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Nearly radial Neumann eigenfunctions on symmetric domains

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Abstract. We study the existence of Neumann eigenfunctions of the Laplacian which do not change sign on the boundary of some special domains. We show that eigenfunctions which are strictly positive on the boundary exist on regular polygons with at least 6 sides, while on equilateral triangles and cubes it is not even possible to find an eigenfunction which is nonnegative on the boundary. This work builds on recent results of Hoffmann-Ostenhof about rectangles.

We use analytic methods combined with symmetry arguments to prove the result for polygons with six or more sides, and combinatorics for equilateral triangles and cubes. The case of the regular pentagon is much harder. Its proof requires deep computational and numerical results which are beyond the scope of the present paper. The pentagonal case, codeveloped with Nilima Nigam and Benjamin Young, will appear in a companion paper.

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1. Introduction

We study the existence of an eigenfunction of the Neumann Laplacian which is positive (or nonnegative) on the boundary of highly symmetric domains. Recently, Hoffmann-Ostenhof [11] proved that on rectangles, any Neumann eigenfunction that is positive on the boundary must be constant. In this paper we prove similar results for regular polygons and higher dimensional boxes.

Schiffer's conjecture (see [21]) states that if a Neumann eigenfunction is constant on the boundary of a domain, then either the eigenfunction is constant in the domain, or the domain must be a disk. This conjecture is still open, although many partial results are known (see e.g. [4, 5, 8]). We relax the boundary restriction (positive instead of constant) and ask if the modified conjecture holds for a special class of domains. All domains considered in our paper are open and simply connected, although the last restriction is not necessary for most arguments.

Our problem has a rather interesting physical interpretation in terms of a sloshing liquid in a cup with a uniform, highly symmetric cross-section (see [12] for a relation between sloshing and Neumann eigenvalue problem). It is nearly obvious that one can disturb a fluid in a round cup so that the created wave is radial. In particular the fluid level rises and lowers simultaneously along the whole cup wall. Hoffmann-Ostenhof [11] proved that it is possible to create a wave in a square cup so that there are a few stationary points along the wall, but it is impossible to make all points move in unison. We prove that no such wave can be created in a triangular cup, even if stationary points are allowed. At the same time, it is possible to create a wave with unison movement along the boundary of regular polygons with at least 6 sides. In summary, we have the following results.

Theorem 1.1. *Any Neumann eigenfunction that is* ***nonnegative*** *on the boundary of an equilateral triangle is constant inside.*

Theorem 1.2 (Hoffmann-Ostenhof [11]). *Any Neumann eigenfunction that is* ***positive*** *on the boundary of a rectangle is constant inside.*

Theorem 1.3. *There exists a Neumann eigenfunction on a regular polygon with* $n \geq 6$ *sides that is* ***positive*** *on the boundary and not constant.*

Remark. The pentagonal case (missing from the above theorems) is similar to $n \geq 6$ cases, but the existing proof is remarkably more complicated, and uses rigorous computer algorithms and validated numerical methods. It will be published as a separate paper [18] coauthored with Nilima Nigam and Benjamin Young. The arguments are also included in our extended preprint [17].

Remark. Squares are in some sense a critical case for regular polygons. An eigenfunction that is positive on the boundary does not exist, yet

$$\varphi(x, y) = -\cos \pi x - \cos \pi y$$

is an eigenfunction of the square $[-1, 1]^2$. It is positive on the boundary, except at the midpoints of all sides (where it equals 0).

We also study higher dimensional boxes. Surprisingly, cubes no longer have nonnegative eigenfunctions.

Theorem 1.4. *Any Neumann eigenfunction that is nonnegative on the boundary of a cube (more generally a box) in dimension* $d > 2$ *is constant inside.*

Careful Finite Element computations suggest the following conjecture.

Conjecture 1.5. *Eigenfunctions which are nonnegative on the boundary of a tetrahedron and octahedron do not exist. However, eigenfunctions which are positive on the boundary exist on dodecahedron and icosahedron.*

The paper uses a variety of methods to handle the different cases. In particular, we use combinatorial and number theoretic results on cubes (Section 4) and equilateral triangles (Section 3). We dissect regular polygons with $n \geq 5$ into congruent right triangles and study their Neumann eigenfunctions. For $n \geq 6$ we can use existing results on the shape of the second Neumann eigenfunction to draw the necessary conclusions (Section 5). However, this last step does not work for regular pentagons ($n = 5$).

