

Quantum ergodicity of Wigner induced random spherical harmonics

Robert Chang

Abstract. We introduce a new notion of a ‘random orthonormal basis of spherical harmonics’ of $L^2(S^2)$ using generalized Wigner ensembles and show that such a random basis is almost surely quantum ergodic. On one hand, because of how our random basis is constructed, this generalizes previous results for random eigenfunctions that are defined using Haar measures on unitary groups. On the other hand, this provides a semi-classical realization of the probabilistic ‘local quantum unique ergodicity’ of Wigner eigenvectors.

Mathematics Subject Classification (2010). Primary: 58J51; Secondary: 60A99.

Keywords. Quantum ergodicity, random orthonormal basis, Wigner ensemble.

1. Introduction

Let (M, g) be a compact Riemannian manifold. Let $\Delta = \Delta_g$ be the Laplace-Beltrami operator and consider the eigenvalue problem $(\Delta - \lambda_k)\varphi_k = 0$ with $0 < \lambda_1 \leq \lambda_2 \leq \dots \uparrow \infty$. The eigenfunctions φ_k are said to be quantum ergodic if for every pseudo-differential operator $A \in \Psi^0(M)$ of degree zero, we have

$$\lim_{\lambda \rightarrow \infty} \frac{1}{\#\{\lambda_k \leq \lambda\}} \sum_{\lambda_k \leq \lambda} |\langle A\varphi_k, \varphi_k \rangle - \omega(A)|^2 = 0, \quad (1)$$

where

$$\omega(A) := \int_{S^*M} \sigma_A d\mu_L \quad (2)$$

is the integral of the principal symbol σ_A of A with respect to the normalized Liouville measure μ_L on the cosphere bundle S^*M . A fundamental result that explains how the mixing properties of a classical system is reflected in the microlocal properties of eigenfunctions is the quantum ergodicity theorem of Shnirelman [6], Zelditch [9], and Colin de Verdière [3]. The theorem states that if the geodesic flow is ergodic, then the Laplacian eigenfunctions φ_k enjoy the quantum ergodic

property (1). In particular, modulo a density zero subsequence, the eigenfunctions become delocalized in phase space in the sense that

$$\langle A\varphi_{k_j}, \varphi_{k_j} \rangle \longrightarrow \omega(A) \quad \text{for all } A \in \Psi^0(M). \tag{3}$$

The asymptotic behavior (3) need not hold when the geodesic flow is no longer assumed to be ergodic. On the sphere, for instance, the geodesic flow is completely integrable and direct computations show that the standard spherical harmonics localize not only on phase space, but also on the base manifold S^2 .

This fact notwithstanding, it is shown in [10] that a random orthonormal basis (defined using Haar measures on unitary groups) of spherical harmonics is almost surely quantum ergodic, a result that is extended (with varying degrees of generality) to random Laplacian eigenfunctions on compact Riemannian manifolds in [11, 12, 5, 2]. The purpose of this paper is to return to the sphere and prove quantum ergodicity for a wider class of ‘random’ spherical harmonics. The setup is as follows. Consider the orthogonal decomposition of $L^2(S^2)$ into a direct sum of subspaces $\mathcal{H}_N = \text{span}\{Y_N^k \mid -N \leq k \leq N\}$ spanned by the standard degree N spherical harmonics. Here, by ‘standard,’ we mean spherical harmonics Y_N^k that are the joint eigenfunctions of the Laplacian $\Delta = \Delta_{S^2}$ and the z -component of the angular momentum operator $L_z = \frac{1}{i} \frac{d}{d\phi}$, that is,

$$\begin{cases} \Delta Y_N^k = -N(N+1)Y_N^k, \\ \frac{1}{i} \frac{\partial}{\partial \phi} Y_N^k = kY_N^k. \end{cases}$$

We write $d_N = \dim \mathcal{H}_N = 2N + 1$ for the dimension of \mathcal{H}_N .

Let $H_N \in \text{Herm}(d_N)$ be a generalized Wigner matrix. (See Section 1.1 for background on random matrix theory.) For $-N \leq k \leq N$, let $u_{N,k} = (u_{N,k}(\alpha))_{\alpha=-N}^N$ be the eigenvectors of H_N . Our object of study is the Wigner induced random basis $\{\psi_{N,k}\}_{k=-N}^N$ for \mathcal{H}_N obtained by ‘transplanting the Wigner eigenvectors onto the sphere’ in the obvious way:

$$\psi_{N,k} := \sum_{\alpha=-N}^N u_{N,k}(\alpha) Y_N^\alpha, \quad -N \leq k \leq N. \tag{4}$$

An equivalent way of thinking about the random basis $\{\psi_{N,k}\}$ is to identify it with a unitary change-of-basis matrix $U_N = (u_{N,k}(\alpha))_{-N \leq k, \alpha \leq N}$ viewed as an element of the probability space $(U(d_N), \mu_N)$. The probability measure μ_N on the unitary group $U(d_N)$ is induced by a generalized Wigner matrix in the following way. Let π be the map from Hermitian matrices to unitary matrices modulo the maximal

torus $U(1)^{d_N}$ defined by

$$\pi: \text{Herm}(d_N) \longrightarrow U(d_N)/U(1)^{d_N}, \quad H_N = U_N^* D(\boldsymbol{\lambda}) U_N \mapsto [U_N],$$

where U_N is a unitary matrix that diagonalizes H_N and $D(\boldsymbol{\lambda})$ is the resulting diagonal matrix. If we write μ_N^{W} for the measure on the Hermitian matrices that describes the generalized Wigner ensemble, then the induced measure μ_N on the unitary group is simply the pushforward of μ_N^{W} under the above map π , that is,

$$\mu_N := \pi_* \mu_N^{\text{W}}. \tag{5}$$

The construction of a Wigner induced random basis (4) for the finite dimensional subspace \mathcal{H}_N extends naturally to all of $L^2(S^2)$. Indeed, let U be the operator that acts block-diagonally on the decomposition $L^2(S^2) = \bigoplus_{N \geq 0} \mathcal{H}_N$ so that the restrictions $U|_{\mathcal{H}_N} = U_N \in U(d_N)$ to the subspaces yield a sequence of independent unitary matrices of the appropriate dimensions. By the preceding paragraph, a Wigner induced random orthonormal basis $\Psi = \{\psi_{N,k}\}_{-N \leq k \leq N, N \geq 0}$ for all of $L^2(S^2)$ may be identified with such an operator U viewed as an element of the product probability space $\prod_{n \geq 0} (U(d_N), \mu_N)$. Henceforth, when the context is clear, we will refer to Ψ simply as a ‘random basis’ with the understanding that it is constructed randomly with respect to the product measure $\prod \mu_N$.

