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The gDMP inverse of Hilbert space operators

Dijana Mosić and Dragan S. Djordjević[1]

Abstract. We define a new generalized inverse (named the gDMP inverse) for a Hilbert space operator using its generalized Drazin inverse and its Moore-Penrose inverse. Thus, we extend the DMP inverse for a square matrix to more general case. Also, we introduce two new classes of operators, g-EP and g-normal operators which include, respectively, EP operators and normal operators. A new binary relation is associated with the gDMP inverse is presented and studied. The notion of core-EP inverse for matrices is extended to generalized Drazin invertible operators on Hilbert space.

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1. Introduction

Let X and Y be arbitrary Hilbert spaces. Denote by $\mathcal{B}(X,Y)$ the set of all bounded linear operators from X to Y. Set $\mathcal{B}(X) = \mathcal{B}(X,X)$. For an operator $A \in \mathcal{B}(X,Y)$, the symbols $N(A)$, $R(A)$, $\sigma(A)$, respectively, will denote the null space, the range and the spectrum of A.

If $A \in \mathcal{B}(X,Y)$ and there exists some $B \in \mathcal{B}(Y,X)$ such that $ABA = A$, then B is an inner generalized inverse of A and the operator A is relatively regular.

An operator $A \in \mathcal{B}(X)$ is called generalized Drazin invertible (or Koliha–Drazin invertible, or quasipolar), if there exists some $B \in \mathcal{B}(X)$ satisfying

$$BAB = B, \quad AB = BA, \quad A - A^2B \text{ is quasinilpotent.}$$

The generalized Drazin inverse B of A is unique and it is denoted by A^d, in the case when it exists (see [6, Theorem 7.5.3], [8]). The set of all generalized Drazin invertible operators of $\mathcal{B}(X)$ is denoted by $\mathcal{B}(X)^d$.

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The Drazin inverse is a special case of the generalized Drazin inverse for which $A - A^2B$ is nilpotent, and it is denoted by A^D [4]. The condition $A - A^2B$ is nilpotent is equivalent to $A^{k+1}B = A^k$, for some non-negative integer k. The smallest k such that $A^{k+1}B = A^k$ holds, is called the index of A and it is denoted by $\mathrm{ind}(A)$. If $\mathrm{ind}(A) \leq 1$, then A is group invertible and A^D is the group inverse of A denoted by $A^\#$.

If A is generalized Drazin invertible, then A^* is generalized Drazin invertible and $(A^*)^d = (A^d)^*$ [9, Lemma 1.3]. It is easy to see that if A is a quasinilpotent operator, then A^d exists and $A^d = 0$. The generalized Drazin inverse of A is in the double commutant of A, that is, for $C \in \mathcal{B}(X)$, $AC = CA$ implies $A^dC = CA^d$.

Recalled that, for $A \in \mathcal{B}(X)$, A^d exists if and only if $0 \notin \mathrm{acc}\,\sigma(A)$. If $A \in \mathcal{B}(X)$ is generalized Drazin invertible, then the spectral idempotent A^π of A corresponding to $\{0\}$ is given by $A^\pi = I - AA^d$. The operator matrix forms of A and A^d with respect to the space decomposition $X = N(A^\pi) \oplus R(A^\pi)$ are given by

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \quad \text{and} \quad A^d = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \tag{1}$$

where A_1 is invertible and A_2 is quasinilpotent. Notice that previous decompositions are not orthogonal. If we denote $C_A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}$ and $Q_A = \begin{bmatrix} 0 & 0 \\ 0 & A_2 \end{bmatrix}$, then $A = C_A + Q_A$ is known as the core-quasinilpotent decomposition of A. The operator C_A is called the core part of A and Q_A is called the quasinilpotent part of A. Notice that $C_A = A^2A^d$ is group invertible, $C_A^\# = A^d$, $Q_A = AA^\pi$ and $C_AQ_A = 0 = Q_AC_A$.

The Moore–Penrose inverse of $A \in \mathcal{B}(X, Y)$ is the operator $B \in \mathcal{B}(Y, X)$ which satisfies the Penrose equations

$$ABA = A, \quad BAB = B, \quad (AB)^* = AB, \quad (BA)^* = BA.$$

The Moore–Penrose inverse of A exists if and only if $R(A)$ is closed in Y. If the Moore–Penrose inverse of A exists, then it is unique, and it is denoted by $A^\dagger$.

Recall that an operator $A \in \mathcal{B}(X)$ is called the EP operator, if $R(A) = R(A^*)$. Also, if A is an EP operator, then $N(A) = N(A^*)$. If $A \in \mathcal{B}(X)$ has a closed range, then A is an EP operator if and only if $A^\dagger = A^\#$.

An operator $A \in \mathcal{B}(X, Y)$ with closed range satisfying $A^* = A^\dagger$ is called a partial isometry.

Recall that a binary relation on a set is called pre-order if satisfies reflexive and transitive properties, and it is called a partial order relation if satisfies reflexive, antisymmetric and transitive properties.

