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A nodal domain theorem and a higher-order Cheeger inequality for the graph p-Laplacian

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Abstract. We consider the nonlinear graph p-Laplacian and the set of eigenvalues and associated eigenfunctions of this operator defined by a variational principle. We prove a nodal domain theorem for the graph p-Laplacian for any $p \geq 1$. While for $p > 1$ the bounds on the number of weak and strong nodal domains are the same as for the linear graph Laplacian ($p = 2$), the behavior changes for $p = 1$. We show that the bounds are tight for $p \geq 1$ as the bounds are attained by the eigenfunctions of the graph p-Laplacian of the path graph. Finally, using the properties of the nodal domains, we prove a higher-order Cheeger inequality for the graph p-Laplacian for $p > 1$. If the eigenfunction associated to the k-th variational eigenvalue of the graph p-Laplacian has exactly k strong nodal domains, then the higher order Cheeger inequality becomes tight as $p \to 1$.

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1. Introduction

In this paper we are concerned with the p-Laplacian operator Δ_p on finite, undirected, weighted graphs $G = (\mathcal{V}, \mathcal{E})$. For simplicity we assume throughout the paper that the graph G is connected. A function f on $\mathcal{V}$ is an eigenfunction of Δ_p corresponding to the eigenvalue λ if it solves the following eigenvalue problem

$$(\Delta_p f)(u) = \lambda\, \mu(u)|f(u)|^{p-2} f(u), \qquad \text{for all } u \in \mathcal{V}. \tag{1}$$

The eigenvalue problem for the linear graph Laplacian is obtained for $p = 2$. The spectrum of Δ_2 has been studied extensively in past decades. In particular

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every eigenvalue of Δ_2 admits a variational characterization in terms of a Rayleigh quotient and several relations between the eigenvalues of Δ_2 and a number of graph invariants have been established [11, 12, 29]. However, for $p \neq 2$, the spectral properties are less well understood. For the general case it is known that the smallest eigenvalue λ_1 of Δ_p is zero, it is simple and any corresponding eigenfunction is constant. It is also known that the smallest nonzero eigenvalue λ_2 admits a variational characterization [3, 4, 6]. The Lusternik-Schnirelman theory allows to generalize the variational characterization of the linear case and to define a sequence of variational eigenvalues of Δ_p for $p \neq 2$.

In this paper we investigate the nodal properties of the eigenfunctions of Δ_p. A strong nodal domain of f is a maximal connected component of $\{u: f(u) \neq 0\}$. Weak nodal domains, instead, can overlap and are defined as the maximal connected components of either $\{u: f(u) \geq 0\}$ or $\{u: f(u) \leq 0\}$. The famous Courant nodal domain theorem provides upper bounds on the number of nodal domains of the eigenfunctions of the continuous Laplacian in $\mathbf{R}^d$. Several authors worked afterwards on a discrete version of the nodal domain theorem for the case of the linear graph Laplacian, see e.g. [16, 17, 22, 26], or the adjacency or modularity matrix [24, 25, 40]. The main contribution of the present work is a unifying generalized version of the Courant nodal domain theorem for the graph p-Laplacian, for any $p \geq 1$. We show that for $p > 1$ the bounds on the number of weak and strong nodal domains are the same as for the linear graph Laplacian whereas the upper bound of the weak nodal domains changes for $p = 1$.

As there are strong relations between the continuous and discrete theory, see e.g. [18, 17, 27], it is worthwhile to quickly review the eigenproblem of the continuous p-Laplacian. If Ω is a bounded domain in $\mathbf{R}^d$, with smooth boundary $\partial\Omega$, the continuous analogous of (1) is

$$-\operatorname{div}(|\nabla f|^{p-2}\nabla f) = \lambda |f|^{p-2} f, \quad \text{in } \Omega, \tag{2}$$

where homogeneous conditions are assumed on $\partial\Omega$. The eigenvalue problem (2) has been studied extensively. When $d = 1$, for instance, the spectrum of Δ_p is completely described [20, 23]. Courant's nodal theorem has been then extended to the eigenfunctions of the continuous p-Laplacian (2) for $p > 1$ [4, 21], where connected components are replaced by connected open subsets. However, note that the difference between strong and weak nodal domains is not considered in the continuous case. Moreover, [21] requires as an assumption what they call the *unique continuation property* to prove the direct generalization of the Courant nodal domain theorem.

As a second main contribution, we provide a higher-order Cheeger inequality relating the k-th variational eigenvalue of the graph p-Laplacian with the k-th isoperimetric or Cheeger constant of the graph. The Cheeger constant $h(G)$ is one of the most important topological invariants of a graph G. The result for the case $k = 2$ was originally proven by Cheeger [10] for compact Riemannian manifolds and the associated Laplace–Beltrami operator.

Several authors transferred afterwards the result of Cheeger to the discrete case, see e.g. [1, 2, 18, 24, 33, 37, 42]. Cheeger's inequality plays an important role in the theory of expander graphs, mixing time of Markov chains, graph coloring but has also applications in computer science such as image segmentation and web search, see e.g. [2, 13, 32, 34] and the references therein. An extension of Cheeger's inequality to the nonlinear graph p-Laplacian has been shown in [3, 6, 28] for $p \geq 1$, where it has been shown that the Cheeger constant $h(G)$ of the graph is the limit of λ_2 of Δ_p as $p \to 1$, with equality for $p = 1$. A number of Cheeger type inequalities have been shown for the continuous p-Laplacian in (2) as well [7, 30, 31].

More recently, a set of higher-order isoperimetric constants $h_k(G)$, $k = 1, 2, 3, \ldots$ alternatively called multi-way Cheeger constants, has been introduced by Miclo [14, 15] in the discrete setting. We provide a Cheeger-type inequality relating the k-th variational eigenvalue of Δ_p with the k-way isoperimetric constant, where the number of strong nodal domains of the eigenfunctions of Δ_p plays a crucial role.

The paper is organized as follows. In Section 2 we fix the notation and provide a number of first results. Section 3 contains the statement of the nodal domain theorem for Δ_p. In Sections 3.1 and 4 we discuss the non-smooth case $p = 1$ and prove the tightness of our results for $p \geq 1$, by studying the eigenfunctions of the graph p-Laplacian for the path graph where the upper bounds on the nodal domain counts are attained. Finally, in Section 5 we discuss a higher-order Cheeger inequality for the graph p-Laplacian.

2. Preliminaries

Let $G = (\mathcal{V}, \mathcal{E})$ be a finite, connected and undirected graph where $\mathcal{V} = (V, \mu)$ and $\mathcal{E} = (E, w)$ are the vertex and edge sets, equipped with positive measures μ and w, respectively. The number of nodes $|V|$ of G is denoted by n. We assume that the vertex weights are strictly positive, $\mu(u) > 0$ for all $u \in V$. An element of E is denoted by uv, where u, v are the nodes connected by uv. We extend the measure function w to the whole product $V \times V$ by letting $w(uv) = 0$ if $uv \notin E$. We write

$u \sim v$ to indicate that there exists an edge $uv \in E$ between those nodes. Similarly, for two sets of nodes $A, B \subseteq V$, we write $A \approx B$ if there exists an edge connecting A and B. Finally, we define $\ell^p(\mathcal{V})$ to be the space of real valued functions on V endowed with the norm $\|f\|_{\ell^p(\mathcal{V})} = (\sum_u \mu(u)|f(u)|^p)^{1/p}$.

Let $\Phi_p: \mathbf{R} \to \mathbf{R}$ be defined as $\Phi_p(x) = |x|^{p-2}x$, then the graph p-Laplacian $\Delta_p: \mathbf{R}^{\mathcal{V}} \to \mathbf{R}^{\mathcal{V}}$ is defined for $p > 1$ as

$$(\Delta_p f)(u) = \sum_{v \in V} w(uv)\Phi_p(f(u) - f(v)), \quad \text{for all } u \in V.$$

The case $p = 1$ will be treated in Section 4, even though parts of the following discussion apply already for the case $p = 1$. A function $f: V \to \mathbf{R}$ is an eigenfunction of Δ_p associated with the eigenvalue λ if the following identity holds

$$(\Delta_p f)(u) = \lambda\, \mu(u)\Phi_p(f(u)), \quad \text{for all } u \in V.$$

We then consider the even functional $\mathcal{R}_p: \mathbf{R}^{\mathcal{V}} \to \mathbf{R}$ defined as

$$\mathcal{R}_p(f) = \frac{1}{2}\frac{\sum_{uv \in E} w(uv)|f(u) - f(v)|^p}{\sum_{u \in V} \mu(u)|f(u)|^p}$$

and the symmetric manifold $\mathcal{S}_p = \{g \in \ell^p(\mathcal{V}): \|g\|_{\ell^p(\mathcal{V})} = 1\}$. It is easy to see that the eigenvalues and eigenfunctions of Δ_p correspond to the critical values and critical points of $\mathcal{R}_p$. Moreover, as $\mathcal{R}_p$ is scale invariant, $\mathcal{R}_p(\alpha f) = \mathcal{R}_p(f)$ for any nonzero $\alpha \in \mathbf{R}$, f is a critical point of $\mathcal{R}_p$ if and only if $f/\|f\|_{\ell^p(\mathcal{V})}$ is a critical point of the restriction $\mathcal{R}_p|_{\mathcal{S}_p}$ of $\mathcal{R}_p$ onto $\mathcal{S}_p$, and they correspond to the same critical value.