For technical reasons, certain indices k need to be excluded from our computations. Let $0 < \nu < \frac{3}{4}$ be a positive constant (guaranteed by Theorem 1.1), and let

$$I_N = [[-N, -N + N^{1/4}]] \cup [[-N + N^{1-\nu}, N - N^{1-\nu}]] \cup [[N - N^{1/4}, N]] \tag{6}$$

be the subset of indices $-N \leq k \leq N$ that are, in the random matrix theory language, ‘in the bulk’ and ‘near the edges.’ We can only work with indices belonging to I_N because the asymptotic normality result of Bourgade and Yau (Theorem 1.1), which we rely on, is established only for $k \in I_N$. (The set I_N displayed above is precisely the set T_N in the statement of Theorem 1.2 in the original paper [1], except that the our indexing convention is $k \in [-N, N]$, and the convention of [1] is $k \in [1, N]$.) It is expected that Theorem 1.1 holds for all indices k (see the remark immediately following Definition 5.1 in [1]). Similar universality results for Wigner eigenvectors are also proved in [4, 7] under additional assumptions on the moments of the entries. For us, the set I_N is sufficient for deriving a quantum ergodicity statement because we are still left with a density one subsequence after discarding indices in the intermediate regime, that is,

$$\frac{|\{k \in I_N\}|}{|\{k \in [-N, N]\}|} \rightarrow 1.$$

Given a pseudo-differential operator $A \in \Psi^0(M)$ of order zero and a random basis Ψ , let $X_N = X_N^A(\{\psi_{N,k}\}) : (U(d_N), \mu_N) \rightarrow \mathbb{R}_{\geq 0}$ be the random variables

$$X_N = X_N^A(\{\psi_{N,k}\}) = \frac{1}{d_N} \sum_{k \in I_N} |\langle A\psi_{N,k}, \psi_{N,k} \rangle - \omega(A)|^2, \tag{7}$$

where $\omega(A)$ is defined in (2). Even though the random variable (7) depends on the choice of a pseudo-differential operator and a random basis, for notational simplicity we will continue to write $X_N := X_N^A(\{\psi_{N,k}\})$. Our quantum ergodicity result is formulated in terms of X_N .

Theorem 1. *Let Ψ be a Wigner induced random orthonormal basis of spherical harmonics for $L^2(S^2)$. Then Ψ is almost surely quantum ergodic with respect to the product probability measure $\prod \mu_N$ in the sense that*

$$\lim_{M \rightarrow \infty} \frac{1}{M} \sum_{N=0}^M X_N = 0 \quad a.s.$$

for every $A \in \Psi^0(S^2)$.

Note that the random variables X_N are independent by construction. Theorem 1 is therefore an easy consequence of the Kolmogorov convergence criterion and Strong Law of Large Numbers once we show that $\mathbb{E}X_N \rightarrow 0$ and $\mathbb{E}X_N^2$ is bounded. Indeed, the following holds.

Theorem 2. *We have $\mathbb{E}X_N = O(d_N^{-\varepsilon_0})$ and $\mathbb{E}X_N^2 = O(d_N^{-\varepsilon'_0})$ for some $\varepsilon_0, \varepsilon'_0 > 0$ guaranteed by Theorem 1.1.*

This is a good place for some remarks. First, since we only work with random spherical harmonics in this paper, we confine ourselves to describing the construction of random bases on S^2 . A similar construction that involves partitioning the spectrum of the Laplacian appropriately can be used to make sense of random bases (defined using either Haar measures or Wigner induced measures on unitary groups) on any compact Riemannian manifold. Readers are referred to [11, 12, 5, 2] for the general construction. A natural next step is to extend our quantum ergodicity result to Wigner induced random bases of Laplacian eigenfunctions or approximate eigenfunctions on other manifolds.

Second, it is known that the eigenvectors of a Gaussian unitary ensemble is distributed by Haar measure on the unitary group. Since the generalized Wigner ensembles contain GUE as a special case, the measure with respect to which

Wigner eigenvectors are distributed (i.e., the Wigner induced measure μ_N) is a vast generalization of Haar measure. It is unknown to the author if such measures can be given an explicit characterization. Nevertheless, universality results from random matrix theory are robust enough for showing that Wigner induced random bases enjoy the same quantum ergodicity property as ‘GUE induced random bases’ (i.e., random bases defined using Haar measure) on the sphere.

Finally, the methods presented in this paper can be used to prove quantum ergodicity of Wigner induced random spherical harmonics on higher dimensional spheres S^p for any $p \geq 2$. It will be clear from the proof that ε_0 and ε'_0 in the statement of Theorem 2 are independent of the dimension p because, in the notation of Theorem 1.1, we have $\varepsilon_0 = \varepsilon_0(Q_1)$ and $\varepsilon'_0 = \varepsilon'_0(Q_2)$ where Q_1, Q_2 are polynomials of the form

$$Q_1(z_1, z_2, z_3, z_4) = z_1 z_2 \bar{z}_3 \bar{z}_4 \quad \text{and} \quad Q_2(z_1, \dots, z_8) = z_1 z_2 z_3 z_4 \bar{z}_5 \bar{z}_6 \bar{z}_7 \bar{z}_8.$$

While $\varepsilon_0, \varepsilon'_0$ remain fixed for all $p \geq 2$, the dimension d_N of the space of degree N spherical harmonics grows like N^{p-1} on S^p . Substituting the asymptotics for d_N into the statement of Theorem 2 gives $\mathbb{E}X_N = O(N^{-\varepsilon_0(p-1)})$ and $\mathbb{E}X_N^2 = O(N^{-\varepsilon'_0(p-1)})$. Observe that, for all p sufficiently large, the Borel–Cantelli lemma becomes applicable and implies the stronger convergence statement that $X_N \rightarrow 0$ almost surely instead of the Cesàro means $\frac{1}{M} \sum_{N=0}^M X_N \rightarrow 0$.

The rest of the paper is organized as follows. Section 1.1 provides a brief summary of random matrix theory that will be used in our proofs. The key result is Theorem 1.1, which states that Wigner eigenvectors (with the appropriate scaling) are asymptotically Gaussian random variables. Section 2 is devoted to proving Proposition 1, which is a special case of Theorem 2. The techniques developed for this special case extends easily to prove the main theorems in Section 3.

The author expresses his sincere gratitude towards Steve Zelditch and Antonio Auffinger for their patience and generous assistance that greatly improved the manuscript.

1.1. Asymptotic normality of Wigner eigenvectors and probabilistic local QUE. We now summarize a universality result for Wigner eigenvectors proved in [1]. In keeping with the indexing convention for spherical harmonics, the indices in this section continue to range from $-N$ to N . Recall also that $d_N = 2N + 1$.

By a generalized Wigner matrix we mean a Hermitian matrix

$$H_N = (h_{jk})_{-N \leq j, k \leq N} \in \text{Herm}(d_N)$$

satisfying

- the entries h_{jk} are independent random variables for $j \leq k$, each with mean zero and variance $\mathbb{E}h_{jk}^2 =: \sigma_{jk}^2$ satisfying the normalization condition $\sum_{j=-N}^N \sigma_{jk}^2 = 1$ for k fixed;
- there exists a constant $c_1 > 0$ independent of N such that $(c_1 N)^{-1} \leq \sigma_{jk}^2 \leq c_1 N$ for all $-N \leq j, k \leq N$;
- there exists a constant $c_2 > 0$ independent of N such that $\mathbb{E}(\mathbf{h}_{jk}^* \mathbf{h}_{jk}) \geq c_2 N^{-1}$ in the sense of inequality between 2×2 positive matrices, where $\mathbf{h}_{jk} := (\operatorname{Re} h_{jk}, \operatorname{Im} h_{jk})$;
- for any $q \in \mathbb{N}$, there exists a constant $C_q > 0$ such that for any N and any $-N \leq j, k \leq N$, we have $\mathbb{E}|\sqrt{d_N} h_{jk}|^q \leq C_q$.

