polynomials which are also eigenfunctions of a fourth-order differential operator. Two of them are orthogonal with respect to positive measures which consist of particular instances of Jacobi weights together with one or two Dirac deltas at the endpoints of the interval of orthogonality. Indeed, consider the Koornwinder measures (see [22])

$$(1-x)^{\alpha}(1+x)^{\beta} + M\delta_{-1} + N\delta_{1}, \quad \alpha, \beta > -1. \tag{1.1}$$

Then, the examples found by Krall correspond with the cases $\alpha = \beta = 0$ and $M = N$, and $\beta = N = 0$ in (1.1), respectively. Krall also discovered a new family satisfying sixth-order differential equations, which corresponds with the case $\alpha = \beta = 0$ in (1.1). But he never published this example which was rediscovered by L. L. Littlejohn forty years later (see [25]).

R. Koekoek proved in 1994 that the Koornwinder polynomials orthogonal with respect to the weight (1.1) for $\alpha = \beta \in \mathbb{N}$ and $M = N$ are also eigenfunctions of a differential operator of order $2\alpha + 4$ (see [20]). F. A. Grünbaum and L. Haine (et al.) proved that polynomials satisfying fourth or higher-order differential equations can be generated by applying Darboux transformations to certain instances of the classical polynomials (see [11, 12, 13]). A. Zhedanov proposed a technique to construct Krall's polynomials and found a differential equation of order $2\alpha + 4$ for the orthogonal polynomials with respect to (1.1) when α is a nonnegative integer and $M = 0$ (see [30]). R. and J. Koekoek proved in 2000 the general case. More precisely, they found a differential operator for the orthogonal polynomials with respect to the weight (1.1) for which they are eigenfunctions; this operator has infinite order except for the following cases, where the order is finite and equals (see [21]):

$$\begin{cases} 2\beta+4 & \text{if } M>0, N=0 \text{ and } \beta\in\mathbb{N}, \\ 2\alpha+4 & \text{if } M=0, N>0 \text{ and } \alpha\in\mathbb{N}, \\ 2\alpha+2\beta+6 & \text{if } M>0, N>0 \text{ and } \alpha,\beta\in\mathbb{N}. \end{cases}$$

Using a different approach, P. Iliev (see [15]) has improved this result by studying the algebra of differential operators associated with Krall-Jacobi orthogonal polynomials.

In 2003, discrete Jacobi–Sobolev orthogonal polynomials which are also common eigenfunctions of a higher-order differential operator entered into the picture. H. Bavinck (see [2]) proved that orthogonal polynomials with respect to the discrete Jacobi–Sobolev inner product

$$\langle p,q\rangle = \int_{-1}^{1} p(x)q(x)(1-x)^{\alpha}(1+x)^{\beta}\,dx \\ + Mp^{(l_1)}(-1)q^{(l_1)}(-1) + Np^{(l_2)}(1)q^{(l_2)}(1),$$

are eigenfunctions of a differential operator of infinite order, except for the following cases, where the order is finite and equals:

$$\begin{cases} 2\beta+4l_1+4 & \text{if } M>0, N=0 \text{ and } \beta\in\mathbb{N}, \\ 2\alpha+4l_2+4 & \text{if } M=0, N>0 \text{ and } \alpha\in\mathbb{N}, \\ 2\alpha+2\beta+4l_1+4l_2+6 & \text{if } M>0, N>0 \text{ and } \alpha,\beta\in\mathbb{N}. \end{cases}$$

For other related papers see [18, 19, 26, 27].

For $\alpha, \beta, \alpha+\beta \neq -1, -2, \ldots$, we use the following definition of the Jacobi polynomials:

$$J_n^{\alpha,\beta}(x) = \frac{(-1)^n(\alpha+\beta+1)_n}{2^n(\beta+1)_n}\sum_{j=0}^{n}\binom{n+\alpha}{j}\binom{n+\beta}{n-j}(x-1)^{n-j}(x+1)^j \\ = \frac{(-1)^n(\alpha+\beta+1)_n(\alpha+1)_n}{n!(\beta+1)_n}\,{}_2F_1\Big(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\Big),$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ denotes the Pochhammer symbol. We use a different normalization of the standard definition of the Jacobi polynomials $P_n^{\alpha,\beta}$ and the equivalence is given by $J_n^{\alpha,\beta}(x) = (-1)^n\frac{(\alpha+\beta+1)_n}{(\beta+1)_n}P_n^{\alpha,\beta}(x)$ (these and the next formulas can be found in [10] pp. 168–173).

We denote by $\mu_{\alpha,\beta}(x)$ the orthogonalizing weight for the Jacobi polynomials normalized so that

$$\int \mu_{\alpha,\beta}(x)dx = 2^{\alpha+\beta+1}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}.$$

Only when $\alpha,\beta > -1$, $\mu_{\alpha,\beta}(x)$, $-1 < x < 1$, is positive, and then

$$\mu_{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}, \quad -1 < x < 1. \tag{1.2}$$

The Jacobi polynomials $(J_n^{\alpha,\beta})_n$ are eigenfunctions of the following second-order differential operator

$$D_{\alpha,\beta} = (x^2-1)\Big(\frac{d}{dx}\Big)^2 + ((\alpha+\beta+2)x - \beta + \alpha)\frac{d}{dx}. \tag{1.3}$$

That is

$$D_{\alpha,\beta}(J_n^{\alpha,\beta}) = \theta_n J_n^{\alpha,\beta}, \quad \theta_n = n(n+\alpha+\beta+1), \quad n \geq 0. \tag{1.4}$$

For $m_1, m_2 \geq 0$ with $m = m_1 + m_2 \geq 1$, let $\boldsymbol{M} = (M_{i,j})_{i,j=0}^{m_1-1}$ and $\boldsymbol{N} = (N_{i,j})_{i,j=0}^{m_2-1}$ be $m_1 \times m_1$ and $m_2 \times m_2$ matrices, respectively. In particular, if $m_1 = 0$ or $m_2 = 0$ we take $\boldsymbol{M} = 0$ or $\boldsymbol{N} = 0$, respectively. We consider the discrete Jacobi–Sobolev bilinear form defined by

$$\begin{aligned}\langle p, q\rangle = &\int_I p(x)q(x)\mu_{\alpha-m_2,\beta-m_1}dx \\ &+ \mathbb{T}_{-1}^{m_1}(p)\boldsymbol{M}\mathbb{T}_{-1}^{m_1}(q)^T + \mathbb{T}_{1}^{m_2}(p)\boldsymbol{N}\mathbb{T}_{1}^{m_2}(q)^T,\end{aligned} \tag{1.5}$$

where for a nonnegative integer k, a real number λ and a polynomial p, we define

$$\mathbb{T}_{\lambda}^{k}(p) = (p(\lambda), p'(\lambda), \dots, p^{(k-1)}(\lambda)).$$

For $\alpha > m_2 - 1$ and $\beta > m_1 - 1$, the measure $\mu_{\alpha-m_2,\beta-m_1}$ in (1.5) is then $(1-x)^{\alpha-m_2}(1+x)^{\beta-m_1}$ and $I = (-1, 1)$.

The purpose of this paper is to prove in a constructive way that if α and β are nonnegative integers with $\alpha \geq m_2$ and $\beta \geq m_1$, then the orthogonal polynomials with respect to (1.5) are eigenfunctions of a finite order differential operator with polynomial coefficients. For discrete Laguerre–Sobolev orthogonal polynomials see [7].

To display our results in full detail we need some notation. For $a, b, c, d, x \in \mathbb{R}$, we define

$$\Gamma_{c,d}^{a,b}(x) = \frac{\Gamma(x+a+1)\Gamma(x+b+1)}{\Gamma(x+c+1)\Gamma(x+d+1)}. \tag{1.6}$$

We next introduce the functions z_l, $1 \leq l \leq m$, in the following way. For $l = 1, \dots, m_1$, z_l is defined by

$$\begin{aligned} z_l(x) = {}& \frac{2^{\alpha+\beta-m_1+l}\Gamma(\beta-m_1+l)}{(m_1-l)!}\Gamma_{0,\alpha+\beta-m_1+l}^{m_1-l,\alpha+\beta}(x) \\ &+ 2^{m_2}\sum_{i=0}^{m_1-1}\left(\sum_{j=l}^{(l+m_2)\wedge m_1} \frac{(j-1)!\binom{m_2}{j-l}M_{i,j-1}}{(-2)^{i+j-l}}\right)\frac{\Gamma_{-i,\alpha}^{\beta,\alpha+\beta+i}(x)}{\Gamma(\beta+i+1)}, \end{aligned} \tag{1.7}$$

where $\wedge$ denotes the minimum between two numbers. For $l = m_1+1, \dots, m$, z_l are defined by

$$\begin{aligned} z_l(x) = {}& \frac{2^{\alpha+\beta-m+l}\Gamma(\alpha-m+l)}{(m-l)!}\Gamma_{0,\alpha+\beta-m+l}^{m-l,\alpha+\beta}(x) \\ &+ \sum_{i=0}^{m_2-1}\left(\sum_{j=l-m_1}^{l\wedge m_2} \frac{(j-1)!\binom{m_1}{l-j}N_{i,j-1}}{(-1)^{l-m_1-1}2^{i+j-l}}\right)\frac{\Gamma_{-i,\beta}^{\alpha,\alpha+\beta+i}(x)}{\Gamma(\alpha+i+1)}. \end{aligned} \tag{1.8}$$

Finally, we also need the polynomials

$$p(x) = \prod_{i=1}^{m_1-1}(-1)^{m_1-i}(x+\alpha-m+1)_{m_1-i}(x+\beta-m_1+i)_{m_1-i}, \tag{1.9}$$

$$q(x) = (-1)^{\binom{m}{2}}\prod_{h=1}^{m-1}\left(\prod_{i=1}^{h}(2(x-m)+\alpha+\beta+i+h)\right). \tag{1.10}$$

Using a general result for discrete Sobolev bilinear forms (see Lemma 2.1 in Section 2), we first characterize the existence of (left) orthogonal polynomials with respect to the Jacobi–Sobolev bilinear form above using the (quasi) Casorati determinant defined by the sequences z_l, $l = 1, \dots, m$. Moreover, we find a closed expression for these orthogonal polynomials in terms of the Jacobi polynomials $(J_n^{\alpha,\beta})_n$ and the sequences z_l, $l = 1, \dots, m$.

Theorem 1.1. *For $m_1, m_2 \geq 0$ with $m = m_1 + m_2 \geq 1$, let $\boldsymbol{M} = (M_{i,j})_{i,j=0}^{m_1-1}$ and $\boldsymbol{N} = (N_{i,j})_{i,j=0}^{m_2-1}$ be $m_1 \times m_1$ and $m_2 \times m_2$ matrices, respectively. For $\alpha \neq m_2 - 1, m_2 - 2, \dots,$ and $\beta \neq m_1 - 1, m_1 - 2, \dots,$ consider the discrete Jacobi–Sobolev bilinear form defined by* (1.5). *If we write*

$$\rho_{x,j}^h = \begin{cases} (-1)^{m-j}\Gamma_{\alpha-m,\beta-j}^{\alpha-j,\beta-1}(x) & \text{for } h = 1, \dots, m_1, \\ 1 & \text{for } h = m_1 + 1, \dots, m, \end{cases} \tag{1.11}$$

then the following conditions are equivalent:

(1) *the discrete Jacobi–Sobolev bilinear form* (1.5) *has a sequence $(q_n)_n$ of (left) orthogonal polynomials*;

(2) *the $m \times m$ (quasi) Casorati determinant*

$$\Lambda(n) = \frac{\begin{vmatrix} \rho_{n,1}^1 z_1(n-1) & \rho_{n,2}^1 z_1(n-2) & \cdots & \rho_{n,m}^1 z_1(n-m) \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n,1}^m z_m(n-1) & \rho_{n,2}^m z_m(n-2) & \cdots & \rho_{n,m}^m z_m(n-m) \end{vmatrix}}{p(n)q(n)}, \tag{1.12}$$

where $z_l, l = 1, \dots, m$, p and q are defined by (1.7), (1.8), (1.9), *and* (1.10) *respectively, does not vanish for $n \geq 0$.*

Moreover, if one of these properties holds, the polynomials defined by

$$q_n(x) = \frac{\begin{vmatrix} J_n^{\alpha,\beta}(x) & -J_{n-1}^{\alpha,\beta}(x) & \cdots & (-1)^m J_{n-m}^{\alpha,\beta}(x) \\ \rho_{n,0}^1 z_1(n) & \rho_{n,1}^1 z_1(n-1) & \cdots & \rho_{n,m}^1 z_1(n-m) \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n,0}^m z_m(n) & \rho_{n,1}^m z_m(n-1) & \cdots & \rho_{n,m}^m z_m(n-m) \end{vmatrix}}{p(n)q(n)}, \tag{1.13}$$

are orthogonal with respect to (1.5) (*as usual for $n < 0$ we take $J_n^{\alpha,\beta} = 0$*).

As in [7], we find the differential properties of the orthogonal polynomials $(q_n)_n$, see (1.13), by using the concept of $\mathcal{D}$-*operators*. This is an abstract concept introduced by one of us in [4] which has shown to be very useful to generate orthogonal polynomials which are also eigenfunctions of differential, difference or q-difference operators (see [1], [4]–[8]). The basic facts about $\mathcal{D}$-operators will be recalled in Section 3. Using the general theory of $\mathcal{D}$-operators and the expression (1.13) for the orthogonal polynomials $(q_n)_n$, we prove in Section 4 the following theorem.

Theorem 1.2. *Assume that any of the two equivalent properties* (1) *and* (2) *in Theorem* 1.1 *holds. If* $\boldsymbol{M}, \boldsymbol{N} \neq 0$*, we assume, in addition, that* α *and* β *are nonnegative integers with* $\alpha \geq m_2$ *and* $\beta \geq m_1$*. If, instead,* $\boldsymbol{M} = 0$*, we assume that only* α *is a positive integer with* $\alpha \geq m_2$*, and if* $\boldsymbol{N} = 0$*, we assume that only* β *is a positive integer with* $\beta \geq m_1$*. Then there exists a finite order differential operator* (*which we construct explicitly*) *for which the orthogonal polynomials* $(q_n)_n$*, see* (1.13)*, are eigenfunctions.*

An important issue will be the explicit calculation of the order of this differential operator in terms of the matrices $\boldsymbol{M}$ and $\boldsymbol{N}$ which define the discrete Jacobi–Sobolev bilinear form (1.5). As in [7], the key concept to calculate that order will be the γ*-weighted rank* associated to a real number γ and a matrix M defined as follows:

Definition 1.3. Let γ and M be a real number and a $m \times m$ matrix, respectively. Write $c_1, \dots, c_m$, for the columns of M and define the numbers η_j, $j = 1, \dots, m$, by

$$\eta_1 = \begin{cases} \gamma + m - 1 & \text{if } c_m \neq 0, \\ 0 & \text{if } c_m = 0; \end{cases}$$

and for $j = 2, \dots, m$,

$$\eta_j = \begin{cases} \gamma + m - j & \text{if } c_{m-j+1} \notin \langle c_{m-j+2}, \dots, c_m \rangle, \\ 0 & \text{if } c_{m-j+1} \in \langle c_{m-j+2}, \dots, c_m \rangle. \end{cases}$$

Denote by $\widetilde{M}$ the matrix whose columns are c_i, with $i \in \{j : \eta_{m-j+1} \neq 0\}$ (i.e., the columns of $\widetilde{M}$ are (from right to left) those columns c_i of M such that $c_i \notin \langle c_{i+1}, \dots, c_m \rangle$). Write $f_1, \dots, f_m$, for the rows of $\widetilde{M}$. We define the numbers τ_j, $j = 1, \dots, m-1$, by

$$\tau_j = \begin{cases} m - j & \text{if } f_j \in \langle f_{j+1}, \dots, f_m \rangle, \\ 0 & \text{if } f_j \notin \langle f_{j+1}, \dots, f_m \rangle. \end{cases}$$

The γ*-weighted rank* of the matrix M, $\gamma\text{-wr}(M)$ in short, is then defined by

$$\gamma\text{-wr}(M) = \sum_{j=1}^{m} \eta_j + \sum_{j=1}^{m-1} \tau_j - \binom{m}{2}.$$

We then have the following result.

