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# On surjectivity and denseness of range of the operator $A + CX$

Dragana S. Cvetković-Ilić[1] and Vladimir Pavlović[2]

**Abstract.** Motivated by a result of Takahashi from *Invertible completions of operator matrices*, Integr. Equ. Oper. Theory 21 (1995) 355–361, in this paper we investigate the problems of characterization of all the pairs of operators $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and $C \in \mathcal{B}(\mathcal{L},\mathcal{K})$ for which there exists some $X \in \mathcal{B}(\mathcal{H},\mathcal{L})$ such that the operator $A + CX$ is surjective/with dense range, for some $X \in \mathcal{B}(\mathcal{H},\mathcal{K})$. We completely solve the former and give some partial results regarding the latter one.

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## 1. Introduction

Let $\mathcal{H}$, $\mathcal{K}$ be separable Hilbert spaces and let $\mathcal{B}(\mathcal{H},\mathcal{K})$ denote the set of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. For simplicity, we also write $\mathcal{B}(\mathcal{H},\mathcal{H})$ as $\mathcal{B}(\mathcal{H})$.

One of the topics of many various currently undergoing investigations and extensively studied problems of operator theory are the so called completion problems of partially given operator matrices (see [1]–[10], [12], [15]–[18]). As a particular instance of this problem, the following question can be found to be of interest: if $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{K})$ and $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ are fixed, does there exist an operator $X \in \mathcal{B}(\mathcal{H},\mathcal{K})$ such that the operator $M_X \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ given by

$$M_X = \begin{bmatrix} A & C \\ X & B \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix},$$

is of some fixed prescribed type?

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The first to ever address such a question (for separable Hilbert spaces not necessarily of finite dimension) was Takahashi. More specifically, in his paper [16] he gave necessary and sufficient conditions for the existence of $X \in \mathcal{B}(\mathcal{H})$ such that $M_X$ is invertible.

The key result obtained in [16] that allowed him to completely solve the problem of completion of $M_X$ to invertibility, was the one that characterizes the pairs of operators $(A, C)$, where $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $C \in \mathcal{B}(\mathcal{L}, \mathcal{K})$, for which the operator $A + CX$ is invertible for some $X \in \mathcal{B}(\mathcal{H}, \mathcal{L})$, which is a result also of interest in connection with the spectrum assignment problem in systems theory. Motivated by this result, in [6] we investigated the similar problem of characterization of all the pairs of operators $(A, C)$ for which the operator $A + CX$ is injective for some $X \in \mathcal{B}(\mathcal{H}, \mathcal{L})$, and completely solved it. The following two questions present themselves as a rather natural continuation of our research: for which operators $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $C \in \mathcal{B}(\mathcal{L}, \mathcal{K})$ does there exist $X \in \mathcal{B}(\mathcal{H}, \mathcal{L})$ such that the operator $A + CX$ is surjective/with dense range? In this paper we will completely answer the former and give some partial answers to the latter one.

## 2. Preliminaries

All Hilbert spaces under consideration in this paper are assumed to be separable. For a given $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, the symbols $\mathcal{N}(A)$ and $\mathcal{R}(A)$ denote the null space and the range of $A$, respectively. Let $n(A) = \dim \mathcal{N}(A)$, $\beta(A) = \text{codim}\mathcal{R}(A)$ and $d(A) = dim\mathcal{R}(A)^{\perp}$. For subspaces $K, L, M \subseteq \mathcal{H}$, by $K \oplus L = M$ we will denote the fact that $K + L = M$ and $K \cap L = \{0\}$, i.e. that the sum is direct.

In this paper by an *operator range* we shall mean a subspace $\mathcal{K} \subseteq \mathcal{H}$ of a separable Hilbert space $\mathcal{H}$ such that $\mathcal{R}(A) = \mathcal{K}$ for some separable Hilbert space $\mathcal{H}_0$ and some $A \in \mathcal{B}(\mathcal{H}_0, \mathcal{H})$.

Two easy facts about operator ranges that we will need are stated as follows.