The Lusternik-Schnirelman theory allows several ways to characterize a sequence of variational eigenvalues of Δ_p. A standard approach, relying on the symmetry of $\mathcal{R}_p$ and $\mathcal{S}_p$, is based on the notion of Krasnoselskii genus.

Definition 2.1. Let A be a closed, symmetric subset of $\mathcal{S}_p$. We define the Krasnoselskii genus of A as

$$\gamma(A) = \begin{cases} 0 & \text{if } A = \emptyset, \\ \inf\{m \mid \text{there exists} A \to \mathbf{R}^m \setminus \{0\}, \text{continuos}, h(-u) = -h(u)\}, \\ \infty & \text{if } \{\dots\} = \emptyset, \text{ in particular if } 0 \in A. \end{cases}$$

Consider the family $\mathcal{F}_k(\mathcal{S}_p) = \{A \subseteq \mathcal{S}_p: A = -A, \text{closed}, \gamma(A) \geq k\}$. As $\mathcal{S}_p$ is compact, it is easy to verify that $\mathcal{R}_p|_{\mathcal{S}_p}$ satisfies the Palais-Smale condition. Then

$$\lambda_k^{(p)} = \min_{A \in \mathcal{F}_k(\mathcal{S}_p)} \max_{f \in A} \mathcal{R}_p(f) \tag{3}$$

defines a sequence of n critical values of the Rayleigh quotient $\mathcal{R}_p$, for $k = 1, \dots, n$. Moreover, it is known that the fact that G is connected implies $0 = \lambda_1^{(p)} < \lambda_2^{(p)} \leq \cdots \leq \lambda_n^{(p)}$. Note that in the definition (3) we can freely use $\mathcal{R}_p$ or its restriction to $\mathcal{S}_p$, as the critical values do not change. From now on we shall call the numbers $\lambda_k^{(p)}$, for $k = 1, \dots, n$, the *variational eigenvalues* of Δ_p. For ease of notation we will often drop the superscript, writing λ_k in place of $\lambda_k^{(p)}$, when the reference to a given p is clear from the context.

Until now we have concentrated on the variational eigenvalues, but the nodal domain theorem requires to consider eigenfunctions. It turns out that one can associate at least one eigenfunction of the graph p-Laplacian to each variational eigenvalue $\lambda_k^{(p)}$. For a function $F\colon \mathcal{S}_p \to \mathbf{R}$ we write $F^c = \{x \in \mathcal{S}_p \colon F(x) \leq c\}$ and $K_\lambda(F) = \{x \in \mathcal{S}_p \colon F(x) = \lambda, \nabla F(x) = 0\}$. The following theorem is part of a more general result known as *deformation theorem*. See for instance [43, Thm. 3.11] or [39, Thm. 4.1.19]

Theorem 2.2. *Let $F\colon \mathcal{S}_p \to \mathbf{R}$ be an even function satisfying the Palais-Smale condition. Then for any $\varepsilon_0 > 0$, $\lambda \in \mathbf{R}$ and any neighbourhood N of $K_\lambda(F)$, there exists $\varepsilon \in (0, \varepsilon_0)$ and an odd homeomorphism $\theta\colon \mathcal{S}_p \to \mathcal{S}_p$ such that*

$$\theta(F^{\lambda+\varepsilon} \setminus N) \subseteq F^{\lambda-\varepsilon}.$$

The deformation theorem allows to show that to each variational eigenvalue belongs at least one corresponding eigenfunction.

Lemma 2.3. *For $k \geq 1$ let $A^* \in \mathcal{F}_k(\mathcal{S}_p)$ be a minimizing set, that is*

$$\lambda_k = \min_{A \in \mathcal{F}_k(\mathcal{S}_p)} \max_{f \in A} \mathcal{R}_p(f) = \max_{f \in A^*} \mathcal{R}_p(f).$$

Then A^* contains at least one critical point of $\mathcal{R}_p$, relative to λ_k.

Proof. Consider the restriction $\mathcal{R}_p|_{\mathcal{S}_p}$. Recall that if F is a function that satisfies the Palais-Smale condition and λ is a critical value of F, then $K_\lambda(F)$ is compact (see f.i. [43, pp. 78–80]). Suppose by contradiction that $A^* \cap K_{\lambda_k}(\mathcal{R}_p|_{\mathcal{S}_p}) = \emptyset$. We already discussed that $\mathcal{R}_p|_{\mathcal{S}_p}$ satisfies the Palais-Smale condition, therefore A^* and $K_{\lambda_k}(\mathcal{R}_p|_{\mathcal{S}_p})$ are compact. Hence there exists a neighborhood N of $K_{\lambda_k}(\mathcal{R}_p|_{\mathcal{S}_p})$ such that $A^* \cap N = \emptyset$. Therefore $A^* = A^* \setminus N$. Since $\max_{f \in A^*} \mathcal{R}_p|_{\mathcal{S}_p}(f) = \lambda_k$, for any $\varepsilon > 0$ we have $A^* \subseteq \mathcal{R}_p|_{\mathcal{S}_p}^{\lambda_k+\varepsilon}$. By the deformation theorem, there exists an odd homeomorphism $\theta\colon \mathcal{S}_p \to \mathcal{S}_p$ such that

$$\theta(A^*) = \theta(A^* \setminus N) \subseteq \theta(\mathcal{R}_p|_{\mathcal{S}_p}^{\lambda_k+\varepsilon} \setminus N) \subseteq \mathcal{R}_p|_{\mathcal{S}_p}^{\lambda_k-\varepsilon}.$$

As θ is an odd homeomorphism we have that $A^* \in \mathcal{F}_k(\mathcal{S}_p)$ implies $\theta(A^*) \in \mathcal{F}_k(\mathcal{S}_p)$. Then

$$\lambda_k \leq \min_{A \in \mathcal{F}_k(\mathcal{S}_p)} \max_{f \in A} \mathcal{R}_p(f) \leq \max_{f \in \theta(A^*)} \mathcal{R}_p(f) \leq \max_{f \in \mathcal{R}_p|_{\mathcal{S}_p}^{\lambda_k - \varepsilon}} \mathcal{R}_p(f) \leq \lambda_k - \varepsilon$$

and we have reached a contradiction. Therefore $A^* \cap K_{\lambda_k}(\mathcal{R}_p|_{\mathcal{S}_p})$ cannot be empty and the lemma is proven. □

We would like to note that the integer valued genus γ is a classical homeomorphism invariant generalization of the concept of dimension. Indeed if A is any symmetric neighborhood of the origin in $\mathbf{R}^k$, then $\gamma(A) = k$, and, vice-versa, if A is any subset such that $\gamma(A) = k$, then A contains at least k mutually orthogonal functions. It follows that, when $p = 2$, the sequence (3) boils down to the Courant-Fischer minimax principle $\lambda_k^{(2)} = \min_{\dim(A)=k} \max_{f \in A} \mathcal{R}_2(f)$. We refer to [39, Ch. 4] or [43, Ch. 2] for an overview.

3. Nodal domain theorem for the graph p-Laplacian

Consider the eigenvalue problem (2) and a continuous function f on Ω. A nodal domain for f is a maximal connected open subset of $\{u \colon f(u) \neq 0\}$. When $p = 2$, Courant's nodal domain theorem states that any eigenfunction for (2) associated to the eigenvalue λ_k has at most k nodal domains.

For graphs, nodal domains induced by a function $f \colon V \to \mathbf{R}$ are commonly defined as follows:

Definition 3.1. Let $f \colon V \to \mathbf{R}$. A subset $A \subseteq V$ is a *strong nodal domain* of G induced by f if the subgraph $G(A)$ induced on G by A is a maximal connected component of either $\{u \colon f(u) > 0\}$ or $\{u \colon f(u) < 0\}$.