Corollary 1.4. *With the assumptions of Theorem 1.2, the minimal order of the differential operators having the orthogonal polynomials* $(q_n)_n$ *as eigenfunctions is at most* $2(\beta\text{-wr}(\boldsymbol{M}) + \alpha\text{-wr}(\boldsymbol{N}) + 1)$.

Actually, for any nonnegative integer $l \geq 0$, we will construct a differential operator of order $2(l + \beta\text{-wr}(\boldsymbol{M}) + \alpha\text{-wr}(\boldsymbol{N}) + 1)$ for which the orthogonal polynomials $(q_n)_n$ are eigenfunctions (see Theorem 4.1). We have computational evidences showing that, except for special values of the parameters α and β and the matrices $\boldsymbol{M}$ and $\boldsymbol{N}$, the minimum order of a differential operator having the orthogonal polynomials $(q_n)_n$ as eigenfunctions seems to be $2(\beta\text{-wr}(\boldsymbol{M}) + \alpha\text{-wr}(\boldsymbol{N}) + 1)$. However this is not true in general. For instance, when $\alpha = \beta, m_1 = m_2 = 1$ and $\boldsymbol{M} = \boldsymbol{N}$, we have $2(\beta\text{-wr}(\boldsymbol{M}) + \alpha\text{-wr}(\boldsymbol{N}) + 1) = 4\alpha + 2$ $(\alpha \geq 1)$. But Koekoek found a differential operator of order $2\alpha + 2$ $(\alpha \geq 1)$ for which these Gegenbauer type orthogonal polynomials are eigenfunctions (see [20]). However, we will show that our method can be adapted to this and other special cases and provides differential operators of order lower than $2(\beta\text{-wr}(\boldsymbol{M}) + \alpha\text{-wr}(\boldsymbol{N}) + 1)$.

We finish pointing out that, as explained above, the approach of this paper is the same as in [7] for discrete Laguerre–Sobolev orthogonal polynomials. Since we work here with two matrices $\boldsymbol{M}$ and $\boldsymbol{N}$ (instead of only one matrix as in [7]), and the sequence of eigenvalues for the Jacobi polynomials (1.4) is a quadratic polynomial in n, the computations are technically more involved. Therefore, we will omit some proofs which are too similar to the corresponding ones in [7].

2. Preliminaries and proof of Theorem 1.1

We say that a sequence of polynomials $(q_n)_n$, with $\deg(q_n) = n, n \geq 0$, is (left) orthogonal with respect to a bilinear form B (not necessarily symmetric) defined in the linear space of real polynomials if $B(q_n, q) = 0$ for all polynomials q with $\deg(q) < n$ and $B(q_n, q_n) \neq 0$. It is clear from the definition that (left) orthogonal polynomials with respect to a bilinear form, if they exist, are unique up to multiplication by nonzero constants. Given a measure ν (positive or not), with finite moments of any order, we consider the bilinear form $B_\nu(p, q) = \int pq d\nu$. We then say that a sequence of polynomials $(q_n)_n$, with $\deg(q_n) = n, n \geq 0$, is orthogonal with respect to the measure ν if it is orthogonal with respect to the bilinear form B_ν.

We will use the following lemma to construct (left) orthogonal polynomials with respect to a discrete Sobolev bilinear form with two nodes. This result is an extension of the Lemma 2.1 of [7].

Lemma 2.1. *For $m_1, m_2 \geq 0$ with $m = m_1 + m_2 \geq 1$, let $\boldsymbol{M} = (M_{i,j})_{i,j=0}^{m_1-1}$ and $\boldsymbol{N} = (N_{i,j})_{i,j=0}^{m_2-1}$ be $m_1 \times m_1$ and $m_2 \times m_2$ matrices, respectively. For a given measure ν and for a couple of real numbers λ and μ ($\lambda \neq \mu$) consider the discrete Sobolev bilinear form defined by*

$$\langle p, q\rangle = \int p(x)q(x)d\nu(x) + \mathbb{T}_\lambda^{m_1}(p)\boldsymbol{M}\mathbb{T}_\lambda^{m_1}(q)^T + \mathbb{T}_\mu^{m_2}(p)\boldsymbol{N}\mathbb{T}_\mu^{m_2}(q)^T, \quad (2.1)$$

where for a nonnegative integer k, a real number λ and a polynomial p, we define

$$\mathbb{T}_\lambda^k(p) = (p(\lambda), p'(\lambda), \ldots, p^{(k-1)}(\lambda)).$$

Assume that the measure $(x-\lambda)^{m_1}(\mu-x)^{m_2}\nu$ has a sequence $(p_n)_n$ of orthogonal polynomials, and write

$$w_{n,i}^1 = \int (x-\lambda)^i(\mu-x)^{m_2} p_n(x)d\nu,$$

$$w_{n,j}^2 = \int (x-\lambda)^{m_1}(\mu-x)^j p_n(x)d\nu.$$

For $l = 1, \ldots, m_1$, define the sequences $(R_l(n))_n$ by

$$R_l(n) = w_{n,l-1}^1 + \sum_{i=0}^{m_1-1}\left(\sum_{j=l}^{(l+m_2)\wedge m_1} \frac{(j-1)!\binom{m_2}{j-l}M_{i,j-1}}{(-1)^{m_2}(\lambda-\mu)^{j-l-m_2}}\right)p_n^{(i)}(\lambda), \quad (2.2)$$

where $\wedge$ denotes the minimum between two numbers. For $l = m_1+1, \ldots, m$, define the sequences $(R_l(n))_n$ by

$$R_l(n) = w_{n,l-m_1-1}^2 + \sum_{i=0}^{m_2-1}\left(\sum_{j=l-m_1}^{l\wedge m_2} \frac{(j-1)!\binom{m_1}{l-j}N_{i,j-1}}{(-1)^{l-m_1-1}(\mu-\lambda)^{j-l}}\right)p_n^{(i)}(\mu). \quad (2.3)$$

Then the following conditions are equivalent:

(1) *for $n \geq m$, the discrete Sobolev bilinear form* (2.1) *has an* (*left*) *orthogonal polynomial q_n with* $\deg(q_n) = n$ *and nonzero norm*;

(2) *the $m \times m$ Casorati determinant $\Lambda(n) = \det(R_i(n-j))_{i,j=1}^m$ does not vanish for $n \geq m$.*

Moreover, if one of these properties holds, the polynomial defined by

$$q_n(x) = \begin{vmatrix} p_n(x) & p_{n-1}(x) & \cdots & p_{n-m}(x) \\ R_1(n) & R_1(n-1) & \cdots & R_1(n-m) \\ \vdots & \vdots & \ddots & \vdots \\ R_m(n) & R_m(n-1) & \cdots & R_m(n-m) \end{vmatrix},$$

has degree n, $n \geq m$, and the sequence $(q_n)_n$ is (*left*) *orthogonal with respect to* (2.1).

Proof. We proceed in the same lines as the proof of Lema 2.1 in [7], but changing the powers $(x-\lambda)^l$, $l = 1, \ldots, m$, to $(x-\lambda)^{i-1}(\mu - x)^{m_2}$, $i = 1, \ldots, m_1$ and $(x-\lambda)^{m_1}(\mu - x)^{j-1}$, $j = 1, \ldots, m_2$. We only have to explain how to find the identities (2.2) and (2.3) for the sequences $R_l(n)$, $l = 1, \ldots, m$.

For $l = 1, \ldots, m_1$, we use $q(x) = (x-\lambda)^{l-1}(\mu - x)^{m_2}$. Then every component i of the vector $\mathbb{T}_\lambda^{m_1}(q)$ is given by

$$[\mathbb{T}_\lambda^{m_1}(q)]_i = \begin{cases} \dfrac{(i-1)!\binom{m_2}{i-l}}{(-1)^{m_2}(\lambda-\mu)^{i-l-m_2}} & \text{if } i = l, \ldots, (l+m_2) \wedge m_1 \\ 0 & \text{if } i = 1, \ldots, l-1 \\ & \text{or } i = l+m_2+1, \ldots, m_1, \end{cases}$$

while $\mathbb{T}_\mu^{m_2} = (0, \ldots, 0)$. Hence, if we write

$$q_n(x) = \sum_{j=0}^{m} \beta_{n,j} \, p_{n-j}(x),$$

where $\beta_{n,0} = 1$, we get (2.2) by evaluating the bilinear form

$$\langle q_n, (x-\lambda)^{l-1}(\mu - x)^{m_2} \rangle = 0, \quad \text{for } l = 1, \ldots, m_1.$$

Similar for $l = m_1 + 1, \dots, m$, where now we use

$$q(x) = (x-\lambda)^{m_1}(\mu - x)^{l-m_1-1}.$$

Then

$$[\mathbb{T}_\mu^{m_2}(q)]_i = \begin{cases} \dfrac{(i-1)!\binom{m_1}{l-i}}{(-1)^{l-m_1-1}(\mu-\lambda)^{i-l}} & \text{if } i = l-m_1, \dots, l \wedge m_2, \\ 0 & \text{if } \quad i = 1, \dots, l-m_1-1 \\ & \text{or } i = l+1, \dots, m_2, \end{cases}$$

while $\mathbb{T}_\lambda^{m_1}(q) = (0, \dots, 0)$. Evaluating $\langle q_n, (x-\lambda)^{m_1}(\mu - x)^{l-m_1-1} \rangle = 0$, for $l = m_1 + 1, \dots, m$, we get (2.3).

Finally, observe that the set of polynomials

$$b_l(x) = \begin{cases} (x-\lambda)^{l-1}(\mu - x)^{m_2}, & l = 1, \dots, m_1, \\ (x-\lambda)^{m_1}(\mu - x)^{l-m_1-1}, & l = m_1+1, \dots, m, \end{cases} \tag{2.4}$$

is linearly independent and has dimension exactly m only when $\lambda \neq \mu$. □

We also need the following combinatorial identities (which can be proved using standard combinatorial techniques).

Lemma 2.2. *Let α, β be non-integers real numbers, such that $\alpha + \beta$ is not an integer. Let m_1, m_2 be nonnegative integers with $m_1 + m_2 \geq 1$ and write $m = m_1 + m_2$.*

(1) *For $k = 1, \dots, m_1 - h - 1$, $h = 0, \dots, m_1 - 2$,*

$$\sum_{l=0}^{m_1-1} \left(-\frac{1}{2}\right)^l \binom{h}{m_1 - l} \binom{l-k}{l} \frac{\Gamma(\alpha - k + 1)\Gamma(\beta - l)}{\Gamma(\alpha + \beta - k - l + 1)} = 0. \tag{2.5}$$

(2) *For $k = 1, \dots, m-1$,*

$$\begin{aligned} &(-1)^k \sum_{l=0}^{m_1-1} \binom{m-l-2}{m_2-1} \binom{l-k}{l} \frac{\Gamma(\alpha - k + 1)\Gamma(\beta - l)}{\Gamma(\alpha + \beta - k - l + 1)} \\ &\quad + \sum_{l=0}^{m_2-1} \binom{m-l-2}{m_1-1} \binom{l-k}{l} \frac{\Gamma(\beta - k + 1)\Gamma(\alpha - l)}{\Gamma(\alpha + \beta - k - l + 1)} = 0. \end{aligned} \tag{2.6}$$

We are now ready to prove Theorem 1.1 in the introduction.

Proof of Theorem 1.1. We proceed in two steps.

First step. Assume $n \geq m$. Notice that for $n \geq m$, $p(n)q(n) \neq 0$, where p and q are the polynomials given by (1.9) and (1.10). Actually, we can remove the normalization $1/(p(n)q(n))$ from the definition of the polynomials q_n (as we will see below, this normalization is going to be useful only for some instances of α and β when $n = 0, \dots, m-1$).

For $n \geq m$, the theorem is a direct consequence of Lemma 2.1 for $\lambda = -1$, $\mu = 1$, the Jacobi measure $\nu = \mu_{\alpha-m_2,\beta-m_1}$, see (1.2), and the Jacobi polynomials $p_n = J_n^{\alpha,\beta}$.

