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Spectral distribution of PDE discretization matrices from isogeometric analysis: the case of L^1 coefficients and non-regular geometry

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Abstract. We consider the matrices arising from the Galerkin B-spline Isogeometric Analysis (IgA) approximation of a d-dimensional second-order Partial Differential Equation (PDE). We compute the singular value and eigenvalue distribution of these matrices under minimal assumptions on the PDE coefficients and the geometry map involved in the IgA discretization. In particular, L^1 coefficients and non-regular geometries are allowed. The mathematical technique used in our derivation is entirely based on the theory of Generalized Locally Toeplitz (GLT) sequences, which is a quite general technique that can also be applied to several other PDE discretization methods.

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1. Introduction

Consider the following second-order diffusion-reaction Partial Differential Equation (PDE) with homogeneous Dirichlet boundary conditions:

$$\begin{cases} -\nabla \cdot K\nabla u + \gamma u = \mathrm{f}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded open Lipschitz domain, $K\colon \Omega \to \mathbb{R}^{d\times d}$ is a symmetric matrix of measurable functions and γ, f are measurable. For the moment, we do not make further assumptions on the PDE coefficients. We consider the discretization matrices arising from the Galerkin B-spline Isogeometric Analysis (IgA) approximation of (1). The spectral distribution of these matrices has been investigated in a series of recent works [3, 4, 5, 6]. Paper [4] addressed the univariate and bivariate constant-coefficient case over a square domain: K was assumed to be the identity, Ω was assumed to be $[0,1]^d$, and $1 \le d \le 2$. In [3, Chapter 4] the analysis of [4] was extended to any dimension $d \ge 1$. Finally, papers [5, 6] settled the general d-dimensional variable-coefficient case over a domain Ω described by a geometry map $\mathbf{G}\colon [0,1]^d \to \overline{\Omega}$, in accordance with the IgA paradigm [2]. It should be noted, however, that in [5] the components of K were assumed to be continuous over $\overline{\Omega}$, and in [6] they were assumed to be in $L^\infty(\Omega)$. Moreover, both [5] and [6] focused on a regular geometry $\mathbf{G}$, i.e., $\mathbf{G} \in C^1([0,1]^d)$ and $\det(J_{\mathbf{G}}) \neq 0$ over $[0,1]^d$ ($J_{\mathbf{G}}$ is the Jacobian matrix of $\mathbf{G}$).

In this paper, we extend the results of [3, 4, 5, 6] by computing the spectral (and singular value) distribution of the Galerkin B-spline IgA discretization matrices under minimal assumptions on the PDE coefficients and the geometry map. These assumptions, which will be detailed in Theorem 1, are all that is necessary for the considered matrices to be meaningful. They do not even ensure that (1) is well-posed. In particular, the geometry $\mathbf{G}$ is not required to be regular and the components of K need not be in $L^\infty(\Omega)$.

Another important aspect of this paper, which is even more important than the results mentioned above, consists in the technique and the arguments used in our derivation. All the main ingredients are based on the theory of Generalized Locally Toeplitz (GLT) sequences [7, 8, 9, 10], and may be grouped under the name of GLT analysis. In our opinion, this kind of analysis is one of the most general and effective tools for computing the spectral distribution of PDE discretization matrices; see [7, Section 1] and [8, Section 1]. A precise target of this paper is to show, in a specific case of interest in Numerical Analysis and Engineering (the IgA case), how the GLT analysis can be applied to obtain spectral distribution results under minimal assumptions on the PDE data (coefficients, geometry, etc.).

In Section 2 we collect all the necessary preliminaries. In Section 3 we describe the Galerkin B-spline IgA approximation of (1). In Section 4 we state our main result (Theorem 1) and we prove it via the GLT analysis.

2. Preliminaries

2.1. Multi-index notation and matrix-sequences. A multi-index $\mathbf{m} \in \mathbb{Z}^d$, also called a d-index, is simply a (row) vector in $\mathbb{Z}^d$; its components are denoted by $m_1, \ldots, m_d$. We indicate by $\mathbf{0}, \mathbf{1}, \mathbf{2}$ the vectors consisting of all zeros, all ones, all twos (their size will be clear from the context). For any d-index $\mathbf{m}$, we set $N(\mathbf{m}) = \prod_{i=1}^d m_i$ and we write $\mathbf{m} \to \infty$ to indicate that $\min(\mathbf{m}) \to \infty$. Inequalities between multi-indices must be interpreted in the componentwise sense. For example, $\mathbf{j} \le \mathbf{k}$ means that $j_i \le k_i$ for every i. If $\mathbf{j}, \mathbf{k}$ are d-indices such that $\mathbf{j} \le \mathbf{k}$, the multi-index range $\mathbf{j}, \dots, \mathbf{k}$ is the set $\{\mathbf{i} \in \mathbb{Z}^d : \mathbf{j} \le \mathbf{i} \le \mathbf{k}\}$. We assume for this set the standard lexicographic ordering:

$$\Big[\dots \big[\, [\, (i_1, \dots, i_d) \,]_{i_d = j_d, \dots, k_d} \, \big]_{i_{d-1} = j_{d-1}, \dots, k_{d-1}} \dots \Big]_{i_1 = j_1, \dots, k_1}. \qquad (2)$$

For instance, if $d = 2$, this ordering is

$$\begin{aligned}
&(j_1, j_2),\ (j_1, j_2 + 1),\ \dots,\ (j_1, k_2),\\
&\quad (j_1 + 1, j_2),\ (j_1 + 1, j_2 + 1),\ \dots,\ (j_1 + 1, k_2),\\
&\qquad \dots\dots\dots,\\
&\qquad\quad (k_1, j_2),\ (k_1, j_2 + 1),\ \dots,\ (k_1, k_2).
\end{aligned}$$

When a d-index $\mathbf{i}$ varies in a multi-index range $\mathbf{j}, \dots, \mathbf{k}$ (this is often written as $\mathbf{i} = \mathbf{j}, \dots, \mathbf{k}$), it is always assumed that $\mathbf{i}$ varies from $\mathbf{j}$ to $\mathbf{k}$ following the ordering (2). In particular, if $\mathbf{m} \in \mathbb{N}^d$ and $X = [x_{\mathbf{i}\mathbf{j}}]_{\mathbf{i},\mathbf{j}=\mathbf{1}}^{\mathbf{m}}$, then X is a $N(\mathbf{m}) \times N(\mathbf{m})$ matrix whose entries are indexed by two d-indices $\mathbf{i}, \mathbf{j}$, both varying in $\mathbf{1}, \dots, \mathbf{m}$ according to (2). $\sum_{\mathbf{i}=\mathbf{j}}^{\mathbf{k}}$ denotes the summation over all multi-indices $\mathbf{i} = \mathbf{j}, \dots, \mathbf{k}$. Operations involving d-indices that have no meaning in $\mathbb{Z}^d$ must be interpreted in the componentwise sense. For example, $\mathbf{j}/\mathbf{k} = (j_1/k_1, \dots, j_d/k_d)$.