Definition 3.2. Let $f \colon V \to \mathbf{R}$. A subset $A \subseteq V$ is a *weak nodal domain* of G induced by f if the subgraph $G(A)$ induced on G by A is a maximal connected component of either $\{u \colon f(u) \geq 0\}$ or $\{u \colon f(u) \leq 0\}$.

For any connected graph G and any $p \geq 1$, $\lambda_1^{(p)} = 0$ is simple and any associated eigenfunction is constant. Thus the strong and weak nodal domain for the eigenfunctions of $\lambda_1^{(p)}$ is V itself.

Fiedler noted in [26, Cor. 3.6] that the number of weak nodal domains induced by any eigenfunction associated to $\lambda_2^{(2)}$ is exactly two. Several authors derived analogous results to the Courant nodal theorem for the higher-order eigenfunctions of Δ_2 [16, 22, 40]. The following nodal domain theorem for the graph Laplacian Δ_2 summarizes their work.

Theorem 3.3. *Let G be connected and $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of Δ_2. Any eigenfunction of λ_k induces at most k weak nodal domains and at most $k + r - 1$ strong nodal domains, where r is the multiplicity of λ_k.*

The authors of [16], in particular, provide examples which show that the bounds for the weak and strong nodal domains are tight. The following theorems show that the results carry over to the p-Laplacian.

Theorem 3.4. *Suppose that G is connected and $p \geq 1$ and denote by $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$ the variational eigenvalues of Δ_p. Let λ be an eigenvalue of Δ_p such that $\lambda < \lambda_k$. Any eigenfunction associated to λ induces at most $k - 1$ strong nodal domains.*

We get as a consequence that, if the variational eigenvalue λ_k has multiplicity r, that is

$$\lambda_{k-1} < \lambda_k = \lambda_{k+1} = \cdots = \lambda_{k+r-1} < \lambda_{k+r}\,,$$

then Theorem 3.4 shows that any eigenfunction of λ_k induces at most $k + r - 1$ strong nodal domains.

Theorem 3.5. *Suppose that G is connected and $p > 1$ and denote by $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$ the variational eigenvalues of Δ_p. Any eigenfunction of λ_k induces at most k weak nodal domains.*

Let us stress that Theorem 3.4 holds for any $p \geq 1$, whereas Theorem 3.5 does not hold in general when $p = 1$. We will discuss the case $p = 1$ in detail in Section 4. As a direct consequence we get the following corollary.

Corollary 3.6. *Suppose that G is connected and let $p > 1$. Any eigenfunction corresponding to the second variational eigenvalue of Δ_p has exactly 2 weak nodal domains.*

Proof. With the definition of Δ_p we have $\sum_u (\Delta_p f)(u) = 0$ for any function f. This implies in particular that for any eigenfunction f of Δ_p with eigenvalue not equal to zero it holds $\sum_u \mu(u)\Phi_p(f(u)) = 0$ which implies that f attains both positive and negative values. As the graph is connected, it holds $\lambda_2 > 0$ and thus any associated eigenfunction has at least two weak nodal domains On the other hand Theorem 3.5 shows that the number of weak nodal domains induced by f is at most 2, and thus it is exactly 2. □

The proof of Theorems 3.4 and 3.5 relies on a number of properties which are of independent interest. Therefore we devote the subsequent discussion to those properties and postpone the proof to the end of the section.

We need, first, the following technical lemma

Lemma 3.7. *Let $p \geq 1$, $a, b, x, y \in \mathbf{R}$ and $xy \leq 0$. Then*

$$|ax - by|^p - (|a|^p|x| + |b|^p|y|)|x-y|^{p-1} \leq 0,$$

where equality holds for $p = 1$ if and only if $xy = 0$ or $ab \geq 0$ and for $p > 1$ if and only if $xy = 0$ or $a = b$.

Proof. We note that with $xy \leq 0$ it holds $|x - y| = |x| + |y|$. It is easy to see that equality holds for $xy = 0$. Thus we assume $xy < 0$ in the following and get

$$\begin{aligned}
&|ax - by|^p - (|a|^p|x| + |b|^p|y|)(|x| + |y|)^{p-1} \\
&\quad \leq (|a||x| + |b||y|)^p - (|a|^p|x| + |b|^p|y|)(|x| + |y|)^{p-1} \\
&\quad = (|x| + |y|)^p \Big[\Big(\frac{|x|}{|x|+|y|}|a| + \frac{|y|}{|x|+|y|}|b|\Big)^p \\
&\qquad\qquad - \frac{|x|}{|x|+|y|}|a|^p - \frac{|y|}{|x|+|y|}|b|^p\Big] \\
&\quad \leq 0,
\end{aligned}$$

where in the last inequality we have used the fact that $f(\lambda) = \lambda^p$ is strictly convex on $\mathbf{R}_+$ for $p > 1$ and convex for $p = 1$. Finally, under the condition $xy < 0$ we have equality in the first inequality if and only if $ab \geq 0$. For the second inequality we note that it is an equality for $p = 1$, whereas, under the condition $xy < 0$, equality holds for $p > 1$ only if $|a| = |b|$, due to the strict convexity of $f(\lambda) = \lambda^p$. Combining the conditions yields the result. $\square$

Given a function $f : V \to \mathbf{R}$ and any $A \subseteq V$ we write $f|_A$ to denote the function $f|_A(u) = f(u)$ if $u \in A$ and $f|_A(u) = 0$ otherwise. The strong and weak nodal spaces of f are defined as the linear span of $f|_{A_1}, \dots, f|_{A_m}$, with A_i being the strong or weak nodal domains of f, respectively. A related version of this result has been proven in [22] for the linear case ($p = 2$). Even though the proof there relied on the the linearity of the operator, it turns out that this requirement is not necessary for the nonlinear generalization.

Lemma 3.8. *Let $p \geq 1$ and let $f : V \to \mathbf{R}$ be any eigenfunction of Δ_p corresponding to the eigenvalue λ. Let F be either the strong or weak nodal space of f. Then for any $g \in F$ it holds $\mathcal{R}_p(g) \leq \lambda$. In the case $p = 1$ the inequality holds with equality for any $g \in F$ with $g \neq 0$.*

Proof. We prove the lemma for the strong nodal domains $A_1, \dots, A_m$. We discuss at the end of the proof how it can be transferred to the weak nodal domains. Note that the strong nodal domains are by construction pairwise disjoint. We denote by $Z = V \setminus \bigcup_{i=1}^m A_i$ the set $Z = \{u \colon f(u) = 0\}$. Let $g = \sum_i \alpha_i f|_{A_i}$ be a function in the strong nodal space F. The statement is trivially true if $g \equiv 0$, therefore we can assume $\sum_i |\alpha_i| > 0$. We have

$$\|g\|^p_{\ell^p(\mathcal{V})} = \sum_{i=1}^m \sum_{u \in A_i} \mu(u) |\alpha_i f|_{A_i}(u)|^p = \sum_{i=1}^m |\alpha_i|^p \|f|_{A_i}\|^p_{\ell^p(\mathcal{V})}. \tag{4}$$

By splitting the summation over V into the sum over $Z, A_1, \dots, A_m$, we get

$$\begin{aligned} &\frac{1}{2} \sum_{u,v \in V} w(uv) |g(u) - g(v)|^p \\ &\quad = \frac{1}{2} \sum_{i=1}^m |\alpha_i|^p \sum_{u,v \in A_i} w(uv) |f|_{A_i}(u) - f|_{A_i}(v)|^p \\ &\quad + \frac{1}{2} \sum_{j \neq i} \sum_{u \in A_j} \sum_{v \in A_i} w(uv) |\alpha_j f|_{A_j}(u) - \alpha_i f|_{A_i}(v)|^p \\ &\quad + \sum_{i=1}^m \sum_{u \in A_i} |\alpha_i f|_{A_i}(u)|^p \sum_{v \in Z} w(uv). \end{aligned} \tag{5}$$

Let A be any strong nodal domain. As f is an eigenfunction of Δ_p corresponding to the eigenvalue λ, for any $u \in A$ we have the chain of equalities $\lambda \mu(u) |f|_A(u)|^p = \lambda \mu(u) f|_A(u) \Phi_p(f(u)) = f|_A(u)(\Delta_p f)(u)$. Therefore

$$\begin{aligned} \lambda \|f|_A\|^p_{\ell^p(\mathcal{V})} &= \sum_{u \in V} f|_A(u)(\Delta_p f)(u) \\ &= \frac{1}{2} \sum_{u,v \in V} w(uv)(f|_A(u) - f|_A(v)) \Phi_p(f(u) - f(v)). \end{aligned}$$