Let $u_{N,k} = (u_{N,k}(\alpha))_{\alpha=-N}^N$ denote the eigenvectors of a generalized Wigner matrix $H_N \in \operatorname{Herm}(d_N)$. The eigenvectors, indexed by $k \in [-N, N]$, are ordered so that the corresponding eigenvalues form a nondecreasing sequence. Of course, an eigenvector is well-defined only up to a phase $e^{i\theta} \in \operatorname{U}(1)$. This phase ambiguity may be eliminated, for instance, by considering instead the equivalence class $[u_{N,k}]$.

Theorem 1.1 (Asymptotic normality for generalized Wigner eigenvectors, [1] Corollary 1.3). *Let $\{H_N\}$ be a sequence of generalized Wigner matrices. Let I_N be the set of indices away from the intermediate regime as defined in (6) (note that I_N depends on a parameter ν). Then there exists $\nu > 0$ such that for any $k \in I_N$ and $J \subset \{-N, \dots, N\}$ with $|J| = m$, we have*

$$\sqrt{d_N}(u_{N,k}(\alpha))_{\alpha \in J} \longrightarrow (\mathcal{N}_j^{(1)} + i\mathcal{N}_j^{(2)})_{j=1}^m$$

in the sense of convergence in moments modulo phases, where $\mathcal{N}_j^{(1)}, \mathcal{N}_j^{(2)}$ are independent standard Gaussians. More precisely, for any polynomial Q in $2m$ variables, there exists $\varepsilon = \varepsilon(Q) > 0$ such that for sufficiently large N we have

$$\sup_{\substack{J \subset \{-N, \dots, N\} \\ |J|=m, k \in I_N}} |\mathbb{E}Q(\sqrt{2N}(e^{i\omega} u_{N,k}(\alpha), e^{-i\omega} \overline{u_{N,k}(\alpha)})_{\alpha \in J}) - \mathbb{E}Q((\mathcal{N}_j^{(1)} + i\mathcal{N}_j^{(2)}), \mathcal{N}_j^{(1)} - i\mathcal{N}_j^{(2)})_{j=1}^m)| \leq d_N^{-\varepsilon}.$$

Here ω a phase independent of H_N and uniform on $(0, 2\pi)$.

In fact, a stronger statement is proved Theorem 1.2 of [1], namely the projection $\langle \mathbf{q}, u_{N,k} \rangle$ of an eigenvector to any unit vector $\mathbf{q} \in \mathbb{R}^{d_N}$ is asymptotically normal. As a corollary, generalized Wigner eigenvectors are ‘locally quantum unique ergodic’ in the following sense. Let $a_N: \{-N, \dots, N\} \rightarrow [-1, 1]$ be a function with $\sum_{\alpha=-N}^N a_N(\alpha) = 0$ and let $|a_N| = \#\{-N \leq \alpha \leq N \mid a_N(\alpha) \neq 0\}$ be the size of its support.

Theorem 1.2 (local QUE for Wigner eigenvectors, [1] Corollary 1.4). *Let $\{H_N\}$ be a sequence of generalized Wigner matrices. Then there exists $\varepsilon > 0$ such that for any $\delta > 0$, there exists a constant $C > 0$ so that for every sequence of functions $\{a_N\}$ as above and $k \in I_N$ we have*

$$\mathbb{P}\left(\left|\frac{d_N}{|a_N|}\langle a_N u_{N,k}, u_{N,k} \rangle\right| > \delta\right) \leq C(d_N^{-\varepsilon} + |a_N|^{-1}), \tag{8}$$

where $\langle a_N u_{N,k}, u_{N,k} \rangle := \sum_{\alpha=-N}^N a_N(\alpha) |u_{N,k}(\alpha)|^2$.

Theorem 1.1 shows that Wigner eigenvectors are asymptotically flat even on small scales by choosing the test functions a_N to have small supports. Note that since the left-hand side of (8) depends only the eigenvectors but not the eigenvalues, the measure used in Theorem 1.2 is precisely the induced measure μ_N defined in (5).

We take this opportunity to remark that on a compact manifold (M, g) , the analogue to the limiting formula (8) given by

$$\int_M f(x) |\varphi_k(x)|^2 dx \longrightarrow \int_M f(x) dx \quad \text{for every } f \in C^\infty(M) \tag{9}$$

is insufficient for concluding that $\{\varphi_k\}$ is quantum ergodic in the sense of (1) or (3). This is because delocalization on the base manifold M is a much weaker condition than diffuseness in the phase space S^*M . For instance, the Laplacian eigenfunctions $e^{i\langle \lambda, x \rangle}$ on a flat torus $\mathbb{R}^n / 2\pi\mathbb{Z}^n$ are delocalized in the sense of (9). But if $\{\lambda_k\}$ is a sequence of lattice points for which the unit vectors $\lambda_k / |\lambda_k|$ tend to a limit vector $\xi \in \mathbb{R}^n$, then the asymptotic formula

$$\langle Ae^{i\langle \lambda_k, x \rangle}, e^{i\langle \lambda_k, x \rangle} \rangle \simeq \int_{\mathbb{R}^n / 2\pi\mathbb{Z}^n} \sigma_A\left(x, \frac{\lambda_k}{|\lambda_k|}\right) dx \quad \text{for every } A \in \Psi^0(\mathbb{R}^n / 2\pi\mathbb{Z}^n)$$

shows that the corresponding weak* limit is a delta mass on the invariant Lagrangian torus $T_\xi \subset S^*M$ for the geodesic flow. Since there always exists a sequence of $\lambda_k / |\lambda_k|$ converging to arbitrary $\xi \in \mathbb{R}^n$, the eigenfunctions $e^{i\langle \lambda, x \rangle}$ are far from diffuse in phase space. Of course, in the random matrix setting it is unclear even how to interpret the phase space when the base manifold is an index set $\{-N, \dots, N\}$. We will need additional tools from semi-classical analysis to show that Theorem 1 holds.

2. Rotationally invariant case

The purpose of this section is to prove Proposition 1 stated below. The difference between the proposition and Theorem 2 is the rotational invariance assumption we

impose on A (and hence on the random variable X_N). This additional assumption allows us to isolate the key computational techniques and exhibit them in a simpler setting.

To clearly distinguish the special case we are currently considering from the general case, let us introduce some new notation. Let $B \in \Psi^0(S^2)$ denote pseudo-differential operators of degree zero that are invariant under z -axis rotations. To these rotationally invariant operators we associate random variables

$$Z_N = Z_N^B(\{\psi_{N,k}\}) = \frac{1}{d_N} \sum_{k \in I_N} |\langle B\psi_{N,k}, \psi_{N,k} \rangle - \omega(B)|^2, \tag{10}$$

where I_N is defined in (6) and $\omega(B)$ is defined in (2). Our goal is to show the following.