Throughout this paper, by a sequence of matrices (or matrix-sequence) we mean a sequence of the form $\{X_{\mathbf{m}}\}_n$, where

- n varies in some infinite subset of $\mathbb{N}$;
- $\mathbf{m} = \mathbf{m}(n)$ is a d-index in $\mathbb{N}^d$ which depends on n, and $\mathbf{m} \to \infty$ as $n \to \infty$;
- $X_{\mathbf{m}}$ is a square matrix of size $N(\mathbf{m})$.

The multi-index $\mathbf{m}$ that parameterizes a matrix-sequence is always assumed to be a d-index.

2.2. Spectral distribution and spectral symbol. We denote by $C_c(\mathbb{C})$ the space of continuous functions $F : \mathbb{C} \to \mathbb{C}$ with bounded support, and by μ_q the Lebesgue measure in $\mathbb{R}^q$. The singular values and eigenvalues of $X \in \mathbb{C}^{m \times m}$ are denoted by $\sigma_j(X)$, $j = 1, \dots, m$, and $\lambda_j(X)$, $j = 1, \dots, m$.

Definition 1. Let $\{X_{\mathbf{m}}\}_n$ be a matrix-sequence, and let $f : D \to \mathbb{C}$ be a measurable function defined on a set $D \subset \mathbb{R}^q$ with $0 < \mu_q(D) < \infty$. We say that $\{X_{\mathbf{m}}\}_n$ has an asymptotic singular value distribution described by f, and we write $\{X_{\mathbf{m}}\}_n \sim_\sigma f$, if, for all $F \in C_c(\mathbb{C})$,

$$\lim_{n\to\infty} \frac{1}{N(\mathbf{m})} \sum_{j=1}^{N(\mathbf{m})} F(\sigma_j(X_{\mathbf{m}})) = \frac{1}{\mu_q(D)} \int_D F(|f(s_1,\dots,s_q)|)\, \mathrm{d}s_1 \dots \mathrm{d}s_q. \quad (3)$$

In this case, f is referred to as the singular value symbol of $\{X_{\mathbf{m}}\}_n$. Similarly, we say that $\{X_{\mathbf{m}}\}_n$ has an asymptotic spectral (or eigenvalue) distribution described by f, and we write $\{X_{\mathbf{m}}\}_n \sim_\lambda f$, if, for all $F \in C_c(\mathbb{C})$,

$$\lim_{n\to\infty} \frac{1}{N(\mathbf{m})} \sum_{j=1}^{N(\mathbf{m})} F(\lambda_j(X_{\mathbf{m}})) = \frac{1}{\mu_q(D)} \int_D F(f(s_1,\dots,s_q))\, \mathrm{d}s_1 \dots \mathrm{d}s_q. \quad (4)$$

In this case, f is referred to as the spectral (or eigenvalue) symbol of $\{X_{\mathbf{m}}\}_n$.

Remark 1. The informal meaning behind (4) is the following. If $N(\mathbf{m})$ is large enough and $\{\mathbf{s}_{j,n},\ j = 1,\dots,N(\mathbf{m})\}$ is an equispaced grid on D, then a suitable ordering of the eigenvalues of $X_{\mathbf{m}}$, say $\lambda_j(X_{\mathbf{m}})$, $j = 1,\dots,N(\mathbf{m})$, is such that the pairs $\{(\mathbf{s}_{j,n}, \lambda_j(X_{\mathbf{m}})),\ j = 1,\dots,N(\mathbf{m})\}$ reconstruct approximately the hypersurface $\{(\mathbf{s}, f(\mathbf{s})),\ \mathbf{s} \in D\}$. In other words, the spectrum of $X_{\mathbf{m}}$, except possibly for $o(N(\mathbf{m}))$ outliers, "behaves" (asymptotically) like a uniform sampling of f over D. For instance, if $q = 1$, $N(\mathbf{m}) = n$, and $D = [a,b]$, then the eigenvalues of $X_{\mathbf{m}}$ are approximately equal to $f(a+i(b-a)/n)$, $i = 1,\dots,n$, for n large enough. Similarly, if $q = 2$, $N(\mathbf{m}) = n^2$, and $D = [a,b] \times [c,d]$, then the eigenvalues of $X_{\mathbf{m}}$ are approximately equal to $f(a+i(b-a)/n, c+j(d-c)/n)$, $i, j = 1,\dots,n$, for n large enough. Eq. (3) has an informal meaning completely analogous to (4).

2.3. GLT sequences. In this section we provide the essentials of the theory of GLT sequences. We first introduce the major building blocks of GLT sequences, i.e., multilevel Toeplitz matrices and multilevel diagonal sampling matrices.

Definition 2. If $\mathbf{m} \in \mathbb{N}^d$ and $f\colon [-\pi,\pi]^d \to \mathbb{C}$ is a function in $L^1([-\pi,\pi]^d)$, the multilevel Toeplitz matrix $T_{\mathbf{m}}(f)$ is the $N(\mathbf{m}) \times N(\mathbf{m})$ matrix defined as

$$T_{\mathbf{m}}(f) = [\hat{f}_{\mathbf{i}-\mathbf{j}}]_{\mathbf{i},\mathbf{j}=\mathbf{1}}^{\mathbf{m}},$$

where

$$\hat{f}_{\mathbf{k}} = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} f(\boldsymbol{\theta})\, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\boldsymbol{\theta}} \mathrm{d}\boldsymbol{\theta}, \quad \mathbf{k} \in \mathbb{Z}^d,$$

are the Fourier coefficients of f.

Definition 3. If $\mathbf{m} \in \mathbb{N}^d$ and $a\colon [0,1]^d \to \mathbb{C}$, the multilevel diagonal sampling matrix $D_{\mathbf{m}}(a)$ is the $N(\mathbf{m}) \times N(\mathbf{m})$ diagonal matrix defined as

$$D_{\mathbf{m}}(a) = \operatorname*{diag}_{\mathbf{j}=\mathbf{1},\ldots,\mathbf{m}} a\Big(\frac{\mathbf{j}}{\mathbf{m}}\Big),$$

where $\mathbf{j}$ varies from $\mathbf{1}$ to $\mathbf{m}$ following the lexicographic ordering (2).

We now turn to the fundamental notion on which the theory of GLT sequences is based: the notion of approximating classes of sequences.

Definition 4. Let $\{X_{\mathbf{m}}\}_n$ be a matrix-sequence. An approximating class of sequences (a.c.s.) for $\{X_{\mathbf{m}}\}_n$ is a sequence of matrix-sequences $\{\{B_{\mathbf{m},k}\}_n\}_k$ with the following property: for every k there exists an n_k such that, for $n \geq n_k$,

$$X_{\mathbf{m}} = B_{\mathbf{m},k} + R_{\mathbf{m},k} + S_{\mathbf{m},k}, \quad \operatorname{rank}(R_{\mathbf{m},k}) \leq \varrho(k) N(\mathbf{m}), \quad \|S_{\mathbf{m},k}\| \leq \omega(k),$$

where n_k, $\varrho(k)$, $\omega(k)$ depend only on k, and $\lim_{k\to\infty} \varrho(k) = \lim_{k\to\infty} \omega(k) = 0$.