Indeed, we use the following expansions (see Theorem 3.21, p.76 of [29], after some computations using Pochhammer symbol properties):

$$J_n^{\alpha,\beta}(x) = \sum_{k=0}^{n} \Bigg[\frac{2k+\alpha+\gamma+1}{k+\alpha+\gamma+1} \frac{1}{\binom{\alpha+\beta}{\beta}\binom{k+\alpha+\gamma}{\alpha}\binom{n}{k}\binom{n+\alpha+\gamma+k+1}{n}} \binom{n+\alpha+\beta}{\alpha}\binom{\alpha+\gamma}{\gamma}\binom{n+\alpha}{n-k} \binom{n+\alpha+\beta+k}{k}\binom{n-k+\beta-\gamma-1}{n-k} \Bigg] J_k^{\alpha,\gamma}(x),$$

and

$$J_n^{\alpha,\beta}(x) = \sum_{k=0}^{n} \Bigg[\frac{2k+\beta+\gamma+1}{k+\beta+\gamma+1} \frac{1}{(-1)^{n-k}\binom{\alpha+\beta}{\beta}\binom{k+\beta+\gamma}{\gamma}\binom{n}{k}\binom{n+\beta+\gamma+k+1}{n}} \binom{n+\alpha+\beta}{\alpha}\binom{\beta+\gamma}{\gamma}\binom{n+\beta}{n-k} \binom{n+\alpha+\beta+k}{k}\binom{n-k+\alpha-\gamma-1}{n-k} \Bigg] J_k^{\gamma,\beta}(x).$$

Therefore we get for $l = 1, \dots, m_1$,

$$\begin{aligned} w^1_{n,l-1} &= \int (x+1)^{l-1} J_n^{\alpha,\beta}(x)\mu_{\alpha,\beta-m_1}(x)dx \\ &= \int J_n^{\alpha,\beta}(x)\mu_{\alpha,\beta-m_1+l-1}(x)dx \\ &= \frac{1}{\Gamma(\alpha+\beta+1)} \\ &\qquad 2^{\alpha+\beta-m_1+l}\Gamma(\beta+1)\Gamma(\beta-m_1+l) \\ &\qquad \Gamma^{\alpha,\alpha+\beta}_{\beta,\alpha+\beta+l-m_1}(n)\binom{n+m_1-l}{n}, \end{aligned}$$

(where we are using the notation (1.6)), and for $l = m_1+1, \dots, m$,

$$\begin{aligned} w^2_{n,l-m_1-1} &= \int (1-x)^{l-m_1-1} J_n^{\alpha,\beta}(x)\mu_{\alpha-m_2,\beta}(x)dx \\ &= \int J_n^{\alpha,\beta}(x)\mu_{\alpha+l-m-1,\beta}(x)dx \\ &= \frac{1}{\Gamma(\alpha+\beta+1)\Gamma(\alpha+\beta+n+l-m+1)} \\ &\qquad (-1)^n 2^{\alpha+\beta-m+l}\Gamma(n+\alpha+\beta+1)\Gamma(\beta+1) \\ &\qquad \Gamma(\alpha-m+l)\binom{n+m-l}{n}. \end{aligned}$$

We also need the following identities (after a combination of formulas (3.94), (3.100), and (3.107) of [29]):

$$(J_n^{\alpha,\beta})^{(i)}(-1) = \frac{(-1)^i i!}{2^i\binom{\alpha+\beta}{\beta}}\binom{n+\alpha+\beta}{\alpha}\binom{n+\beta}{n-i}\binom{n+\alpha+\beta+i}{i},$$

$$(J_n^{\alpha,\beta})^{(i)}(1) = \frac{(-1)^n i!}{2^i\binom{\alpha+\beta}{\beta}}\binom{n+\alpha+\beta}{\alpha}\binom{n+\alpha}{n-i}\binom{n+\alpha+\beta+i}{i}.$$

If we replace these identities in (2.2) and (2.3), we get, after some computations, the expressions

$$R_l(n) = \frac{\Gamma(\beta+1)\Gamma(n+\alpha+1)}{\Gamma(\alpha+\beta+1)\Gamma(n+\beta+1)} \left[\frac{2^{\alpha+\beta-m_1+l}\Gamma(\beta-m_1+l)}{(m_1-l)!}\Gamma^{m_1-l,\alpha+\beta}_{0,\alpha+\beta-m_1+l}(n) + 2^{m_2}\sum_{i=0}^{m_1-1}\left(\sum_{j=l}^{(l+m_2)\wedge m_1} \frac{(j-1)!\binom{m_2}{j-l}M_{i,j-1}}{(-2)^{i+j-l}}\right)\frac{\Gamma^{\beta,\alpha+\beta+i}_{-i,\alpha}(n)}{\Gamma(\beta+i+1)}\right], \tag{2.7}$$

for $l = 1, \dots, m_1$, and for $l = m_1+1, \dots, m$,

$$R_l(n) = \frac{(-1)^n\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \left[\frac{2^{\alpha+\beta-m+l}\Gamma(\alpha-m+l)}{(m-l)!}\Gamma^{m-l,\alpha+\beta}_{0,\alpha+\beta-m+l}(n) + \sum_{i=0}^{m_2-1}\left(\sum_{j=l-m_1}^{l\wedge m_2} \frac{(j-1)!\binom{m_1}{l-j}N_{i,j-1}}{(-1)^{l-m_1-1}2^{i+j-l}}\right)\frac{\Gamma^{\alpha,\alpha+\beta+i}_{-i,\beta}(n)}{\Gamma(\alpha+i+1)}\right]. \tag{2.8}$$

Comparing (2.7) and (2.8) with (1.7) and (1.8), we get

$$R_l(n) = \begin{cases} \dfrac{\Gamma(\beta+1)\Gamma(n+\alpha+1)}{\Gamma(\alpha+\beta+1)\Gamma(n+\beta+1)} z_l(n), & l = 1, \dots, m_1, \\[2ex] \dfrac{(-1)^n\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} z_l(n), & l = m_1+1, \dots, m. \end{cases}$$

And therefore

$$R_l(n-j) = \begin{cases} \dfrac{(-1)^m\Gamma(\beta+1)\Gamma(n+\alpha+1-m)}{\Gamma(\alpha+\beta+1)\Gamma(n+\beta)}(-1)^j\rho^l_{n,j} z_l(n-j), \\ \qquad\qquad\qquad\qquad l = 1, \dots, m_1, \\[2ex] \dfrac{(-1)^n\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(-1)^j\rho^l_{n,j} z_l(n-j), \qquad l = m_1+1, \dots, m, \end{cases}$$

where $\rho^l_{n,j}, l = 1, \dots, m$, was defined by (1.11).

Since $n \geq m$, the hypothesis on α and β shows that $n+\alpha+1-m, n+\beta \neq 0, -1, -2, \dots$ and hence Theorem 2.1 gives that the polynomial q_n, $n \geq m$, is orthogonal with respect to the Jacobi Sobolev bilinear form defined by (1.5).

Second step. Assume $n = 0, 1, \ldots, m-1$. When α and β are integers, $p(n)q(n)$ can vanish for some $n = 0, \ldots, m-1$. Hence, we first prove that even if for some $n = 0, \ldots, m-1$, $p(n)q(n) = 0$, the ratio Λ_n (1.12) and the polynomial q_n (1.13) are well defined (and hence q_n has degree n if and only if $\Lambda_n \neq 0$).

Indeed, assume first that $p(n) = 0$. From (1.9) we have that n should be either $n = -\alpha + m - i$ or $n = -\beta + i$, $i = 1, \ldots, m_1 - 1$. Consider first the case $n = -\alpha + m - i$. Write $\widetilde{\Lambda}$ for the $m \times (m+1)$ matrix function

$$\widetilde{\Lambda}(x) = \begin{pmatrix} \rho^1_{x,0} z_1(x) & \rho^1_{x,1} z_1(x-1) & \cdots & \rho^1_{x,m} z_1(x-m) \\ \vdots & \vdots & \ddots & \vdots \\ \rho^m_{x,0} z_m(x) & \rho^m_{x,1} z_m(x-1) & \cdots & \rho^m_{x,m} z_m(x-m) \end{pmatrix}.$$

For $j \geq 1$, the function $\rho^h_{x,j}$, $h = 1, \ldots, m_1$, (1.11) is actually a polynomial: $\rho^h_{x,j} = (-1)^{m-j}(x+\alpha-m+1)_{m-j}(x+\beta-j+1)_{j-1}$. Hence $n = -\alpha+m-i$ is a root of $\rho^h_{x,j}$, for $j = 0, \ldots, m-i$. Therefore, we have that rank $\widetilde{\Lambda}(-\alpha+m-i) \leq m_2+i$. So for $h = 1, \ldots, m+1$, 0 is an eigenvalue of $\widetilde{\Lambda}_h(-\alpha+m-i)$ of geometric multiplicity at least $m-m_2-i = m_1-i$, where $\widetilde{\Lambda}_h$ is the square matrix obtained by removing the h-th column of $\widetilde{\Lambda}$. We then deduce that 0 is an eigenvalue of $\widetilde{\Lambda}_h(-\alpha+m-i)$ of algebraic multiplicity at least m_1-i. This implies that $x = -\alpha+m-i$ is a root of $\det \widetilde{\Lambda}_h(x)$ of multiplicity at least $m_1 - i$, which it is precisely the multiplicity of $-\alpha+m-i$ as a zero of $p(x)$. A similar result can be proved for the other zeros of p and the zeros of q. This proves that the ratios $\det \widetilde{\Lambda}_h(x)/(p(x)q(x))$ are well defined even when $p(x)q(x) = 0$. Hence for $n = 0, \ldots, m-1$, the ratio Λ_n (1.12) and the polynomial q_n (1.13) are well defined and q_n has degree n if and only if $\Lambda_n \neq 0$.

We now prove that q_n are orthogonal for $n = 0, 1, \ldots, m-1$. We need first to introduce some notation. For $l = 1, \ldots, m$, we define

$$\lambda_l(n) = \begin{cases} \dfrac{(-1)^m \Gamma(\alpha+\beta+1)\Gamma(\beta+n)}{\Gamma(\beta+1)\Gamma(\alpha+n+1-m)}, & l = 1, \ldots, m_1, \\[2ex] \dfrac{(-1)^n \Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}, & l = m_1+1, \ldots, m. \end{cases} \tag{2.9}$$

Write $F_n(x)$, $f_l(n)$, $l = 1, \ldots, m$, for the row vectors of size $m+1$ whose entries are

$$F_{n,j}(x) = (-1)^{j-1} J^{\alpha,\beta}_{n-j+1}(x), \quad f_{l,j}(n) = \rho^l_{n,j-1} z_l(n-j+1) \tag{2.10}$$

for $j = 1, \ldots, m+1$ (in particular, $F_{n,j} = 0$, $j = n+2, \ldots, m+1$). Hence

$$q_n(x) = \frac{1}{p(n)q(n)} \begin{vmatrix} F_n(x) \\ f_1(n) \\ \vdots \\ f_m(n) \end{vmatrix}.$$

Consider the basis of $\mathbb{P}_{n-1}$ given by $v_h(x) = (1-x)^h$, $h = 0, \ldots, n-1$. Since $\deg(v_h) = h$, it is enough to prove that $\langle q_n, v_h \rangle = 0$, $h = 0, \ldots, n-1$. Since $b_l(x)$, $l = 1, \ldots m$, (see (2.4)) is a basis of $\mathbb{P}_{m-1}$ and $n-1 \le m-1$, we have $v_h(x) = \sum_{l=1}^m a_{h,l} b_l(x)$, and then

$$\langle q_n, v_h \rangle = \frac{1}{p(n)q(n)} \begin{vmatrix} \left(\sum_{l=1}^m a_{h,l} \langle F_{n,j}, b_l \rangle, \, j = 1, \ldots, m+1 \right) \\ f_1(n) \\ \vdots \\ f_m(n) \end{vmatrix}. \tag{2.11}$$

Notice that $p(n)q(n)$ and each entry of the rows $f_l(n)$ (2.10) are meromorphic functions of α or β. It was shown in the first step that

$$\int_{-1}^1 J_n^{\alpha,\beta}(x) b_l(x) d\mu_{\alpha-m_2,\beta-m_1}(x) dx$$

and $(J_n^{\alpha,\beta})^{(i)}(\pm 1)$ are also meromorphic functions of α or β, and then $\langle F_{n,j}, b_l \rangle$ is also meromorphic. This shows that $\langle q_n, v_h \rangle$ is a meromorphic function of α or β. It is then enough to prove that $\langle q_n, v_h \rangle = 0$ assuming that α, β and $\alpha + \beta$ are non-integers real numbers. Hence we have $p(n)q(n), \lambda_l(n) \neq 0$, $l = 1, \ldots, m$, see (2.9).

Proceeding as in the proof of the first step, we can prove that if $\deg(b_l(x)) < n$ then

$$f_{l,j}(n) = \lambda_l(n) \langle F_{n,j}, b_l \rangle, \quad j = 1, \ldots, m+1. \tag{2.12}$$

On the other hand, if $\deg(b_l(x)) \ge n$ then

$$f_{l,j}(n) = \lambda_l(n) \langle F_{n,j}, b_l \rangle, \quad j = 1, \ldots, n+1. \tag{2.13}$$

Write $u = (\sum_{l=1}^m a_{h,l} \langle F_{n,j}, b_l \rangle, \, j = 1, \ldots, m+1)$, i.e., u is the first row in the determinant (2.11). Since $p(n)q(n), \lambda_l(n) \neq 0$, $\langle q_n, v_h \rangle = 0$ will follow if we prove that $u = \sum_{l=1}^m \frac{a_{h,l}}{\lambda_l(n)} f_l(n)$, i.e. a linear combination of the rest of the rows.

For $j = 1, \dots, n+1$, we have from (2.12) and (2.13)

$$u_j = \sum_{l=1}^{m} a_{h,l} \langle F_{n,j}, b_l \rangle = \sum_{l=1}^{m} \frac{a_{h,l}}{\lambda_l(n)} f_{l,j}(n).$$

For $j = n+2, \dots, m+1$, $F_{n,j} = 0$, see (2.10), and then $u_j = 0$. Taking into account the definition of f_l, see (2.10), it is then enough to prove that

$$\sum_{l=1}^{m} \frac{a_{h,l}}{\lambda_l(n)} \rho_{n,j-1}^{l} z_l(n-j+1) = 0, \tag{2.14}$$

for $h = 0, \dots, n-1$, $j = n+2, \dots, m+1$ and $n = 0, \dots, m-1$.

Since $n - j + 1 < 0$, we have from (1.7) and (1.8)

$$z_l(n-j+1) = \begin{cases} \dfrac{\Gamma(\beta - m_1 + l)\Gamma_{0,\alpha+\beta-m_1+l}^{m_1-l,\alpha+\beta}(n-j+1)}{2^{-\alpha-\beta+m_1-l}(m_1-l)!}, & l = 1, \dots, m_1, \\ \dfrac{\Gamma(\alpha - m + l)\Gamma_{0,\alpha+\beta-m+l}^{m-l,\alpha+\beta}(n-j+1)}{2^{-\alpha-\beta+m-l}(m-l)!}, & l = m_1+1, \dots, m. \end{cases}$$

Using (1.11) and (2.10), we get

$$\frac{\rho_{n,j-1}^{l} z_l(n-j+1)}{\lambda_l(n)} = \begin{cases} c_n \dfrac{\Gamma(\beta - m_1 + l)\Gamma_{0,\alpha+\beta-m_1+l}^{\alpha,m_1-l}(n-j+1)}{(-1)^{j-1}2^{m_1-l}(m_1-l)!}, & l = 1, \dots, m_1, \\ c_n \dfrac{\Gamma(\alpha - m + l)\Gamma_{0,\alpha+\beta-m+l}^{\beta,m-l}(n-j+1)}{(-1)^{n}2^{m-l}(m-l)!}, & l = m_1+1, \dots, m, \end{cases}$$

where

$$c_n = \frac{2^{\alpha+\beta}\Gamma(\beta+1)\Gamma(n-j+\alpha+\beta+2)}{\Gamma(\alpha+\beta+1)\Gamma(n-j+\beta+2)}.$$

Inserting them into (2.14), we get

$$\begin{aligned} &(-1)^{j-1} \sum_{l=1}^{m_1} \frac{a_{h,l}\Gamma(\beta - m_1 + l)\Gamma_{0,\alpha+\beta-m_1+l}^{\alpha,m_1-l}(n-j+1)}{2^{m_1-l}(m_1-l)!} \\ &\quad + (-1)^{n} \sum_{l=1}^{m_2} \frac{a_{h,m_1+l}\Gamma(\alpha - m_2 + l)\Gamma_{0,\alpha+\beta-m_2+l}^{\beta,m_2-l}(n-j+1)}{2^{m_2-l}(m_2-l)!} = 0. \end{aligned} \tag{2.15}$$

If we write $k=-n+j-1$, after some computations we can rewrite (2.15) in the simpler form

$$\begin{aligned}(-1)^k\sum_{l=0}^{m_1-1}\binom{l-k}{l}\frac{a_{h,m_1-l}\Gamma(\alpha-k+1)\Gamma(\beta-l)}{2^l\Gamma(\alpha+\beta-k-l+1)}\\+\sum_{l=0}^{m_2-1}\binom{l-k}{l}\frac{a_{h,m-l}\Gamma(\beta-k+1)\Gamma(\alpha-l)}{2^l\Gamma(\alpha+\beta-k-l+1)}=0.\end{aligned}\tag{2.16}$$

for $k=1,\dots,m-h-1$, $h=0,\dots,m-2$. This identity is then equivalent to (2.14). Then we finish the proof by proving (2.16). For $h=0$, we have that

$$a_{0,l}=\begin{cases}\dfrac{\binom{l+m_2-2}{m_2-1}}{2^{l+m_2-1}}, & l=1,\dots,m_1,\\[2ex] \dfrac{\binom{l-2}{l-m_1-1}}{2^{l-1}}, & l=m_1+1,\dots,m.\end{cases}$$

Inserting them into (2.16), we get the identity (2.6) in Lemma 2.2.