**Lemma 2.1.** 1) *If $M \subseteq \mathcal{K}$ is an operator range and $\mathcal{H}$ is an infinitedimensional Hilbert space, then $M = \mathcal{R}(A)$ for some $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$.*

2) *If $M \subseteq \mathcal{H}$ is closed subspace and $K \subseteq \mathcal{H}$ an operator range then there is an operator range $S \subseteq K$ such that $K = (M \cap K) \oplus S$.*

If $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is such that $\mathcal{R}(A)$ is closed and $n(A) < \infty$, then $A$ is said to be a upper semi-Fredholm operator. If $\beta(A) < \infty$, then $A$ is called a lower semi-Fredholm operator. A semi-Fredholm operator is one which is either upper semi-Fredholm or lower semi-Fredholm. An operator $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is called Fredholm if it is both lower semi-Fredholm and upper semi-Fredholm. The subset of $\mathcal{B}(\mathcal{H},\mathcal{K})$ consisting of all Fredholm operators is denoted by $\Phi(\mathcal{H},\mathcal{K})$. By $\Phi_+(\mathcal{H},\mathcal{K})$ ($\Phi_-(\mathcal{H},\mathcal{K})$) we denote the set of all upper (lower) semi-Fredholm operators from $\mathcal{B}(\mathcal{H},\mathcal{K})$. If $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is a semi-Fredholm operator, the index of $A$ is defined by $\mathrm{ind}(A) = n(A) - d(A)$.

Below we list some facts that will be used throughout the paper.

**Theorem 2.1** ([13]). *If $\mathcal{H}$ and $\mathcal{K}$ are infinitedimensional Banach spaces and $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is a semi-Fredholm operator, then $A + K$ is semi-Fredholm and* $\mathrm{ind}(A + K) = \mathrm{ind}(K)$ *for every compact operator $K \in \mathcal{B}(\mathcal{H},\mathcal{K})$.*

**Theorem 2.2** ([13]). *If $A \in \Phi_+(\mathcal{H},\mathcal{K})$ and $M \subseteq \mathcal{H}$ is a closed subspace, then the subspace $A[M]$ is also closed.*

**Lemma 2.2.** *Let $A, T \in \mathcal{B}(H,\mathcal{K})$. If $\mathcal{R}(T)$ is finitedimensional then the following are equivalent*:

1) $\mathcal{R}(A + T)$ *is closed,*
2) $\mathcal{R}(A) + \mathcal{R}(T)$ *is closed,*
3) $\mathcal{R}(A)$ *is closed.*

**Lemma 2.3** ([11]). *If the operator range $M$ of a separable Hilbert space is not closed, then there exists a closed infinitedimensional subspace $F$ such that $F \cap M = \{0\}$.*

**Lemma 2.4** ([8]). *Let $\mathcal{H},\mathcal{K}$ be separable Hilbert spaces, $M, L \subseteq \mathcal{K}$ operator ranges and $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ such that $\mathcal{R}(T) \subseteq M + L$. Then $T = T_1 + T_2$ for some $T_1, T_2 \in \mathcal{B}(\mathcal{H},\mathcal{K})$ with $\mathcal{R}(T_1) \subseteq M$, $\mathcal{R}(T_2) \subseteq L$.*

**Theorem 2.3** ([8]). *If $\mathcal{H}$ is a separable Hilbert space and $M, L \subseteq \mathcal{H}$ operator ranges such that $M + L = \mathcal{H}$, then there are closed subspaces $M_0 \subseteq M$ and $L_0 \subseteq L$ such that $M_0 \oplus L_0 = \mathcal{H}$.*

## 3. Surjectivity of the operator $A + CX$

In this section we address the question for which operators $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and $C \in \mathcal{B}(\mathcal{L},\mathcal{K})$ there exists an operator $X \in \mathcal{B}(\mathcal{H},\mathcal{L})$ such that the operator $A+CX$ is surjective and completely answer it. The following theorem presents the main result of this section.

**Theorem 3.1.** *Let $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and $C \in \mathcal{B}(\mathcal{L},\mathcal{K})$. Then there exists an operator $X \in \mathcal{B}(\mathcal{H},\mathcal{L})$ such that $A + CX$ is surjective if and only if $\mathcal{R}(A) + \mathcal{R}(C) = \mathcal{K}$ and one of the following two conditions holds*:

(1) *$\mathcal{R}(A)$ is not closed, or*

(2) *$\mathcal{R}(A)$ is closed and either $d(A) \leq n(A)$ or $A^{-1}[\mathcal{R}(C)]$ contains a closed infinitedimensional subspace.*

The proof of Theorem 3.1 will be given in a form of a series of results to be presented below. Using a result of Douglas from [7] we make the following observation that will be used in the sequel without any explicit mention: if $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and $C \in \mathcal{B}(\mathcal{L},\mathcal{K})$, then there exists $X \in \mathcal{B}(\mathcal{H},\mathcal{L})$ such that $A+CX$ is surjective if and only if $A+T$ is surjective for some $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ with $\mathcal{R}(T) \subseteq \mathcal{R}(C)$. Clearly a necessary condition for this to hold is that $\mathcal{R}(A) + \mathcal{R}(C) = \mathcal{K}$.