Proposition 1. *In the above notation, we have $\mathbb{E}Z_N = O(d_N^{-\varepsilon})$ and $\mathbb{E}Z_N^2 = O(d_N^{-\varepsilon'})$ for some $\varepsilon, \varepsilon' > 0$ guaranteed by Theorem 1.1.*

Proof of Proposition 1. Note that the rotational invariance hypothesis implies that the matrix elements $\langle BY_N^\alpha, Y_N^\beta \rangle$ vanish whenever $\alpha \neq \beta$. Rewriting the random basis elements $\psi_{N,k}$ in terms of spherical harmonics Y_N^α using (4), the expression (10) becomes

$$\begin{aligned} Z_N &= \frac{1}{d_N} \sum_{k \in I_N} \left| \sum_{\alpha, \beta = -N}^N \langle BY_N^\alpha, Y_N^\beta \rangle u_{N,k}(\alpha) u_{N,k}(\beta) - \omega(B) \right|^2 \\ &= \frac{1}{d_N} \sum_{k \in I_N} \left| \sum_{\alpha} \langle BY_N^\alpha, Y_N^\alpha \rangle |u_{N,k}(\alpha)|^2 - \omega(B) \right|^2 \\ &= S_1 + S_2, \end{aligned}$$

where

$$\begin{aligned} S_1 &= \frac{1}{d_N} \sum_{k \in I_N} \sum_{\alpha, \beta} \langle BY_N^\alpha, Y_N^\alpha \rangle \langle BY_N^\beta, Y_N^\beta \rangle |u_{N,k}(\alpha)|^2 |u_{N,k}(\beta)|^2, \\ S_2 &= -\frac{2\omega(B)}{d_N} \sum_{k \in I_N} \sum_{\alpha} \langle BY_N^\alpha, Y_N^\alpha \rangle |u_{N,k}(\alpha)|^2 + \frac{1}{d_N} \sum_{k \in I_N} \omega(B)^2. \end{aligned}$$

We use the Weingarten formula [8] to compute the expectation $\mathbb{E}Z_N = \mathbb{E}S_1 + \mathbb{E}S_2$. Let $(u_{N,k}(\alpha))_{-N \leq k, \alpha \leq N} \in U(d_N)$ be a unitary matrix and for $1 \leq j \leq m$, let $k_j, k'_j, \alpha_j, \alpha'_j \in [-N, N]$ be indices. The Weingarten formula

states that the integral

$$I_N(m) := \int_{U(d_N)} u_{N,k_1}(\alpha_1) \cdots u_{N,k_m}(\alpha_m) \overline{u_{N,k'_1}(\alpha'_1)} \cdots \overline{u_{N,k'_m}(\alpha'_m)} dU_N$$

of a polynomial in the entries of $(u_{N,k}(\alpha))$ with respect to Haar measure dU_N has an asymptotic formula in terms of the Kronecker delta functions on the indices:

$$I_N(m) = d_N^{-m} \sum \delta_{k_1 k'_1} \delta_{\alpha_1 \alpha'_1} \cdots \delta_{k_\ell k'_\ell} \delta_{\alpha_\ell \alpha'_\ell} + O(d_N^{-m-1}), \tag{11}$$

where the sum is over all choices of j_1, \dots, j_m as a permutation of $1, \dots, m$. Let Q be the polynomial in $2m$ variables defined by $Q((z_j, w_j)_{j=1}^m) := z_1 \cdots z_m \bar{w}_1 \cdots \bar{w}_m$. Then, in the notation of Theorem 1.1, direct computation with Gaussian random variables shows that

$$\left| \frac{1}{d_N^m} \mathbb{E} Q((\mathcal{N}_j^{(1)} + i\mathcal{N}_j^{(2)}, \mathcal{N}_j^{(1)} - i\mathcal{N}_j^{(2)})_{j=1}^m) - I_N(m) \right| = O(d_N^{-m-1}) \tag{12}$$

Putting together (11), (12), and Theorem 1.1 proves the following key lemma.

Lemma 2.1. *Let $(u_{N,k}(\alpha)) \in U(d_N)$ be a unitary matrix. Let I_N be the set defined in (6). Then for indices $k_1, \dots, k_m, k'_1, \dots, k'_m \in I_N$ and $\alpha_1, \dots, \alpha_m, \alpha'_1, \dots, \alpha'_m \in [-N, N]$, we have*

$$\begin{aligned} & \mathbb{E}(u_{N,k_1}(\alpha_1) \cdots u_{N,k_m}(\alpha_m) \overline{u_{N,k'_1}(\alpha'_1)} \cdots \overline{u_{N,k'_m}(\alpha'_m)}) \\ &= d_N^{-m} \sum \delta_{k_1 k'_1} \delta_{\alpha_1 \alpha'_1} \cdots \delta_{k_m k'_m} \delta_{\alpha_m \alpha'_m} + O(d_N^{-m-\varepsilon}) \end{aligned} \tag{13}$$

for some $\varepsilon = \varepsilon(Q) > 0$ guaranteed by Theorem 1.1.

Returning to the quantity $\mathbb{E}Z_N = \mathbb{E}S_1 + \mathbb{E}S_2$, we find that (13) implies

$$\mathbb{E}(|u_{N,k}(\alpha)|^2 |u_{N,k}(\beta)|^2) = d_N^{-2}(1 + \delta_{\alpha\beta}) + O(d_N^{-2-\varepsilon_1}) \quad \text{for } k \in I_N,$$

which gives

$$\begin{aligned} \mathbb{E}S_1 &= \frac{1}{d_N} \sum_{k \in I_N} \sum_{\alpha, \beta} \langle BY_N^\alpha, Y_N^\alpha \rangle \langle BY_N^\beta, Y_N^\beta \rangle \mathbb{E}(|u_{N,k}(\alpha)|^2 |u_{N,k}(\beta)|^2) \\ &= \sum_{\alpha, \beta} \langle BY_N^\alpha, Y_N^\alpha \rangle \langle BY_N^\beta, Y_N^\beta \rangle \left(\frac{1}{d_N^2} (1 + \delta_{\alpha\beta}) + O(d_N^{-2-\varepsilon_1}) \right) \\ &= \left(\frac{1}{d_N} \sum_{\alpha} \langle BY_N^\alpha, Y_N^\alpha \rangle \right)^2 + \frac{1}{d_N^2} \sum_{\alpha} \langle BY_N^\alpha, Y_N^\alpha \rangle^2 + O(d_N^{-\varepsilon_1}). \end{aligned} \tag{14}$$