Roughly speaking, $\{\{B_{\mathbf{m},k}\}_n\}_k$ is an a.c.s. for $\{X_{\mathbf{m}}\}_n$ if $X_{\mathbf{m}}$ is equal to $B_{\mathbf{m},k}$ plus a small-rank matrix (with respect to the matrix size $N(\mathbf{m})$) plus a small-norm matrix. A useful criterion to identify an a.c.s. is provided by the next lemma [8, Section 4.3]. For any $X \in \mathbb{C}^{m\times m}$, we denote by $\|X\|_1$ the trace-norm (or Schatten 1-norm) of X, i.e., the sum of all the singular values of X.

Lemma 1. *Let $\{X_{\mathbf{m}}\}_n$ be a matrix-sequence and let $\{\{B_{\mathbf{m},k}\}_n\}_k$ be a sequence of matrix-sequences. Assume that for every k there exists an n_k such that, for $n \geq n_k$,*

$$\|X_{\mathbf{m}} - B_{\mathbf{m},k}\|_1 \leq \varepsilon(k) N(\mathbf{m}),$$

where $\lim_{k\to\infty} \varepsilon(k) = 0$. Then $\{\{B_{\mathbf{m},k}\}_n\}_k$ is an a.c.s. for $\{X_{\mathbf{m}}\}_n$.

Let us now turn to GLT sequences. A GLT sequence $\{X_{\mathbf{m}}\}_n$ is a specific matrix-sequence equipped with a measurable function $\chi\colon [0,1]^d \times [-\pi,\pi]^d \to \mathbb{C}$. This function is referred to as the symbol of $\{X_{\mathbf{m}}\}_n$. We write $\{X_{\mathbf{m}}\}_n \sim_{\mathrm{GLT}} \chi$ to indicate that $\{X_{\mathbf{m}}\}_n$ is a GLT sequence with symbol χ. The following properties can be found in [8, Section 7.7].

GLT 1. If $\{X_{\mathbf{m}}\}_n \sim_{\mathrm{GLT}} \chi$, then $\{X_{\mathbf{m}}\}_n \sim_\sigma \chi$. If moreover the matrices $X_{\mathbf{m}}$ are Hermitian, then $\{X_{\mathbf{m}}\}_n \sim_\lambda \chi$.

GLT 2. We have

- $\{T_{\mathbf{m}}(f)\}_n \sim_{\mathrm{GLT}} \chi(\hat{\mathbf{x}},\boldsymbol{\theta}) = f(\boldsymbol{\theta})$ if $f \in L^1([-\pi,\pi]^d)$;
- $\{D_{\mathbf{m}}(a)\}_n \sim_{\mathrm{GLT}} \chi(\hat{\mathbf{x}},\boldsymbol{\theta}) = a(\hat{\mathbf{x}})$ if $a\colon [0,1]^d \to \mathbb{C}$ is Riemann-integrable;
- $\{Z_{\mathbf{m}}\}_n \sim_{\mathrm{GLT}} \chi(\hat{\mathbf{x}},\boldsymbol{\theta}) = 0$ if and only if $\{Z_{\mathbf{m}}\}_n \sim_\sigma 0$.

GLT 3. If $X_{\mathbf{m}} = \sum_{i=1}^r c_i \prod_{j=1}^{q_i} X_{\mathbf{m}}^{(i,j)}$, where $r, q_1, \ldots, q_r \in \mathbb{N}$, $c_1, \ldots, c_r \in \mathbb{C}$, and $\{X_{\mathbf{m}}^{(i,j)}\}_n \sim_{\mathrm{GLT}} \chi_{ij}$, then $\{X_{\mathbf{m}}\}_n \sim_{\mathrm{GLT}} \chi = \sum_{i=1}^r c_i \prod_{j=1}^{q_i} \chi_{ij}$.

GLT 4. $\{X_{\mathbf{m}}\}_n \sim_{\mathrm{GLT}} \chi$ if and only if there exist GLT sequences $\{B_{\mathbf{m},k}\}_n \sim_{\mathrm{GLT}} \chi_k$ such that $\chi_k \to \chi$ in measure over $[0,1]^d \times [-\pi,\pi]^d$ and $\{\{B_{\mathbf{m},k}\}_n\}_k$ is an a.c.s. for $\{X_{\mathbf{m}}\}_n$.

GLT 1 states the main distribution results for GLT sequences. **GLT 2** lists the fundamental examples of GLT sequences, from which one can construct, via **GLT 3**, many other GLT sequences. From **GLT 3** we see that GLT sequences form an algebra. More precisely, for any fixed sequence $\{\mathbf{m} = \mathbf{m}(n)\}_n \subseteq \mathbb{N}^d$ such that $\mathbf{m} \to \infty$ as $n \to \infty$, the set

$$\mathcal{A} = \{\{X_{\mathbf{m}}\}_n \colon \{X_{\mathbf{m}}\}_n \sim_{\mathrm{GLT}} \chi \text{ for some measurable } \chi\colon [0,1]^d \times [-\pi,\pi]^d \longrightarrow \mathbb{C}\}$$

is an algebra on $\mathbb{C}$, the so-called GLT algebra [8, Section 7.5]. **GLT 4** can be informally rephrased as follows: if a sequence of GLT sequences $\{B_{\mathbf{m},k}\}_n$ "converges" to a matrix-sequence $\{X_{\mathbf{m}}\}_n$ (in the sense of Definition 4), and if the corresponding sequence of symbols χ_k converges to a function χ (in measure), then $\{X_{\mathbf{m}}\}_n$ is a GLT sequence with symbol χ.

Remark 2. Any matrix-sequence $\{Z_{\mathbf{m}}\}_n$ such that $\{Z_{\mathbf{m}}\}_n \sim_\sigma 0$ is said to be a zero-distributed sequence. By [8, Theorem 2.10], if $\lim_{n\to\infty} N(\mathbf{m})^{-1}\|Z_{\mathbf{m}}\|_1 = 0$ then $\{Z_{\mathbf{m}}\}_n$ is zero-distributed.

3. Isogeometric Galerkin B-spline approximation

In this section, we describe the Galerkin B-spline IgA approximation of (1). Our description is purely formal, since the assumptions on the PDE data K, γ, f are too weak (only measurability). These hypotheses do not even ensure that (1) is well-posed, and the weak form of (1) is actually not defined. Let us however proceed formally for the moment, keeping in mind that our derivation is correct if, say, γ, f and the components of K belong to $L^\infty(\Omega)$. Specific assumptions will be added later on, in our main result (Theorem 1); yet they will be minimal. In this way, we aim at showing that the GLT analysis presented in Section 4 allows one to obtain spectral distribution results for PDE discretization matrices under very weak assumptions on the PDE data.