The rest of the proof proceeds by induction on m_2. We first consider $m_2=0$, for which $b_l(x)=(1+x)^{l-1}$, $l=0,\dots,m_1$. A simple computation gives

$$a_{h,l}=(-1)^l2^{h-l}\binom{h}{l}.$$

Inserting this identity in (2.16), we get

$$\sum_{l=0}^{m_1-1}\left(-\frac{1}{2}\right)^l\binom{h}{m_1-l}\binom{l-k}{l}\frac{\Gamma(\alpha-k+1)\Gamma(\beta-l)}{\Gamma(\alpha+\beta-k-l+1)}=0.$$

But this is the identity (2.5) in Lemma 2.2.

From now on, we write $b_l^{m_2}$, $l=1,\dots,m$, for the basis (2.4) corresponding to the nonnegative integers m_1 and m_2, and $b_l^{m_2+1}$, $l=1,\dots,m+1$, for the basis (2.4) corresponding to m_1 and m_2+1. We also write $a_{h,l}^{m_2}$, $a_{h,l}^{m_2+1}$ for the corresponding coefficients of v_h with respect to $b_l^{m_2}$, $b_l^{m_2+1}$, respectively. For $h=1,\dots,m+1$, we have the following relationship between $a_{h,l}^{m_2}$ and $a_{h,l}^{m_2+1}$

$$a_{h,l}^{m_2+1}=\begin{cases}a_{h-1,l}^{m_2}, & l=1,\dots,m_1,\\ 0, & l=m_1+1,\\ a_{h-1,l-1}^{m_2}, & l=m_1+2,\dots,m+1.\end{cases}$$

This shows that the identity (2.16) for $m_2 + 1$ and h reduces to (2.16) for m_2 and $h - 1$, and hence the induction hypothesis says that (2.16) holds for $m_2 + 1$ and $h = 1, \dots, m + 1$. □

3. $\mathcal{D}$-operators

The $\mathcal{D}$-operator concept was introduced by one of us in [4]. In [1], [4]–[8], it was shown that $\mathcal{D}$-operators are an extremely useful tool of an unified method for generating families of polynomials that are eigenfunctions of higher-order differential, difference or q-difference operators. Hence, we start by recalling the concept of $\mathcal{D}$-operator.

The starting point is a sequence of polynomials $(p_n)_n$, $\deg(p_n) = n$, and an algebra of operators $\mathcal{A}$ that act in the linear space of polynomials $\mathbb{P}$. For the Jacobi polynomials we consider the algebra $\mathcal{A}$ formed by all differential operators of finite order which do not increase the degree of polynomials, i.e.

$$\mathcal{A} = \Big\{ \sum_{j=0}^{s} f_j \Big(\frac{d}{dx}\Big)^j : f_j \in \mathbb{P}, \deg(f_j) \le j, j = 0, \dots, s, s \in \mathbb{N} \Big\}. \tag{3.1}$$

If $f_s \neq 0$ then the *order* of such differential operator is s. In addition, we assume that the polynomials p_n, $n \ge 0$, are eigenfunctions of a certain operator $D_p \in \mathcal{A}$. We write $(\theta_n)_n$ for the corresponding eigenvalues such that $D_p(p_n) = \theta_n p_n$, $n \ge 0$. For the Jacobi polynomials, θ_n is a polynomial in n of degree 2 (see (1.4)), but we do not assume any constraint on the sequence $(\theta_n)_n$ in this section.

Given two sequences of numbers, $(\varepsilon_n)_n$ and $(\sigma_n)_n$, a $\mathcal{D}$-operator associated with the algebra $\mathcal{A}$ and the sequence of polynomials $(p_n)_n$ is defined as follows. First, we consider the operator $\mathcal{D}\colon \mathbb{P} \to \mathbb{P}$ defined by linearity from

$$\mathcal{D}(p_n) = -\frac{1}{2}\sigma_{n+1} p_n + \sum_{j=1}^{n} (-1)^{j+1} \sigma_{n-j+1} \varepsilon_n \cdots \varepsilon_{n-j+1} p_{n-j}, \quad n \ge 0.$$

Then, we say that $\mathcal{D}$ is a $\mathcal{D}$-operator if $\mathcal{D} \in \mathcal{A}$. In [4] this type of $\mathcal{D}$-operator was designated as type 2, whereas $\mathcal{D}$-operators of type 1 appear when the sequence $(\sigma_n)_n$ is constant. $\mathcal{D}$-operators of type 1 are simpler but they are only useful when the sequence of eigenvalues $(\theta_n)_n$ is linear in n; this is the reason why we used $\mathcal{D}$-operators of type 1 in [7] for discrete Laguerre–Sobolev polynomials, but we have to use $\mathcal{D}$-operators of type 2 in this paper for discrete Jacobi–Sobolev polynomials.

Let us now provide a couple of examples of $\mathcal{D}$-operators for the Jacobi polynomials. We now consider the algebra $\mathcal{A}$ of differential operators defined by (3.1). The two $\mathcal{D}$-operators for the Jacobi polynomials are defined by the sequences $(\varepsilon_{n,h})_n$ and $(\sigma_{n,h})_n$, $h = 1, 2$, given by

$$\varepsilon_{n,1} = -\frac{n+\alpha}{n+\beta}, \quad \sigma_{n,1} = \sigma_n = 2n + \alpha + \beta - 1, \tag{3.2}$$

$$\varepsilon_{n,2} = 1, \qquad \sigma_{n,2} = -\sigma_n = -(2n + \alpha + \beta - 1). \tag{3.3}$$

As proved in Lemma A.7 of [4], these sequences define two $\mathcal{D}$-operators $\mathcal{D}_1$ and $\mathcal{D}_2$ for the Jacobi polynomials. Moreover

$$\mathcal{D}_1 = -\frac{\alpha+\beta+1}{2} I + (1-x)\frac{d}{dx} \quad \text{and} \quad \mathcal{D}_2 = \frac{\alpha+\beta+1}{2} I + (1+x)\frac{d}{dx}. \tag{3.4}$$

We next show how to use $\mathcal{D}$-operators to construct new sequences of polynomials $(q_n)_n$ such that there exists an operator $D_q \in \mathcal{A}$ for which they are eigenfunctions (we follow the same lines as Section 3 in [8]).

Consider a combination of $m + 1$, $m \geq 1$, consecutive p_n's. We also use m arbitrary polynomials $Y_1, Y_2, \ldots, Y_m$, and m $\mathcal{D}$-operators, $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_m$, (which are not necessarily different) defined by the pairs of sequences $(\varepsilon_n^h)_n$, $(\sigma_n^h)_n$, $h = 1, \ldots, m$:

$$\mathcal{D}_h(p_n) = -\frac{1}{2}\sigma_{n+1}^h p_n + \sum_{j=1}^{n} (-1)^{j+1} \sigma_{n-j+1}^h \varepsilon_n^h \cdots \varepsilon_{n-j+1}^h p_{n-j}. \tag{3.5}$$

For $h = 1, 2, \ldots, m$, we assume that the sequences $(\varepsilon_n^h)_n$ and $(\sigma_n^h)_n$ are rational functions in n. We write $\xi_{x,i}^h$, $i \in \mathbb{Z}$ and $h = 1, 2, \ldots, m$, for the auxiliary functions defined by

$$\xi_{x,i}^h = \prod_{j=0}^{i-1} \varepsilon_{x-j}^h, \quad i \geq 1, \qquad \xi_{x,0}^h = 1, \qquad \xi_{x,i}^h = \frac{1}{\xi_{x-i,-i}^h}, \quad i \leq -1. \tag{3.6}$$

We consider the $m \times m$ (quasi) Casorati determinant defined by

$$\Omega(x) = \det(\xi_{x-j,m-j}^l Y_l(\theta_{x-j}))_{l,j=1}^m. \tag{3.7}$$

Then, we have the following result.

Theorem 3.1 (Theorem 3.1 of [8]). *Let $\mathcal{A}$ and $(p_n)_n$ be an algebra of operators that act in the linear space of polynomials and a sequence of polynomials $(p_n)_n$,* $\deg(p_n) = n$*, respectively. We assume that $(p_n)_n$ are eigenfunctions of an operator $D_p \in \mathcal{A}$, i.e., the numbers $\theta_n, n \geq 0$, exist such that $D_p(p_n) = \theta_n p_n$, $n \geq 0$. We also have m pairs of sequences of numbers $(\varepsilon_n^h)_n$, $(\sigma_n^h)_n$, $h = 1, \dots, m$, which define m $\mathcal{D}$-operators $\mathcal{D}_1, \dots, \mathcal{D}_m$ (not necessarily different) for $(p_n)_n$ and $\mathcal{A}$ (see (3.5)) and for $h = 1, 2, \dots, m$, we assume that each one of the sequences $(\varepsilon_n^h)_n$, $(\sigma_n^h)_n$ is a rational function in n.*

Let $Y_1, Y_2, \dots, Y_m$, be m arbitrary polynomials that satisfy $\Omega(n) \neq 0$, $n \geq 0$, where Ω is the Casorati determinant defined by (3.7).

Consider the sequence of polynomials $(q_n)_n$ defined by

$$q_n(x) = \begin{vmatrix} p_n(x) & -p_{n-1}(x) & \cdots & (-1)^m p_{n-m}(x) \\ \xi_{n,m}^1 Y_1(\theta_n) & \xi_{n-1,m-1}^1 Y_1(\theta_{n-1}) & \cdots & Y_1(\theta_{n-m}) \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{n,m}^m Y_m(\theta_n) & \xi_{n-1,m-1}^m Y_m(\theta_{n-1}) & \cdots & Y_m(\theta_{n-m}) \end{vmatrix}. \tag{3.8}$$

For a rational function S, we define the function λ_x by

$$\lambda_x - \lambda_{x-1} = S(x)\Omega(x),$$

and for $h = 1, \dots, m$, we define the function $M_h(x)$ by

$$M_h(x) = \sum_{j=1}^{m} (-1)^{h+j} \xi_{x,m-j}^h S(x+j) \det(\xi_{x+j-r,m-r}^l Y_l(\theta_{x+j-r}))_{l \in \mathbb{I}_h; r \in \mathbb{I}_j}, \tag{3.9}$$

where $\mathbb{I}_h = \{1, 2, \dots, m\} \setminus \{h\}$.

We assume the following:

$$S(x)\Omega(x) \text{ is a polynomial in } x; \tag{3.10}$$

there exists polynomials $\widetilde{M}_1, \dots, \widetilde{M}_m$ such that

$$M_h(x) = \sigma_{x+1}^h \widetilde{M}_h(\theta_x), \quad h = 1, \dots, m; \tag{3.11}$$

there exist a polynomial P_S such that

$$P_S(\theta_x) = 2\lambda_x + \sum_{h=1}^{m} Y_h(\theta_x) M_h(x). \tag{3.12}$$

Then, there exists an operator $D_{q,S} \in \mathcal{A}$ such that

$$D_{q,S}(q_n) = \lambda_n q_n, \quad n \geq 0.$$

Moreover, the operator $D_{q,S}$ is defined by

$$D_{q,S} = \frac{1}{2} P_S(D_p) + \sum_{h=1}^{m} \tilde{M}_h(D_p)\mathcal{D}_h Y_h(D_p), \tag{3.13}$$

where $D_p \in \mathcal{A}$ is the operator for which the polynomials $(p_n)_n$ are eigenfunctions.

Remark 3.2. For the particular cases of Laguerre, Jacobi, or Askey–Wilson polynomials, we can find Casorati determinants similar to (3.8) in [12]–[17].

Remark 3.3. According to Remark 3.2 in [8], the polynomial P_S (3.12) also satisfies

$$P_S(\theta_x) - P_S(\theta_{x-1}) = S(x)\Omega(x) + S(x+m)\Omega(x+m).$$

Remark 3.4. The operator D_p in the theorem above does not depent on n. This implies that the polynomials p_n satisfy a second-order differential equation of the form $a_2(x)p_n''(x) + a_1(x)p_n'(x) = \theta_n p_n(x)$, where a_2 and a_1 are independent of n. The application of the $\mathcal{D}$-operator method to other families of polynomials p_n which satisfy second-order differential equations of the form $a_{2,n}(x)p_n''(x) + a_{1,n}(x)p_n'(x) = \theta_n p_n(x)$ with polynomial differential coefficients depending on n (like the Bernstein–Szegő polynomials for instance) remains as a challenge.

The assumptions (3.10), (3.11) and (3.12) turn out to be straightforward for $\mathcal{D}$-operators of type 1 (we can then take $\theta_x = x$) but they need to be checked when we use $\mathcal{D}$-operators of type 2. The rest of this section will be devoted to check these three assumptions for the $\mathcal{D}$-operators (3.4) associated to the Jacobi polynomials.