The proof for the regular pentagon, contained in the preprint [17] and an upcoming companion paper [18], is more complicated. In there, we show that the nodal line for the second Neumann eigenfunction of a right triangle must connect two longest sides, but this seemingly simple fact is extremely hard to prove. Similar results for obtuse triangles have been obtained by Atar and Burdzy [1] using very sophisticated probabilistic techniques. Some theoretical results needed in [18] are contained in Appendix A. They are included in the present paper as they could be of independent interest in Spectral Theory. Nevertheless, due to strongly computational nature of the proof for the pentagon, and invaluable input from Nilima Nigam and Benjamin Young, the core of the argument will appear as a separate manuscript in a journal devoted to computational aspects of mathematics.

2. Definitions and auxiliary results

The Neumann eigenvalue problem can be approached classically, by solving the partial differential equation

$$\begin{aligned} \Delta u_n &= -\mu_n u_n \quad && \text{in } D, \\ \partial_\nu u_n &= 0 && \text{on } \partial D. \end{aligned}$$

However, it is often more useful to work with the variational weak formulation

$$\mu_n = \inf_{\substack{S \subset H^1(D) \\ \dim S = n}} \sup_{u \in S} \frac{\int_D |\nabla u|^2}{\int_D u^2}, \tag{1}$$

where $H^1(D)$ is the Sobolev space of all functions $u \in L^2(D)$ such that $\nabla u \in (L^2(D))^2$. The right side of (1) is commonly called the *Rayleigh–Ritz* quotient. In this context the minimizers of the Rayleigh–Ritz quotient are the eigenfunctions. For Lipschitz domains (and even more general domains for which appropriate Sobolev embeddings exist) the two approaches lead to the same eigenvalues and the same eigenfunctions (via elliptic regularity considerations). For a broad overview on this topic see [3] and [6].

Note that the variational characterization lacks any obvious boundary conditions. This is a consequence of the Neumann (also called natural) boundary condition being automatically enforced by the Sobolev spaces. In contrast, to enforce Dirichlet boundary condition one seeks minimizers of the Rayleigh–Ritz quotient over a subset of $H^1(D)$ consisting of functions with zero trace on the appropriate part of the boundary of the domain.

In general it is true that

$$0 = \mu_1 < \mu_2 \leq \mu_3 \leq \cdots \leq \mu_n \to \infty.$$

However in some special cases one can show that μ_2 is simple. In particular,

Lemma 2.1 ([19, Theorem 1]). *For non-equilateral triangles μ_2 is simple.*

We will also work with a mixed Dirichlet–Neumann eigenvalue problem:

$$\begin{aligned} \Delta u_n &= -\lambda_n u_n && \text{in } D, \\ u_n &= 0 && \text{on } B \subset \partial D, \\ \partial_\nu u_n &= 0 && \text{on } \partial D \setminus B. \end{aligned}$$

Note that eigenfunctions satisfy Dirichlet boundary conditions on B, and the appropriate variational formulation must include the same restriction.

$$\lambda_n = \inf_{\substack{S \subset H_B^1(D) \\ \dim S = n}} \sup_{u \in S} \frac{\int_D |\nabla u|^2}{\int_D u^2},$$

where $H_B^1(D)$ is a subspace of $H^1(D)$ consisting of all functions satisfying $u = 0$ on B. If $meas(B) > 0$, we have

$$0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \to \infty.$$

In what follows we need a few geometric results from [19].

Lemma 2.2 ([19, Lemma 4]). *Suppose D is a domain with a line of symmetry. Then there cannot be two orthogonal antisymmetric eigenfunctions in the span of the eigenspaces of μ_2 and μ_3 (note that μ_2 might equal μ_3).*

Lemma 2.3 (Special case of [19, Lemma 5]). *The nodal line for the second Neumann eigenfunction on a triangle must start on one side and end on another side or vertex connecting the other two sides.*

Let us introduce the following naming convention for isosceles triangles.

Definition. A triangle is superequilateral (subequilateral) if it is isosceles with aperture angle larger (smaller) than $\pi/3$.

Extensive numerical studies suggest that Lemma 2.3 can be strengthened as follows.