3.1. Isogeometric Galerkin method. The weak form of (1) consists in finding $u \in H_0^1(\Omega)$ such that, for all $v \in H_0^1(\Omega)$,

$$a(u, v) = \mathrm{F}(v),$$

where $a(u, v) = \int_\Omega ((\nabla u)^T K \nabla v + \gamma u v)$ and $\mathrm{F}(v) = \int_\Omega \mathrm{f} v$. In the standard Galerkin method, we look for an approximation $u_{\mathcal{W}}$ of u by choosing a finite dimensional approximation space $\mathcal{W} \subset H_0^1(\Omega)$ and by solving the following (Galerkin) problem: find $u_{\mathcal{W}} \in \mathcal{W}$ such that, for all $v \in \mathcal{W}$,

$$a(u_{\mathcal{W}}, v) = \mathrm{F}(v).$$

If $\{\varphi_1, \ldots, \varphi_N\}$ is a basis of $\mathcal{W}$, then $u_{\mathcal{W}} = \sum_{j=1}^N u_j \varphi_j$ for a unique vector $\mathbf{u} = (u_1, \ldots, u_N)^T$, and, by linearity, the computation of $u_{\mathcal{W}}$ is equivalent to solving the linear system

$$A\mathbf{u} = \mathbf{f},$$

where

$$A = \left[a(\varphi_j, \varphi_i)\right]_{i,j=1}^N = \left[\int_\Omega \left((\nabla \varphi_j)^T K \nabla \varphi_i + \gamma \varphi_j \varphi_i\right)\right]_{i,j=1}^N \tag{5}$$

is the stiffness matrix and $\mathbf{f} = [\mathrm{F}(\varphi_i)]_{i=1}^N$.

Suppose that the physical domain Ω can be described by a global geometry function $\mathbf{G}\colon [0,1]^d \to \overline{\Omega}$, which is invertible and satisfies $\mathbf{G}(\partial([0,1]^d)) = \partial\overline{\Omega}$. Let

$$\{\hat{\varphi}_1, \ldots, \hat{\varphi}_N\} \tag{6}$$

be a set of basis functions defined on the reference (parametric) domain $[0,1]^d$ and vanishing on the boundary $\partial([0,1]^d)$. In the Galerkin IgA approach, the approximation space is chosen as $\mathcal{W} = \langle \varphi_i : i = 1, \ldots, N \rangle$, with

$$\varphi_i(\mathbf{x}) = \hat{\varphi}_i(\mathbf{G}^{-1}(\mathbf{x})) = \hat{\varphi}_i(\hat{\mathbf{x}}), \quad \mathbf{x} = \mathbf{G}(\hat{\mathbf{x}}). \tag{7}$$

The resulting stiffness matrix A is given by (5), with the basis functions φ_i defined as in (7). By applying standard differential calculus, one obtains the following expression for A in terms of $\mathbf{G}$ and $\hat{\varphi}_i$, $i = 1, \ldots, N$:

$$A = \left[\int_{[0,1]^d} \left((\nabla \hat{\varphi}_j)^T K_{\mathbf{G}} \nabla \hat{\varphi}_i + \gamma(\mathbf{G}) \hat{\varphi}_j \hat{\varphi}_i\right) |\det(J_{\mathbf{G}})|\right]_{i,j=1}^N, \tag{8}$$

where

$$K_{\mathbf{G}} = (J_{\mathbf{G}})^{-1} K(\mathbf{G})(J_{\mathbf{G}})^{-T}, \tag{9}$$

and $J_{\mathbf{G}}$ is the Jacobian matrix of $\mathbf{G}$, i.e.,

$$J_{\mathbf{G}} = \left[\frac{\partial G_i}{\partial \hat{x}_j}\right]_{i,j=1}^{d} = \left[\frac{\partial x_i}{\partial \hat{x}_j}\right]_{i,j=1}^{d}.$$

In the context of IgA, the functions $\hat{\varphi}_i$ are usually tensor-product B-splines or their rational versions, the so-called Non-Uniform Rational B-Splines (NURBS). In this paper, the role of the $\hat{\varphi}_i$ will be played by tensor-product B-splines over uniform knot sequences.

3.2. B-spline basis functions and IgA Galerkin matrices. We now detail the explicit construction of our basis functions $\hat{\varphi}_i$. For $p, n \geq 1$, consider the uniform knot sequence

$$t_1 = \cdots = t_{p+1} = 0 < t_{p+2} < \cdots < t_{p+n} < 1 = t_{p+n+1} = \cdots = t_{2p+n+1},$$

where

$$t_{i+p+1} = \frac{i}{n}, \quad i = 0, \dots, n.$$

The B-splines of degree p on this knot sequence are denoted by

$$N_{i,[p]}\colon [0,1] \to \mathbb{R}, \quad i = 1, \dots, n+p,$$

and are defined recursively as follows [1]: for $1 \leq i \leq n + 2p$,

$$N_{i,[0]}(t) = \begin{cases} 1 & \text{if } t \in [t_i, t_{i+1}), \\ 0 & \text{otherwise;} \end{cases}$$

for $1 \leq k \leq p$ and $1 \leq i \leq n + 2p - k$,

$$N_{i,[k]}(t) = \frac{t - t_i}{t_{i+k} - t_i} N_{i,[k-1]}(t) + \frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+1}} N_{i+1,[k-1]}(t),$$

where we assume that a fraction with zero denominator is zero. We know from [1] that the functions $N_{1,[p]}, \dots, N_{n+p,[p]}$ form a basis for the spline space

$$\left\{s \in C^{p-1}([0,1])\colon s_{|\left[\frac{i}{n}, \frac{i+1}{n}\right)} \in \mathbb{P}_p,\ i = 0, \dots, n-1\right\},$$

where $\mathbb{P}_p$ is the space of polynomials of degree less than or equal to p. Moreover, $N_{1,[p]}, \dots, N_{n+p,[p]}$ possess the following properties [1].

- Local support property:

$$\mathrm{supp}(N_{i,[p]}) = [t_i, t_{i+p+1}], \quad i = 1, \dots, n+p.$$

- Vanishment on the boundary:

$$N_{i,[p]}(0) = N_{i,[p]}(1) = 0, \quad i = 2, \dots, n+p-1.$$

- Nonnegative partition of unity:

$$N_{i,[p]}(t) \geq 0, \quad t \in [0,1],\ i = 1, \dots, n+p, \tag{10}$$

$$\sum_{i=1}^{n+p} N_{i,[p]}(t) = 1, \quad t \in [0,1]. \tag{11}$$

- Bound for derivatives:

$$\sum_{i=1}^{n+p} |N'_{i,[p]}(t)| \leq 2pn, \quad t \in [0,1]. \tag{12}$$

Note that the derivatives $N'_{1,[p]}(t), \dots, N'_{n+p,[p]}(t)$ may not be defined at some of the points $\frac{1}{n}, \dots, \frac{n-1}{n}$ when $p = 1$. In the summation (12), the undefined values are counted as 0.