We need to introduce some notation. We write $N_{1;x}^{\alpha;j}$ and $N_{2;x}^{\beta;j}$, $j \in \mathbb{N}$ and $x \in \mathbb{R}$, for the following functions:

$$N_{1;x}^{\alpha;j} = (-1)^j (x - j + \alpha + 1)_j, \quad N_{2;x}^{\beta;j} = (x - j + \beta + 1)_j.$$

The following properties hold easily by definition

$$N_{1;x-i}^{\alpha;m-i} = N_{1;x-i}^{\alpha;j} N_{1;x-i-j}^{\alpha;m-i-j}, \quad N_{2;x-i}^{\beta;m-i} = N_{2;x-i}^{\beta;j} N_{2;x-i-j}^{\beta;m-i-j}.$$

Assume now that, as in Theorem 3.1, we have m $\mathcal{D}$-operators $\mathcal{D}_i$, $i = 1,\dots,m$, associated to the Jacobi polynomials and defined by the sequences $(\varepsilon_n^h)_n$ and $(\sigma_n^h)_n$, $h = 1,\dots,m$. Assume also that they correspond with the two $\mathcal{D}$-operators defined by the sequences (3.2) and (3.3). More precisely,

$$\varepsilon_n^h = \begin{cases} \varepsilon_{n,1}, & \text{for } h = 1,\dots,m_1, \\ \varepsilon_{n,2}, & \text{for } h = m_1+1,\dots,m, \end{cases} \qquad \sigma_n^h = \begin{cases} \sigma_{n,1}, & \text{for } h = 1,\dots,m_1, \\ \sigma_{n,2}, & \text{for } h = m_1+1,\dots,m. \end{cases} \tag{3.14}$$

where the sequences $(\varepsilon_{n,1})_n$, $(\sigma_{n,1})_n$, and $(\varepsilon_{n,2})_n$, $(\sigma_{n,2})_n$, are defined by (3.2) and (3.3), respectively. The functions $\xi_{x,j}^h$ defined in (3.6) can then be written as

$$\xi_{x,j}^h = \begin{cases} \dfrac{N_{1;x}^{\alpha;j}}{N_{2;x}^{\beta;j}} = \dfrac{(-1)^j (x-j+\alpha+1)_j}{(x-j+\beta+1)_j}, & \text{for } h = 1,\dots,m_1, \\ 1, & \text{for } h = m_1+1,\dots,m. \end{cases} \tag{3.15}$$

We also need to introduce the polynomials p and q defined by

$$p(x) = \prod_{i=1}^{m_1-1} N_{1;x-m_2-i}^{\alpha;m_1-i} N_{2;x-1}^{\beta;m_1-i}, \tag{3.16}$$

$$q(x) = (-1)^{\binom{m}{2}} \prod_{h=1}^{m-1} \Big(\prod_{i=1}^{h} \sigma_{x-m+\frac{i+h+1}{2}} \Big), \tag{3.17}$$

where as in (3.2) $\sigma_x = 2x+\alpha+\beta-1$. It is easy to check that the polynomial p in (3.16) is the same as the polynomial p defined in (1.9), as well as the polynomial q in (3.17) is the same as the polynomial q defined in (1.10).

The key concept in order to check the assumptions (3.10), (3.11), and (3.12) in Theorem 3.1 for the Jacobi polynomials is an *involution* that characterizes the subring $\mathbb{R}[\theta_x]$ in $\mathbb{R}[x]$, where $\theta_x = x(x+\alpha+\beta+1)$ are the eigenvalues for the Jacobi polynomials. This involution is given by

$$\mathcal{I}^{\alpha+\beta}(f(x)) = f(-(x+\alpha+\beta+1)), \quad f \in \mathbb{R}[x]. \tag{3.18}$$

Clearly, we have $\mathcal{I}^{\alpha+\beta}(\theta_x) = \theta_x$. Hence every polynomial in θ_x is invariant under the action of $\mathcal{I}^{\alpha+\beta}$. Conversely, if $f \in \mathbb{R}[x]$ is invariant under $\mathcal{I}^{\alpha+\beta}$, then $f \in \mathbb{R}[\theta_x]$.

We also have that if $f \in \mathbb{R}[x]$ is skew invariant, i.e., $\mathcal{I}^{\alpha+\beta}(f) = -f$, then f is divisible by $\theta_{x-1/2} - \theta_{x+1/2}$ and the quotient belongs to $\mathbb{R}[\theta_x]$. We remark that in the case of Jacobi polynomials and from the definition of θ_x and σ_x we have that $\sigma_{x+1} = \theta_{x-1/2} - \theta_{x+1/2}$. We observe that σ_{x+1} is itself skew invariant.

According to the definition (3.18) we have the following properties:

$$\mathfrak{I}^{\alpha+\beta+i}(\theta_{x-j}) = \theta_{x+i+j}, \qquad \mathfrak{I}^{\alpha+\beta+i}(\sigma_{x-j}) = -\sigma_{x+i+j+2}, \tag{3.19}$$

$$\mathfrak{I}^{\alpha+\beta+i}(N^{\alpha;m-h}_{1;x-j-h}) = N^{\beta;m-h}_{2;x+m+i+j}, \quad \mathfrak{I}^{\alpha+\beta+i}(N^{\beta;m-h}_{2;x-j-h}) = N^{\alpha;m-h}_{1;x+m+i+j}. \tag{3.20}$$

We are now ready to establish that the three assumptions (3.10), (3.11), and (3.12) in Theorem 3.1 hold for the two $\mathcal{D}$-operators associated with the Jacobi polynomials.

Lemma 3.5. *Let $\mathcal{A}$ and $(p_n)_n$ be the algebra of differential operators (3.1) and the sequence of Jacobi polynomials $p_n = J_n^{\alpha,\beta}$, respectively. Let $D_{\alpha,\beta}$ be the second-order differential operator (1.3) such that $\theta_n = n(n+\alpha+\beta+1)$ and $D_{\alpha,\beta}(J_n^{\alpha,\beta}) = \theta_n J_n^{\alpha,\beta}$. For $j = 1, 2$, we also have m_j $\mathcal{D}$-operators defined by the sequences $(\varepsilon_{n,j})_n$, $(\sigma_{n,j})_n$ (see (3.2) and (3.3)). Then, we write $m = m_1 + m_2$ and we let Ξ be a polynomial in x, which is invariant under the action of $\mathfrak{I}^{\alpha+\beta-m-1}$. We define the rational function S by*

$$S(x) = \frac{\sigma_{x-\frac{m-1}{2}}\,\Xi(x)(N^{\beta;m-1}_{2;x-1})^{m_1}}{p(x)q(x)}, \tag{3.21}$$

where p and q are the polynomials defined by (3.16) and (3.17), respectively. Then, the three assumptions (3.10), (3.11), and (3.12) in Theorem 3.1 hold for any polynomials Y_l, $l = 1, \dots, m$.

The proof is quite technical at certain points and is given separately in the appendix.

4. Differential properties for the discrete Jacobi–Sobolev polynomials

In this section we will study differential properties of orthogonal polynomials with respect to the discrete Jacobi–Sobolev bilineal form (1.5). They are a consequence of the determinantal representation (1.13) and Theorem 3.1.

Comparing (1.11) with (3.15), we get

$$\rho^h_{n,j} = \begin{cases} N^{\alpha,m-j}_{1;n-j} N^{\beta;j-1}_{2;n-1} = (\beta+n-m+1)_{m-1}\xi^h_{n-j,m-j}, & l = 1, \dots, m_1, \\ \xi^h_{n-j,m-j}, & l = m_1+1, \dots, m. \end{cases}$$

Theorem 1.1 gives then the following determinantal expression for the orthogonal polynomials with respect to the discrete Jacobi–Sobolev bilineal form (1.5)

$$q_n(x) = \frac{(\beta+n-m+1)_{m-1}^{m_1}}{p(n)q(n)} \begin{vmatrix} J_n^{\alpha,\beta}(x) & -J_{n-1}^{\alpha,\beta}(x) & \cdots & (-1)^m J_{n-m}^{\alpha,\beta}(x) \\ \xi_{n,m}^1 z_1(n) & \xi_{n-1,m-1}^1 z_1(n-1) & \cdots & z_1(n-m) \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{n,m}^m z_m(n) & \xi_{n-1,m-1}^m z_m(n-1) & \cdots & z_m(n-m) \end{vmatrix}. \tag{4.1}$$

Observe that the functions $z_l, l = 1, \ldots, m$, (see (1.7) and (1.8)) are not polynomials in θ_x (not even polynomials in x) as they should be if we want to apply Theorem 3.1. But it turns out that if α and β are nonnegative integers satisfying $\alpha \geq m_2$ and $\beta \geq m_1$, then the functions $z_l, l = 1, \ldots, m$, are polynomials in θ_x. Using that we prove the following

Theorem 4.1. *Assume that any of the two equivalent properties* (1) *and* (2) *in Theorem* 1.1 *hold. If* $\boldsymbol{M}, \boldsymbol{N} \neq 0$, *we assume, in addition, that* α *and* β *are nonnegative integers with* $\alpha \geq m_2$ *and* $\beta \geq m_1$. *If, instead,* $\boldsymbol{M} = 0$, *we assume that only* α *is a positive integer with* $\alpha \geq m_2$, *and if* $\boldsymbol{N} = 0$, *we assume that only* β *is a positive integer with* $\beta \geq m_1$. *Consider a polynomial* Ξ *invariant under the action of* $\mathfrak{I}^{\alpha+\beta-m-1}$ (*see* (3.18)) *and the associated rational function* S (*see* (3.21)). *Then there exists a finite order differential operator* D_S (*which can be constructed using* (3.13)) *for which the orthogonal polynomials* $(q_n)_n$, *see* (1.13), *are eigenfunctions. Moreover, up to an additive constant, the corresponding eigenvalues* $(\lambda_n)_n$ *of* D_S *are* $\lambda_n = P_S(\theta_n)$, *where* P_S *is the polynomial defined by the difference equation*

$$P_S(\theta_x) - P_S(\theta_{x-1}) = S(x)\Omega(x) + S(x+m)\Omega(x+m), \tag{4.2}$$

where Ω *is the* (*quasi*) *Casorati determinant*

$$\Omega(x) = \det(\xi_{x-j,m-j}^l z_l(x-j))_{l,j=1}^m. \tag{4.3}$$

Moreover, the order of the differential operator D_S *is*

$$\deg \Xi + 2(\beta\text{-wr}(\boldsymbol{M}) + \alpha\text{-wr}(\boldsymbol{N}) + 1),$$

where α-wr *and* β-wr *are the* α *and* β *weighted rank introduced in Definition* 1.3.

Theorem 1.2 and Corollary 1.4 in the Introduction are then consequences of Theorem 4.1 (for the particular case of $\Xi = 1$).

Proof. For the two $\mathcal{D}$-operators associated to the Jacobi polynomials, Lemma 3.5 guarantees that the assumptions in Theorem 3.1 hold for each rational function S defined by (3.21) and any polynomials Y_l, $l = 1, \dots, m$. If we prove that there exist polynomials Y_l, $l = 1, \dots, m$, such that $z_l(x) = Y_l(\theta_x)$, where z_l are the functions defined by (1.7) and (1.8), the first part of Theorem 4.1 will follow as a consequence of the determinantal representation (4.1) and Theorem 3.1 (possibly after a renormalization constant).

If β is a nonnegative integer with $\beta \geq m_1$, we can rewrite the functions z_l, $l = 1, \dots, m_1$, in the form (see (1.7))

$$
\begin{aligned}
z_l(x) = {} & \frac{2^{\alpha+\beta-m_1+l}\Gamma(\beta - m_1 + l)}{(m_1 - l)!} u^\alpha_{m_1-l}(x) \\
& + 2^{m_2} \sum_{i=0}^{m_1-1} \left(\sum_{j=l}^{(l+m_2)\wedge m_1} \frac{(j-1)!\binom{m_2}{j-l} M_{i,j-1}}{(-2)^{i+j-l}} \right) \frac{u^0_{\beta+i}(x)}{\Gamma(\beta+i+1)},
\end{aligned}
\tag{4.4}
$$

where u_j^λ, $\lambda \in \mathbb{R}$ and $j \in \mathbb{N}$, is the polynomial of degree $2j$ defined by

$$
u_j^\lambda(x) = (x + \alpha - \lambda + 1)_j (x + \beta + \lambda - j + 1)_j. \tag{4.5}
$$

Analogously, if α is a nonnegative integer with $\alpha \geq m_2$, we can rewrite the functions z_l, $l = m_1 + 1, \dots, m$, in the form (see (1.8))

$$
\begin{aligned}
z_l(x) = {} & \frac{2^{\alpha+\beta-m+l}\Gamma(\alpha - m + l)}{(m - l)!} u^\alpha_{m-l}(x) \\
& + \sum_{i=0}^{m_2-1} \left(\sum_{j=l-m_1}^{l\wedge m_2} \frac{(j-1)!\binom{m_1}{l-j} N_{i,j-1}}{(-1)^{l-m_1-1} 2^{i+j-l}} \right) \frac{u^{\alpha-\beta}_{\alpha+i}(x)}{\Gamma(\alpha+i+1)}.
\end{aligned}
\tag{4.6}
$$

But it is easy to see that $u_j^\lambda(x) \in \mathbb{R}[\theta_x]$:

$$
u_j^\lambda(x) = \prod_{i=1}^{j} [(\alpha - \lambda + i)(\beta + \lambda - i + 1) + \theta_x].
$$

Hence if β is a nonnegative integer with $\beta \geq m_1$, for $l = 1, \dots, m_1$, there exists a polynomial Y_l, such that $z_l(x) = Y_l(\theta_x)$, and analogously, if α is a nonnegative integer with $\alpha \geq m_2$, for $l = m_1 + 1, \dots, m$, there also exists a polynomial Y_l, such that $z_l(x) = Y_l(\theta_x)$. This finishes the proof of the first part of the Theorem.

Now we have to prove that the order of D_S is exactly $\deg \Xi + 2(\beta\text{-wr}(\boldsymbol{M}) + \alpha\text{-wr}(\boldsymbol{N}) + 1)$. This proof is quite technical at certain points, so it will be given separately in the appendix. □

4.1. Examples

1. Consider $\boldsymbol{M} = (M_{i,j})_{i,j=0}^{m_1-1}$ and $\boldsymbol{N} = (N_{i,j})_{i,j=0}^{m_2-1}$ in the discrete Jacobi–Sobolev bilinear form (1.5) as the symmetric matrices with entries

$$M_{i,j} = \begin{cases} M_{i+j}, & i+j \le m_1 - 1, \\ 0, & i+j > m_1 - 1, \end{cases} \quad \text{and} \quad N_{i,j} = \begin{cases} N_{i+j}, & i+j \le m_2 - 1, \\ 0, & i+j > m_2 - 1, \end{cases}$$

where $M_{m_1-1}, N_{m_2-1} \neq 0$. Then the bilinear form reduces to the bilinear form defined by the moment functional

$$\mu_{\alpha-m_2,\beta-m_1}(x) + \sum_{i=0}^{m_1-1} M_i \delta_{-1}^{(i)} + \sum_{i=0}^{m_2-1} N_i \delta_1^{(i)}. \tag{4.7}$$

We observe that in this case we can calculate directly the degrees of the polynomials z_l defined by (4.4) and (4.6). Indeed

$$\deg z_l = \begin{cases} 2(\beta + m_1 - l), & l = 1, \dots, m_1, \\ 2(\alpha + m - l), & l = m_1 + 1, \dots, m. \end{cases}$$

Then the degree of the polynomial P defined by (A.3) is given by (A.4). Using Lemma A.3 and Corollary 1.4 we deduce that the minimal order of the differential operators having the orthogonal polynomials with respect to (4.7) as eigenfunctions is at most $2(m_1\beta + m_2\alpha + 1)$. For the case of $m_1 = m_2 = 1$ we recover Koekoeks' result [21].