The star partial order was defined by Drazin [3]. Dolinar and Marovt [5] generalized the definition of the star partial order to $\mathcal{B}(X)$ and proved that the star order $(A \leq^* B)$ is a partial order on $\mathcal{B}(X)$. For $A, B \in \mathcal{B}(X)$,

$$A \leq^* B \iff (A^*A = A^*B \quad \text{and} \quad AA^* = BA^*).$$

If A has a closed range, than this order may be characterized as

$$A \leq^* B \iff (A^\dagger A = A^\dagger B \quad \text{and} \quad AA^\dagger = BA^\dagger).$$

Šemrl [16] extended the definition of the minus partial order to $\mathcal{B}(X)$. In [14], the minus partial order was defined only for relatively regular operators as: let $A, B \in \mathcal{B}(X, Y)$ be relatively regular, then A is said to be below B under the minus partial order $(A \leq^- B)$ if there exists an inner generalized inverse A^- of A such that

$$A^-A = A^-B \quad \text{and} \quad AA^- = BA^-.$$

The minus partial order is a partial order on the set of all relatively regular operators from $\mathcal{B}(X, Y)$.

For $A, B \in \mathcal{B}(X)$ such that $\operatorname{ind}(A) \leq 1$, the sharp order is defined by

$$A \leq^\# B \iff A^\# A = A^\# B \quad \text{and} \quad AA^\# = BA^\#.$$

The sharp order is a partial order on the set of operators $\{A \in \mathcal{B}(X) : \operatorname{ind}(A) \leq 1\}$.

The generalized Drazin pre-order as an extension of Drazin order for complex matrices [11], and it was defined in [12]. Let $A, B \in \mathcal{B}(X)$ be the generalized Drazin invertible such that $A = C_A + Q_A$ and $B = C_B + Q_B$ are the core-quasinilpotent decompositions of A and B respectively. The operator A is said to be below B under the generalized Drazin relation $(A \leq^d B)$ if $C_A \leq^\# C_B$. The generalized Drazin relation is a pre-order on $\mathcal{B}(X)^d$.

Theorem 1.1. [12] *Let $A, B \in \mathcal{B}(X)$ be generalized Drazin invertible such that $A = C_A + Q_A$ and $B = C_B + Q_B$ are the core-quasinilpotent decompositions of A and B respectively. Then $A \leq^d B$ if and only if*

$$A^dA = A^dB \quad \text{and} \quad AA^d = BA^d.$$

Malik and Thome [10] introduced a new generalized inverse called DMP inverse for a square matrix A of index m using the Drazin inverse A^D and the Moore-Penrose $A^\dagger$ of A as $A^{D,\dagger} = A^DAA^\dagger$. This generalized inverse extends the notion of the core inverse, presented by Baksalary and Trenkler in [1] while they necessarily require $m \leq 1$.

In [13], the core-EP inverse is introduced for a square matrix over an arbitrary field which. The matrix is not essentially of index one, so the core inverse is extended.

We define a new generalized inverse, the gDMP inverse for a generalized Drazin invertible operator $A \in \mathcal{B}(X)$ with a closed range using its generalized Drazin inverse and its Moore-Penrose inverse as an extension of the DMP inverse for a square matrix introduced in [10]. As a generalization of EP operators, we investigate g-EP operators. g-normal operators, which extend normal operators, are presented, and we study their relations with g-EP operators. Some properties of a new binary relation via the gDMP inverse are given with respect to the generalized Drazin pre-order and the star partial order. We present the core-EP inverse of an operator on Hilbert space as a generalization of core-EP inverse for matrix.

2. gDMP inverse

In this section, we introduce the gDMP inverse of a Hilbert space operator using its generalized Drazin inverse and its Moore–Penrose inverse.

First, we investigate a new generalized inverse from a geometrical point of view.

Theorem 2.1. *Let $A \in \mathcal{B}(X)$ be generalized Drazin invertible such that $R(A)$ is closed. The system of conditions*

$$AB = P_{R(AA^d A), N(A^d A^\dagger)}, \quad R(B) \subset R(AA^d), \tag{2}$$

is consistent and it has the unique solution $B = A^d AA^\dagger$.

Proof. We know that $AA^d AA^\dagger$ is a projector onto $R(AA^d A)$ along $N(A^d A^\dagger)$ and $R(A^d AA^\dagger) \subset R(AA^d)$. So, $B = A^d AA^\dagger$ satisfies conditions (2).

If two operators B_1 and B_2 satisfy conditions (2), then

$$A(B_1 - B_2) = P_{R(AA^d A), N(A^d A^\dagger)} - P_{R(AA^d A), N(A^d A^\dagger)} = 0.$$

So, $R(B_1 - B_2) \subset N(A) \subset N(A^d A)$. By $R(B_1) \subset R(AA^d)$ and $R(B_2) \subset R(AA^d)$, we conclude that $R(B_1 - B_2) \subset R(AA^d) \cap N(AA^d) = \{0\}$ implying $B_1 = B_2$. Hence, only one B satisfies (2). □

Definition 2.1. Let $A \in \mathcal{B}(X)$ be generalized Drazin invertible such that $R(A)$ is closed. The gDMP inverse of A is defined as

$$A^{d,\dagger} = A^d AA^\dagger.$$

Let $A \in \mathcal{B}(X)$ be generalized Drazin invertible such that $R(A)$ is closed. Consider the system of equations

$$BAB = B, \quad BA = A^d A. \tag{3}$$

This system of equations is obviously consistent, since $B = A^d AA^\dagger$ is one of its solutions.