Let $\mathbf{p}, \mathbf{n}$ be d-indices such that $p_i, n_i \geq 1$, $i = 1, \dots, d$. The tensor-product B-splines that vanish on the boundary $\partial([0,1]^d)$ are defined as

$$N_{\mathbf{i},[\mathbf{p}]} = N_{i_1,[p_1]} \otimes \cdots \otimes N_{i_d,[p_d]}, \quad \mathbf{i} = \mathbf{2}, \dots, \mathbf{n}+\mathbf{p}-\mathbf{1}, \tag{13}$$

where the tensor product of d functions $f_i\colon E_i \to \mathbb{C}$, $i = 1, \dots, d$, is the function

$$f_1 \otimes \cdots \otimes f_d\colon E_1 \times \cdots \times E_d \longrightarrow \mathbb{C}$$

given by

$$(f_1 \otimes \cdots \otimes f_d)(\xi_1, \dots, \xi_d) = f_1(\xi_1) \dots f_d(\xi_d), \quad (\xi_1, \dots, \xi_d) \in E_1 \times \cdots \times E_d.$$

In this paper, the functions $\hat{\varphi}_1, \dots, \hat{\varphi}_N$ in (6) are chosen as the tensor-product B-splines in (13). Note that $N = N(\mathbf{n}+\mathbf{p}-\mathbf{2})$. We adopt for the tensor-product B-splines (13) the lexicographic ordering (2). This ordering is followed when assembling the stiffness matrix (8), which from now on will be denoted by $A^{[\mathbf{p}]}_{\mathbf{G},\mathbf{n}}$, in order to emphasize its dependence on $\mathbf{p}, \mathbf{n}, \mathbf{G}$. In multi-index notation, we have

$$A^{[\mathbf{p}]}_{\mathbf{G},\mathbf{n}} = \Bigg[\int_{[0,1]^d} \big((\nabla N_{\mathbf{j}+\mathbf{1},[\mathbf{p}]})^T K_{\mathbf{G}} \nabla N_{\mathbf{i}+\mathbf{1},[\mathbf{p}]} + \gamma(\mathbf{G}) N_{\mathbf{j}+\mathbf{1},[\mathbf{p}]} N_{\mathbf{i}+\mathbf{1},[\mathbf{p}]}\big) |\det(J_{\mathbf{G}})|\Bigg]_{\mathbf{i},\mathbf{j}=\mathbf{1}}^{\mathbf{n}+\mathbf{p}-\mathbf{2}}.$$

Note that

$$A^{[\mathbf{p}]}_{\mathbf{G},\mathbf{n}} = K^{[\mathbf{p}]}_{\mathbf{G},\mathbf{n}} + R^{[\mathbf{p}]}_{\mathbf{G},\mathbf{n}}, \tag{14}$$

where

$$K^{[\mathbf{p}]}_{\mathbf{G},\mathbf{n}} = \left[\int_{[0,1]^d} (\nabla N_{\mathbf{j}+\mathbf{1},[\mathbf{p}]})^T K_{\mathbf{G}} |\det(J_{\mathbf{G}})|\, \nabla N_{\mathbf{i}+\mathbf{1},[\mathbf{p}]}\right]_{\mathbf{i},\mathbf{j}=\mathbf{1}}^{\mathbf{n}+\mathbf{p}-\mathbf{2}}$$

is the matrix resulting from the discretization of the higer-order (diffusion) term in (1), and

$$R^{[\mathbf{p}]}_{\mathbf{G},\mathbf{n}} = \left[\int_{[0,1]^d} \gamma(\mathbf{G}) |\det(J_{\mathbf{G}})| N_{\mathbf{j}+\mathbf{1},[\mathbf{p}]} N_{\mathbf{i}+\mathbf{1},[\mathbf{p}]}\right]_{\mathbf{i},\mathbf{j}=\mathbf{1}}^{\mathbf{n}+\mathbf{p}-\mathbf{2}}$$

is the matrix resulting from the discretization of the lower-order (reaction) term.

4. GLT analysis of the IgA Galerkin matrices

We denote by $\mathbb{Q}$ the field of rational numbers. Let $\mathbb{Q}^d_+ = \{(r_1, \ldots, r_d) \in \mathbb{Q}^d : r_i > 0, i = 1, \ldots, d\}$ and fix $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_d) \in \mathbb{Q}^d_+$. From now on, we assume that $n_j = \nu_j n$ for each $j = 1, \ldots, d$, i.e., $\mathbf{n} = \boldsymbol{\nu} n$. The discretization parameter n is assumed to vary in an infinite subset of $\mathbb{N}$ such that $\mathbf{n} = \boldsymbol{\nu} n \in \mathbb{N}^d$. Following the multi-index notation, we set $N(\boldsymbol{\nu}) = \prod_{i=1}^d \nu_i$. In Theorem 1, we consider the sequence of normalized IgA Galerkin matrices $\{n^{d-2} A^{[\mathbf{p}]}_{\mathbf{G},\mathbf{n}}\}_n$, and we compute its singular value and eigenvalue distributions. The technique used for proving Theorem 1 relies on the theory of GLT sequences; we refer to this technique as the GLT analysis.

We start with some preliminary definitions. For $p \ge 0$, let $\phi_{[p]}$ be the cardinal B-spline of degree p, which is defined recursively over the uniform knot sequence $\{0, 1, \ldots, p + 1\}$ as follows [1]:

$$\phi_{[0]}(t) = \begin{cases} 1 & \text{if } t \in [0, 1), \\ 0 & \text{elsewhere,} \end{cases}$$

$$\phi_{[p]}(t) = \frac{t}{p}\phi_{[p-1]}(t) + \frac{p+1-t}{p}\phi_{[p-1]}(t-1), \quad p \ge 1.$$

The cardinal B-spline $\phi_{[p]}$ is a non-negative piecewise polynomial of degree p and of class $C^{p-1}(\mathbb{R})$. The support of $\phi_{[p]}$ is the interval $[0, p + 1]$, and $\phi_{[p]}$ is symmetric around the point $t = \frac{p+1}{2}$, i.e.,

$$\phi_{[p]}\Big(\frac{p+1}{2} + t\Big) = \phi_{[p]}\Big(\frac{p+1}{2} - t\Big), \quad t \in \mathbb{R}. \tag{15}$$

Let $\dot\phi_{[p]}$ and $\ddot\phi_{[p]}$ be the first and second derivative of $\phi_{[p]}$. Let $H_{\mathbf{p}}$ be the $d \times d$ symmetric matrix of continuous functions given by

$$(H_{\mathbf{p}})_{ij} = \begin{cases} \left(\bigotimes_{r=1}^{i-1} h_{p_r}\right) \otimes f_{p_i} \otimes \left(\bigotimes_{r=i+1}^{d} h_{p_r}\right), & i = j, \\ \left(\bigotimes_{r=1}^{i-1} h_{p_r}\right) \otimes g_{p_i} \otimes \left(\bigotimes_{r=i+1}^{j-1} h_{p_r}\right) \otimes g_{p_j} \otimes \left(\bigotimes_{r=j+1}^{d} h_{p_r}\right), & i < j, \\ \left(\bigotimes_{r=1}^{j-1} h_{p_r}\right) \otimes g_{p_j} \otimes \left(\bigotimes_{r=j+1}^{i-1} h_{p_r}\right) \otimes g_{p_i} \otimes \left(\bigotimes_{r=i+1}^{d} h_{p_r}\right), & i > j, \end{cases}$$

where $h_p,\ g_p,\ f_p\colon [-\pi,\pi] \to \mathbb{R}$ are defined for all $p \geq 1$ by

$$h_p(\theta) = \phi_{[2p+1]}(p+1) + 2\sum_{k=1}^{p} \phi_{[2p+1]}(p+1-k)\cos(k\theta), \tag{16}$$

$$g_p(\theta) = -2\sum_{k=1}^{p} \dot\phi_{[2p+1]}\,(p+1-k)\sin(k\theta), \tag{17}$$

$$f_p(\theta) = -\ddot\phi_{[2p+1]}(p+1) - 2\sum_{k=1}^{p} \ddot\phi_{[2p+1]}(p+1-k)\cos(k\theta). \tag{18}$$

We denote by $\circ$ the componentwise (Hadamard) product of matrices.