2. Consider

$$\boldsymbol{M} = \operatorname{diag}(M_0, \dots, M_{m_1-1}), \quad M_{m_1-1} \neq 0$$

and

$$\boldsymbol{N} = \operatorname{diag}(N_0, \dots, N_{m_2-1}), \quad N_{m_2-1} \neq 0$$

in the discrete Jacobi–Sobolev bilinear form (1.5). The degrees of the polynomials z_l in (4.4) and (4.6) are now given by

$$\deg z_l = \begin{cases} 2(\beta + m_1 - 1), & l = 1, \dots, m_1, \\ 2(\alpha + m_2 - 1), & l = m_1 + 1, \dots, m. \end{cases}$$

In this case we can not apply Lemma A.1 to calculate the degree of the polynomial P in (A.3). We have to use Definition 1.3 and Corollary 1.4 to calculate the order of the differential operator. But we already know how to calculate α-wr($\boldsymbol{N}$) and β-wr($\boldsymbol{M}$) if $\boldsymbol{M}$ and $\boldsymbol{N}$ are diagonal matrices (see p. 86 of [7]). Call

$$I =\{j : 1 \leq j \leq m_1, M_{j-1} = 0\}, \qquad J = \{j : 1 \leq j \leq m_2, N_{j-1} = 0\},$$
$$s = \#\{j : 1 \leq j \leq m_1, M_{j-1} \neq 0\}, \qquad t = \#\{j : 1 \leq j \leq m_2, N_{j-1} \neq 0\}.$$

Then the minimal order of the differential operators having the orthogonal polynomials as eigenfunctions is at most

$$2\left[t(\alpha - m_2 - 1) + s(\beta - m_1 - 1) + 2\sum_{i=1}^{2}\binom{m_i+1}{2} - 2\sum_{j\in I} j - 2\sum_{j\in J} j + 1\right].$$

For the special case of $I = \{1,\dots,m_1 - 1\}$ and $J = \{1,\dots,m_2 - 1\}$ we have that $s = t = 1$. Therefore the order of the differential operator is given by $2(\alpha + \beta + m_1 + m_2 - 1)$ and we recover Bavinck's result [2].

3. As we mentioned in the Introduction there are some special situations where it is possible to find a differential operator of order lower than the one given by Theorem 4.1. In this theorem the differential operator is obtained from the rational function S (see (3.21)) by taking $\Xi = 1$. However for special values of the parameters α and β and the matrices $\boldsymbol{M}$ and $\boldsymbol{N}$, a *better* rational function S can be considered satisfying the three assumptions (3.10), (3.11), and (3.12) in Theorem 3.1 in such a way that the order of the differential operator constructed using this new S is less than $2(\beta\text{-wr}(\boldsymbol{M}) + \alpha\text{-wr}(\boldsymbol{N}) + 1)$. Here we consider a couple of examples of this situation. In both examples we assume that $m_1 = m_2$ and $\alpha = \beta \in \mathbb{N}$, $\alpha \geq m_1$.

3.1. Take $m_1 = m_2 = 1$. Then the matrices $\boldsymbol{M}$ and $\boldsymbol{N}$ reduce to numbers which, in addition, we assume they are equal, i.e. $\boldsymbol{M} = \boldsymbol{N} > 0$. The polynomials $(q_n)_n$ are then orthogonal with respect to the Gegenbauer type positive measure

$$(1 - x^2)^{\alpha-1}\chi_{[-1,1]} + \boldsymbol{M}(\delta_{-1} + \delta_1). \tag{4.8}$$

Our assumptions imply that (see (3.15))

$$\xi^1_{x,j} = (-1)^j, \quad \xi^2_{x,j} = 1, \tag{4.9}$$

and

$$z_1(x) = z_2(x) = 4^\alpha(\alpha - 1)! + \frac{2\boldsymbol{M}(x + 1)_{2\alpha}}{\alpha!}, \tag{4.10}$$

where z_1 and z_2 are the polynomials defined by (4.4) and (4.6), respectively. Hence we have for the polynomials Y_1 and Y_2 satisfying $Y_1(\theta_x) = z_1(x)$, $Y_2(\theta_x) = z_2(x)$, respectively, that $Y_1 = Y_2$ and both have degree exactly α. In particular, we have

$$\Omega(x) = -2z_1(x-1)z_1(x-2), \tag{4.11}$$

where Ω is the determinant defined by (4.3).

The rational function S (see (3.21)) in Theorem 4.1 is now a polynomial of degree 1; more precisely $S(x) = -\frac{1}{2}\sigma_{x-1/2} = -(x+\alpha-1)$. For $\Xi = 1$, the associated differential operator D_S in Theorem 4.1 has order $2(\beta\text{-wr}(\boldsymbol{M})+\alpha\text{-wr}(\boldsymbol{N})+1) = 4\alpha+2$. However, for this example there is a better choice for the function S, in the sense that one can construct from the new S a differential operator of order $2\alpha + 2$ for which the orthogonal polynomials $(q_n)_n$ are eigenfunctions. Indeed, consider the rational function

$$S(x) = \frac{\sigma_{x-1/2}R(x)}{\Omega(x)}, \tag{4.12}$$

where R is the polynomial defined by

$$R(x) = 4^{\alpha-1}(\alpha-1)! + \frac{\boldsymbol{M}(x-1)_\alpha(x+\alpha)_\alpha}{2\alpha!}.$$

We now check the three assumptions (3.10), (3.11), and (3.12) in Theorem 3.1. The first assumption (3.10) is trivial since $S(x)\Omega(x) = \sigma_{x-1/2}R(x)$, which is obviously a polynomial. A simple computation shows that the polynomial R satisfies the difference equation

$$2\sigma_{x+1/2}R(x+1) + 2\sigma_{x+3/2}R(x+2) = \sigma_{x+1}z_1(x).$$

Using this difference equation together with (4.9), (4.10), and (4.11) one gets

$$M_1(x) = M_2(x) = \frac{1}{4}\sigma_{x+1} = \frac{2x+2\alpha+1}{4}, \tag{4.13}$$

where M_1 and M_2 are the functions defined by (3.9). The second assumption (3.11) is now straightforward by taking $\widetilde{M}_1(x) = \widetilde{M}_2(x) = 1/4$.

In this case, the difference equation $\lambda_x - \lambda_{x-1} = S(x)\Omega(x)$ can be easily solved to get

$$\lambda_x = 4^{\alpha-1}(\alpha-1)!x(x+2\alpha-1) + \frac{\boldsymbol{M}(x-1)_{2\alpha+2}}{2(\alpha+1)!}. \tag{4.14}$$

If we write

$$Q(x) = 2\lambda_x + z_1(x)M_1(x) + z_2(x)M_2(x),$$

equations (4.10), (4.13), and (4.14) give

$$Q(x) = 2 \cdot 4^{\alpha-1}(\alpha-1)!(\theta_x + 2\alpha + 1) + \frac{\boldsymbol{M}(x+1)_{2\alpha}}{(\alpha+1)!}\left[\theta_x + (\alpha+1)(2\alpha+1)\right].$$

Therefore $I^{\alpha+\beta}(Q) = Q$ and hence Q is actually a polynomial in θ_x. That is, there exists a polynomial P_S such that $P_S(\theta_x) = Q(x)$, and hence the third assumption (3.12) holds. Moreover, since Q has degree $2\alpha + 2$, the polynomial P_S has degree exactly $\alpha + 1$.

Theorem 3.1 gives that the orthogonal polynomials $(q_n)_n$ are eigenfunctions of the differential operator given by $D_{q,S}$ (see (3.13)). Since in this example P_S is a polynomial of degree $\alpha + 1$, $\tilde{M}_1(x) = \tilde{M}_2(x) = 1/4$, $Y_1(x) = Y_2(x)$ is a polynomial of degree α and the $\mathcal{D}$-operators for the Jacobi polynomials (see (3.4)) are both differential operators of order 1, we deduce that the differential operator $D_{q,S}$ (see (3.13)) has order equal to $2\alpha + 2$. Hence, for this example the rational function S (see (4.12)) provides for the orthogonal polynomials $(q_n)_n$ a differential operator of order less than the one constructed from the function S in Theorem 4.1. That orthogonal polynomials $(q_n)_n$ with respect to the Gegenbauer type measure (4.8) are eigenfunctions of a differential operator of order $2\alpha + 2$ was first proved by R. Koekoek in 1994 [20] (the case $\alpha = 1$ was discovered by H. Krall in 1940 [24]).

3.2. The following example is new, as far as the authors know. Consider $m_1 = m_2 = 2$ and $\alpha = \beta \in \mathbb{N}, \alpha \geq 2$. Then we have 2×2 matrices $\boldsymbol{M}$ and $\boldsymbol{N}$. Consider for simplicity the case when

$$\boldsymbol{M} = \begin{pmatrix} \boldsymbol{M}_0 & \boldsymbol{M}_1 \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{N} = \begin{pmatrix} \boldsymbol{M}_0 & -\boldsymbol{M}_1 \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{M}_1 \neq 0, \quad \boldsymbol{M}_0 \neq \boldsymbol{M}_1.$$

The polynomials $(q_n)_n$ are then (left) orthogonal with respect to the bilinear form (see (1.5))

$$\begin{aligned}\langle p, q \rangle = &\int_{-1}^{1} p(x)q(x)(1-x^2)^{\alpha-2}dx \\ &+ [p(-1)q(-1) + p(1)q(1)]\boldsymbol{M}_0 \\ &+ [p(-1)q'(-1) - p(1)q'(1)]\boldsymbol{M}_1.\end{aligned}$$

Again, our assumptions imply (4.9) and

$$z_1(x) = z_3(x) = \frac{4^\alpha(\alpha-2)!}{2}(x+1)(x+2\alpha) + \frac{4(\boldsymbol{M}_0-\boldsymbol{M}_1)}{\alpha!}(x+1)_{2\alpha}, \quad (4.15)$$

$$z_2(x) = z_4(x) = 4^\alpha(\alpha-1)! + \frac{4\boldsymbol{M}_1}{\alpha!}(x+1)_{2\alpha}, \quad (4.16)$$

where z_1, z_2 and z_3, z_4 are the polynomials defined by (4.4) and (4.6), respectively. Hence we have for the polynomials $Y_i, i = 1,2,3,4$, satisfying $Y_i(\theta_x) = z_i(x), i = 1,2,3,4$, that Y_i has degree exactly α for all $i = 1,2,3,4$.

The associated differential operator D_S in Theorem 4.1 has order $2(\beta\text{-wr}(\boldsymbol{M}) + \alpha\text{-wr}(\boldsymbol{N})+1) = 4\alpha+2$. However, again, there is a better choice for the function S, in the sense that one can construct from this new S a differential operator of order $2\alpha+2$ for which the orthogonal polynomials $(q_n)_n$ are eigenfunctions. Indeed, consider the rational function

$$S(x) = \frac{\sigma_{x-3/2}R(x)}{\Omega(x)},$$

where R is the polynomial defined by

$$R(x) = 16^{\alpha-1}(\alpha-1)!(\alpha-2)! + 2\cdot 4^{\alpha-1}\boldsymbol{M}_0(x-1)_{\alpha-1}(x+\alpha-1)_{\alpha-1} - 4^{\alpha-1}\frac{\boldsymbol{M}_1}{\alpha}(x-2)_\alpha(x+\alpha-1)_\alpha.$$

The three assumptions (3.10), (3.11), and (3.12) in Theorem 3.1 can be easily checked using that

$$M_1(x) = M_3(x) = \sigma_{x+1}\Big(-\frac{\boldsymbol{M}_1}{(\alpha-1)!}(x+3)_{2\alpha-4}[\theta_x + (\alpha+1)(2\alpha-1)]\Big),$$

$$M_2(x) = M_4(x) = \sigma_{x+1}\Big(2\cdot 4^{\alpha-2}(\alpha-2)! + \frac{\boldsymbol{M}_0-\boldsymbol{M}_1}{(\alpha-1)!}(x+3)_{2\alpha-4}[\theta_x + (\alpha+1)(2\alpha-1)]\Big),$$

$$\widetilde{M}_1(x) = \widetilde{M}_3(x) = -\frac{\boldsymbol{M}_1}{(\alpha-1)!}(x+3)_{2\alpha-4}[\theta_x + (\alpha+1)(2\alpha-1)],$$

$$\widetilde{M}_2(x) = \widetilde{M}_4(x) = 2\cdot 4^{\alpha-2}(\alpha-2)! + \frac{\boldsymbol{M}_0-\boldsymbol{M}_1}{(\alpha-1)!}(x+3)_{2\alpha-4}[\theta_x + (\alpha+1)(2\alpha-1)],$$

$$\lambda_x = 16^{\alpha-1}(\alpha-1)!(\alpha-2)!x(x+2\alpha-3) + 2\frac{4^{\alpha-1}\boldsymbol{M}_0}{\alpha}(x-1)_{2\alpha} - \frac{4^{\alpha-1}\boldsymbol{M}_1}{\alpha(\alpha+1)}(x-2)_{2\alpha+2}.$$

For the polynomial $Q(x) = 2\lambda_x + \sum_{h=1}^{4} z_h(x) M_h(x)$, we have

$$\begin{aligned} Q(x) &= 2\cdot 16^{\alpha-1}(\alpha-1)!(\alpha-2)!(\theta_x + 2(\alpha+1)) \\ &\quad + \Big[\frac{4^\alpha}{\alpha}(x+3)_{2\alpha-4}\Big(\theta_x^2 + 2(4\alpha^2+\alpha-1)\theta_x + \frac{1}{2}(2\alpha-1)_4\Big)\Big]\boldsymbol{M}_0 \\ &\quad - \Big[2\frac{4^{\alpha-1}}{\alpha(\alpha+1)}(x+2)_{2\alpha-2}\Big(\theta_x^2 + 2(4\alpha^2+7\alpha+2)\theta_x + \frac{1}{2}(2\alpha)_4\Big)\Big]\boldsymbol{M}_1. \end{aligned}$$

Therefore $I^{\alpha+\beta}(Q) = Q$ and hence Q is actually a polynomial in θ_x. That is, there exists a polynomial P_S such that $P_S(\theta_x) = Q(x)$, and hence (3.12) holds. Moreover, since Q has degree $2\alpha+2$, the polynomial P_S has degree just $\alpha+1$.

Theorem 3.1 gives that the orthogonal polynomials $(q_n)_n$ are eigenfunctions of the differential operator given by $D_{q,S}$ (3.13). From the definition of $\mathcal{D}$-operators for the Jacobi polynomials (see (3.4)) it follows that

$$\sum_{h=1}^{4} \widetilde{M}_h(D_p)\mathcal{D}_h Y_h(D_p) = 2\widetilde{M}_1(D_p)\frac{d}{dx}Y_1(D_p) + 2\widetilde{M}_2(D_p)\frac{d}{dx}Y_2(D_p).$$

Now, using the definition of $\widetilde{M}_i(x), Y_i(x), i = 1,2$, we have that the degree of $\widetilde{M}_1(x)Y_1(x) + \widetilde{M}_2(x)Y_2(x)$ is at most 2α. Therefore, the order of the differential operator above is at most $2\alpha+1$. That means that, since P_S is a polynomial of degree $\alpha+1$, the differential operator $D_{q,S}$ (3.13) has order equal to $2\alpha+2$.