Theorem 2.2. *If A is generalized Drazin invertible, $R(A)$ is closed, and B satisfies (3), then $A^2(B - A^\dagger)$ is quasinilpotent.*

Proof. Notice the following:

$$\sigma(A^2(B - A^\dagger)) \cup \{0\} = \sigma((A(B - A^\dagger)A) \cup \{0\} = \sigma(AA^d A - A) \cup \{0\} = \{0\},$$

since $AA^d A - A$ is quasinilpotent. □

The gDMP inverse of a Hilbert space operator can be seen as a generalization of the DMP inverse of a complex square matrix presented in [10] as an extension of generalized inverse introduce in [15] for matrices of index $m \leq 1$, and it is also an extension of the core–inverse [1].

Let the generalized Drazin invertible operator $A \in \mathcal{B}(X)$ have a closed range. By [2, Lemma 1.2], the operator A has the following matrix representation with respect to the orthogonal sums $X = R(A) \oplus N(A^*)$:

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(A) \\ N(A^*) \end{bmatrix} \longrightarrow \begin{bmatrix} R(A) \\ N(A^*) \end{bmatrix}, \tag{4}$$

where $D = A_1 A_1^* + A_2 A_2^*$ maps $R(A)$ into itself and $D > 0$ (meaning $D \geq 0$ invertible). Observe that A_1, A_2 and D are linear bounded operators. Also,

$$A^\dagger = \begin{bmatrix} A_1^* D^{-1} & 0 \\ A_2^* D^{-1} & 0 \end{bmatrix}. \tag{5}$$

Suppose that

$$A^d = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$$

is the generalized Drazin inverse of A. Since $A^d A = AA^d$ is equivalent to

$$B_1 A_1 = A_1 B_1 + A_2 B_3,$$

$$B_1 A_2 = A_1 B_2 + A_2 B_4,$$

$$B_3A_1 = 0,$$
$$B_3A_2 = 0,$$

then $A^dAA^d = A^d$ implies

$$B_3 = 0 = B_4,$$
$$B_1A_1B_1 = B_1,$$
$$B_1A_1B_2 = B_2.$$

So, $B_1A_1 = A_1B_1$ and $B_1A_2 = A_1B_2$. Now, it follows

$$B_2 = B_1A_1B_2 = B_1B_1A_2 = (B_1)^2A_2.$$

Because

$$A - A^2A^d = \begin{bmatrix} A_1 - A_1^2B_1 & A_2 - A_1^2B_2 \\ 0 & 0 \end{bmatrix}$$

is quasinilpotent and $\sigma(A_1 - A_1^2B_1) \subset \sigma(A - A^2B) \cup \{0\} = \{0\}$, we deduce that $A_1 - A_1^2B_1$ is quasinilpotent. Hence, $B_1 = A_1^d$ and

$$A^d = \begin{bmatrix} A_1^d & (A_1^d)^2A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(A) \\ N(A^*) \end{bmatrix} \longrightarrow \begin{bmatrix} R(A) \\ N(A^*) \end{bmatrix}. \quad (6)$$

The gDMP inverse of A is given by

$$A^{d,\dagger} = A^dAA^\dagger = \begin{bmatrix} A_1^d & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(A) \\ N(A^*) \end{bmatrix} \longrightarrow \begin{bmatrix} R(A) \\ N(A^*) \end{bmatrix}.$$

So, we just proved the following theorem.

Theorem 2.3. *Let $A \in \mathcal{B}(X)$ be the generalized Drazin invertible such that $R(A)$ is closed and A is written as in (4). Then*

$$A^{d,\dagger} = \begin{bmatrix} A_1^d & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(A) \\ N(A^*) \end{bmatrix} \longrightarrow \begin{bmatrix} R(A) \\ N(A^*) \end{bmatrix}. \quad (7)$$

The canonical form for the DMP inverse of a square matrix was present in [10] using the Hartwig–Spindelböck decomposition [7] which is a powerful tool to investigate various classes of complex square matrices. We use the matrix form of a linear bounded operator (4) which is induced by some natural decompositions of Hilbert spaces.

Notice that we can define the outer inverse $A^{\dagger,d} = A^\dagger AA^d$ of the operator A given by (4) and it has the following decomposition:

$$A^{\dagger,d} = \begin{bmatrix} A_1^*D^{-1}A_1A_1^d & A_1^*D^{-1}A_1^dA_2 \\ A_2^*D^{-1}A_1A_1^d & A_2^*D^{-1}A_1^dA_2 \end{bmatrix}.$$

Some properties of the gDMP inverse are given in the next result.

Theorem 2.4. *Let $A \in \mathcal{B}(X)$ be generalized Drazin invertible such that $R(A)$ is closed. The following statements hold*:

(a) $AA^{d,\dagger}$ *is a projector onto* $R(AA^dA)$ *along* $N(A^dA^\dagger)$.

(b) $A^{d,\dagger}A = A^dA$ *is a projector onto* $R(A^dA)$ *along* $N(A^dA)$.

We also prove the following result.

Theorem 2.5. *Let $A \in \mathcal{B}(X)$ be generalized Drazin invertible such that $R(A)$ is closed. Then $A^{d,\dagger}A = AA^{d,\dagger}$ if and only if $A^{d,\dagger} = A^d$.*

Proof. Notice that, by (6) and (7),

$$A^{d,\dagger}A = AA^{d,\dagger} \iff A_1^dA_2 = 0 \iff (A_1^d)^2A_2 = 0 \iff A^{d,\dagger} = A^d. \qquad \square$$

3. g-EP operators

We define g-EP operators as an extension of EP operators. In this section we investigate properties of g-EP operators.