Let $A, B \subset V$ be two distinct strong nodal domains. If $uv \in E$, $u \in A$ and $v \in B$, then $f(u)f(v) < 0$, as the strong nodal domains are maximal connected components. This implies that, for such u and v,

$$\operatorname{sign}(f(u) - f(v)) = \operatorname{sign}(f|_A(u)) = -\operatorname{sign}(f|_B(v)).$$

Thus

$$
\begin{aligned}
&\lambda \, \|f|_A\|^p_{\ell^p(\mathcal{V})} \\
&\quad = \frac{1}{2}\sum_{u,v\in A} w(uv)|f|_A(u) - f|_A(v)|^p + \sum_{u\in A} |f|_A(u)|^p \sum_{v\in Z} w(uv) \\
&\qquad + \frac{1}{2}\sum_{B:B\neq A}\sum_{u\in A}\sum_{v\in B}(w(uv)|f|_A(u)| + w(vu)|f|_A(v)|)\,|f|_A(u) - f|_B(v)|^{p-1},
\end{aligned}
$$

where the summation over B runs over all the nodal domains different from A. Combining the preceding formula with (4) and (5) yields

$$
\frac{1}{2}\sum_{u,v\in V} w(uv)|g(u)-g(v)|^p - \lambda \|g\|^p_{\ell^p(\mathcal{V})} = \frac{1}{2}\sum_{i\neq j}\sum_{u\in A_i}\sum_{v\in A_j} w(uv) F_{ij}(u,v), \quad (6)
$$

where

$$
\begin{aligned}
F_{ij}(u,v) &= |\alpha_i f|_{A_i}(u) - \alpha_j f|_{A_j}(v)|^p \\
&\quad - (|\alpha_i|^p |f|_{A_i}(u)| + |\alpha_j|^p |f|_{A_j}(v)|)|f|_{A_i}(u) - f|_{A_j}(v)|^{p-1}.
\end{aligned}
$$

By Lemma 3.7 each of the quantities $F_{ij}(u,v)$ is nonpositive. Since for distinct domains A and B, $w(uv) > 0$ holds if and only if $f|_A(u) f|_B(v) < 0$, we deduce that the quantity in (6) is nonpositive as well. As g is not identically zero we conclude that $\mathcal{R}_p(g) \leq \lambda$. Also note that, by Lemma 3.7, we have the equality $\mathcal{R}_p(g) = \lambda$ when $p = 1$.

The proof can be transferred to the weak nodal domains $A_1, \dots, A_m$ by considering instead the sets $B_i = A_i \cap \{u\colon f(u) \neq 0\}$, $i = 1, \dots, m$ and noting that

$$
\sum_{k=1}^m \alpha_k \, f|_{A_k} = \sum_{k=1}^m \alpha_k \, f|_{B_k}.
$$

As for the strong nodal domains, the sets $B_1, \dots, B_m$ are pairwise disjoint and together with $Z = V \setminus \bigcup_{i=1}^m B_i = \{u\colon f(u) = 0\}$, form a partition of V. Replacing $A_1, \dots, A_m$ with $B_1, \dots, B_m$ in the argument above, all the steps remain true. $\square$

We are now ready to prove the nodal domain theorem for the graph p-Laplacian. The proof is given here assuming $p > 1$. The case $p = 1$ is discussed in Section 4.

Proof of Theorem 3.4. Let $\lambda_1 \leq \cdots \leq \lambda_n$ be the variational eigenvalues of Δ_p, and let λ be any eigenvalue such that $\lambda < \lambda_k$. Consider any eigenfunction f corresponding to λ. Let $A_1, \dots, A_m$ be the strong nodal domains of f and let F be the corresponding strong nodal space. Lemma 3.8 implies that $\max_{g \in F} \mathcal{R}_p(g) \leq \lambda$. As the functions $f|_{A_1}, \dots, f|_{A_m}$ are linear independent we have $\gamma(F \cap \mathcal{S}_p) = m$. In particular $F \cap \mathcal{S}_p \in \mathcal{F}_m(\mathcal{S}_p)$ and by the definition of λ_m we get

$$\lambda_m \leq \max_{g \in F \cap \mathcal{S}_p} \mathcal{R}_p(g) \leq \lambda < \lambda_k \,. \tag{7}$$

As a consequence we have $\lambda_m < \lambda_k$ which implies $m \leq k - 1$. □

For the weak nodal domains we need a few additional remarks. Let $A_1, \dots, A_m$ be the weak nodal domains of f. Since $\cup_i A_i = V$ and G is connected, then for any i there exists j such that $A_i \approx A_j$. Moreover, the following lemma holds

Lemma 3.9. *Let A and B be two weak nodal domains induced by the non-constant eigenfunction $f : V \to \mathbf{R}$ and such that $A \approx B$. Then there exist $u \in A$ and $v \in B \setminus A$ such that $u \sim v$.*

Proof. If $A \cap B = \emptyset$ the statement is straightforward. Assume that $A \cap B \neq \emptyset$. By definition we have $f(u) = 0$, for any $u \in A \cap B$, thus for any such u it holds $0 = \lambda\, \mu(u)\Phi_p(f(u)) = \sum_{v \in V} w(uv)\Phi_p(f(u) - f(v)) = \sum_{v \in V} w(uv)\Phi_p(f(v))$. Note that, by definition, as $u \in A \cap B$, then any v such that $v \sim u$ is either in A or in B. As $w(uv) > 0$ when $u \sim v$, the values $\Phi_p(f(v))$ have to be either all zero or both positive and negative. However, the maximality of the nodal domains implies that $\Phi_p(f(v))$ can not be zero for all $v \sim u$ and all $u \in A \cap B$. Then there exists $v \in A \cup B$ such that $v \sim u$ and $f(v) \neq 0$. This concludes the proof. □

It is clear that adjacent nodal domains have different sign. Then we deduce from the above lemma that, given any two adjacent weak nodal domains $A \approx B$ of an eigenfunction f, two cases are possible:

P1. there exist $u \in A$ and $v \in B$ such that $u \sim v$ and $f(u)f(v) < 0$;

P2. $f(u)f(v) = 0$ for all $u \in A$ and $v \in B$ such that $u \sim v$, and there exist $u \in A$ and $v \in B$ such that $u \sim v$, $f(u) = 0$ and $f(v) \neq 0$.

Proof of Theorem 3.5. Let f be an eigenfunction of λ_k and let $A_1, \dots, A_m$ be the weak nodal domains of f. Suppose by contradiction that $m > k$. We deduce from Lemma 3.8 that inequality (7) holds also for the weak nodal domains. Namely, for any g in the weak nodal space F of f, we have $\max_{g \in F} \mathcal{R}_p(g) = \max_{g \in F \cap \mathcal{S}_p} \mathcal{R}_p(g) \leq \lambda_k$. Observe that, as $m > k$ and $f \in F$, we have

$F = \mathrm{span}\{f\} \oplus H$, for some H such that $\dim H \geq k$. In particular $m = \gamma(F \cap \mathcal{S}_p)$ and $k \leq \gamma(H \cap \mathcal{S}_p)$. As a consequence $H \cap \mathcal{S}_p \in \mathcal{F}_k(\mathcal{S}_p)$ and we get

$$\lambda_k \geq \max_{g \in F \cap \mathcal{S}_p} \mathcal{R}_p(g) \geq \max_{g \in H \cap \mathcal{S}_p} \mathcal{R}_p(g) \geq \min_{X \in \mathcal{F}_k(\mathcal{S}_p)} \max_{g \in X} \mathcal{R}_p(g) = \lambda_k\,.$$

Thus the relations above hold with equality and we deduce that $H \cap \mathcal{S}_p$ is a minimizing set, and by Lemma 2.3 there exists an eigenfunction $g = \sum_{s=1}^m \alpha_s f|_{A_s} \in H$.