Conjecture 2.4. *The nodal line for the second Neumann eigenfunction ends in a vertex only for superequilateral triangles. For all other non-equilateral triangles the nodal line connects the two longest sides.*

As part of the proof of Theorem 1.3 we show that the conjecture holds for right triangles with the smallest angle no larger than $\pi/6$. In our companion paper [18] we give a proof for the case of the smallest angle equal approximately $\pi/5$. Note that an even stronger conjecture was posed by Atar and Burdzy [1, Conjecture 3.2] for obtuse triangles, where the nodal line would be confined in the right triangle bounded by two long sides and the altitude perpendicular to the longest side.

3. Equilateral triangle: proof of Theorem 1.1

The Neumann spectrum for an equilateral triangle can be split into symmetric modes $\varphi_{m,n}$ and antisymmetric modes $\psi_{m,n}$, forming eigenspaces of eigenvalues $\lambda_{m,n}$, $n \geq m \geq 0$. Note that $\lambda_{m,n}$ might be equal for different pairs of integers m, n. This means that there exist eigenfunctions of an equilateral triangle that combine many different modes. For more details see McCartin [15].

Theorem 8.1 from [15] states that the symmetric modes never vanish, while the antisymmetric modes degenerate when $m = n$. Furthermore, Theorem 8.2 from [15] gives that both types of modes are rotationally symmetric if and only if $m \equiv n \pmod 3$. Finally, any symmetric mode that is not rotationally symmetric can be written as a sum of two rotated (by 120 and 240 degrees) antisymmetric modes. More precisely, denoting the rotations of $\psi_{m,n}$ by $\psi_{m,n,120}$ and $\psi_{m,n,240}$ we have $\varphi_{m,n} = \psi_{m,n,120} + \psi_{m,n,240}$.

Suppose f is an eigenfunction for some λ for an equilateral triangle with horizontal side s_1. Then f is a linear combination of the symmetric and antisymmetric modes (with respect to the altitude a_1 perpendicular to s_1). If $\varphi_{m,n}$ is symmetric but not rotationally symmetric, we rewrite it using two antisymmetric modes. Therefore

$$f = \sum_{m,k\geq 0} a_{m,m+3k}\varphi_{m,m+3k} + \sum_{m\not\equiv n \pmod 3} a_{m,n}(\psi_{m,n,120} + \psi_{m,n,240}) + \sum_{m\neq n} b_{m,n}\psi_{m,n}.$$

Suppose f is nonnegative on the boundary; let G be the group of isometries of the equilateral triangle. Then $f \circ U$, $U \in G$ is also nonnegative on the boundary. Furthermore, the orbit of G on any antisymmetric mode ψ has size 6 (all possible rotations and reflections are different) or length 2 (if ψ is rotationally symmetric). At the same time, the sum of the reflections of ψ along the line of antisymmetry, and in particular at the midpoints of the boundary edges, cancel out. Therefore the function

$$F = \sum_{U\in G} f \circ U = 6\sum_{m,k\geq 0} a_{m,m+3k}\varphi_{m,m+3k}. \tag{2}$$

is also nonnegative on the boundary. It follows that Theorem 1.1 needs to be proved only for eigenfunctions of the form F.

Consider the equilateral triangle with vertices $(0,0)$, $(1,0)$ and $(1/2, \sqrt{3}/2)$. We have the following symmetric modes (see e.g. [15])

$$\begin{aligned}\varphi_{m,n}(x,y) &= (-1)^{m+n}\cos\left(\frac{1}{3}\pi(2x-1)(m-n)\right)\cos\left(\frac{2}{\sqrt{3}}\pi(m+n)y\right)\\ &\quad + (-1)^m\cos\left(\frac{1}{3}\pi(2x-1)(m+2n)\right)\cos\left(\frac{2}{\sqrt{3}}\pi my\right)\\ &\quad (-1)^n\cos\left(\frac{1}{3}\pi(2x-1)(2m+n)\right)\cos\left(\frac{2}{\sqrt{3}}\pi my\right),\end{aligned} \tag{3}$$

with corresponding eigenvalues

$$\lambda_{m,n} = \frac{16\pi^2}{9}(m^2+mn+n^2).$$

Note that an eigenvalue λ might have a high multiplicity, since many different pairs of integers (m,n) might produce the same value λ. Therefore more than one pair of integers might belong to a particular eigenvalue λ.