Theorem 1. *Consider the differential problem* (1). *Assume that* $\gamma(\mathbf{G})|\det(J_{\mathbf{G}})|$ *and the components of* $K_{\mathbf{G}}|\det(J_{\mathbf{G}})|$ *belong to* $L^1([0,1]^d)$. *Then the sequence of normalized IgA Galerkin matrices* $\{n^{d-2}A^{[\mathbf{p}]}_{\mathbf{G},\mathbf{n}}\}_n$, *with* $\mathbf{n} = \boldsymbol{\nu} n$, *is a GLT sequence with symbol*

$$f^{(\boldsymbol{\nu})}_{\mathbf{G},\mathbf{p}}(\hat{\mathbf{x}},\boldsymbol{\theta}) = \frac{\boldsymbol{\nu}\left(|\det(J_{\mathbf{G}}(\hat{\mathbf{x}}))|K_{\mathbf{G}}(\hat{\mathbf{x}}) \circ H_{\mathbf{p}}(\boldsymbol{\theta})\right)\boldsymbol{\nu}^T}{N(\boldsymbol{\nu})},$$

i.e.,

$$\{n^{d-2}A^{[\mathbf{p}]}_{\mathbf{G},\mathbf{n}}\}_n \sim_{\mathrm{GLT}} f^{(\boldsymbol{\nu})}_{\mathbf{G},\mathbf{p}}. \tag{19}$$

Moreover,

$$\{n^{d-2}A^{[\mathbf{p}]}_{\mathbf{G},\mathbf{n}}\}_n \sim_{\sigma} f^{(\boldsymbol{\nu})}_{\mathbf{G},\mathbf{p}}, \quad \{n^{d-2}A^{[\mathbf{p}]}_{\mathbf{G},\mathbf{n}}\}_n \sim_{\lambda} f^{(\boldsymbol{\nu})}_{\mathbf{G},\mathbf{p}}. \tag{20}$$

Remark 3. Without the integrability assumption on $\gamma(\mathbf{G})|\det(J_{\mathbf{G}})|$ and the components of $K_{\mathbf{G}}|\det(J_{\mathbf{G}})|$, the IgA Galerkin matrix $A^{[\mathbf{p}]}_{\mathbf{G},\mathbf{n}}$ may not be defined. This assumption can therefore be considered as minimal.

Remark 4. The integrability assumption on $\gamma(\mathbf{G})|\det(J_{\mathbf{G}})|$ and the components of $K_{\mathbf{G}}|\det(J_{\mathbf{G}})|$ is satisfied if, for example,

(a) γ and the components of K belong to $L^1(\Omega)$, and $\mathbf{G}$ is regular, i.e., $\mathbf{G} \in C^1([0,1]^d)$ and $\det(J_{\mathbf{G}}) \neq 0$ over $[0,1]^d$;

(b) $\gamma \in L^1(\Omega)$, the components of K belong to $L^\infty(\Omega)$, $\mathbf{G} \in C^1([0,1]^d)$, and $|\det(J_{\mathbf{G}})|^{-1} \in L^1([0,1]^d)$.

Hence, (19) and (20) hold in both these cases. Note that (b) can be satisfied even if $\mathbf{G}$ is not regular.

Proof of Theorem 1. The proof consists of the following four steps, which may be grouped under the name of GLT analysis. Throughout this proof, the letter C will denote a generic constant independent of n. Moreover, for every $s,t = 1,\ldots,d$, E_{st} will denote the $d \times d$ matrix having 1 in position (s,t) and 0 elsewhere. We shall repeatedly use the following trace-norm inequality:

$$\|X\|_1 \le \sum_{i,j=1}^{m} |x_{ij}|, \qquad X \in \mathbb{C}^{m\times m}. \tag{21}$$

The proof is simple. Let $X = U\Sigma V^*$ be a singular value decomposition of X. Then, setting $Q = VU^*$, the matrix Q is unitary and

$$\begin{aligned}\|X\|_1 &= \mathrm{trace}(\Sigma) = \mathrm{trace}(U^*XV) = \mathrm{trace}(XQ)\\ &\le \sum_{i=1}^{m}\sum_{k=1}^{m} |x_{ik}q_{ki}| \le \sum_{i=1}^{m} \max_{k=1,\ldots,m} |q_{ki}| \sum_{k=1}^{m} |x_{ik}| \le \sum_{i=1}^{m}\sum_{k=1}^{m} |x_{ik}|.\end{aligned}$$

Step 1. We show that

$$\|R^{[\mathbf{p}]}_{\mathbf{G},\mathbf{n}}\|_1 \le C. \tag{22}$$

This inequality, in combination with Remark 2 and the equation $\mathbf{n} = \boldsymbol{\nu} n$, implies $\{n^{d-2}R^{[\mathbf{p}]}_{\mathbf{G},\mathbf{n}}\}_n \sim_\sigma 0$, and it follows from (14), **GLT 2** and **GLT 3** that

$$\{n^{d-2}A^{[\mathbf{p}]}_{\mathbf{G},\mathbf{n}}\}_n \sim_{\mathrm{GLT}} f^{(\boldsymbol{\nu})}_{\mathbf{G},\mathbf{p}} \quad \text{if and only if} \quad \{n^{d-2}K^{[\mathbf{p}]}_{\mathbf{G},\mathbf{n}}\}_n \sim_{\mathrm{GLT}} f^{(\boldsymbol{\nu})}_{\mathbf{G},\mathbf{p}}. \tag{23}$$

In this way, the analysis of the sequence $\{n^{d-2}A^{[\mathbf{p}]}_{\mathbf{G},\mathbf{n}}\}_n$ is reduced to the analysis of its diffusive part $\{n^{d-2}K^{[\mathbf{p}]}_{\mathbf{G},\mathbf{n}}\}_n$.