**Theorem 3.2.** *Let $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $C \in \mathcal{B}(\mathcal{L},\mathcal{K})$ and assume that $A^{-1}[\mathcal{R}(C)]$ contains a closed infinitedimensional subspace $M$. Then there exists an operator $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ such that $\mathcal{R}(T) \subseteq \mathcal{R}(C)$ and $\mathcal{R}(A+T) = \mathcal{R}(A) + \mathcal{R}(C)$.*

**Proof.** Let $M \subseteq A^{-1}[\mathcal{R}(C)]$ be a closed infinitedimensional subspace. Then there exists $S \in \mathcal{B}(M,\mathcal{K})$ such that $\mathcal{R}(S) = \mathcal{R}(C)$. If

$$A = \begin{pmatrix} A_1 & A_2 \end{pmatrix} : \begin{pmatrix} M \\ M^{\perp} \end{pmatrix} \longrightarrow \mathcal{K}$$

define $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ by

$$T = \begin{pmatrix} S - A_1 & 0 \end{pmatrix} : \begin{pmatrix} M \\ M^{\perp} \end{pmatrix} \longrightarrow \mathcal{K}.$$

Since $\mathcal{R}(A_1) = A[M] \subseteq \mathcal{R}(C)$, it is straightforward that $\mathcal{R}(T) \subseteq \mathcal{R}(C)$, as well as that $\mathcal{R}(A+T) = \mathcal{R}(A) + \mathcal{R}(C)$.

As an immediate corollary of Theorem 3.2 we have the following:

**Theorem 3.3.** *Let $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $C \in \mathcal{B}(\mathcal{L},\mathcal{K})$ be such that $\mathcal{R}(A) + \mathcal{R}(C) = \mathcal{K}$ and assume that $A^{-1}[\mathcal{R}(C)]$ contains a closed infinitedimensional subspace. Then there exists an operator $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ such that $\mathcal{R}(T) \subseteq \mathcal{R}(C)$ and $A+T$ is surjective.*

Theorem 3.2 directly solves our problem also in the case when $\mathcal{R}(A)$ is not closed. More precisely, we have the following theorem.

**Theorem 3.4.** *Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $C \in \mathcal{B}(\mathcal{L}, \mathcal{K})$ be such that $\mathcal{R}(A) + \mathcal{R}(C) = \mathcal{K}$. If $\mathcal{R}(A)$ is not closed, then $A^{-1}[\mathcal{R}(C)]$ contains a closed infinitedimensional subspace $M$ and consequently there exists an operator $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $\mathcal{R}(T) \subseteq \mathcal{R}(C)$ and $A + T$ is surjective.*

*Proof.* By Theorem 2.3 we can write $\mathcal{K} = F_A \oplus F_C$, where $F_A, F_C$ are closed subspaces and $F_A \subseteq \mathcal{R}(A)$, $F_C \subseteq \mathcal{R}(C)$. For $M := \mathcal{R}(A) \cap F_C$ we have $\mathcal{R}(A) = F_A \oplus M$. The subspace $M_0 := A^{-1}[M] = A^{-1}[F_C]$ is closed. $M$ is infinitedimensional because otherwise $\mathcal{R}(A)$ would be closed, thus so is $M_0$. Clearly $A[M_0] \subseteq \mathcal{R}(C)$. The rest follows by Theorem 3.3. □

The next theorem is a partial converse of Theorem 3.2.