Definition 3.1. Let $A \in \mathcal{B}(X)$ be generalized Drazin invertible such that $R(A)$ is closed. The operator A is called g-EP if the following holds:

$$\lim_{m\to\infty}(A^\dagger A^{m+1}A^d - A^{m+1}A^dA^\dagger) = 0.$$

Observe that, an operator A is g-EP if and only if A^* is g-EP. Obviously, if A is an EP operator, then A is g-EP. Also, any invertible operator is g-EP. Clearly, any quasinilpotent operator with the closed range is g-EP operator.

Theorem 3.1. *Let $A \in \mathcal{B}(X)$ be generalized Drazin invertible such that $R(A)$ is closed. If the notations from (4) are retained, then A is g-EP if and only if the following conditions hold*:

(i) $\lim_{m\to\infty}(A_1^*D^{-1}A_1^{m+1}A_1^d - A_1^mA_1^d) = 0,$

(ii) $\lim_{m\to\infty} A_2^*D^{-1}A_1^{m+1}A_1^d = 0,$

(iii) $\lim_{m\to\infty} A_1^mA_1^dA_2 = 0.$

Proof. Since the decomposition $X = R(A) \oplus N(B^*)$ is orthogonal, we use the fact that a sequence of block matrices converges if and only if every sequence of entries converges.

If A is written as in (4), then $A^\dagger$ and A^d are written as in (5) and (6), respectively. Also, we have

$$A^{m+1} = \begin{bmatrix} A_1^{m+1} & A_1^m A_2 \\ 0 & 0 \end{bmatrix}.$$

Now, $\lim_{m\to\infty} (A^\dagger A^{m+1} A^d - A^{m+1} A^d A^\dagger) = 0$ is equivalent to each of the following:

$$\lim_{m\to\infty} (A_1^* D^{-1} A_1^{m+1} A_1^d - A_1^m A_1^d) = 0,$$

$$\lim_{m\to\infty} A_1^* D^{-1} A_1^m A_1^d A_2 = 0,$$

$$\lim_{m\to\infty} A_2^* D^{-1} A_1^{m+1} A_1^d = 0,$$

$$\lim_{m\to\infty} A_2^* D^{-1} A_1^m A_1^d A_2 = 0.$$

We only need to prove that the second and the forth equation are equivalent to (iii). By

$$\begin{aligned} \|A_1^m A_1^d A_2\| &= \|D D^{-1} A_1^m A_1^d A_2\| \\ &\le \|A_1 A_1^* D^{-1} A_1^m A_1^d A_2\| + \|A_2 A_2^* D^{-1} A_1^m A_1^d A_2\| \\ &\le \|A_1\| \|A_1^* D^{-1} A_1^m A_1^d A_2\| + \|A_2\| \|A_2^* D^{-1} A_1^m A_1^d A_2\|, \end{aligned}$$

the second and the forth equation imply (iii). From $\lim_{m\to\infty} A_1^m A_1^d A_2 = 0$ and

$$\|A_1^* D^{-1} A_1^m A_1^d A_2\| \le \|A_1^* D^{-1}\| \|A_1^m A_1^d A_2\|,$$

we get the second equation. In the same way we obtain that (iii) implies the forth equation. □

Now, we investigate some properties of g-EP operators.

Theorem 3.2. *Let $A \in \mathcal{B}(X)$ be generalized Drazin invertible such that $R(A)$ is closed. Then A is g-EP if and only if*

$$\lim_{m\to\infty} (A^\dagger A^{m+2} A^d - A^{m+1} A^d) = 0 \quad \textit{and} \quad \lim_{m\to\infty} (A^{m+2} A^d A^\dagger - A^{m+1} A^d) = 0.$$

Proof. Assume that A is g-EP. By

$$\lim_{m\to\infty}(A^{\dagger}A^{m+1}A^{d} - A^{m+1}A^{d}A^{\dagger}) = 0,$$

we get

$$A^{\dagger}A^{m+2}A^{d} - A^{m+1}A^{d} = (A^{\dagger}A^{m+1}A^{d} - A^{m+1}A^{d}A^{\dagger})A$$

and

$$A^{m+2}A^{d}A^{\dagger} - A^{m+1}A^{d} = A(A^{m+1}A^{d}A^{\dagger} - A^{\dagger}A^{m+1}A^{d}),$$

we have

$$\lim_{m\to\infty}(A^{\dagger}A^{m+2}A^{d} - A^{m+1}A^{d}) = 0$$

and

$$\lim_{m\to\infty}(A^{m+2}A^{d}A^{\dagger} - A^{m+1}A^{d}) = 0.$$

If

$$\lim_{m\to\infty}(A^{\dagger}A^{m+2}A^{d} - A^{m+1}A^{d}) = 0$$

and

$$\lim_{m\to\infty}(A^{m+2}A^{d}A^{\dagger} - A^{m+1}A^{d}) = 0,$$

then we get

$$\lim_{m\to\infty}(A^{\dagger}A^{m+2}A^{d} - A^{m+2}A^{d}A^{\dagger}) = 0,$$

which implies that A is g-EP. □

Theorem 3.3. *Let $A \in \mathcal{B}(X)$ be g-EP and a partial isometry. Then*

$$\lim_{m\to\infty}((A^{\dagger})^{d}(A^{\dagger})^{m+1}A - A(A^{\dagger})^{m+1}(A^{\dagger})^{d}) = 0.$$