To prove (22), we note that, by (10), (11), and (13), $\sum_{\mathbf{i}=\mathbf{1}}^{\mathbf{n}+\mathbf{p}-\mathbf{2}} |N_{\mathbf{i}+\mathbf{1},[\mathbf{p}]}| \le 1$ over $[0,1]^d$. Hence, by (21),

$$\begin{aligned}\|R^{[\mathbf{p}]}_{\mathbf{G},\mathbf{n}}\|_1 &\le \sum_{\mathbf{i},\mathbf{j}=\mathbf{1}}^{\mathbf{n}+\mathbf{p}-\mathbf{2}} |(R^{[\mathbf{p}]}_{\mathbf{G},\mathbf{n}})_{\mathbf{i}\mathbf{j}}|\\ &\le \sum_{\mathbf{i},\mathbf{j}=\mathbf{1}}^{\mathbf{n}+\mathbf{p}-\mathbf{2}} \int_{[0,1]^d} |\gamma(\mathbf{G})\det(J_{\mathbf{G}})|\,|N_{\mathbf{j}+\mathbf{1},[\mathbf{p}]}|\,|N_{\mathbf{i}+\mathbf{1},[\mathbf{p}]}|\\ &\le \int_{[0,1]^d} |\gamma(\mathbf{G})\det(J_{\mathbf{G}})|.\end{aligned}$$

Step 2. Let $L^1([0,1]^d, \mathbb{R}^{d\times d})$ be the space of functions $L\colon [0,1]^d \to \mathbb{R}^{d\times d}$ such that $L_{ij} \in L^1([0,1]^d)$ for all $i, j = 1, \dots, d$. Consider the linear operator

$$\mathscr{L}_{\mathbf{n}}^{[\mathbf{p}]}(\cdot)\colon L^1([0,1]^d, \mathbb{R}^{d\times d}) \longrightarrow \mathbb{R}^{N(\mathbf{n}+\mathbf{p}-\mathbf{2})\times N(\mathbf{n}+\mathbf{p}-\mathbf{2})},$$

$$\mathscr{L}_{\mathbf{n}}^{[\mathbf{p}]}(L) = \left[\int_{[0,1]^d} (\nabla N_{\mathbf{j}+\mathbf{1},[\mathbf{p}]})^T L \,\nabla N_{\mathbf{i}+\mathbf{1},[\mathbf{p}]}\right]_{\mathbf{i},\mathbf{j}=\mathbf{1}}^{\mathbf{n}+\mathbf{p}-\mathbf{2}}.$$

In Steps 3–4 we show that, for all $L \in L^1([0,1]^d, \mathbb{R}^{d\times d})$,

$$\{n^{d-2}\mathscr{L}_{\mathbf{n}}^{[\mathbf{p}]}(L)\}_n \sim_{\mathrm{GLT}} \frac{\boldsymbol{\nu}(L(\hat{\mathbf{x}}) \circ H_{\mathbf{p}}(\boldsymbol{\theta}))\boldsymbol{\nu}^T}{N(\boldsymbol{\nu})}. \tag{24}$$

Once this is proved, from $K_{\mathbf{G},\mathbf{n}}^{[\mathbf{p}]} = \mathscr{L}_{\mathbf{n}}^{[\mathbf{p}]}(K_{\mathbf{G}}|\det(J_{\mathbf{G}})|)$ we get

$$\{n^{d-2}K_{\mathbf{G},\mathbf{n}}^{[\mathbf{p}]}\}_n \sim_{\mathrm{GLT}} \frac{\boldsymbol{\nu}\big(K_{\mathbf{G}}(\hat{\mathbf{x}})|\det(J_{\mathbf{G}}(\hat{\mathbf{x}}))| \circ H_{\mathbf{p}}(\boldsymbol{\theta})\big)\boldsymbol{\nu}^T}{N(\boldsymbol{\nu})} = f_{\mathbf{G},\mathbf{p}}^{(\boldsymbol{\nu})}(\hat{\mathbf{x}}, \boldsymbol{\theta}),$$

and the proof is completed, because (19)–(20) follow from (23) and **GLT 1** (note that the matrices $A_{\mathbf{G},\mathbf{n}}^{[\mathbf{p}]}$ are symmetric because K is symmetric).

Step 3. We first prove (24) in the case where $L(\hat{\mathbf{x}}) = a(\hat{\mathbf{x}})E_{st}$ for some $a \in L^1([0,1]^d)$ and some pair of indices $1 \le s, t \le d$. In this case, (24) has already been proved in [5, Section 4] under the additional assumption that $a \in C([0,1]^d)$. Here, we prove that (24) holds for every $a \in L^1([0,1]^d)$. Take a sequence $\{a_k\}_k$ such that $a_k \in C([0,1]^d)$ and $a_k \to a$ in $L^1([0,1]^d)$. Since $a_k \in C([0,1]^d)$, we have

$$\{n^{d-2}\mathscr{L}_{\mathbf{n}}^{[\mathbf{p}]}(a_k(\hat{\mathbf{x}})E_{st})\}_n \sim_{\mathrm{GLT}} \frac{\boldsymbol{\nu}(a_k(\hat{\mathbf{x}})E_{st} \circ H_{\mathbf{p}}(\boldsymbol{\theta}))\boldsymbol{\nu}^T}{N(\boldsymbol{\nu})} \tag{25}$$

for all k. Since $a_k \to a$ in $L^1([0,1]^d)$, it is clear that

$$\frac{\boldsymbol{\nu}(a_k(\hat{\mathbf{x}})E_{st} \circ H_{\mathbf{p}}(\boldsymbol{\theta}))\boldsymbol{\nu}^T}{N(\boldsymbol{\nu})} \to \frac{\boldsymbol{\nu}(a(\hat{\mathbf{x}})E_{st} \circ H_{\mathbf{p}}(\boldsymbol{\theta}))\boldsymbol{\nu}^T}{N(\boldsymbol{\nu})} \quad \text{in measure.} \tag{26}$$

We show that

$$\{\{n^{d-2}\mathscr{L}_{\mathbf{n}}^{[\mathbf{p}]}(a_k(\hat{\mathbf{x}})E_{st})\}_n\}_k \quad \text{is an a.c.s. for} \quad \{n^{d-2}\mathscr{L}_{\mathbf{n}}^{[\mathbf{p}]}(a(\hat{\mathbf{x}})E_{st})\}_n. \tag{27}$$

Once this is proved, (24) follows from (25)–(27) and **GLT 4**.

For every $\mathbf{i}, \mathbf{j} = \mathbf{1}, \dots, \mathbf{n}+\mathbf{p}-\mathbf{2}$,

$$\begin{aligned}\big|\big(\mathscr{L}_{\mathbf{n}}^{[\mathbf{p}]}(a(\hat{\mathbf{x}})E_{st}) - \mathscr{L}_{\mathbf{n}}^{[\mathbf{p}]}(a_k(\hat{\mathbf{x}})E_{st})\big)_{\mathbf{ij}}\big| &= \left|\int_{[0,1]^d} (a-a_k)\frac{\partial N_{\mathbf{j}+\mathbf{1},[\mathbf{p}]}}{\partial \hat{x}_s}\frac{\partial N_{\mathbf{i}+\mathbf{1},[\mathbf{p}]}}{\partial \hat{x}_t}\right| \\ &\le \int_{[0,1]^d} |a-a_k| \left|\frac{\partial N_{\mathbf{j}+\mathbf{1},[\mathbf{p}]}}{\partial \hat{x}_s}\right| \left|\frac{\partial N_{\mathbf{i}+\mathbf{1},[\mathbf{p}]}}{\partial \hat{x}_t}\right|.\end{aligned}$$

By (12), (13), and (21), we obtain

$$\|\mathcal{L}_{\mathbf{n}}^{[\mathbf{p}]}(a(\hat{\mathbf{x}})E_{st}) - \mathcal{L}_{\mathbf{n}}^{[\mathbf{p}]}(a_k(\hat{\mathbf{x}})E_{st})\|_1 \\ \leq \sum_{\mathbf{i},\mathbf{j}=\mathbf{1}}^{\mathbf{n}+\mathbf{p}-\mathbf{2}} \int_{[0,1]^d} |a - a_k| \left|\frac{\partial N_{\mathbf{j}+\mathbf{1},[\mathbf{p}]}}{\partial \hat{x}_s}\right| \left|\frac{\partial N_{\mathbf{i}+\mathbf{1},[\mathbf{p}]}}{\partial \hat{x}_t}\right| \leq 4 p_s p_t n_s n_t \int_{[0,1]^d} |a - a_k|.$$

In view of the equation $\mathbf{n} = \boldsymbol{\nu} n$, we arrive at

$$\|n^{d-2}\mathcal{L}_{\mathbf{n}}^{[\mathbf{p}]}(a(\hat{\mathbf{x}})E_{st}) - n^{d-2}\mathcal{L}_{\mathbf{n}}^{[\mathbf{p}]}(a_k(\hat{\mathbf{x}})E_{st})\|_1 \leq C\, N(\mathbf{n}+\mathbf{p}-\mathbf{2}) \int_{[0,1]^d} |a - a_k|,$$

and (27) follows from Lemma 1.