As $\mathcal{R}_p(g)$ is the maximum of the Rayleigh quotient on H we deduce from the proof of Lemma 3.8 that $\sum_{i \neq j} \sum_{u \in A_i} \sum_{v \in A_j} w(uv) F_{ij}(u, v) = 0$, where

$$\begin{aligned} F_{ij}(u, v) = {} & |\alpha_i f|_{A_i}(u) - \alpha_j f|_{A_j}(v)|^p \\ & - (|\alpha_i|^p |f|_{A_i}(u)| + |\alpha_j|^p |f|_{A_j}(v)|)|f|_{A_i}(u) - f|_{A_j}(v)|^{p-1}. \end{aligned}$$

By Lemma 3.7 each of the summands $w(uv)F_{ij}(u, v)$ is nonpositive, then all of them have to vanish individually. Choose any pair of adjacent sets $A_s \approx A_r$. If they satisfy property P1 above, then there exist $u \in A_s$ and $v \in A_r$ such that $w(uv) > 0$ and $f|_{A_s}(u) f|_{A_r}(v) < 0$. Therefore $w(uv)F_{sr}(u, v) = 0$ implies $\alpha_s = \alpha_r$, by virtue of Lemma 3.7.

If P1 does not hold, then P2 holds. Since g is an eigenfunction of Δ_p, for any $\beta \in \mathbf{R}$, we have the following entrywise equations

$$\lambda_k \mu(u) \Phi_p(\beta f(u)) = \sum_{v \in V} w(uv) \Phi_p(\beta f(u) - \beta f(v)), \quad u \in V,$$

$$\lambda_k \mu(u) \Phi_p(g(u)) = \sum_{v \in V} w(uv) \Phi_p(g(u) - g(v)), \quad u \in V.$$

As P2 holds for A_s and A_r, then there exist $u \in A_s$ and $v \in A_r$ such that $u \sim v$, $f(u) = 0$ and $f(v) \neq 0$. Then $\beta f(u) = g(u) = 0$ and the previous equations imply

$$\sum_{v \in V} w(uv) \{\Phi_p(\beta f(v)) - \Phi_p(g(v))\} = 0\,.$$

The quantities $w(uv)$ are zero unless $v \sim u$. Since $f(u) = 0$, the maximality of the nodal domains implies that all the vertices v adjacent to u are either in A_s or in A_r. We have

$$\begin{aligned} & \sum_{v \in A_s} w(uv) \{\Phi_p(\alpha_s f|_{A_s}(v)) - \Phi_p(\beta f|_{A_s}(v))\} \\ & \quad = \sum_{v \in A_r} w(uv) \{\Phi_p(\beta f|_{A_r}(v)) - \Phi_p(\alpha_r f|_{A_r}(v))\}. \end{aligned}$$

Thus choosing $\beta = \alpha_s$ we get $\{\Phi_p(\alpha_s) - \Phi_p(\alpha_r)\} \sum_{v \in A_r} w(uv)\Phi_p(f|_{A_r}(v)) = 0$. Since $w(uv) \geq 0$ for all $v \in A_r$, there exists $x \in A_r$ such that $w(ux) > 0$, $f(x) \neq 0$, and the entries of $f|_{A_r}$ have same sign. Then the previous identity implies $\Phi_p(\alpha_s) - \Phi_p(\alpha_r) = 0$, that is $\alpha_s = \alpha_r$.

We finally conclude that, if $A_s \approx A_r$, then $\alpha_s = \alpha_r$. The connectedness of the graph implies then $\alpha = \alpha_1 = \cdots = \alpha_m$, and we obtain $g = \sum_{s=1}^m \alpha_s f|_{A_s} = \alpha f$. This gives a contradiction as by construction $g \in H$ is linear independent with respect to f. □

We show in the following that the bounds cannot be improved in general, by discussing the nodal domain structure of an example graph.

3.1. Nodal domains of the eigenfunctions of the path graph. It is well known that for $p = 2$, the upper bounds shown in the nodal theorem are tight, for any k. Simple examples where the those bounds are achieved for $p = 2$ are the line segment, in the continuous setting, and the path graph

$$P_n = \circ\!\!-\!\!\!-\!\!\!-\!\!\circ\!\!-\!\!\!-\!\!\!-\!\!\circ\!\!-\!\!\!-\!\!\!-\!\!\circ \cdots\cdots\cdots\cdots \circ$$

in the discrete case. For convenience, throughout this section we identify V with the integers set $\{1, \ldots, n\}$, and we fix both the vertex and the edge measures to be constantly one.

The eigenfunctions $f_k(x)$ of the continuous p-Laplacian on the line segment are known to be given for $p > 1$ by $f_k(x) = \sin_p(kx)$, where $\sin_p(x)$ is a special periodic function [23, 38]. However, dissimilar to the case $p = 2$, a direct computation reveals that the functions obtained by evaluating $f_k(x)$ on a uniform grid, are not the eigenfunctions of Δ_p on P_n for $p \neq 2$. The reason is that when $p \neq 2$, there is no addition formula relating $\sin_p$ and its derivative [35]. Since an explicit formula for the eigenfunctions of Δ_p on P_n when $p \neq 2$ is out of reach, we devote the remaining part of this section to show that the variational eigenpairs of the p-Laplacian on P_n have several special properties, and in particular we prove that the number of nodal domains induced by the eigenfunction of the variational eigenvalue λ_k on P_n, is exactly k.

For a function $f: V \to \mathbf{R}$ let us define $\tilde{f}: \mathbf{Z} \to \mathbf{R}$ as follows: first we define g by $g(i) = f(-i+1)$ for $i = 0, -1, \ldots, -n+1$ and $g = f$ over V; then we define $\tilde{f}$ by extending g periodically over $\mathbf{Z}$. The extension $\tilde{f}$ allows us to recast the eigenvalue equation (1) as the infinite system of nonlinear equations

$$\mathcal{H}(\lambda, \tilde{f}, k) = D\Phi_p D\tilde{f}(k) - \lambda\, \Phi_p(\tilde{f}(k+1)) = 0, \quad k \in \mathbf{Z}, \tag{8}$$

where D is the forward difference operator defined by $Df(k) = f(k+1) - f(k)$. One easily verifies that

$$(\Delta_p f)(u) = \lambda\, \Phi_p(f(u)), \quad \text{for all } u \in V \iff \mathcal{H}(\lambda, \tilde{f}, k) = 0, \text{ for all } k \in \mathbf{Z}. \tag{9}$$

It turns out that (8) is a particular version of a famous non-linear difference equation that has been studied quite intensively in the difference and differential equations literature (see e.g. [19, Chap. 3]). In the following any interval $[a, b]$ is meant to be discrete, i.e. $[a, b] = \{x \in \mathbf{Z}: a \le x \le b\}$. We shall say that $(a, a+1]$ is a generalized zero for f if $f(a) \neq 0$ and $f(a)f(a+1) \le 0$. Equation (9) is said to be disconjugate on $[a, b]$ provided that any solution of this equation has at most one generalized zero on $(a, b+1]$ and the solution f satisfying $f(a) = 0$ has no generalized zeros on $(a, b+1]$. The following generalized version of Sturm's comparison theorem is due to Rehák [41, Thm. 2].

Theorem 3.10. *Let $p > 1$, $\eta \ge \lambda$ and let $\tilde{f}, \tilde{g}$ be sequences such that $\mathcal{H}(\lambda, \tilde{f}, k) = \mathcal{H}(\eta, \tilde{g}, k) = 0$ for $s \le k \le t$. If $\tilde{g}$ is disconjugate on $[s, t]$ then $\tilde{f}$ is disconjugate on $[s, t]$ as well.*

We get as a direct consequence the following lemma.

Lemma 3.11. *Let (λ, f) and (η, g) be two eigenpairs of Δ_p on P_n with $\eta \ge \lambda$. If f and g have the same number of generalized zeros, then the generalized zeros of f and g coincide.*

Proof. The proof is a direct consequence of Theorem 3.10. We briefly sketch the argument. Let $(a_1, a_1+1], \dots, (a_k, a_k+1]$ and $(b_1, b_1+1], \dots, (b_k, b_k+1]$ be the generalized zeros of g and f respectively, ordering them so that $2 \le a_i + 1 \le a_{i+1} \le n-1$ and $2 \le b_i + 1 \le b_{i+1} \le n-1$, for $i = 1, \dots, k$. Using the symmetry of $\tilde{f}$ and $\tilde{g}$ one observes that $b_1 \ge a_1$, as otherwise Theorem 3.10 would be contradicted. This implies that $b_2 \ge a_2$, as $b_2 < a_2$ would imply that g is disconjugate on $[b_1, a_2+1]$, while f is not. Proceeding by induction we have $b_i \ge a_i$ for $i = 1, \dots, k$. A similar argument shows that $b_k \le a_k$, thus $b_i \le a_i$ for $i = 1, \dots, k$. □

Let us make a few further remarks. Let f be an eigenfunction on P_n. Then $f(1)f(n) \neq 0$. Indeed $f(1) = 0$ implies $0 = \mathcal{H}(\lambda, \tilde{f}, 0) = \Phi_p\big(f(2)\big)$ and thus $f(i) = 0$ for all $i \in V$. Similarly for $f(n)$. Moreover, the next lemma shows that all variational eigenvalues of P_n are distinct.