The first sum in (14) can be rewritten using semi-classical analysis. Let

$$\Pi_N: L^2(S^2) \rightarrow \mathcal{H}_N$$

denote the spectral projection onto the eigenspace of degree N spherical harmonics. Let $A \in \Psi^0(S^2)$ be any pseudo-differential operator of degree zero (not necessarily rotationally invariant), then Weyl’s law states that

$$\frac{1}{d_N} \sum_{\alpha} \langle AY_N^{\alpha}, Y_N^{\alpha} \rangle = \frac{1}{d_N} \text{tr}(\Pi_N A \Pi_N) = \omega(A) + O(d_N^{-1}). \tag{15}$$

For the second sum in (14), it suffices to note that the squares $\langle AY_N^{\alpha}, Y_N^{\alpha} \rangle^2$ of the matrix elements are uniformly bounded in N because the pseudo-differential operator $A \in \Psi^0(S^2)$ (again, not necessarily rotationally invariant) is a bounded operator from $L^2(S^2)$ to itself. Since we are summing over $-N \leq \alpha \leq N$ (i.e., summing d_N number of terms) and dividing by d_N^2 , the second sum has only a lower order contribution:

$$\frac{1}{d_N^2} \sum_{\alpha} \langle BY_N^{\alpha}, Y_N^{\alpha} \rangle^2 = O(d_N^{-1}). \tag{16}$$

Combining (14), (15), and (16) yields

$$\mathbb{E}S_1 = (\omega(B) + O(d_N^{-1}))^2 + O(d_N^{-1}) + O(d_N^{-\varepsilon_1}) = \omega(B)^2 + O(d_N^{-\varepsilon_1}).$$

The asymptotics for $\mathbb{E}S_2$ is similarly computed. By (13), we have

$$\mathbb{E}|u_{N,k}(\alpha)|^2 = d_N^{-1} + O(d_N^{-1-\varepsilon_2}) \quad \text{for } k \in I_N,$$

whence

$$\begin{aligned} \mathbb{E}S_2 &= -\frac{2\omega(B)}{d_N} \sum_{k \in I_N} \sum_{\alpha} \langle BY_N^{\alpha}, Y_N^{\alpha} \rangle \mathbb{E}|u_{N,k}(\alpha)|^2 + \frac{1}{d_N} \sum_{k \in I_N} \omega(B)^2 \\ &= -2\omega(B) \sum_{\alpha} \langle BY_N^{\alpha}, Y_N^{\alpha} \rangle \left(\frac{1}{d_N} + O(d_N^{-1-\varepsilon_2}) \right) + \omega(B)^2 \\ &= -2\omega(B)^2 + \omega(B)^2 + O(d_N^{-\varepsilon_2}), \end{aligned}$$

where the last equality follows from Weyl’s law (15). Adding together the expressions for $\mathbb{E}S_1$ and $\mathbb{E}S_2$ shows that $\mathbb{E}Z_N = O(d_N^{-\min\{\varepsilon_1, \varepsilon_2\}}) = O(d_N^{-\varepsilon})$ as the factors of $\omega(B)^2$ cancel exactly. This proves the first part of Proposition 1.

The computations for the second moment $\mathbb{E}Z_N^2$ is more tedious, but no new techniques are required. Write a second copy of the random variable Z_N with the indices j, η, ξ in place of k, α, β , then direct computation shows $\mathbb{E}Z_N^2 = T_1 + T_2 + \dots + T_5$ with

$$\begin{aligned}
 T_1 &= \frac{1}{d_N^2} \sum_{k,j \in I_N} \sum_{\alpha, \beta, \eta, \xi} \langle BY_N^\alpha, Y_N^\alpha \rangle \langle BY_N^\beta, Y_N^\beta \rangle |u_{N,k}(\alpha)|^2 |u_{N,k}(\beta)|^2 \\
 &\quad \langle BY_N^\eta, Y_N^\eta \rangle \langle BY_N^\xi, Y_N^\xi \rangle |u_{N,j}(\eta)|^2 |u_{N,j}(\xi)|^2, \\
 T_2 &= -\frac{4\omega(B)}{d_N^2} \sum_{k,j \in I_N} \sum_{\alpha, \beta, \eta} \langle BY_N^\alpha, Y_N^\alpha \rangle \langle BY_N^\beta, Y_N^\beta \rangle |u_{N,k}(\alpha)|^2 |u_{N,k}(\beta)|^2 \\
 &\quad \langle BY_N^\eta, Y_N^\eta \rangle |u_{N,j}(\eta)|^2, \\
 T_3 &= \frac{2\omega(B)^2}{d_N^2} \sum_{k,j \in I_N} \sum_{\alpha, \beta} \langle BY_N^\alpha, Y_N^\alpha \rangle \langle BY_N^\beta, Y_N^\beta \rangle |u_{N,k}(\alpha)|^2 |u_{N,k}(\beta)|^2, \\
 T_4 &= \frac{4\omega(B)^2}{d_N^2} \sum_{k,j \in I_N} \sum_{\alpha, \eta} \langle BY_N^\alpha, Y_N^\alpha \rangle |u_{N,k}(\alpha)|^2 \langle BY_N^\eta, Y_N^\eta \rangle |u_{N,j}(\eta)|^2, \\
 T_5 &= -\frac{4\omega(B)^3}{d_N^2} \sum_{k,j \in I_N} \sum_{\alpha} \langle BY_N^\alpha, Y_N^\alpha \rangle |u_{N,k}(\alpha)|^2 + \frac{1}{d_N^2} \sum_{k,j \in I_N} \omega(B)^4.
 \end{aligned}$$

We work out the asymptotics for $\mathbb{E}T_1$ in detail. Thanks again to (13), we have $\mathbb{E}(|u_{N,k}(\alpha)|^2 |u_{N,k}(\beta)|^2 |u_{N,j}(\eta)|^2 |u_{N,j}(\xi)|^2) = d_N^{-4} (C_1 + \delta_{kj} C_2) + O(d_N^{-4-\varepsilon'_1})$, (17)

where

$$\begin{aligned}
 C_1 &= C_1(\alpha, \beta, \eta, \xi) = (1 + \delta_{\alpha\beta})(1 + \delta_{\eta\xi}), \\
 C_2 &= C_2(\alpha, \beta, \eta, \xi) = \delta_{\alpha\eta}(1 + \delta_{\beta\xi} + 2\delta_{\eta\xi}) + \delta_{\alpha\xi}(1 + \delta_{\beta\eta} + 2\delta_{\beta\xi}) \\
 &\quad + \delta_{\beta\eta}(1 + 2\delta_{\alpha\beta}) + \delta_{\beta\xi}(1 + 2\delta_{\eta\xi}) + 6\delta_{\alpha\beta}\delta_{\beta\xi}\delta_{\eta\xi}.
 \end{aligned}$$

These imply

$$\begin{aligned}
 \mathbb{E}T_1 &= \frac{1}{d_N^4} \sum_{\alpha, \beta, \eta, \xi} C_1(\alpha, \beta, \eta, \xi) \langle BY_N^\alpha, Y_N^\alpha \rangle \langle BY_N^\beta, Y_N^\beta \rangle \langle BY_N^\eta, Y_N^\eta \rangle \langle BY_N^\xi, Y_N^\xi \rangle \\
 &\quad + \frac{1}{d_N^5} \sum_{\alpha, \beta, \eta, \xi} C_2(\alpha, \beta, \eta, \xi) \langle BY_N^\alpha, Y_N^\alpha \rangle \langle BY_N^\beta, Y_N^\beta \rangle \langle BY_N^\eta, Y_N^\eta \rangle \langle BY_N^\xi, Y_N^\xi \rangle \\
 &\quad + O(d_N^{-\varepsilon'_1}).
 \end{aligned}
 \tag{18}$$

Notice that the leading orders of C_1 and C_2 are different because there is a factor of δ_{kj} in front of C_2 but not C_1 in (17).