Proof. Since A is g-EP, then

$$\lim_{m\to\infty}(A^{\dagger}A^{m+1}A^{d} - A^{m+1}A^{d}A^{\dagger})^{*} = 0.$$

From $A^{*} = A^{\dagger}$, we get

$$\begin{aligned}(A^{\dagger}A^{m+1}A^{d} - A^{m+1}A^{d}A^{\dagger})^{*} &= (A^{d})^{*}(A^{m})^{*}A^{\dagger}A - AA^{\dagger}(A^{m})^{*}(A^{d})^{*}\\ &= (A^{\dagger})^{d}(A^{\dagger})^{m+1}A - A(A^{\dagger})^{m+1}(A^{\dagger})^{d}.\end{aligned}$$

Thus, the proof is completed. □

4. g–normal operators

In this section, we present g–normal operators and study their relation with g-EP operators.

Definition 4.1. Let $A \in \mathcal{B}(X)$ be generalized Drazin invertible. The operator A is called g–normal if the following holds:

$$\lim_{m\to\infty}(A^*A^{m+1}A^d - A^{m+1}A^dA^*) = 0.$$

We prove the next result.

Theorem 4.1. *Let $A \in \mathcal{B}(X)$ be generalized Drazin invertible such that $R(A)$ is closed. If we retain the notations from (4), then A is g–normal if and only if the following conditions hold*:

(i) $\lim_{m\to\infty}(A_1^*A_1^{m+1}A_1^d - A_1^mA_1^dD) = 0$,

(ii) $\lim_{m\to\infty} A_2^*A_1^{m+1}A_1^d = 0$,

(iii) $\lim_{m\to\infty} A_1^mA_1^dA_2 = 0$.

Proof. In the similar way as in the proof of Theorem 3.1, we can show this theorem. □

We prove that under some conditions g–normal and g-EP operators coincide in the following theorem.

Theorem 4.2. *Let $A \in \mathcal{B}(X)$ be the generalized Drazin invertible such that $R(A)$ is closed. If we retain the notations from (4), then A is g–normal if and only if A is g-EP and*

$$\lim_{m\to\infty}(DA_1^{m+1}A_1^d - A_1^{m+1}A_1^dD) = 0.$$

Proof. Suppose that A is g–normal. By Theorem 4.1,

$$\lim_{m\to\infty}(A_1^*A_1^{m+1}A_1^d - A_1^mA_1^dD) = 0$$

and

$$\lim_{m\to\infty} A_2^*A_1^{m+1}A_1^d = 0.$$

From

$$\begin{aligned}\|DA_1^{m+1}A_1^d - A_1^{m+1}A_1^d D\| &\leq \|A_1A_1^*A_1^{m+1}A_1^d - A_1^{m+1}A_1^d D\| \\ &\quad + \|A_2A_2^*A_1^{m+1}A_1^d\| \\ &\leq \|A_1\|\|A_1^*A_1^{m+1}A_1^d - A_1^m A_1^d D\| \\ &\quad + \|A_2\|\|A_2^*A_1^{m+1}A_1^d\|,\end{aligned}$$

we obtain

$$\lim_{m\to\infty}(DA_1^{m+1}A_1^d - A_1^{m+1}A_1^d D) = 0. \tag{8}$$

Since

$$\|A_2^*A_1^{m+1}A_1^d - A_2^*D^{-1}A_1^{m+1}A_1^d D\| \leq \|A_2^*D^{-1}\|\|DA_1^{m+1}A_1^d - A_1^{m+1}A_1^d D\|,$$

by (8),

$$\lim_{m\to\infty}(A_2^*A_1^{m+1}A_1^d - A_2^*D^{-1}A_1^{m+1}A_1^d D) = 0.$$

Now, using

$$\|A_2^*D^{-1}A_1^{m+1}A_1^d D\| \leq \|A_2^*A_1^{m+1}A_1^d\| + \|A_2^*A_1^{m+1}A_1^d - A_2^*D^{-1}A_1^{m+1}A_1^d D\|,$$

we deduce that

$$\lim_{m\to\infty} A_2^*D^{-1}A_1^{m+1}A_1^d D = 0$$

which yields the condition (ii) of Theorem 3.1:

$$\lim_{m\to\infty} A_2^*D^{-1}A_1^{m+1}A_1^d = 0.$$

The equalities (8) and

$$\begin{aligned}&A_1^*A_1^{m+1}A_1^d - A_1^m A_1^d D - (A_1^*D^{-1}A_1^{m+1}A_1^d D - A_1^m A_1^d D) \\ &\quad = A_1^*D^{-1}(DA_1^{m+1}A_1^d - A_1^{m+1}A_1^d D)\end{aligned}$$

imply

$$\lim_{m\to\infty}[A_1^*A_1^{m+1}A_1^d - A_1^m A_1^d D - (A_1^*D^{-1}A_1^{m+1}A_1^d D - A_1^m A_1^d D)] = 0.$$

We can check that

$$\lim_{m\to\infty}(A_1^*D^{-1}A_1^{m+1}A_1^d D - A_1^m A_1^d D) = 0$$

which gives condition (i) of Theorem 3.1:

$$\lim_{m\to\infty}(A_1^*D^{-1}A_1^{m+1}A_1^d - A_1^m A_1^d) = 0.$$

So, by Theorem 3.1, we conclude that A is g-EP.