**Theorem 3.5.** *Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $C \in \mathcal{B}(\mathcal{L}, \mathcal{K})$ be such that $\mathcal{R}(A) + \mathcal{R}(C) = \mathcal{K}$, $\mathcal{R}(A)$ is closed and $n(A) < d(A)$. If there exists an operator $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $\mathcal{R}(T) \subseteq \mathcal{R}(C)$ and $A + T$ is surjective, then $A^{-1}[\mathcal{R}(C)]$ contains a closed infinitedimensional subspace $M$.*

*Proof.* Using Lemma 2.1 we can write $\mathcal{K} = \mathcal{R}(A) \oplus F$, for some operator range $F \subseteq \mathcal{R}(C)$. Also, by Lemma 2.4 we have $T = T_A + T_F$ for some $T_A, T_F \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $\mathcal{R}(T_A) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(T_F) \subseteq F$. From $\mathcal{R}(T) \subseteq \mathcal{R}(C)$ it follows $\mathcal{R}(T_A) \subseteq \mathcal{R}(A) \cap \mathcal{R}(C)$. We have $T_A = AW$ for some $W \in \mathcal{B}(\mathcal{H})$; clearly $\mathcal{R}(W) \subseteq A^{-1}[\mathcal{R}(C)]$.

There is equality $\mathcal{R}(A(1 + W)) = \mathcal{R}(A)$. Indeed, for $z \in \mathcal{R}(A)$ we have $\mathcal{R}(A) \ni z = (A + T)x = A(1 + W)x + T_F x$, for some $x \in \mathcal{H}$, which gives $z = A(1 + W)x$, since $A(1 + W)x \in \mathcal{R}(A)$ and $T_F x \in F$. From this it directly follows that $\mathcal{R}(1 + W) + \mathcal{N}(A) = \mathcal{H}$.

We will show that $W$ cannot be a compact operator, from which the existence will follow of a closed subspace $M \subseteq \mathcal{R}(W) \subseteq A^{-1}[\mathcal{R}(C)]$ of infinite dimension, so the proof of the theorem will be completed. To prove this, we first note that $T_F[(1+W)^{-1}[\mathcal{N}(A)]] = F$. Indeed, let $z \in F$. Then $F \ni z = A(1+W)x + T_F x$, for some $x \in \mathcal{H}$, whence $x \in (1 + W)^{-1}[\mathcal{N}(A)]$ and $T_F x = z$.

Now if $W$ were compact then, letting

$$P := \mathcal{R}(1 + W) \cap \mathcal{N}(A), \quad P_0 := (1 + W)^{-1}[P] \cap \mathcal{N}(1 + W)^{\perp}$$

and keeping in mind that $\mathcal{R}(1+W)+\mathcal{N}(A)=\mathcal{H}$, we would have

$$\begin{aligned}\dim(1+W)^{-1}[\mathcal{N}(A)] &= \dim P_0 + \dim \mathcal{N}(1+W)\\ &= \dim P + \dim \mathcal{R}(1+W)^{\perp}\\ &= \dim \mathcal{N}(A)\\ &< \dim F,\end{aligned}$$

by one of the assumptions of the theorem, therefore contradicting the hypothesis $T_F[(1+W)^{-1}[\mathcal{N}(A)]] = F$. □

The analysis below completes the proof of Theorem 3.1.

**Theorem 3.6.** *Let $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $C \in \mathcal{B}(\mathcal{L},\mathcal{K})$ be such that $\mathcal{R}(A)$ is closed and $\mathcal{R}(A)+\mathcal{R}(C)=\mathcal{K}$. If $d(A) \leq n(A)$ then there exists an operator $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ such that $\mathcal{R}(T) \subseteq \mathcal{R}(C)$ and $A+T$ is surjective.*

**Proof.** As before we can write $\mathcal{K}=\mathcal{R}(A)\oplus F$, for some operator range $F \subseteq \mathcal{R}(C)$. Since $d(A) \leq n(A)$ we can fix an operator $T_0 \in \mathcal{B}(\mathcal{N}(A),\mathcal{K})$ such that $\mathcal{R}(T_0)=F$. For $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ defined by

$$T = \begin{pmatrix} 0 & T_0 \end{pmatrix} : \begin{pmatrix} \mathcal{N}(A)^{\perp} \\ \mathcal{N}(A) \end{pmatrix} \longrightarrow \mathcal{K}$$

it is easy to verify to meet all the requirements of the theorem.

## 4. Denseness of range of the operator $A + CX$

In [6] we completely solved the problem of characterization of all the pairs of operators $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and $C \in \mathcal{B}(\mathcal{L},\mathcal{K})$ for which the operator $A+CX$ is injective for some $X \in \mathcal{B}(\mathcal{H},\mathcal{L})$. In view of the main result of the previous section, to sort of complete this line of investigation it naturally remains to answer the question for which operators $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and $C \in \mathcal{B}(\mathcal{L},\mathcal{K})$ there exists $X \in \mathcal{B}(\mathcal{H},\mathcal{L})$ such that the operator $A+CX$ is with dense range. Unfortunately, unlike the first two problems, we have not been able to fully solve this one, but have rather given some partial answers to it. In particular, we show that if $A \notin \Phi_+(\mathcal{H},\mathcal{K})$ then there always exists such an operator $X$. Also, we give necessary and sufficient conditions for the existence of such an operator in the case when $A \in \Phi(\mathcal{H},\mathcal{K})$.