Let us make some comments about the general case of the matrices

$$\boldsymbol{M} = \begin{pmatrix} M_{00} & M_{01} \\ M_{10} & M_{11} \end{pmatrix}, \quad \boldsymbol{N} = \begin{pmatrix} N_{00} & N_{01} \\ N_{10} & N_{11} \end{pmatrix}.$$

We have been able to find a differential operator of lower order in the following situation:

$$\alpha = \beta, \quad z_1(x) = z_3(x), \quad z_2(x) = z_4(x).$$

In that case the polynomials $z_1(x)$ and $z_2(x)$ are given by

$$\begin{aligned} z_1(x) &= \frac{4^\alpha(\alpha-2)!}{2}(x+1)(x+2\alpha) + \frac{4(M_{00}-M_{01})}{\alpha!}(x+1)_{2\alpha} \\ &\quad - \frac{2(M_{10}-M_{11})}{(\alpha+1)!}(x)_{2\alpha+2}, \end{aligned}$$

$$z_2(x) = 4^\alpha(\alpha-1)! + \frac{4M_{01}}{\alpha!}(x+1)_{2\alpha} - \frac{2M_{11}}{(\alpha+1)!}(x)_{2\alpha+2},$$

while the matrices $\boldsymbol{M}$ and $\boldsymbol{N}$ become

$$\boldsymbol{M} = \begin{pmatrix} M_{00} & M_{01} \\ M_{10} & M_{11} \end{pmatrix}, \quad \boldsymbol{N} = \begin{pmatrix} M_{00} & -M_{01} \\ -M_{10} & M_{11} \end{pmatrix}.$$

If we call again $Y_i, i = 1,2$, the polynomials satisfying $Y_i(\theta_x) = z_i(x), i = 1,2$, we see that the degree of $Y_i, i = 1,2$, depends now on the parameters of the matrices $\boldsymbol{M}$ and $\boldsymbol{N}$. In fact we have

$$\deg Y_1(x) = \begin{cases} \alpha + 1 & \text{if } M_{10} \neq M_{11}, \\ \alpha & \text{if } M_{10} = M_{11} \text{ and } M_{00} \neq M_{01}, \\ 1 & \text{if } M_{10} = M_{11} \text{ and } M_{00} = M_{01}, \end{cases}$$

and

$$\deg Y_2(x) = \begin{cases} \alpha + 1 & \text{if } M_{11} \neq 0, \\ \alpha & \text{if } M_{11} = 0 \text{ and } M_{01} \neq 0, \\ 0 & \text{if } M_{11} = 0 \text{ and } M_{01} = 0. \end{cases}$$

There several combinations, but basically we can summarize them in 3 nontrivial cases.

Case 1. There are three possible situations:

(1) if $\deg Y_1(x) = 1$ and $\deg Y_2(x) = \alpha$, then $M_{11} = M_{10} = 0$ and $M_{00} = M_{01}$;

(2) if $\deg Y_1(x) = \alpha = \deg Y_2(x)$, then $M_{11} = M_{10} = 0$, $M_{01} \neq 0$ and $M_{00} \neq M_{01}$ (this case is the one that we studied in detail above);

(3) if $\deg Y_1(x) = \alpha + 1$ and $\deg Y_2(x) = 0$, then $M_{11} = M_{01} = 0$ and $M_{10} \neq 0$.

In any of the three situations above, we have that $2 \deg P_S(x) = 4\alpha + 2$ but we can construct a differential operator of order $2\alpha + 2$ for which the corresponding Jacobi–Sobolev orthogonal polynomials are eigenfunctions.

Case 2. There are two possible situations:

(1) if $\deg Y_1(x) = 1$ and $\deg Y_2(x) = \alpha + 1$, then $M_{11} \neq 0$, $M_{00} = M_{01}$ and $M_{10} = M_{11}$;

(2) if $\deg Y_1(x) = \alpha + 1 = \deg Y_2(x)$ and $\det(\boldsymbol{M}) = 0$, then $M_{11} \neq 0$, $M_{10} \neq M_{11}$ and $M_{00}M_{11} = M_{01}M_{10}$.

In any of the two situations above, we have that $2 \deg P_S(x) = 4\alpha + 6$ but we can construct a differential operator of order $2\alpha + 4$ for which the corresponding Jacobi–Sobolev orthogonal polynomials are eigenfunctions.

Case 3. There are three possible situations:

(1) if $\deg Y_1(x) = \alpha$ and $\deg Y_2(x) = \alpha + 1$, then $M_{11} \neq 0$, $M_{10} = M_{11}$ and $M_{00} \neq M_{01}$;

(2) if $\deg Y_1(x) = \alpha + 1$ and $\deg Y_2(x) = \alpha$, then $M_{11} = 0$ and $M_{01}, M_{10} \neq 0$;

(3) if $\deg Y_1(x) = \alpha + 1 = \deg Y_2(x)$ and $\det(\boldsymbol{M}) \neq 0$, then $M_{11} \neq 0$, $M_{10} \neq M_{11}$ and $M_{00}M_{11} \neq M_{01}M_{10}$.

In any of the three situations above, we have that $2 \deg P_S(x) = 4\alpha + 10$ but we can construct a differential operator of order $2\alpha + 6$ for which the corresponding Jacobi–Sobolev orthogonal polynomials are eigenfunctions.

For higher dimensions computational evidences indicate that we can find the same phenomenon of lowering the order of the differential operator when

$$m_1 = m_2, \quad \alpha = \beta, \quad \text{and} \quad z_l(x) = z_{m_1+l}(x), \quad l = 1, \ldots, m_1.$$

The matrices $\boldsymbol{M}$ and $\boldsymbol{N}$ are then related in the following way:

$$\boldsymbol{M} = (M_{ij})_{i,j=0}^{m_1-1}, \quad \boldsymbol{N} = ((-1)^{i+j} M_{ij})_{i,j=0}^{m_1-1}.$$

The situation gets more complicated and with many more possibilities. The complete study of the general situation is out of the scope of this paper and it will be pursued elsewhere.

Appendix

In this appendix we will give the proof of Lemma 3.5 and the proof of the last part of Theorem 4.1.

Proof of Lemma 3.5. In this subsection we will use the following notation. Given a finite set of positive integers $F = \{f_1, \ldots, f_m\}$ (hence $f_i \neq f_j$, $i \neq j$), the expression

$$\begin{bmatrix} z_{f,j} \\ f \in F \end{bmatrix}^{j=1,\ldots,m} \tag{A.1}$$

inside a matrix or a determinant denotes the submatrix defined by

$$\begin{pmatrix} z_{f_1,1} & z_{f_1,2} & \cdots & z_{f_1,m} \\ \vdots & \vdots & \ddots & \vdots \\ z_{f_m,1} & z_{f_m,2} & \cdots & z_{f_m,m} \end{pmatrix}.$$

Given m numbers u_i, $i = 1, \dots, m$, and two nonnegative integers m_1 and m_2 with $m_1 + m_2 = m$, we form the pair $\mathcal{U} = (U_1, U_2)$, where U_1 is the m_1-tuple $U_1 = (u_1, \dots, u_{m_1})$ and U_2 is the m_2-tuple $U_2 = (u_{m_1+1}, \dots, u_m)$. We also write $\mathbb{U}_1$ and $\mathbb{U}_2$ for the sets

$$\mathbb{U}_1 = \{1, \dots, m_1\}, \quad \mathbb{U}_2 = \{m_1 + 1, \dots, m\}. \tag{A.2}$$

The proof of Lemma 3.5 is based in the following technical Lemma.

Lemma A.1. *Let $Y_1, Y_2, \dots, Y_m$, be nonzero polynomials satisfying* $\deg Y_i = u_i$, $i = 1, \dots, m$. *Write r_i for the leading coefficient of Y_i, $1 \le i \le m$. For real numbers α, β, consider the rational function P defined by*

$$P(x) = \frac{\left| \begin{matrix} \begin{bmatrix} N_{1;x-j}^{\alpha;m-j} N_{2;x-1}^{\beta;j-1} Y_i(\theta_{x-j}) \\ i \in \mathbb{U}_1 \end{bmatrix} \\ \begin{bmatrix} Y_i(\theta_{x-j}) \\ i \in \mathbb{U}_2 \end{bmatrix} \end{matrix} \right|^{j=1,\dots,m}}{p(x)q(x)}, \tag{A.3}$$

where p and q are the polynomials (3.16) and (3.17), respectively. The determinant (A.3) should be understood in the manner explained above (see (A.1)). Then P is a polynomial of degree at most

$$d = 2\sum_{u \in U_1, U_2} u - 2\sum_{i=1}^{2} \binom{m_i}{2}. \tag{A.4}$$

Moreover, if the elements in U_1 and U_2 are different (i.e. $u_i \ne u_j$, for $i \ne j$, $i, j \in \{1, \dots, m_1\}$, and $u_i \ne u_j$, for $i \ne j$, $i, j \in \{m_1 + 1, \dots, m\}$), then P is a polynomial of degree exactly (A.4) with leading coefficient given by

$$r = V_{U_1} V_{U_2} \prod_{i=1}^{m} r_i,$$

where V_X denotes the Vandermonde determinant associated to the set $X = \{x_1, \dots, x_K\}$ defined by

$$V_X = \prod_{i<j} (x_j - x_i).$$

Proof. The Lemma can be proved using the same approach as in the proof of Lemma 3.3 in [5]. □

Proof of Lemma 3.5. Consider the sets $\mathbb{U}_j$, $j = 1, 2$, given by (A.2). By construction (see (3.14)), we have that for $h \in \mathbb{U}_j$, the $\mathcal{D}$-operator $\mathcal{D}_h$ is defined by the sequence $(\varepsilon_{n,j})_n$ (see (3.2) and (3.3)).

Since the polynomial Ξ is invariant under the action of $\mathcal{I}^{\alpha+\beta-m-1}$, we have

$$\mathcal{I}^{\alpha+\beta+i}(\Xi(x-j)) = \Xi(x+m+i+j+1). \tag{A.5}$$

As a consequence of (3.19) and (3.20) we have

$$\mathcal{I}^{\alpha+\beta+i}(q(x-j)) = (-1)^{\binom{m}{2}} q(x+i+j+m+1), \tag{A.6}$$

and

$$\mathcal{I}^{\alpha+\beta+i}(p(x-j)) = p(x+i+j+m+1), \tag{A.7}$$

where the polynomials p and q are defined by (3.16) and (3.17), respectively.

Now, we check the first assumption (3.10) in Theorem 3.1, i.e.: $S(x)\Omega(x)$ is a polynomial in x. From the definition of $S(x)$ in (3.21) and $\Omega(x)$ in (3.7) it follows, using (3.15), that

$$S(x)\Omega(x) = \frac{\sigma_{x-\frac{m-1}{2}}\,\Xi(x)}{p(x)q(x)} \left| \begin{matrix} \begin{bmatrix} N_{1;x-j}^{\alpha;m-j} N_{2;x-1}^{\beta;j-1} Y_i(\theta_{x-j}) \\ i \in \mathbb{U}_1 \end{bmatrix} \\ \begin{bmatrix} Y_i(\theta_{x-j}) \\ i \in \mathbb{U}_2 \end{bmatrix} \end{matrix} \right|^{j=1,\dots,m}. \tag{A.8}$$

Therefore,

$$S(x)\Omega(x) = \sigma_{x-\frac{m-1}{2}}\,\Xi(x)P(x), \tag{A.9}$$

where P is the rational function (A.3) defined in Lemma A.1. According to this lemma, P is actually a polynomial and thus $S(x)\Omega(x)$ is also a polynomial.

Now we check the second assumption (3.11) in Theorem 3.1, i.e.: polynomials $\widetilde{M}_1, \dots, \widetilde{M}_m$, exist such that

$$M_h(x) = \sigma_{x+1}\widetilde{M}_h(\theta_x), \quad h = 1, \dots, m.$$

We first prove that

$$\mathcal{I}^{\alpha+\beta}(M_h(x)) = -M_h(x), \quad h = 1, \dots, m,$$

where $M_h(x), h = 1, \dots, m$, are defined in (3.9). Hence, $M_h(x), h = 1, \dots, m$, according to the discussion after (3.18), is divisible by σ_{x+1} and the quotient belongs to $\mathbb{R}[\theta_x]$.

We assume that the h-th $\mathcal{D}$-operator is $\mathcal{D}_1$ (similar proof for $\mathcal{D}_2$). As before, we can remove all the denominators in $M_h(x)$ in this case and rearrange the determinant to obtain

$$M_h(x) = \sum_{j=1}^{m} (-1)^{h+j} \frac{\sigma_{x+j-\frac{m-1}{2}} \Xi(x+j)}{p(x+j)q(x+j)} N_{1;x}^{\alpha;m-j} N_{2;x+j-1}^{\beta;j-1} \begin{vmatrix} \begin{bmatrix} N_{1;x+j-r}^{\alpha;m-r} N_{2;x+j-1}^{\beta;r-1} Y_l(\theta_{x+j-r}) \\ l \in \mathbb{U}_1 \setminus \{h\} \end{bmatrix}^{l \neq h, r \neq j} \\ \begin{bmatrix} Y_l(\theta_{x+j-r}) \\ l \in \mathbb{U}_2 \end{bmatrix} \end{vmatrix}.$$

Hence, using (3.20), (A.5), (A.6), and (A.7), we have

$$\begin{aligned} \mathcal{J}^{\alpha+\beta}(M_h(x)) &= -\sum_{j=1}^{m} (-1)^{h+j} (-1)^{\binom{m}{2}} \frac{\sigma_{x+\frac{m-1}{2}-j+2} \Xi(x+m-j+1)}{p(x+m-j+1)q(x+m-j+1)} \\ &\quad N_{2;x+m-j}^{\beta;m-j} N_{1;x}^{\alpha;j-1} \begin{vmatrix} \begin{bmatrix} N_{2;x+m-j}^{\beta;m-r} N_{1;x+r-j}^{\alpha;r-1} Y_l(\theta_{x-j+r}) \\ l \in \mathbb{U}_1 \setminus \{h\} \end{bmatrix}^{l \neq h, r \neq j} \\ \begin{bmatrix} Y_l(\theta_{x-j+r}) \\ l \in \mathbb{U}_2 \end{bmatrix} \end{vmatrix} \\ &= -(-1)^{\binom{m}{2}} (-1)^{m-1} (-1)^{\binom{m-1}{2}} \sum_{j=1}^{m} (-1)^{h+j} \frac{\sigma_{x+j-\frac{m-1}{2}} \Xi(x+j)}{p(x+j)q(x+j)} \\ &\quad N_{1;x}^{\alpha;m-j} N_{2;x+j-1}^{\beta;j-1} \begin{vmatrix} \begin{bmatrix} N_{1;x+j-r}^{\alpha;m-r} N_{2;x+j-1}^{\beta;r-1} Y_l(\theta_{x+j-r}) \\ l \in \mathbb{U}_1 \setminus \{h\} \end{bmatrix}^{l \neq h, r \neq j} \\ \begin{bmatrix} Y_l(\theta_{x+j-r}) \\ l \in \mathbb{U}_2 \end{bmatrix} \end{vmatrix} \\ &= -(-1)^{m(m-1)} M_h(x) \\ &= -M_h(x). \end{aligned}$$

renamed the index j ($j \to m-j+1$) in the second step and that we interchanged all columns in the determinant ($r \to m-r+1$), thus the corresponding change of signs. A computation using (3.15) and Lemma 3.3 of [9] shows that M_h is actually a polynomial in x. And so the assumption (3.11) follows.