Consider the first line of the expression (18) (i.e., the part that involves only C_1). Recall that $C_1 = (1 + \delta_{\alpha\beta})(1 + \delta_{\eta\xi}) = 1 + \delta_{\alpha\beta} + \delta_{\eta\xi} + \delta_{\alpha\beta}\delta_{\eta\xi}$ contains four terms. We claim that only the constant term has a top order contribution when computing the asymptotics of $\mathbb{E}T_1$; the other three terms containing Kronecker delta functions all have lower order contributions. Indeed, notice that

$$\frac{1}{d_N^4} \sum_{\alpha, \beta, \eta, \xi} \delta_{\alpha\beta} \langle BY_N^\alpha, Y_N^\alpha \rangle \langle BY_N^\beta, Y_N^\beta \rangle \langle BY_N^\eta, Y_N^\eta \rangle \langle BY_N^\xi, Y_N^\xi \rangle$$

is equal to

$$\frac{1}{d_N^4} \sum_{\alpha, \eta, \xi} \langle BY_N^\alpha, Y_N^\alpha \rangle^2 \langle BY_N^\eta, Y_N^\eta \rangle \langle BY_N^\xi, Y_N^\xi \rangle = O(d_N^{-1}),$$

which is a lower order term because we are summing d_N^3 number of uniformly bounded products of matrix elements but dividing by d_N^4 .

We now turn our attention to the second line of the expression (18) (i.e., the part that involves only C_2). Notice that each term of C_2 contains at least one Kronecker delta function on the indices α, β, η, ξ . At the same time, we are dividing the sum by d_N^5 . Therefore, the entire second line is of order at most $O(d_N^{-2})$. These observations imply that the expected value of T_1 has the simple asymptotics

$$\begin{aligned} \mathbb{E}T_1 &= \frac{1}{d_N^4} \sum_{\alpha, \beta, \eta, \xi} \langle BY_N^\alpha, Y_N^\alpha \rangle \langle BY_N^\beta, Y_N^\beta \rangle \langle BY_N^\eta, Y_N^\eta \rangle \langle BY_N^\xi, Y_N^\xi \rangle + O(d_N^{-\epsilon'_1}) \\ &= \omega(B)^4 + O(d_N^{-\epsilon'_1}). \end{aligned}$$

Similar arguments show that

$$\begin{aligned} \mathbb{E}T_2 &= -\frac{4\omega(B)}{d_N^2} \sum_{k, j \in I_N} \sum_{\alpha, \beta, \eta} \langle BY_N^\alpha, Y_N^\alpha \rangle \langle BY_N^\beta, Y_N^\beta \rangle \langle BY_N^\eta, Y_N^\eta \rangle \\ &\quad - \frac{1}{d_N^3} (1 + \delta_{\alpha\beta} + \delta_{kj}(\delta_{\alpha\eta} + \delta_{\beta\eta} + 2\delta_{\alpha\beta}\delta_{\beta\eta})) + O(d_N^{-\epsilon'_2}) \\ &= -4\omega(B)^4 + O(d_N^{-\epsilon'_2}), \end{aligned}$$

$$\begin{aligned} \mathbb{E}T_3 &= \frac{2\omega(B)^2}{d_N^2} \sum_{k, j \in I_N} \sum_{\alpha, \beta} \langle BY_N^\alpha, Y_N^\alpha \rangle \langle BY_N^\beta, Y_N^\beta \rangle \frac{1}{d_N^2} (1 + \delta_{\alpha\beta}) + O(d_N^{-\epsilon'_3}) \\ &= 2\omega(B)^4 + O(d_N^{-\epsilon'_3}), \end{aligned}$$

$$\begin{aligned} \mathbb{E}T_4 &= \frac{4\omega(B)^2}{d_N^2} \sum_{k,j \in I_N} \sum_{\alpha,\eta} \langle BY_N^\alpha, Y_N^\alpha \rangle \langle BY_N^\eta, Y_N^\eta \rangle \frac{1}{d_N^2} (1 + \delta_{kj} \delta_{\alpha\eta}) + O(d_N^{-\varepsilon'_3}) \\ &= 4\omega(B)^4 + O(d_N^{-\varepsilon'_3}), \\ \mathbb{E}T_5 &= -\frac{4\omega(B)^3}{d_N^2} \sum_{k,j \in I_N} \sum_{\alpha} \langle BY_N^\alpha, Y_N^\alpha \rangle \frac{1}{d_N} + \frac{1}{d_N^2} \sum_{k,j \in I_N} \omega(B)^4 + O(d_N^{-\varepsilon'_4}) \\ &= -4\omega(B)^4 + \omega(B)^4 + O(d_N^{-\varepsilon'_4}). \end{aligned}$$

As before, the factors of $\omega(B)^4$ cancel exactly, and we are left with

$$\mathbb{E}Z_N^2 = \mathbb{E}T_1 + \dots + \mathbb{E}T_5 = O(d_N^{-\min\{\varepsilon'_1, \dots, \varepsilon'_4\}}) = O(d_N^{-\varepsilon'}).$$

This concludes the proof of Proposition 1. □

3. Proof of main theorems

We now return to Theorem 1 and Theorem 2, which do not have invariance assumptions on the operator $A \in \Psi^0(S^2)$. This means that we can no longer assume *a priori* (as we did in the previous section) that the matrix elements $\langle AY_N^\alpha, Y_N^\beta \rangle$ vanish for $\alpha \neq \beta$. We will show, however, that by taking a Fourier series representation of the operator A and using orthogonality properties of the spherical harmonics, the general case reduces to the rotationally invariant case.

3.1. Reduction to Fourier coefficients. The goal of this section is to obtain a Fourier series representation for a general pseudo-differential operator. Let r_θ denote rotation about the z -axis by angle θ , that is, if we write a point $x = (\cos \tau \sin \phi, \sin \tau \sin \phi, \cos \phi) \in S^2$ in spherical coordinates, then

$$r_\theta(x) := (\cos(\tau - \theta) \sin \phi, \sin(\tau - \theta) \sin \phi, \cos \phi).$$

Given $A \in \Psi^0(S^2)$, form a new operator

$$A_\theta := r_\theta^* A r_{-\theta}^* \in \Psi^0(S^2),$$

where $(r_\theta^* \varphi)(x) := \varphi(r_\theta(x))$ for any smooth function $\varphi \in C^\infty(S^2)$. For $n \in \mathbb{Z}$, the Fourier coefficients $\hat{A}(n)$ of A_θ are defined by

$$\hat{A}(n) := \int_{S^1} e^{-in\theta} A_\theta d\theta \in \Psi^0(S^2). \tag{19}$$

These new operators are related to the original operator A in the following way.