If A is g-EP and

$$\lim_{m\to\infty}(DA_1^{m+1}A_1^d - A_1^{m+1}A_1^d D) = 0,$$

by Theorem 3.1,

$$\lim_{m\to\infty}(A_1^* D^{-1} A_1^{m+1}A_1^d - A_1^m A_1^d) = 0$$

and

$$\lim_{m\to\infty} A_2^* D^{-1} A_1^{m+1} A_1^d = 0.$$

Then, by

$$\begin{aligned}\|A_2^* A_1^{m+1} A_1^d\| &\le \|A_2^* D^{-1} D A_1^{m+1} A_1^d - A_2^* D^{-1} A_1^{m+1} A_1^d D\| \\ &\quad + \|A_2^* D^{-1} A_1^{m+1} A_1^d D\| \\ &\le \|A_2^* D^{-1}\| \|DA_1^{m+1}A_1^d - A_1^{m+1}A_1^d D\| \\ &\quad + \|A_2^* D^{-1} A_1^{m+1} A_1^d\| \|D\|,\end{aligned}$$

we have that the condition (ii) of Theorem 4.1 $\lim\limits_{m\to\infty} A_2^* A_1^{m+1} A_1^d = 0$ holds.

Observe that, from

$$\begin{aligned}&\|A_1^* A_1^{m+1} A_1^d - A_1^m A_1^d D - (A_1^* D^{-1} A_1^{m+1} A_1^d D - A_1^m A_1^d D)\| \\ &\quad \le \|A_1^* D^{-1} D A_1^{m+1} A_1^d - A_1^* D^{-1} A_1^{m+1} A_1^d D\| \\ &\quad \le \|A_1^* D^{-1}\| \|DA_1^{m+1} A_1^d - A_1^{m+1} A_1^d D\|,\end{aligned}$$

it follows

$$\lim_{m\to\infty}[A_1^* A_1^{m+1} A_1^d - A_1^m A_1^d D - (A_1^* D^{-1} A_1^{m+1} A_1^d D - A_1^m A_1^d D)] = 0.$$

Further, by

$$\begin{aligned}&\|A_1^* A_1^{m+1} A_1^d - A_1^m A_1^d D\| \\ &\quad \le \|A_1^* A_1^{m+1} A_1^d - A_1^m A_1^d D - (A_1^* D^{-1} A_1^{m+1} A_1^d D - A_1^m A_1^d D)\| \\ &\qquad + \|A_1^* D^{-1} A_1^{m+1} A_1^d - A_1^m A_1^d\| \|D\|,\end{aligned}$$

notice that the condition (i) of Theorem 4.1

$$\lim_{m\to\infty}(A_1^* A_1^{m+1} A_1^d - A_1^m A_1^d D) = 0$$

is satisfied. Using Theorem 4.1, we deduce that A is g–normal. □

Notice that a partial isometry is a g-EP operator if and only if it is g–normal operator.

Next, we verify that a g–normal operator is also g-EP in general.

Corollary 4.1. *Let $A \in \mathcal{B}(X)$ be generalized Drazin invertible such that $R(A)$ is closed. If A is g–normal, then A is g-EP.*

Proof. If A is g–normal, by definition,

$$\lim_{m\to\infty}(A^*A^{m+1}A^d - A^{m+1}A^dA^*) = 0,$$

which gives

$$\lim_{m\to\infty} A^\dagger A(A^*A^{m+1}A^d - A^{m+1}A^dA^*)(A^\dagger)^* = 0$$

and

$$\lim_{m\to\infty}(A^*A^{m+1}A^d - A^{m+1}A^dA^*)(A^\dagger)^* = 0.$$

Thus,

$$\lim_{m\to\infty}(A^*A^{m+1}A^d(A^\dagger)^* - A^\dagger A^{m+2}A^d) = 0$$

and

$$\lim_{m\to\infty}(A^*A^{m+1}A^d(A^\dagger)^* - A^{m+1}A^d) = 0.$$

Then, from

$$\begin{aligned}\|A^\dagger A^{m+2}A^d - A^{m+1}A^d\| \le \|A^\dagger A^{m+2}A^d - A^*A^{m+1}A^d(A^\dagger)^*\| \\ + \|A^*A^{m+1}A^d(A^\dagger)^* - A^{m+1}A^d\|,\end{aligned}$$

we deduce that

$$\lim_{m\to\infty}(A^\dagger A^{m+2}A^d - A^{m+1}A^d) = 0.$$

Similarly, we can prove that

$$\lim_{m\to\infty}(A^{m+2}A^dA^\dagger - A^{m+1}A^d) = 0.$$

Hence, by Theorem 3.2, A is g-EP. □

5. Binary relation via gDMP inverse

We introduce a new binary relation associated with the gDMP inverse.