Finally, we check the third assumption (3.12) in Theorem 3.1, i.e.: a polynomial P_S exists such that

$$P_S(\theta_x) = 2\lambda_x + \sum_{h=1}^{m} Y_h(\theta_x) M_h(x).$$

As it was pointed out in [8] (see (5.8)) it is sufficient to see that

$$\mathfrak{I}^{\alpha+\beta-1}(S(x)\Omega(x)) = -(S(x+m)\Omega(x+m)).$$

From (A.8), using (3.20), (A.5), (A.6), and (A.7) again, we have

$$\begin{aligned}
\mathfrak{I}^{\alpha+\beta-1}(S(x)\Omega(x)) &= -(-1)^{\binom{m}{2}} \frac{\sigma_{x+\frac{m+1}{2}}\,\Xi(x+m)}{p(x+m)q(x+m)} \\
&\qquad \begin{vmatrix} \begin{bmatrix} N^{\beta;m-j}_{2;x+m-1} N^{\alpha;j-1}_{1;x+j-1} Y_i(\theta_{x+j-1}) \\ i \in \mathbb{U}_1 \end{bmatrix}^{j=1,\dots,m} \\ \begin{bmatrix} Y_i(\theta_{x+j-1}) \\ i \in \mathbb{U}_2 \end{bmatrix} \end{vmatrix} \\
&= -(-1)^{\binom{m}{2}}(-1)^{\binom{m}{2}} \frac{\sigma_{x+m-\frac{m-1}{2}}\,\Xi(x+m)}{p(x+m)q(x+m)} \\
&\qquad \begin{vmatrix} \begin{bmatrix} N^{\beta;j-1}_{2;x+m-1} N^{\alpha;m-j}_{1;x+m-j} Y_i(\theta_{x+m-j}) \\ i \in \mathbb{U}_1 \end{bmatrix}^{j=1,\dots,m} \\ \begin{bmatrix} Y_i(\theta_{x+m-j}) \\ i \in \mathbb{U}_2 \end{bmatrix} \end{vmatrix} \\
&= -S(x+m)\Omega(x+m). \qquad \square
\end{aligned}$$

Proof of the last part of Theorem 4.1. It remained to prove the computation of the order of the operator D_S in Theorem 4.1. For that, we will give two auxiliary lemmas. We need first to introduce some notation. Given m arbitrary polynomials $Y_1, \dots, Y_m$, we will denote by $\mathcal{Y}$ the m-tuple of polynomials $(Y_1, \dots, Y_m)$. The m-tuple formed by interchanging the polynomials Y_i and Y_j in $\mathcal{Y}$ is denoted by $\mathcal{Y}_{i\leftrightarrow j}$; the m-tuple formed by changing the polynomial Y_i to $aY_i + bY_j$ in $\mathcal{Y}$, where a and b are real numbers, is denoted by $\mathcal{Y}_{i\leftrightarrow ai+bj}$; and the m-tuple formed by removing the polynomial Y_i in $\mathcal{Y}$ is denoted by $\mathcal{Y}_{\{i\}}$.

Lemma A.2. *Given m arbitrary polynomials $Y_1, \dots, Y_m$, we form the m-tuple of polynomials $\mathcal{Y} = (Y_1, \dots, Y_m)$ and consider the operator $D_{q,S} = D_{q,S}(\mathcal{Y})$ (3.13). Then, for any numbers $a, b \in \mathbb{R}$ we have*

$$D_{q,S}(\mathcal{Y}) = -D_{q,S}(\mathcal{Y}_{i\leftrightarrow j}), \tag{A.10}$$

$$D_{q,S}(\mathcal{Y}_{i\leftrightarrow ai+bj}) = aD_{q,S}(\mathcal{Y}). \tag{A.11}$$

Proof. It is analogous to the proof of Lemma 3.4 in [7]. □

Lemma A.3. *For $m_1, m_2 \geq 0$ with $m = m_1 + m_2 \geq 1$, let $\boldsymbol{M} = (M_{i,j})_{i,j=0}^{m_1-1}$ and $\boldsymbol{N} = (N_{i,j})_{i,j=0}^{m_2-1}$ be $m_1 \times m_1$ and $m_2 \times m_2$ matrices, respectively. If $\boldsymbol{M}, \boldsymbol{N} \neq 0$, we assume, in addition, that α and β are nonnegative integers with $\alpha \geq m_2$ and $\beta \geq m_1$. If, instead, $\boldsymbol{M} = 0$, we assume that only α is a positive integer with $\alpha \geq m_2$, and if $\boldsymbol{N} = 0$, we assume that only β is a positive integer with $\beta \geq m_1$. For $j = 1, \dots, m$, define the polynomials Y_l, $Y_l(\theta_x) = z_l(x)$, where z_l is defined by (4.4) and (4.6). Then the degree of the polynomial P defined by (A.3) is $2(\beta\text{-wr}(\boldsymbol{M}) + \alpha\text{-wr}(\boldsymbol{N}))$.*

Proof. It is analogous to the proof Lemma 4.1 in [7]. □

Proof of the last part of Theorem 4.1. We will compute the order of the operator D_S in Theorem 4.1. Given m numbers u_i, $i = 1, \dots, m$, and two nonnegative integers m_1 and m_2 with $m_1 + m_2 = m$, we form the pair $\mathcal{U} = (U_1, U_2)$, where U_1 is the m_1-tuple $U_1 = (u_1, \dots, u_{m_1})$ and U_2 is the m_2-tuple $U_2 = (u_{m_1+1}, \dots, u_m)$. Given a polynomial Ξ which is invariant under the action of $\mathfrak{I}^{\alpha+\beta-m-1}$, we associate the rational function S as in (3.21). Given m polynomials Y_i, $i = 1, \dots, m$, with $\deg Y_i = u_i$, we consider the (quasi) Casorati determinant Ω and the polynomial P as in (A.3). As established in the proof of Lemma 3.5 (see (A.9)), we have $S(x)\Omega(x) = \sigma_{x-\frac{m-1}{2}}\Xi(x)P(x)$, and then

$$\text{the degree of } S\Omega \text{ is } \deg \Xi + d + 1, \text{ where } d \text{ is the degree of } P. \tag{A.12}$$

Consider now the polynomials Y_l, defined by $Y_l(\theta_x) = z_l(x)$, $l = 1, \dots, m$, where z_l are the polynomials (4.4) and (4.6). The operator $D_{q,S}$ (3.13) is the sum of the operators $T_1 = \frac{1}{2}P_S(D_p)$ and $T_2 = \sum_{h=1}^{m} \widetilde{M}_h(D_p)\mathfrak{D}_h Y_h(D_p)$. Since the order of the differential operator D_p is 2, it is clear from the definition of the polynomial P_S that the order of $P_S(D_p)$ is just $2\deg P_S$. According to (4.2), we have that $2\deg P_S = \deg(S\Omega) + 1$. Using (A.12), we get $\deg P_S = \deg(\Xi)/2 + d/2 + 1$. Hence

$$\text{the order of } T_1 \text{ is } \deg(\Xi) + d + 2, \tag{A.13}$$

where d is the degree of the polynomial P associated to the polynomials Y_l (see (A.3)). Using Lemma A.3, we then get that the order of T_1 is $\deg(\Xi) + 2(\beta\text{-wr}(\boldsymbol{M}) + \alpha\text{-wr}(\boldsymbol{N})) + 2$. It is now enough to prove that the order of the operator T_2 is less than the order of T_1.

To stress the dependence of P_S, P, M_h, $\widetilde{M}_h$, Ω and the operator $D_{q,S}$ on the m-tuple of polynomials $\mathcal{Y} = (Y_1, \dots, Y_m)$, we write $P_S = P_S(\mathcal{Y})$, $P = P(\mathcal{Y})$, $M_h = M_h(\mathcal{Y})$, $\widetilde{M}_h = \widetilde{M}_h(\mathcal{Y})$, $\Omega = \Omega(\mathcal{Y})$ and $D_{q,S} = D_{q,S}(\mathcal{Y})$.

Interchanging and using linear combinations of two polynomials, we can get from the polynomials Y_i, $i = 1, \dots, m$, new polynomials $\widehat{Y}_i$, $i = 1, \dots, m$, satisfying the following two conditions.

(∗) $\deg \widehat{Y}_i \neq \deg \widehat{Y}_j$, $i \neq j$, $1 \le i, j \le m_1$, or $m_1 + 1 \le i, j \le m$. $\deg \widehat{Y}_i$ is increasing from $i = 1, \dots, m_1$, and from $i = m_1 + 1, \dots, m$.

(∗∗) Fixing h, $1 \le h \le m_1$, define g_h as the first nonnegative integer such that $g_h \notin \{\deg \widehat{Y}_i, i = 1, \dots, m_1\}$. Then for $0 \le g < g_h$, $\widehat{Y}_g = x^g$. The same for $m_1 + 1 \le h \le m$.

Write $\widehat{U}_1 = \{\deg \widehat{Y}_i, 1 \le i \le m_1\}$ and $\widehat{U}_2 = \{\deg \widehat{Y}_i, m_1 + 1 \le i \le m\}$.

Using the invariance properties (A.10) and (A.11), we then have

$$P_S = P_S(\mathcal{Y}) = P_S(\widehat{\mathcal{Y}}),$$

$$D_{q,S} = D_{q,S}(\mathcal{Y}) = D_{q,S}(\widehat{\mathcal{Y}}),$$

where $\widehat{\mathcal{Y}} = (\widehat{Y}_1, \dots, \widehat{Y}_m)$. If we write $\widehat{M}_h = M_h(\widehat{\mathcal{Y}})$, $\widehat{\tilde{M}}_h = \tilde{M}_h(\widehat{\mathcal{Y}})$, $h = 1, \dots, m$, we then have

$$T_2 = \sum_{h=1}^{m} \widehat{\tilde{M}}_h(D_p)\mathcal{D}\widehat{Y}_h(D_p). \tag{A.14}$$

Since $T_1 = P_S(\mathcal{Y}) = P_S(\widehat{\mathcal{Y}})$, we have as before that the order of T_1 is $\deg(\Xi) + d + 2$, where d is the degree of the polynomial $P(\widehat{\mathcal{Y}}) = P(\mathcal{Y})$, which according to Lemma A.1 is ((∗∗) says that the elements in $\widehat{U}_1$ and $\widehat{U}_2$ are different, respectively)

$$d = 2\sum_{u \in \widehat{U}_1, \widehat{U}_2} u - 2\sum_{i=1}^{2} \binom{m_i}{2}. \tag{A.15}$$

Using (A.14) it follows that the order of the operator T_2 is less than or equal to

$$\max\{2\deg \widehat{\tilde{M}}_h + 2\deg \widehat{Y}_h + 1, h = 1, \dots, m\}.$$

From (3.11), we get that $\deg \widehat{\tilde{M}}_h = (\deg \widehat{M}_h - 1)/2$. Hence the order of the operator T_2 is less than or equal to

$$\max\{\deg \widehat{M}_h + 2\deg \widehat{Y}_h, h = 1, \dots, m\}.$$

Fixed now h, $1 \le h \le m_1$ (the same for $m_1 + 1 \le h \le m$). We write $\widehat{\mathcal{Y}}_g^h$ for the m-tuple formed by changing the polynomial $\widehat{Y}_h$ to x^g in $\widehat{\mathcal{Y}}$, and write $\widehat{\Omega}_g^h = \Omega(\widehat{\mathcal{Y}}_g^h)$. Using (∗), we get that $\widehat{\Omega}_g^h = 0$ for $0 \le g < g_h$ (there are two equal rows) and hence $S\widehat{\Omega}_g^h = 0$ for $0 \le g < g_h$. Using (A.12), we have that

$\deg(S\widehat{\Omega}_g^h) = \deg \Xi + \deg P(\widehat{\mathcal{Y}}_g^h) + 1$. ($**$) and ($*$) show that the degrees of the m_1 first polynomials in $\mathcal{Y}_{g_h}^h$ are different, as well as those of the m_2 last polynomials. Then Lemma A.1 gives

$$\deg P(\widehat{\mathcal{Y}}_{g_h}^h) = 2 \sum_{u \in (\widehat{U}_1 \setminus \{\deg \widehat{Y}_h\}, \widehat{U}_2)} u + 2g_h - 2 \sum_{i=1}^{2} \binom{m_i}{2}. \tag{A.16}$$

For $g > g_h$, Lemma A.1 also gives

$$\begin{aligned} \deg P(\widehat{\mathcal{Y}}_g^h) &\le 2 \sum_{u \in (\widehat{U}_1 \setminus \{\deg \widehat{Y}_h\}, \widehat{U}_2)} u + 2g - 2 \sum_{i=1}^{2} \binom{m_i}{2} \\ &= \deg P(\widehat{\mathcal{Y}}_{g_h}^h) + 2(g - g_h). \end{aligned}$$

Using (A.12), we get that

$$\deg(S\widehat{\Omega}_g^h) \le 2(g - g_h) + \deg(S\widehat{\Omega}_{g_h}^h)$$

for $g \ge g_h$.

Using now Lemma 3.3 of [9], we get that the degree of the polynomial $\widehat{M}_h$ is less than or equal to $\deg(S\widehat{\Omega}_{g_h}^h) - 2g_h$. Hence, using (A.12), (A.15), and (A.16), we have

$$\begin{aligned} \deg \widehat{M}_h + 2 \deg \widehat{Y}_h &\le \deg(S\widehat{\Omega}_{g_h}^h) - 2g_h + 2 \deg \widehat{Y}_h \\ &= \deg \Xi + \deg P(\widehat{\mathcal{Y}}_{g_h}^h) + 1 - 2g_h + 2 \deg \widehat{Y}_h \\ &= \deg \Xi + 2 \sum_{u \in (\widehat{U}_1, \widehat{U}_2)} u - 2 \sum_{i=1}^{2} \binom{m_i}{2} + 1 \\ &= \deg \Xi + d + 1. \end{aligned}$$

Comparing with (A.13), this gives that the order of the operator T_2 is less than the order of T_1. This completes the proof of Theorem 4.1. □

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