Lemma 3.1. *The partial sums $\sum_{|n| \leq N} \widehat{A}(n)$ converge in the operator norm to A as $N \rightarrow \infty$.*

Proof of Lemma 3.1. Let D_θ denote the generator of z-axis rotation so that $r_\theta^* = e^{-i\theta D_\theta}$. Then, since D_θ and r_θ^* commute, we have

$$\frac{\partial}{\partial \theta} A_\theta = \frac{1}{i}(D_\theta A_\theta - A_\theta D_\theta) = \frac{1}{i} \text{ad}_{D_\theta}(A_\theta) \in \Psi^0(M).$$

This implies that the map $\theta \mapsto A_\theta$ is differentiable, and by elementary properties of convolution with the Dirichlet kernel $D_N(\theta) = \sum_{n=-N}^N e^{in\theta}$ we get uniform convergence

$$\sum_{n=-N}^N \widehat{A}(n) = \sum_{n=-N}^N \int_{S^1} e^{-in\theta} A_\theta d\theta = \int_{S^1} D_N(\theta) A_\theta d\theta \longrightarrow A_0 = A. \quad \square$$

Lemma 3.2. *For $n \neq 0$, we have $\|\widehat{A}(n)\| = O(n^{-\ell})$ for every $\ell \geq 1$.*

Proof of Lemma 3.2. Integrating (19) by parts gives

$$n\widehat{A}(n) = \frac{i}{2\pi} e^{-in\theta} A_\theta \Big|_{\theta=0}^{2\pi} - \int_{S^1} e^{-in\theta} \text{ad}_{D_\theta}(A_\theta) d\theta = - \int_{S^1} e^{-in\theta} \text{ad}_{D_\theta}(A_\theta) d\theta.$$

It follows that integrating by parts ℓ times yields

$$(-n)^\ell \widehat{A}(n) = \int_{S^1} e^{-in\theta} (\text{ad}_{D_\theta})^\ell(A_\theta) d\theta.$$

Since $(\text{ad}_{D_\theta})^\ell(A_\theta) \in \Psi^0(S^2)$ for all $\ell \geq 1$, we conclude that $n^\ell \|\widehat{A}(n)\| = O(1)$. □

These lemmas allow us to replace A with finite sums of the form $\sum_{|n| \leq N} \widehat{A}(n)$. We record several facts about the operators $\widehat{A}(n)$. First, conjugating by rotation $A \mapsto r_\theta^* A r_{-\theta}^* = A_\theta$ changes the principal symbol of A by the canonical transformation on the cosphere bundle:

$$\sigma_{A_\theta}(x, \xi) = \sigma_A(r_\theta(x), (Dr_{-\theta}(x))^{-1}\xi).$$

From definition (19) of $\widehat{A}(n)$ we see

$$\begin{aligned} \omega(\widehat{A}(n)) &= \int_{S^*M} \int_{S^1} e^{-in\theta} \sigma_A(r_\theta(x), (Dr_{-\theta}(x))^{-1}\xi) d\theta d\mu_L \\ &= \begin{cases} \omega(A) & \text{if } n = 0, \\ 0 & \text{if } n \neq 0, \end{cases} \end{aligned} \tag{20}$$

with the second equality following from interchanging the order of integration and using the fact that the Liouville measure μ_L is invariant under canonical transformations.

Second, from the definition of spherical harmonics, for each fixed n the matrix elements of $\widehat{A}(n)$ are related to those of A by the identity

$$\langle \widehat{A}(n) Y_N^\alpha, Y_N^\beta \rangle = \begin{cases} \langle AY_N^\alpha, Y_N^{\alpha-n} \rangle & \text{if } \alpha = \beta + n \\ 0 & \text{if } \alpha \neq \beta + n \end{cases} \quad \text{simultaneously for all } N. \tag{21}$$

In other words, the infinite block-diagonal matrix with blocks of the form

$$(\langle \widehat{A}(n) Y_N^\alpha, Y_N^\beta \rangle)_{\alpha, \beta = -N}^N$$

is obtained from the infinite block diagonal matrix with blocks

$$(\langle AY_N^\alpha, Y_N^\beta \rangle)_{\alpha, \beta = -N}^N$$

by replacing all the entries except those on the n th diagonal above (or below, depending on the sign of n) the main diagonal by zeros.

3.2. Computations with Fourier coefficients. Having defined Fourier coefficients $\widehat{A}(n)$ and discussed their properties, we proceed to compute the expected value and second moment of the associated random variables

$$\begin{aligned} W_{n,N} &:= \frac{1}{d_N} \sum_{k \in I_N} |\langle \widehat{A}(n) \psi_{N,k}, \psi_{N,k} \rangle - \omega(\widehat{A}(n))|^2 \\ &= \begin{cases} \frac{1}{d_N} \sum_{k \in I_N} \left| \sum_{\alpha = -N+n}^N \langle AY_N^\alpha, Y_N^\alpha \rangle u_{N,k}(\alpha) \overline{u_{N,k}(\alpha)} - \omega(A) \right|^2 & \text{if } n = 0, \\ \frac{1}{d_N} \sum_{k \in I_N} \left| \sum_{\alpha = -N+n}^N \langle AY_N^\alpha, Y_N^{\alpha-n} \rangle u_{N,k}(\alpha) \overline{u_{N,k}(\alpha-n)} \right|^2 & \text{if } n \neq 0, \end{cases} \end{aligned}$$

where the second equality is obtained by first writing $\psi_{N,k}$ in terms of Y_N^α using (4), and then applying (20) and (21). We make the crucial observation that the discussion following (21) implies the identity

$$X_N = \sum_{n \in \mathbb{Z}} W_{n,N} \quad \text{for each } N = 0, 1, 2, \dots \tag{22}$$

The asymptotics for $\mathbb{E}W_{n,N}$ and $\mathbb{E}W_{n,N}^2$ can be easily computed.

Lemma 3.3. *For each fixed $n \in \mathbb{Z}$, we have $\mathbb{E}W_{n,N} = O(d_N^{-\varepsilon})$ and $\mathbb{E}W_{n,N}^2 = O(d_N^{-\varepsilon'})$ for some $\varepsilon, \varepsilon' > 0$ guaranteed by Theorem 1.1.*

Proof of Lemma 3.3. Thanks to (21), we recognize that $\widehat{A}(0)$ is a rotationally invariant operator of the kind considered in Section 2. Thus, when $n = 0$ the statement of the lemma follows from Proposition 1.