Definition 5.1. Let $A, B \in \mathcal{B}(X)$ and let $A^{d,\dagger}$ be the gDMP inverse of A. Then $A \leq^{d,\dagger} B$ if

$$AA^{d,\dagger} = BA^{d,\dagger} \quad \text{and} \quad A^{d,\dagger}A = A^{d,\dagger}B.$$

Obviously, the relation "$\leq^{d,\dagger}$" is reflexive, but this relation is not transitive as we will see in the next example. Thus, this relation is neither a pre-order nor a partial order on $\mathcal{B}(X)$.

Example 5.1. Consider complex 3×3 matrices

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $A^d = A^\# = A$,

$$A^\dagger = \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}, \quad A^{d,\dagger} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$B^d = B^D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}, \quad B^\dagger = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}, \quad B^{d,\dagger} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Since $A^{d,\dagger}A = A = A^{d,\dagger}B$, $AA^{d,\dagger} = A^{d,\dagger} = BA^{d,\dagger}$, $CB^{d,\dagger} = B^{d,\dagger} = BB^{d,\dagger}$, $B^{d,\dagger}C = B^D = B^{d,\dagger}B$ and $A^{d,\dagger}A = A \neq A^{d,\dagger} = A^{d,\dagger}C$, we deduce that $A \leq^{d,\dagger} B$, $B \leq^{d,\dagger} C$ and $A \not\leq^{d,\dagger} C$.

The relation between the "$\leq^{d,\dagger}$" and the generalized Drazin pre-order are given in the following theorems.

Corollary 5.1. *Let $A, B \in \mathcal{B}(X)$ such that A has a closed range, and (using notations from (4)) $(A_1^d)^2 A_2 = 0$. Then $A \leq^{d,\dagger} B$ if and only if $A \leq^d B$.*

Proof. The hypothesis $(A_1^d)^2 A_2 = 0$ gives $A^{d,\dagger} = A^d$. $\square$

Theorem 5.1. *Let $A \in \mathcal{B}(X)$ be the generalized Drazin invertible such that $R(A)$ is closed and $N(A^*) \subset N(A^d)$. If $B \in \mathcal{B}(X)$, then $A \leq^{d,\dagger} B$ if and only if $A \leq^d B$.*

Proof. From $R(I - AA^\dagger) = N(A^*) \subset N(A^d)$, we obtain $A^{d,\dagger} = A^d AA^\dagger = A^d$. □

Theorem 5.2. *Let $A, B \in \mathcal{B}(X)$ such that A is g-EP and $\|A^d\| \leq 1$. Then $A \leq^{d,\dagger} B$ if and only if $A \leq^d B$.*

Proof. Suppose that $A \leq^{d,\dagger} B$. From $AA^{d,\dagger} = BA^{d,\dagger}$, i.e. $AA^d AA^\dagger = BA^d AA^\dagger$, note that

$$AA^d = AA^d AA^\dagger AA^d = BA^d AA^\dagger AA^d = BA^d.$$

Since $A^{d,\dagger} A = A^{d,\dagger} B$, then $A^d A = A^d AA^\dagger B$ which gives

$$\begin{aligned} A^d A - A^d B &= (A^d)^{m+1} A^{m+1} - (A^d)^{m+1} A^m B \\ &= (A^d)^m [A^m AA^d - A^\dagger A^{m+1} A^d B] \\ &= (A^d)^m [A^m A^d AA^\dagger - A^\dagger A^{m+1} A^d] B. \end{aligned}$$

As A is *g-EP*, we get

$$\|A^d A - A^d B\| \leq \|(A^d)^m\| \|A^{m+1} A^d A^\dagger - A^\dagger A^{m+1} A^d\| \|B\| \longrightarrow 0,$$

when $m \to \infty$. Hence, $A^d A = A^d B$ implying $A \leq^d B$.

If $A \leq^d B$, then $A^d A = A^d B = BA^d$. We now get

$$AA^{d,\dagger} = AA^d AA^\dagger = BA^d AA^\dagger = BA^{d,\dagger}.$$

Also, we have $A^{m+2} A^d = A^{m+1} A^d A = A^{m+1} A^d B$ which yields

$$\begin{aligned} A^d AA^\dagger A - A^d AA^\dagger B &= (A^d)^{m+1} [A^{m+2} A^d - A^{m+2} A^d A^\dagger B] \\ &= (A^d)^{m+1} [A^{m+1} A^d - A^{m+2} A^d A^\dagger] B. \end{aligned}$$

Because A is *g-EP*, by Theorem 3.2,

$$\|A^{d,\dagger} A - A^{d,\dagger} B\| \leq \|(A^d)^{m+1}\| \|A^{m+1} A^d - A^{m+2} A^d A^\dagger\| \|B\| \longrightarrow 0,$$

when $m \to \infty$. Thus, $A^{d,\dagger} A = A^{d,\dagger} B$ and $A \leq^{d,\dagger} B$. □

Remark. By the proof of Theorem 5.2, observe that for a generalized Drazin invertible $A \in \mathcal{B}(X)$ such that $R(A)$ is closed and for $B \in \mathcal{B}(X)$ the following statements hold:

(a) If $A \leq^{d,\dagger} B$, then $AA^d = BA^d$;

(b) if $A \leq^{d} B$, then $AA^{d,\dagger} = BA^{d,\dagger}$;

(c) if $AA^d = BA^d$ and $A^\dagger B = A^\dagger A$, then $A \leq^{d,\dagger} B$;

(d) if $A \leq^{d,\dagger} B$ and the implication $A^\dagger(A-B)A^d = 0 \implies A^\dagger(A-B) = 0$ is satisfied, then $A^\dagger B = A^\dagger A$;

(e) if $A \leq^{*} B$, then $A^{d,\dagger}A = A^{d,\dagger}B$;

(f) if $A \leq^{*} B$ and $AA^dB = BAA^d$, then $A \leq^{d,\dagger} B$.