Step 4. To prove (24) for an arbitrary $L \in L^1([0,1]^d, \mathbb{R}^{d\times d})$, it suffices to observe that, by the linearity of $\mathcal{L}_{\mathbf{n}}^{[\mathbf{p}]}(\cdot)$,

$$\mathcal{L}_{\mathbf{n}}^{[\mathbf{p}]}(L) = \mathcal{L}_{\mathbf{n}}^{[\mathbf{p}]}\left(\sum_{s,t=1}^{d} L_{st}(\hat{\mathbf{x}})E_{st}\right) = \sum_{s,t=1}^{d} \mathcal{L}_{\mathbf{n}}^{[\mathbf{p}]}(L_{st}(\hat{\mathbf{x}})E_{st}).$$

Hence, (24) follows from Step 3 and **GLT 3**. □

Remark 5. Steps 2–3 in the proof of Theorem 1 show that

- Propositions 4.1–4.2 in [6] continue to hold even if the hypothesis "$a \in L^\infty([0,1]^d)$" is replaced by "$a \in L^1([0,1]^d)$";
- Lusin's theorem is actually unnecessary in [6], because all the results obtained therein by means of this theorem can also be derived from the arguments used in Steps 2–3.

Example. Consider the differential problem (1) in the unidimensional case $d = 1$, with

$$\Omega = (0,1), \quad \gamma = 0, \quad K(x) = -r \log x, \quad r > 0.$$

Note that $K \notin L^\infty(\Omega)$. Assume that the geometry map is given by

$$G\colon [0,1] \longrightarrow [0,1], \quad G(\hat{x}) = \hat{x}^q, \quad q > 1.$$

It is clear that G is non-regular as $G'(0) = 0$. The mapping of the uniform mesh $\frac{i}{n}$, $i = 0,\dots,n$, through the function G is the non-uniform grid $\left(\frac{i}{n}\right)^q$, $i = 0,\dots,n$, whose points rapidly accumulate at $x = 0$. This induces a local refinement around the site $x = 0$, and the choice of G is then a way to better approximate the solution in a neighboorhood of $x = 0$, where the coefficient K diverges to infinity.

In view of (9), the function $K_G|G'|$ appearing in Theorem 1 is given by

$$K_G(\hat{x})|G'(\hat{x})| = \frac{K(G(\hat{x}))}{|G'(\hat{x})|} = \frac{-r\log\hat{x}}{\hat{x}^{q-1}},$$

and it belongs to $L^1([0,1])$ for $q < 2$. In this case, the Galerkin B-spline IgA stiffness matrix (14) is well-defined as

$$A_{G,n}^{[p]} = K_{G,n}^{[p]} = \left[\int_{[0,1]} \frac{-r\log\hat{x}}{\hat{x}^{q-1}} N'_{j+1,[p]}(\hat{x}) N'_{i+1,[p]}(\hat{x})\,\mathrm{d}\hat{x}\right]_{i,j=1}^{n+p-2},$$

and Theorem 1 reads as follows:

$$\{n^{-1}A_{G,n}^{[p]}\}_n \sim_{\mathrm{GLT}} f_{G,p}^{(1)} \tag{28}$$

and

$$\{n^{-1}A_{G,n}^{[p]}\}_n \sim_\sigma f_{G,p}^{(1)}, \quad \{n^{-1}A_{G,n}^{[p]}\}_n \sim_\lambda f_{G,p}^{(1)}, \tag{29}$$

where the symbol $f_{G,p}^{(1)}$ is given by

$$f_{G,p}^{(1)}(\hat{x},\theta) = \frac{-r\log\hat{x}}{\hat{x}^{q-1}} f_p(\theta).$$

Here, f_p is the function (18), which coincides with $H_{\mathbf{p}}$ for $d = 1$ and $\mathbf{p} = p$.

Since f_p is symmetric around $\theta = 0$ by (15), a direct computation shows that the distribution relations (29) remain true if we consider as the domain of $f_{G,p}^{(1)}$ the square $[0,1]\times[0,\pi]$ instead of $[0,1]\times[-\pi,\pi]$. According to Remark 1, if the size $n+p-2 = \ell^2$ is large enough, the eigenvalues of $n^{-1}A_{G,n}^{[p]}$ are approximately given by the uniform samples

$$f_{G,p}^{(1)}\Big(\frac{i}{\ell}, \frac{j\pi}{\ell}\Big), \quad i,j = 1,\dots,\ell. \tag{30}$$

This is illustrated in Figure 1, where we see that the majority of the eigenvalues matches the symbol already for $n = 100$. Note that the six largest eigenvalues have been cut from the figure, so as to improve the quality of the plot.

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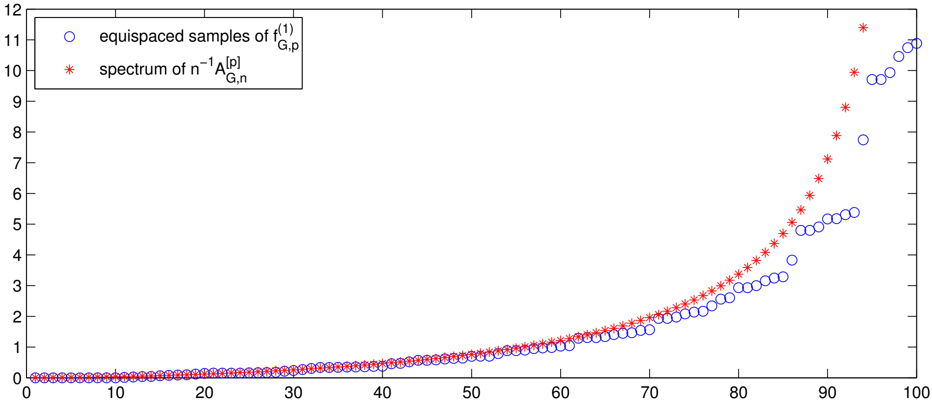

Figure 1. Comparison between the equispaced samples (30) and the eigenvalues of $n^{-1}A^{[p]}_{G,n}$ in the case where $r = 1$, $q = 1.5$, $p = 2$ and $n + p - 2 = n = \ell^2 = 100$.

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