We find that the rotationally symmetric modes satisfy

$$\varphi_{m,m+3k}(x,0) = \cos(2\pi kx) + \cos(2\pi(m+2k)x) + \cos(2\pi(m+k)x). \tag{4}$$

Note that we only need to consider the values of $\varphi_{m,m+3k}$ on one side, so we assumed $y = 0$. This last quantity integrates to 0 on (0, 1), the side of the triangle, unless $k = 0$. This means that a pair (M, M) must belong to λ if the eigenfunction associated to λ has nonnegative boundary values. This implies that all other pairs (m, n) for the same λ satisfy $3M^2 = m^2 + mn + n^2$.

3.1. Eigenfunctions positive on the boundary. Note that if both m and n are even, then m^2+mn+n^2 is even, otherwise it is odd. Therefore an even M implies even m, n. By induction, any pair (m, n) belonging to eigenvalue λ must have both m and n divisible by 2^s and at least one not divisible by 2^{s+1}, whenever 4^s divides M^2, but $2 \cdot 4^s$ does not. Hence there exists s such that for any pair $(m, m + 3k)$ that belongs to λ we must have $m = 2^s m_1$ and $k = 2^s k_1$, where at most one of the k_1 and m_1 is even. Therefore

$$\begin{aligned}
\varphi_{m,m+3k}(2^{-s-1}, 0) &= \cos(\pi k_1) + \cos(\pi(m_1 + 2k_1)) + \cos(\pi(m_1 + k_1)) \\
&= (-1)^{k_1} + (-1)^{m_1} + (-1)^{m_1+k_1} \\
&= -1, \\
\varphi_{m,m+3k}(0, 0) &= 3.
\end{aligned}$$

We have proved that all $\varphi_{m,m+3k}$ that belongs to λ must equal -1 at the same point on the boundary (and 3 at vertices). Hence any linear combination of such eigenfunctions with $\sum_{m,k} a_{m,m+3k} \neq 0$ in F must change sign. Also, unconditionally the eigenfunction cannot be strictly positive.

This argument is very similar to the one used by Hoffmann-Ostenhof [11] on squares to prove nonexistence of eigenfunctions positive on the boundary. It is however impossible to rule out the existence of eigenfunctions nonnegative on the boundary using this method. We might be able to prove that the linear combination must equal 0 at many points, but not that it changes sign.

3.2. Eigenfunctions nonnegative on the boundary. Here we develop an improved method based on the fact that Neumann eigenfunction cannot vanish on an open subset of the boundary (it would satisfy Dirichlet condition). Therefore, if we find an open set on which eigenfunction integrates to 0, it must change sign on that set.

We only need to work with $\lambda = \frac{16\pi^2}{9} 3M^2$ ($m = M, k = 0$ is admissible in (4), see comment below that formula). It is also possible that other pairs (m, k) with $k > 0$ give the same λ. In that case we have

$$3M^2 = 3m^2 + 3mk + k^2.$$

It is easy to check that

$$M < m + k < m + 2k < 2M, \quad 0 < k < M. \tag{5}$$

Consider points

$$x_i = \frac{2i+1}{2M}, \quad 0 \le i < M,$$

and integrals over symmetric intervals around these points

$$\begin{aligned}\sum_{i=0}^{M-1} \int_{x_i-a}^{x_i+a} \cos(\alpha x)\, dx &= \frac{2}{\alpha} \sin(\alpha a) \sum_{i=0}^{M-1} \cos(\alpha x_i) \\ &= \frac{2}{\alpha} \sin(\alpha a) \cos(\alpha/2) \sin(\alpha/2) \csc(\alpha/2M)\end{aligned}$$

by [9, Section 1.341, Formula 3], as long as $\sin(\alpha/2M) \neq 0$.