By Theorem 5.2, we see that the following corollaries hold.

Corollary 5.2. *The relation "$\leq^{d,\dagger}$" is a pre-order on the set of operators*

$$\{A \text{ is } g\text{-}EP\colon \|A^d\| \leq 1\}.$$

Corollary 5.3. *The relation "$\leq^{d,\dagger}$" is a pre-order on the set of operators*

$$\{A \text{ is } g\text{-}normal\colon \|A^d\| \leq 1\}.$$

Now, we consider the relation between the "$\leq^{d,\dagger}$" and the star partial order.

Theorem 5.3. *Let $A, B \in \mathcal{B}(X)$ such that A is generalized Drazin invertible with a closed range. If the notations from (4) are retained, $A_1^d = A_1^* D^{-1}$ and $A_2 = 0$, then $A \leq^{d,\dagger} B$ if and only if $A \leq^{*} B$.*

Proof. From $A_1^d = A_1^* D^{-1}$ and $A_2 = 0$, we conclude that $A^{d,\dagger} = A^\dagger$. □

Theorem 5.4. *Let $A \in \mathcal{B}(X)$ be generalized Drazin invertible such that $R(A)$ is closed and $R(A^*) \subset R(A^d)$. If $B \in \mathcal{B}(X)$, then $A \leq^{d,\dagger} B$ if and only if $A \leq^{*} B$.*

Proof. By $R(A^\dagger) = R(A^*) \subset R(A^d) = R(A^dA)$ and A^dA is a projector, we have $A^{d,\dagger} = A^dAA^\dagger = A^\dagger$. □

As a consequence, we have the next result.

Corollary 5.4. *The relation "$\leq^{d,\dagger}$" is a partial order on the set of operators*

$$\{A \in \mathcal{B}(X)^d\colon R(A) \text{ is closed and } R(A^*) \subset R(A^d)\}.$$

6. Core-EP inverse

In this section, the core-EP inverse and the *core-EP inverse are presented for generalized Drazin invertible operators extending the core-EP inverse and the *core-EP inverse, respectively, which are defined in [13] for matrices.

Definition 6.1. Let $A \in \mathcal{B}(X)$ be generalized Drazin invertible. An operator $B \in \mathcal{B}(X)$ is a core-EP inverse of A if

$$BAB = B, \quad R(B) = R(B^*) = R(AA^d).$$

Definition 6.2. Let $A \in \mathcal{B}(X)$ be the generalized Drazin invertible. An operator $B \in \mathcal{B}(X)$ is a *core-EP inverse of A if

$$BAB = B, \quad R(B) = R(B^*) = R((AA^d)^*).$$

We characterize the core-EP inverse of operators in the following theorem.

Theorem 6.1. *Let $A \in \mathcal{B}(X)$ be generalized Drazin invertible. Then $B \in \mathcal{B}(X)$ is a core-EP inverse of A if and only if*

$$BAB = B, \quad (AB)^* = AB, \quad (BA - I)AA^d = 0, \quad R(B) \subset R(AA^d).$$

Proof. Suppose that $B \in \mathcal{B}(X)$ is a core-EP inverse of A. Since $BAB = B$, we obtain $R(BA) = R(B) = R(AA^d)$ which yields $BAAA^d = AA^d$. Also, we have

$$R((AB)^*) = R(B^*A^*) = R(B^*) = R(AA^d)$$

and

$$R(AB) = AR(B) = AR(AA^d) = R(A^2A^d) = R(AA^d).$$

Hence, $R((AB)^*) = R(AB)$, that is, idempotent AB is an EP operator. So, $(AB)^* = AB$.

Conversely, by $BAAA^d = AA^d$ and $R(B) \subset R(AA^d)$, we deduce that $R(B) \subset R(AA^d) \subset R(B)$, i.e. $R(B) = R(AA^d)$. The assumptions $BAB = B$ and $(AB)^* = AB$ give

$$R(B^*) = R(B^*A^*) = R(AB) = AR(B) = R(A^2A^d) = R(AA^d).$$

Thus, B is a core-EP inverse of A. □

Notice that, if A is group invertible in Theorem 6.1, then we obtain $BA^2 = A$ and also $ABA = A$.

In the similar way as in the proof of Theorem 6.1, we can verify the next result.

Theorem 6.2. *Let $A \in \mathcal{B}(X)$ be generalized Drazin invertible. Then $B \in \mathcal{B}(X)$ is a *core-EP inverse of A if and only if*

$$BAB = B, \quad (BA)^* = BA, \quad AA^d(AB - I) = 0, \quad R(B^*) \subset R((AA^d)^*).$$

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Dijana Mosić, Faculty of Sciences and Mathematics, University of Niš, P.O. Box 224, 18000 Niš, Serbia

e-mail: dijana@pmf.ni.ac.rs

Dragan S. Djordjević, Faculty of Sciences and Mathematics, University of Niš, P.O. Box 224, 18000 Niš, Serbia

e-mail: dragan@pmf.ni.ac.rs