As before, the claim that there exists an operator $X \in \mathcal{B}(\mathcal{H}, \mathcal{L})$ such that $A + CX$ is with dense range amounts to saying that $\mathcal{R}(A + T)$ is dense for some $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ with $\mathcal{R}(T) \subseteq \mathcal{R}(C)$. Clearly a necessary condition for this to hold is that the subspace $\mathcal{R}(A) + \mathcal{R}(C)$ is dense in $\mathcal{K}$ or, equivalently, that $\mathcal{N}(A^*) \cap \mathcal{N}(C^*) = \{0\}$.

A simple observation combined with Theorem 3.1 yields the following proposition.

**Proposition 4.1.** *Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $C \in \mathcal{B}(\mathcal{L}, \mathcal{K})$ be such that $\mathcal{R}(A)$ is closed and $\mathcal{R}(C)$ is finitedimensional. The following are equivalent*:

(1) *there exists an operator $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $\mathcal{R}(T) \subseteq \mathcal{R}(C)$ and $\mathcal{R}(A+T)$ is dense in $\mathcal{K}$*;

(2) *there exists an operator $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $\mathcal{R}(T) \subseteq \mathcal{R}(C)$ and $\mathcal{R}(A+T)$ is surjective*;

(3) $\mathcal{R}(A) + \mathcal{R}(C) = \mathcal{K}$ *and* $d(A) \leq n(A)$.

**Proof.** By Lemma 2.2 the subspace $\mathcal{R}(A + T)$ is closed for every $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $\mathcal{R}(T) \subseteq \mathcal{R}(C)$. For this reason (1) and (2) say the same thing. To see that (3) is equivalent to these, in view of Theorem 3.1, it suffices to show that if there exists an operator $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $\mathcal{R}(T) \subseteq \mathcal{R}(C)$ and $\mathcal{R}(A + T)$ is surjective, then it must be that $d(A) \leq n(A)$. By Theorem 3.1 the inequality $n(A) < d(A)$ would imply the existence of an infinitedimensional subspace $M \subseteq \mathcal{H}$ such that $A[M] \subseteq \mathcal{R}(C)$. Given that $n(A) < \infty$, we can infer from this that $M \cap \mathcal{N}(A)^\perp$ must also be of infinite dimension, which would mean that $A[M]$, and thus $\mathcal{R}(C)$ as well, is infinitedimensional – a contradiction.

As a second partial result we have the following conclusion which immediately follows from Theorem 3.2.

**Theorem 4.1.** *Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $C \in \mathcal{B}(\mathcal{L}, \mathcal{K})$ be such that $\mathcal{R}(A) + \mathcal{R}(C)$ is dense in $\mathcal{K}$. If $A^{-1}[\mathcal{R}(C)]$ contains a closed infinitedimensional subspace, then there exists an operator $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $\mathcal{R}(T) \subseteq \mathcal{R}(C)$ and $\mathcal{R}(A + T)$ is dense in $\mathcal{K}$.*

Due to the next theorem, in our further considerations we need only consider the case when $A$ is upper semi-Fredholm.

**Theorem 4.2.** *Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $C \in \mathcal{B}(\mathcal{L}, \mathcal{K})$ be such that $\mathcal{R}(A) + \mathcal{R}(C)$ is dense in $\mathcal{K}$. If $\beta(A^*) = \infty$, then there exists an operator $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $\mathcal{R}(T) \subseteq \mathcal{R}(C)$ and $\mathcal{R}(A + T)$ is dense in $\mathcal{K}$.*