Lemma 3.12. *Let $p > 1$ and let $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$ be the variational eigenvalues of Δ_p on P_n. Then $0 < \lambda_2 < \lambda_3 < \cdots < \lambda_n$.*

Proof. Suppose λ and η are two variational eigenvalues with $\lambda = \eta$ and let f be any eigenfunction of λ. Arguing as in the proof of Theorem 3.5, there exists an eigenfunction g of η which is linear independent with respect to f. We can assume w.l.o.g. that $f(1) = g(1) = 1$. Then

$$\mathcal{H}(\lambda, \tilde{f}, 0) = \Phi_p(1 - f(2)) - \lambda = \Phi_p(1 - g(2)) - \lambda = \mathcal{H}(\lambda, \tilde{g}, 0),$$

implying $f(2) = g(2)$. By induction we get $f = g$, leading to a contradiction. □

The following theorem, finally, gives a complete description of the p-Laplacian nodal domains of the path graph, for any $p > 1$.

Theorem 3.13. *Let $p > 1$ and let f be an eigenfunction corresponding to the variational eigenvalue λ_k of Δ_p on P_n. Then the number of zero entries of f is at most $k - 1$, and it induces exactly k weak and strong nodal domains.*

Proof. Let us write $\nu(f)$ to denote the number of weak nodal domains of f. Observe that an eigenfunction of Δ_p on the path graph cannot vanish on two consecutive entries as otherwise it would be constantly zero. Indeed, if i is such that $f(i) \neq 0$ and $f(i+1) = 0$, then by (9) we have $\Phi_p(f(i+2)) = -\Phi_p(f(i))$. This implies that the number of zero entries of f is at most $\nu(f) - 1$.

Let us show that $\nu(f) = k$. The statement is true for $k = 1, 2$ due to Corollary 3.6. We proceed by induction. For $k > 2$ assume that $\nu(f) = k - 1$ for any eigenfunction f of λ_{k-1}, and let g be an eigenfunction of λ_k. Note that, as the multiplicity of each λ_k is one, by the nodal theorem if follows that $\nu(g) \leq k$. Arguing as in Lemma 3.11, using Theorem 3.10, one observes that the the overall number of generalized zeros of g cannot be less than the one of f. If follows that $\nu(g) \geq k - 1$. To complete the proof we show that $\nu(g) \neq k - 1$. To this end, we assume that $\nu(g) = k - 1$ and we show that this implies a contradiction. Equation (9) for f and g becomes

$$\lambda_{k-1}\, \Phi_p(f(i)) = \Phi_p(f(i) - f(i+1)) - \Phi_p(f(i-1) - f(i)), \qquad (10)$$

$$\lambda_k\, \Phi_p(g(i)) = \Phi_p(g(i) - g(i+1)) - \Phi_p(g(i-1) - g(i)). \qquad (11)$$

Consider the set $V_+ = \{i \in V : f(i)g(i) \neq 0\}$. We show by induction that the following inequalities hold

$$\Phi_p\Big(1 - \frac{f(i+1)}{f(i)}\Big) < \Phi_p\Big(1 - \frac{g(i+1)}{g(i)}\Big), \quad \text{for all } i \in V_+ \setminus \{n\}. \qquad (12)$$

As $k-1 = \nu(f) = \nu(g)$, then Lemma 3.11 implies that f and g have the same generalized zeros. Since $\tilde{f}(0) = f(1)$ and $\tilde{g}(0) = g(1)$, from $\lambda_{k-1} < \lambda_k$, (10) and (11) we get $\Phi_p(1-f(2)/f(1)) < \Phi_p(1-g(2)/g(1))$. We have $1 \in V_+$ and (12) holds for $i = 1$. We assume that $i-1 \in V_+$ satisfies (12), and show that the same holds for the next index in V_+. There are two possible cases: either $i \in V_+$, which we discuss next, or $i \notin V_+$, which we discuss below.

If $i \in V_+$ then $f(i)g(i) \neq 0$ and we derive from $\lambda_{k-1} < \lambda_k$, (10) and (11) that

$$\begin{aligned}\Phi_p\Big(1-\frac{f(i+1)}{f(i)}\Big) - \Phi_p\Big(1-\frac{g(i+1)}{g(i)}\Big)\\ < \Phi_p\Big(\frac{f(i-1)}{f(i)}-1\Big) - \Phi_p\Big(\frac{g(i-1)}{g(i)}-1\Big)\end{aligned} \tag{13}$$

Note that, as f and g have the same generalized zeros and

$$f(i)g(i)f(i-1)g(i-1) \neq 0,$$

then f and g have the same sign on $[i-1,i]$. Therefore $i-1 \in V_+$ and (13) imply that (12) holds for $i \in V_+$.

Now let us discus the case $i \notin V_+$. Note that, as (12) holds for $i-1 \in V_+$, $f(i)$ and $g(i)$ can not be both zero.

The case $g(i) = 0$ and $f(i) \neq 0$ is not possible. In fact, as (12) holds for $i-1 \in V_+$, then $\Phi_p\,(1-f(i)/f(i-1)) < 1$, showing that $f(i)f(i-1) > 0$. On the other hand $g(i) = 0$ implies that g has a generalized zero on $(i-1,i]$, yielding a contradiction.

Finally, if $f(i) = 0$ and $g(i) \neq 0$, then as (12) holds for $i-1$ we have $g(i-1)g(i) < 0$. Therefore $(i-1,i]$ is a generalized zero for g. Now note that $f(i) = 0$ implies that $(i,i+1]$ is not a generalized zero of f. Thus, by Theorem 3.10, $g(i)g(i+1) > 0$. We deduce that $i+1 \in V_+$ and from $\lambda_{k-1} < \lambda_k$, (10) and (11), we get

$$\Phi_p\Big(1-\frac{f(i+2)}{f(i+1)}\Big) - \Phi_p\Big(1-\frac{g(i+2)}{g(i+1)}\Big) < -1 - \Phi_p\Big(\frac{g(i)}{g(i+1)}-1\Big). \tag{14}$$

As $g(i)g(i+1) > 0$ we have $\Phi_p\,(g(i)/g(i+1)-1) > -1$ and we obtain from (14) that (12) holds for $i+1 \in V_+$.

Now observe that we can proceed the other way round to show that the following sequence of inequalities holds as well

$$\Phi_p\Big(1-\frac{f(i-1)}{f(i)}\Big) < \Phi_p\Big(1-\frac{g(i-1)}{g(i)}\Big), \quad \text{for all } i \in V_+ \setminus \{1\}. \tag{15}$$

In fact, since $f(n) = \tilde{f}(n+1) \neq 0$ and $g(n) = \tilde{g}(n+1) \neq 0$, then $n \in V_+$ and (15) holds for $i = n$. Thus we have the basis for the induction and we can repeat the same argument as before. To conclude we observe that there exist two consecutive indices m and $m+1$ in V_+, thus by plugging $i = m$ into (12) and $i = m+1$ into (15) we obtain a contradiction. To this end note that, as $f(1) \neq 0$, if there are no consecutive indices in V_+, then $f(i) = 0$ for all even indices i. Therefore (10) implies that $f(i)$ is nonzero for i odd, and we get $\lambda_{k-1}\Phi_p(f(1)) = \Phi_p(f(1))$ and $\lambda_{k-1}\Phi_p(f(3)) = 2\Phi_p(f(3))$, which is not possible. $\square$

4. Nodal properties of the 1-Laplacian

We devote this section to discuss the non-smooth case of the 1-Laplacian which becomes a set-valued operator. With the set-valued sign operator $\mathrm{Sign}(x) = \{1\}$ if $x > 0$, $\mathrm{Sign}(x) = \{-1\}$, if $x < 0$ and $\mathrm{Sign}(x) = [-1, 1]$ for $x = 0$, it is then straightforward to verify that the 1-Laplacian is the operator realizing the following entrywise identity [28],

$$(\Delta_1 f)(u) = \Big\{ \sum_{v \in V} w(uv) z(uv) \;\Big|\; z(uv) = -z(vu),\ z(uv) \in \mathrm{Sign}(f(u) - f(v)) \Big\}.$$

The corresponding eigenequation [28, 9] reads

$$0 \in (\Delta_1 f)(u) - \lambda\, \mu(u) \mathrm{Sign}(f(u)).$$

It has been shown that this is a necessary condition for a critical point of the associated non-smooth Rayleigh quotient $\mathcal{R}_1$ [28] via the Clarke subdifferential, and more recently also to be sufficient [9].