When $n \neq 0$, expanding the square yields

$$W_{n,N} = \frac{1}{d_N} \sum_{k \in I_N} \sum_{\alpha, \beta} \langle AY_N^\alpha, Y_N^{\alpha-n} \rangle \langle AY_N^\beta, Y_N^{\beta-n} \rangle u_{N,k}(\alpha) u_{N,k}(\beta) \overline{u_{N,k}(\alpha-n) u_{N,k}(\beta-n)}.$$

Appealing once again to the asymptotic formula (13), we find

$$\begin{aligned} & \mathbb{E}(u_{N,k}(\alpha) u_{N,k}(\beta) \overline{u_{N,k}(\alpha-n) u_{N,k}(\beta-n)}) \\ &= d_N^{-2} (\delta_{\alpha, \alpha-n} \delta_{\beta, \beta-n} + \delta_{\alpha, \beta-n} \delta_{\beta, \alpha-n}) + O(d_N^{-2-\varepsilon}). \end{aligned}$$

Since $n \neq 0$ by hypothesis, by what is now a standard argument we conclude that all the terms in the expression of $\mathbb{E}W_{n,N}$ that contain Kronecker delta functions are of order at most $O(d_N^{-1})$, so $\mathbb{E}W_{n,N} = O(d_N^{-\varepsilon})$.

The second moment computation is equally straightforward. Indeed, we have

$$\begin{aligned} W_{n,N}^2 = & \frac{1}{d_N^2} \sum_{k,j \in I_N} \sum_{\alpha, \beta, \eta, \xi} \langle AY_N^\alpha, Y_N^{\alpha-n} \rangle \langle AY_N^\beta, Y_N^{\beta-n} \rangle \langle AY_N^\eta, Y_N^{\eta-n} \rangle \langle AY_N^\xi, Y_N^{\xi-n} \rangle \\ & u_{N,k}(\alpha) u_{N,k}(\beta) \overline{u_{N,k}(\alpha-n) u_{N,k}(\beta-n)} \\ & u_{N,j}(\eta) u_{N,j}(\xi) \overline{u_{N,j}(\eta-n) u_{N,j}(\xi-n)}. \end{aligned}$$

It is easy to verify using (13) that the expected value of the product of eigenvector components is asymptotically zero because every term in the asymptotic formula contains a factor of $\delta_{\alpha, \alpha-n}$ for $n = 1, \dots, 4$. □

3.3. Approximation argument. We finish the computations for $\mathbb{E}X_N$ and $\mathbb{E}X_N^2$ by an approximation argument.

Proof of Theorem 2. Fix some small constant $\omega > 0$, then by (22) there exists $M > 0$ such that $\sum_{|n| > M} W_{n,N} < \omega$. Using Lemma 3.3 for the asymptotics of $\mathbb{E}W_{n,N}$ yields

$$\mathbb{E}X_N \leq \mathbb{E} \left(\sum_{|n| \leq M} W_{n,N} + \omega \right) = \sum_{|n| \leq M} \mathbb{E}W_{n,N} + \omega = O(d_N^{-\varepsilon}) + \omega.$$

The asymptotics for the second moment is similarly computed using the elementary inequality $(a_1 + \dots + a_m)^2 \leq m(a_1^2 + \dots + a_m^2)$ and Lemma 3.3:

$$\begin{aligned} \mathbb{E}X_N^2 &\leq \mathbb{E}\left(\sum_{|n|\leq M} W_{n,N} + \omega\right)^2 \\ &\leq (2M + 1) \sum_{|n|\leq M} \mathbb{E}W_{n,N}^2 + 2\omega \sum_{|n|\leq M} \mathbb{E}W_{n,N} + \omega^2 \\ &= O(d_N^{-\varepsilon'}) + O(d_N^{-\varepsilon}) + \omega^2. \end{aligned}$$

Since ω is arbitrary, Theorem 2 is proved with $\varepsilon_0 = \varepsilon$ and $\varepsilon'_0 = \min\{\varepsilon, \varepsilon'\}$. \square

Proof of Theorem 1. Let $\sigma_N^2 := \mathbb{E}X_N^2 - (\mathbb{E}X_N)^2$ be the variance of the random variable X_N . Theorem 1 shows that the sequence $\{X_N\}$ satisfies Kolmogorov's convergence criterion, that is, $\sum_{N=1}^\infty \sigma_N^2/N^2 < \infty$. We may therefore invoke the Strong Law of Large Numbers to conclude that the partial sums $\frac{1}{M} \sum_{N=0}^M X_N$ converge to its expected value almost surely. But $\mathbb{E}X_N = O(d_N^{-\varepsilon})$, which implies that the expected values of the partial sums converge to zero, finishing the proof of Theorem 1. \square

References

- [1] P. Bourgade and H.-T. Yau, The eigenvector moment flow and local quantum unique ergodicity. Preprint, 2013. [arXiv:1312.1301](#) [math.PR]
- [2] N. Burq and G. Lebeau, Injections de Sobolev probabilistes et applications. *Ann. Sci. Éc. Norm. Supér.* (4) **46** (2013), no. 6, 917–962. [MR 3134684 Zbl 1296.46031](#)
- [3] Y. Colin de Verdière, Ergodicité et fonctions propres du laplacien. *Comm. Math. Phys.* **102** (1985), no. 3, 497–502. [MR 818831 Zbl 0592.58050](#)
- [4] A. Knowles and J. Yin, Eigenvector distribution of Wigner matrices. *Probab. Theory Related Fields* **155** (2013), no. 3-4, 543–582. [MR 3034787 Zbl 1268.15033](#)
- [5] K. Maples, Quantum unique ergodicity for random bases of spectral projections. *Math. Res. Lett.* **20** (2013), no. 6, 1115–1124. [MR 3228625 Zbl 1298.58020](#)
- [6] A. I. Šnirel'man, Ergodic properties of eigenfunctions. *Uspehi Mat. Nauk* **29** (1974), no. 6(180), 181–182. In Russian. [MR 0402834 Zbl 0324.58020](#)
- [7] T. Tao and V. Vu, Random matrices: universal properties of eigenvectors. *Random Matrices Theory Appl.* **1** (2012), no. 1, 1150001. [MR 2930379 Zbl 1248.15031](#)
- [8] D. Weingarten, Asymptotic behavior of group integrals in the limit of infinite rank. *J. Mathematical Phys.* **19** (1978), no. 5, 999–1001. [MR 0471696 Zbl 0388.28013](#)
- [9] S. Zelditch, Uniform distribution of eigenfunctions on compact hyperbolic surfaces. *Duke Math. J.* **55** (1987), no. 4, 919–941. [MR 916129 Zbl 0643.58029](#)

- [10] S. Zelditch, Quantum ergodicity on the sphere. *Comm. Math. Phys.* **146** (1992), no. 1, 61–71. [MR 1163667](#) [Zbl 0759.58049](#)
- [11] S. Zelditch, A random matrix model for quantum mixing. *Internat. Math. Res. Notices* (1996), no. 3, 115–137. [MR 1383753](#) [Zbl 0858.58048](#)
- [12] S. Zelditch, Quantum ergodicity of random orthonormal bases of spaces of high dimension. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **372** (2014), no. 2007, 20120511. [MR 3151085](#) [Zbl 1322.81052](#)

Received December 12, 2015; revised March 30, 2016

Robert Chang, Department of Mathematics, Northwestern University,
2033 Sheridan Road, Evanston, IL 60208, USA

e-mail: hchang@math.northwestern.edu