**Proof.** From $\beta(A^*) = \infty$ it follows that there exists a closed infinitedimensional subspace $F \subseteq \mathcal{H}$ such that $\mathcal{R}(A^*) \cap F = \{0\}$ (use Lemma 2.3 if $\mathcal{R}(A^*)$ is not closed). Thus we can fix an injective operator $Y \in \mathcal{B}(\mathcal{L}, \mathcal{H})$ such that $\mathcal{R}(Y) \subseteq F$. We claim that $A^* + YC^*$ is one-to-one. Indeed suppose $x \in \mathcal{N}(A^*)^\perp$ and $y \in \mathcal{N}(A^*)$ are such that $(A^* + YC^*)(x + y) = 0$. $\mathcal{R}(A^*) \cap F = \{0\}$ then gives $A^*x = 0$ and $YC^*(x + y) = 0$. But $x \in \mathcal{N}(A^*)^\perp$ and $Y$ is injective, so $x = 0$ and $C^*y = 0$. Given that $y \in \mathcal{N}(A^*)$, it now follows from $\mathcal{N}(A^*) \cap \mathcal{N}(C^*) = \{0\}$ that $y = 0$.

The following auxiliary lemma is needed in the proof of Theorem 4.3.

**Lemma 4.1.** *Let $A \in \Phi_+(\mathcal{H}, \mathcal{K})$ be such that* $\mathrm{ind}(A) < 0$. *If $C \in \mathcal{B}(\mathcal{L}, \mathcal{K})$ is compact, then there is no $X \in \mathcal{B}(\mathcal{H}, \mathcal{L})$ such that $\mathcal{R}(A + CX)$ is dense in $\mathcal{K}$.*

**Proof.** Suppose towards a contradiction that $\mathcal{R}(A + CX)$ is dense in $\mathcal{K}$ for some $X \in \mathcal{B}(\mathcal{H}, \mathcal{L})$. If $\mathcal{H}$ is finitedimensional, then $\mathcal{R}(A + CX)$ is finitedimensional and $\mathcal{R}(A + CX) = \mathcal{K}$, so $dim\mathcal{K} < \infty$. But by Theorem 4.1 we must then have $\mathrm{ind}(A) \geq 0$, contrary to our assumption.

Thus $\mathcal{H}$ must be infinitedimensional. Since $n(A) < \infty$, this also implies that $dim\mathcal{K} = \infty$. $C$ is compact, so the operator $A + CX$ is semi-Fredholm and $\mathrm{ind}(A + CX) < 0$, by Theorem 2.1. But this means that $\mathcal{R}(A + CX)$ is closed and that also $\mathcal{R}(A + CX) \neq \mathcal{K}$, which contradicts our assumption again.

In view of Theorem 3.1, the following result completely answers our question in the case when $A \in \Phi(\mathcal{H}, \mathcal{K})$.

**Theorem 4.3.** *Suppose $A \in \Phi(\mathcal{H}, \mathcal{K})$ and $C \in \mathcal{B}(\mathcal{L}, \mathcal{K})$. Then the following are equivalent*:

(1) *there exists an operator $X \in \mathcal{B}(\mathcal{H}, \mathcal{L})$ such that $\mathcal{R}(A + CX)$ is dense in $\mathcal{K}$*;

(2) *there exists an operator $X \in \mathcal{B}(\mathcal{H}, \mathcal{L})$ such that $A + CX$ is surjective*;

(3) *$\mathcal{R}(A) + \mathcal{R}(C) = \mathcal{K}$ and either* $\mathrm{ind}(A) \geq 0$ *or $\mathcal{R}(A) \cap \mathcal{R}(C)$ contains a closed infinitedimensional subspace.*

*Proof.* (1) $\Longrightarrow$ (2): $\mathcal{R}(A) + \mathcal{R}(C)$ is dense in $\mathcal{K}$, but is also closed, being of finite codimension, so in fact $\mathcal{R}(A) + \mathcal{R}(C) = \mathcal{K}$. If $\mathrm{ind}(A) \geq 0$, then (2) follows directly from Theorem 3.6. Thus assume that $\mathrm{ind}(A) < 0$.

By Lemma 4.1 the operator $C$ is not compact, so there exists a closed infinitedimensional subspace $F \subseteq \mathcal{R}(C)$. From $d(A) < \infty$ it follows that $F_1 := F \cap \mathcal{R}(A)$ is closed and infinitedimensional as well. Thus the subspace $F_2 := A^{-1}[F_1]$ is closed and infinitedimensional; clearly $A[F_2] \subseteq \mathcal{R}(C)$. Now (2) follows from Theorem 3.3.

Since the implication (2) $\implies$ (1) is trivial, it remains to see that (2) $\iff$ (3). But this is easily seen to be a direct consequence of Theorem 3.1, in view of Theorem 2.2 and the fact that $n(A) < \infty$. □

We end the paper with a sufficient condition for our question to have an affirmative answer.