The classical Lusternik-Schnirelman theory can be extended to the case of a locally Lipschitz functional (see [8, Sec. 3] and [9]) and provides a variational characterization of the spectrum of Δ_1. In particular, as for $p > 1$, the following sequence

$$\lambda_k^{(1)} = \min_{A \in \mathcal{F}_k(\mathcal{S}_1)} \max_{f \in A} \mathcal{R}_1(f), \quad k = 1, \ldots, n,$$

defines a set of n variational eigenvalues of Δ_1.

Although the continuity of $\mathcal{R}_p$, with respect to p, implies $\lambda_k^{(p)} \to \lambda_k^{(1)}$, as p decreases to 1, the nodal domains structure of the corresponding eigenfunctions slightly changes for the limit case $p = 1$. In fact, our nodal domain theorems carry over to the case $p = 1$, but in a weaker form. The main difference is that the number of weak nodal domains of the k-th variational eigenfunction is upper bounded by $k + r - 1$ (where r is the multiplicity of the corresponding eigenvalue

$\lambda_k^{(1)}$) instead of k, as for $p > 1$. Note indeed that the proof of Theorem 3.4 holds unchanged if $p = 1$. This is not the case for the weak nodal domains. Thus, the nodal domain theorem for the 1-Laplacian reads as follows

Theorem 4.1. *Let G be connected and $0 = \lambda_1 < \cdots \leq \lambda_n$ be the variational eigenvalues of the 1-Laplacian. If λ_k has multiplicity r, then any eigenfunction of λ_k induces at most $k + r - 1$ strong and weak nodal domains.*

We finally show that Theorem 4.1 is tight by discussing the eigenfunctions for $p = 1$ of the unweighted path graph. For the sake of simplicity we consider this time the path graph P_3 on three vertices

$$P_3 = (v_1) \text{———} (v_2) \text{———} (v_3)\ .$$

With $z(v_2v_1) \in \mathrm{Sign}(f(v_2) - f(v_1))$ and $z(v_2v_3) \in \mathrm{Sign}(f(v_2) - f(v_3))$ we get the following system of equations for the eigenvalues and eigenfunctions of Δ_1, when $\mu(u) = \sum_v w(uv)$

$$\begin{cases} -z(v_2v_1) \in \lambda\,\mathrm{Sign}(f(v_1)), \\ z(v_2v_1) + z(v_2v_3) \in 2\,\lambda\,\mathrm{Sign}(f(v_2)), \\ -z(v_2v_3) \in \lambda\,\mathrm{Sign}(f(v_3)). \end{cases}$$

We show in the following that any non-constant eigenfunction has eigenvalue $\lambda = 1$. To this end we make a case distinction. If $f(v_1) > 0$, we have the cases

- $f(v_2) < f(v_1)$ implies $z(v_2v_1) = -1$, thus $\lambda = 1$,
- $f(v_2) > f(v_1)$ implies $z(v_2v_1) = 1$, thus $\lambda = -1$, which is a contradiction as $\lambda \geq 0$,
- $f(v_2) = f(v_1) > 0$ implies $z(v_2v_1) = -\lambda$ and thus $z(v_2v_3) = 3\lambda$ together with $-z(v_2v_3) \in \lambda\mathrm{Sign}(f(v_3))$ leads to a contradiction for $\lambda > 0$.

Similarly, if $f(v_1) = 0$ we have following the cases.

- $f(v_2) > f(v_1) = 0$ implies $1 + z(v_2v_3) = 2\lambda$. If $f(v_3) \leq 0$ one has $z(v_2v_3) = 1$ and this yields $\lambda = 1$. If $f(v_3) > 0$, then $z(v_2v_3) = -\lambda$ and thus $1 = z(v_2v_1) = 3\lambda$ which together with $z(v_2v_1) \in (-\lambda, \lambda)$ leads to a contradiction.
- $f(v_2) = f(v_1) = 0$ and $f(v_3) > 0$ yields $z(v_2v_3) = -1$ and thus $\lambda = 1$.

These are, up to sign, all the cases ones has to consider. In all the cases one gets the eigenvalue $\lambda = 1$. Thus the variational eigenvalues have to be $\lambda_2^{(1)} = \lambda_3^{(1)} = 1$.

One eigenfunction f for the eigenvalue $\lambda = 1$ is given by $f(v_1) = -f(v_2) = f(v_3) = 1$. This eigenfunction has three weak and strong nodal domains and thus the result for $p > 1$ that the number of weak nodal domains of the k-th eigenvalue with multiplicity r is upper bounded by k does not hold for the case $p = 1$. Moreover, our bound of $k + r - 1 = 2 + 2 - 1 = 3$ is tight for the given example.

5. A higher order Cheeger inequality via nodal domains

A set of multi-way Cheeger constants $h_k(G)$, $k = 2, 3, \ldots$ alternatively called high-order isoperimetric constants, has been recently studied by [14, 34, 36]. For $A \subseteq V$, let $E(A, \overline{A})$ be the set of edges having one endpoint in A and one in the complement of A, denoted as $\overline{A}$. Consider the quantity

$$c(A) = \frac{w(E(A, \overline{A}))}{\mu(A)},$$

where the measure of a discrete set is given by the sum of the weights of the elements in the set. Finally let $\mathcal{D}_k(G)$ be the set of k non-empty, mutually disjoint subsets of V, $\mathcal{D}_k(G) = \{\emptyset \neq A_1, \ldots, A_k \subseteq V : A_i \cap A_j = \emptyset\}$. The higher-order isoperimetric constants $h_k(G)$ are defined as

$$h_k(G) = \min_{\mathcal{A} \in \mathcal{D}_k(G)} \max_{A \in \mathcal{A}} c(A).$$

We conclude the paper by exploiting the relation among the high-order isoperimetric constants, the variational eigenvalues of Δ_p and their nodal domains.

Theorem 5.1. *For $p > 1$, let $f : V \to \mathbf{R}$ be an eigenfunction of Δ_p corresponding to the variational eigenvalue $\lambda_k^{(p)}$, and let m be the number of its strong nodal domains. Then*

$$\left(\frac{2}{\tau(G)}\right)^{p-1} \left(\frac{h_m(G)}{p}\right)^p \leq \lambda_k^{(p)} \leq 2^{p-1}\, h_k(G),$$

where $\tau(G) = \max_{u \in V} \frac{d(u)}{\mu(u)}$ and $d(u) = \sum_{v \in V} w(uv)$ is the degree of the vertex u.

This theorem is a direct generalization of Theorem 5 in Daneshgar et al [14], where the result has been proven for the linear graph Laplacian ($p = 2$) and $\mu(u) = d(u)$. Before discussing the proof of the theorem, let us briefly comment on the sharpness of the proposed Cheeger inequality. For $k = 2$ the result has been shown in [3, 6] for $p > 1$ and it has been noted there that the inequality becomes tight as $p \to 1$ as the second eigenfunction always has two strong nodal domains given that the graph is connected. The equality for $p = 1$ and $k = 2$ has been shown in [28], see also [9]. For $k > 2$ the situation changes as now an extra condition is required in order that the higher order Cheeger inequality becomes tight for $p \to 1$. Namely, as p approaches one, the number of strong nodal domains of the eigenfunction corresponding to the variational eigenvalue λ_k has to become equal to k. As discussed in the preceding section, the unweighted path graph is a graph with this property. However it is known that, when G is not a tree, the number of nodal domains of the eigenfunctions of $\lambda_k^{(2)}$ is in general less than k. In fact, for $p = 2$, the number of strong nodal domains induced by any eigenfunction f of $\lambda_k^{(2)}$ is at least $k + r - 1 - \ell - z$, where z is the number of vertices where f is zero, and ℓ the minimal number of edges that need to be removed from G in order to turn it into a tree [5, 44]. It remains an interesting open problem to generalize these lower bounds on the number of nodal domains to the nonlinear case $p \neq 2$.

The proof of Theorem 5.1 relies on the following Lemma which is of independent interest.