**Theorem 4.4.** *Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $C \in \mathcal{B}(\mathcal{L}, \mathcal{K})$ be such that $\mathcal{R}(A) + \mathcal{R}(C)$ is dense in $\mathcal{K}$. If $C \in \Phi_-(\mathcal{L}, \mathcal{K})$ and $dim\mathcal{R}(A) = \infty$, then there is some $X \in \mathcal{B}(\mathcal{H}, \mathcal{L})$ such that $A + CX$ is with dense range.*

**Proof.** There exists $X \in \mathcal{B}(\mathcal{H}, \mathcal{L})$ such that $A + CX$ is with dense range if and only if there exists $Y \in \mathcal{B}(\mathcal{L}, \mathcal{H})$ such that $A^* + YC^*$ is injective. We will establish the existence of such an operator $Y$.

Since $n(C^*) < \infty$, the subspace $\mathcal{N}(A^*) \oplus \mathcal{N}(C^*)$ is closed and consequently we have $\mathcal{K} = \mathcal{N}(A^*) \oplus \mathcal{N}(C^*) \oplus U$, where $U := (\mathcal{N}(A^*) \oplus \mathcal{N}(C^*))^{\perp}$. Since $\mathcal{R}(C^*)$ is closed, it now follows that the subspaces $V_1 := C^*[\mathcal{N}(A^*)]$ and $V_2 := C^*[U]$ are both closed and we have $\mathcal{R}(C^*) = V_1 \oplus V_2$. The closed subspace $U$ is infinitedimensional, since $n(C^*) < \infty$ and $\dim \mathcal{R}(A^*) = \infty$. Thus there are closed subspaces $S_1, S_2 \subseteq U$ such that $\dim S_1 = \dim V_1$, $\dim S_2 = \infty$ and $U = S_1 \oplus S_2$. Let $P \in \mathcal{B}(U, S_2)$ be the projection onto $S_2$ parallel to $S_1$. The subspace $S_2$ being of infinite dimension, we can fix an injective operator $B_0 \in \mathcal{B}(U, S_2)$ and set $B := P + B_0$. The restriction $C_0 \in \mathcal{B}(U, V_2)$ of the operator $C^*$ is invertible and for $D_2 := BC_0^{-1} \in \mathcal{B}(V_2, S_2)$ we now clearly have $D_2 C^* x \neq Px$ for every nonzero $x \in U$. Since $\dim S_1 = \dim V_1$, there is an injective operator $D_1 \in \mathcal{B}(V_1, S_1)$. Define $D \in \mathcal{B}(\mathcal{L}, \mathcal{K})$ by

$$D = \begin{bmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} V_1 \\ V_2 \\ \mathcal{R}(C^*)^{\perp} \end{bmatrix} \longrightarrow \begin{bmatrix} S_1 \\ S_2 \\ \mathcal{N}(A^*) \oplus \mathcal{N}(C^*) \end{bmatrix}.$$

For $Y = -A^* D$ we claim that the operator $A^* + YC^*$ is one-to-one. To see this let $z \in \mathcal{N}(A^*)$, $y \in \mathcal{N}(C^*)$ and $x \in U$ be such that $(A^* + YC^*)(z + y + x) = 0$, i.e. such that $w := (1 - DC^*)(z + y + x) \in \mathcal{N}(A^*)$. From $z + y + x = DC^* z + DC^* x + w$ and $DC^* z = D_1 C^* z \in S_1$, $DC^* x = D_2 C^* x \in S_2$, it now follows $y = 0$ and $x = D_2C^*x + D_1C^*z$ and consequently $D_2C^*x = Px$. As observed before this implies that $x = 0$. But then we have $D_1C^*z = 0$ whence, by injectivity of $D_1$, $C^*z = 0$. Finally, $\mathcal{N}(A^*) \cap \mathcal{N}(C^*) = \{0\}$ now gives $z = 0$.

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Dragana Cvetković-Ilić, Department of Mathematics,
Faculty of Science and Mathematics, University of Nis, 18000 Nis, Serbia

e-mail: dragana@pmf.ni.ac.rs

Vladimir Pavlović, Department of Mathematics,
Faculty of Science and Mathematics, University of Nis, 18000 Nis, Serbia

e-mail: vlada@pmf.ni.ac.rs