Lemma 5.2. *For any $f: V \to \mathbf{R}$ and any $p > 1$, there exists $A \subseteq \{u: f(u) \neq 0\}$ such that*

$$\mathcal{R}_p(f) \geq \Big(\frac{2}{\tau(G)}\Big)^{p-1}\Big(\frac{c(A)}{p}\Big)^p.$$

Proof. Consider the sets $E_0 = \{uv \in E: |f(u)|^p - |f(v)|^p = 0\}$, $E_+ = \{uv \in E: |f(u)|^p - |f(v)|^p > 0\}$ and, for $\lambda \geq 0$, $A_\lambda = \{u \in V: |f(u)|^p > \lambda\}$. By changing the order of summation and integration, and by the definition of $\mu(A_\lambda)$ we have

$$\int_0^\infty \mu(A_\lambda)d\lambda = \int_0^\infty \sum_{u\in A_\lambda} \mu(u)d\lambda = \sum_{u\in V}\int_0^{|f(u)|^p} \mu(u)d\lambda = \|f\|^p_{\ell^p(\mathcal{V})}. \tag{16}$$

Now we derive an upper bound for $\int_0^\infty w(E(A_\lambda, \overline{A_\lambda}))d\lambda$. Exchanging the role of integration and summation, as before, we have

$$\begin{aligned}\int_0^\infty w(E(A_\lambda, \overline{A_\lambda}))d\lambda &= \sum_{uv\in E_+} w(uv)\int_{|f(v)|^p}^{|f(u)|^p} d\lambda \\ &= \frac{1}{2}\sum_{uv\in E} w(uv)\big||f(u)|^p - |f(v)|^p\big|.\end{aligned} \tag{17}$$

Moreover, if q is the Hölder conjugate of p, then Hölder's inequality implies

$$\begin{aligned}
&\frac{1}{2}\sum_{uv\in E} w(uv)\big||f(u)|^p-|f(v)|^p\big| \\
&\quad=\sum_{uv\notin E_0}\frac{w(uv)}{2}|f(u)-f(v)|\left|\frac{|f(u)|^p-|f(v)|^p}{f(u)-f(v)}\right| \\
&\quad\leq\left\{\frac{1}{2}\sum_{uv\in E} w(uv)|f(u)-f(v)|^p\right\}^{\frac{1}{p}}\left\{\sum_{uv\notin E_0}\frac{w(uv)}{2}\left|\frac{|f(u)|^p-|f(v)|^p}{f(u)-f(v)}\right|^q\right\}^{\frac{1}{q}}.
\end{aligned} \tag{18}$$

We use now the following inequality from [3], which holds for any $x, y \in \mathbf{R}$ and $p > 1$,

$$\left(\frac{1}{p}\left|\frac{|x|^p-|y|^p}{x-y}\right|\right)^q \leq \left(\frac{1}{p}\left|\frac{|x|^p-|y|^p}{|x|-|y|}\right|\right)^q \leq \frac{|x|^p+|y|^p}{2}$$

to get

$$\begin{aligned}
\sum_{uv\notin E_0}\frac{w(uv)}{2}\left|\frac{|f(u)|^p-|f(v)|^p}{f(u)-f(v)}\right|^q &\leq \frac{p^q}{4}\sum_{uv\in E} w(uv)(|f(u)|^p+|f(v)|^p) \\
&\leq \frac{p^q\,\tau(G)\,\|f\|^p_{\ell^p(\mathcal{V})}}{2}.
\end{aligned}$$

Thus, together with (16), (17), and (18), we finally get the inequality

$$\frac{\displaystyle\int_0^\infty w(E(A_\lambda,\overline{A_\lambda}))d\lambda}{\displaystyle\int_0^\infty \mu(A_\lambda)d\lambda} \leq p\left(\frac{\frac{1}{2}\sum_{uv} w(uv)|f(u)-f(v)|^p}{\sum_u \mu(u)|f(u)|^p}\right)^{1/p}\left(\frac{\tau(G)}{2}\right)^{1/q}.$$

Since w ad μ are positive functions, we get

$$\frac{\displaystyle\int_0^\infty w(E(A_\lambda,\overline{A_\lambda}))d\lambda}{\displaystyle\int_0^\infty \mu(A_\lambda)d\lambda} \geq \inf_{\lambda\geq 0}\frac{w(E(A_\lambda,\overline{A_\lambda}))}{\mu(A_\lambda)},$$

which shows in turn that there exists $\lambda_* \in [0,\infty)$ such that

$$c(A_{\lambda_*}) \leq p\,\mathcal{R}_p(f)^{1/p}\left(\frac{\tau(G)}{2}\right)^{1/q}.$$

Finally, as $A_{\lambda_*} \subseteq \{u\colon f(u) \neq 0\}$ by construction, the statement follows. □

Proof of Theorem 5.1. Let $A_1, \ldots, A_m$ be the strong nodal domains of f. Lemma 3.8 implies $\lambda_k^{(p)} \geq \mathcal{R}_p(f|_{A_i})$, for any $i = 1, \ldots, m$. Moreover, by applying Lemma 5.2, we deduce that for any i there exists $B_i \subseteq A_i$ such that

$$\mathcal{R}_p(f|_{A_i}) \geq (2/\tau(G))^{p-1} (c(B_i)/p)^p \ .$$

As the nodal domains are disjoint and non-empty, they belong to $\mathcal{D}_m(G)$. We get

$$\begin{aligned}\max_{i=1,\ldots,m} \mathcal{R}_p(f|_{A_i}) &\geq \min_{\{B_i\}\in\mathcal{D}_m(G)} \max_{i=1,\ldots,m} \Big(\frac{2}{\tau(G)}\Big)^{p-1} \Big(\frac{c(B_i)}{p}\Big)^p \\ &= \Big(\frac{2}{\tau(G)}\Big)^{p-1} \Big(\frac{h_m(G)}{p}\Big)^p,\end{aligned}$$

which finishes the proof of the first inequality in the statement. For the second one, let χ_A denote the indicator function of $A \subseteq V$. Note that $\mathcal{R}_p(\chi_A) = c(A)$, and let $\{A_1^*, \ldots, A_k^*\} \subseteq \mathcal{D}_k(G)$ be such that $h_k(G) = \max_{i=1,\ldots,k} c(A_i^*)$. Let $\mathcal{X}$ be the span of $\chi_{A_1^*}, \ldots, \chi_{A_k^*}$. For any $g \in \mathcal{X}$, that is $g(u) = \sum_{i=1}^k \alpha_k \chi_{A_i}(u)$, we have

$$\sum_{u\in V} \mu(u)|g(u)|^p = \sum_{i=1}^k \sum_{u\in A_i^*} \mu(u)|\alpha_i \chi_{A_i^*}(u)|^p = \sum_{i=1}^k |\alpha_i|^p \sum_{u\in V} \mu(u)|\chi_{A_i^*}(u)|^p.$$

Using the fact that $A_i^* \cap A_j^* = \emptyset$ for $i \neq j$, we get

$$\begin{aligned}|g(u) - g(v)|^p &= \Big|\sum_{i=1}^k \alpha_i (\chi_{A_i^*}(u) - \chi_{A_i^*}(v))\Big|^p \\ &\leq 2^{p-1} \sum_{i=1}^k |\alpha_i|^p |\chi_{A_i^*}(u) - \chi_{A_i^*}(v)|^p\end{aligned}$$

and we obtain as a consequence

$$\begin{aligned}\mathcal{R}_p(g) &\leq 2^{p-1} \frac{\sum_{i=1}^k |\alpha_i|^p \sum_{uv} w(uv)|\chi_{A_i^*}(u) - \chi_{A_i^*}(v)|^p}{\sum_{i=1}^k |\alpha_i|^p \sum_u \mu(u)|\chi_{A_i^*}(u)|^p} \\ &\leq 2^{p-1} \max_{i=1,\ldots,k} \mathcal{R}_p(\chi_{A_i^*})\end{aligned}$$

where we have used the inequality $(\sum_i a_i)/(\sum_i b_i) \leq \max_i a_i/b_i$, which holds for $a_i, b_i \geq 0$. Finally note that $\gamma(\mathcal{X} \cap \mathcal{S}_p) = k$ by construction, therefore $\mathcal{X} \cap \mathcal{S}_p \in \mathcal{F}_k(\mathcal{S}_p)$ and the latter inequality implies $\lambda_k^{(p)} \leq \max_{g\in\mathcal{X}\cap\mathcal{S}_p} \mathcal{R}_p(g) \leq 2^{p-1} h_k(G)$, concluding the proof. $\square$

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