Examining formula (4) we find $\alpha = 2\pi k$, $2\pi(m+k)$ and $2\pi(m+2k)$. In each case $\sin(\alpha/2M) \neq 0$ due to (5). Furthermore $\sin(\alpha/2) = 0$, hence the eigenfunctions $\varphi_{m,m+3k}$ with $k > 0$ integrate to 0 over the union of $(x_i - a, x_i + a)$. We only need to show the same property for for the eigenfunction with $k = 0$:

$$\varphi_{M,M}(x, 0) = 1 + 2\cos(2\pi M x). \tag{6}$$

We have

$$\begin{aligned}\sum_{i=0}^{M-1} \int_{x_i+a}^{x_i-a} \varphi_{M,M}(x, 0)\, dx &= \sum_{i=0}^{M-1} \left(2a + \frac{2}{\pi M} \sin(2\pi M a) \cos((2i+1)\pi)\right) \\ &= 2aM - \frac{2}{\pi} \sin(2\pi M a).\end{aligned}$$

Let $z_0 = 2\pi M a_0$ and find a positive solution of $z_0 = 2\sin(z_0)$. We get $z_0 < \pi$ and $a_0 < 1/2M$. Hence intervals $(x_i - a_0, x_i + a_0)$ fit inside $(0, 1)$, the side of the equilateral triangle. At the same time, any linear combination of eigenfunctions from (4) integrates to 0 over the union of these intervals. Hence it must change sign, as it cannot satisfy both Dirichelt and Neumann conditon on any interval (be identically 0 on any interval).

4. Cubes: proof of Theorem 1.4

4.1. Fully symmetric eigenfunctions. Consider the cube $C = [-1,1]^n$. Suppose it has a Neumann eigenfunction that is positive (nonnegative) on the boundary. We can symmetrize this eigenfunction by applying all isometries of the cube and summing the resulting eigenfunctions (as in the equilateral triangle case). We will get a new eigenfunction that is positive (nonnegative) on the boundary, symmetric with respect to $x_i = 0$ for any i and invariant under arbitrary permutation of variables x_i. We only need to prove that this fully symmetric eigenfunction cannot be positive (nonnegative) on the boundary.

Any symmetric eigenfunction can be written as a sum of simple eigenfunctions of the form

$$(-1)^{\sum m_i} \prod_{i=1}^{n} \cos(m_i \pi x_i).$$

The factor $(-1)^{\sum m_i}$ ensures positivity in all vertices ($x_i = \pm 1$). Invariance under permutations of variables gives

$$\varphi_\lambda(x) = \sum_{\substack{\sum m_i^2 = \lambda \\ M = \{m_1 \le \dots \le m_n\}}} a_M (-1)^{\sum m_i} \sum_{\sigma_n} \prod_{i=1}^{n} \cos(m_{\sigma_n(i)} \pi x_i), \tag{7}$$

where σ_n denotes any permutation of $\{1,\dots,n\}$ and a_M are arbitrary coefficients depending on the nondecreasing sequence of nonnegative integers m_i. We require that $\sum m_i^2 = \lambda$ to ensure all terms belong to the same eigenvalue. Formula (7) gives the most general form of the eigenfunction that is invariant under the group of the isometries of the cube C. We need to show that this eigenfunction is negative somewhere on the boundary of the cube, regardless of the choice of λ. Due to symmetry we only need to consider one face.

Note also, that φ_λ is also a linear combination of eigenfunctions of the lower dimensional Laplacian on a face. Indeed, fixing $x_1 = 1$ gives a sum of products of cosines, hence again a symmetric function. However, due the presence of the permutations σ_n, we drop different m_i in different terms, and hence we get a sum of eigenfunctions for various eigenvalues. Every non-constant Neumann eigenfunction is orthogonal to the constant eigenfunction. Hence φ_λ integrates to 0 on each face, unless an eigenfunction which is constant on the face is a part of φ_λ, cf. the discussion below (4) pertaining to equilateral triangles.

Therefore the sequence $m_1 = \cdots = m_{n-1} = 0$, $m_n = m = \sqrt{\lambda}$ gives one of the terms in φ_λ. Consequently, $\lambda = m^2$ for some integer m. Otherwise φ_λ integrates to 0 over any face, hence it must change sign on each face.

4.2. Positive eigenfunctions. We begin with a special case to illustrate the approach. Suppose $\lambda = m^2$ with odd m. Recall that φ_λ is a sum over all sequences $m_1 \le m_2 \le \cdots \le m_n$ such that

$$m_1^2 + \cdots + m_n^2 = \lambda = m^2.$$

Hence at least one m_i is odd. Consider a discrete set of points:

$$X = \{(x_1 \ge x_2 \ge \cdots \ge x_n) : x_i \in \{0, 1\}\}.$$

These points correspond to the center of the cube $(0, \dots, 0)$, the center of the face $(1, 0, \dots, 0)$, the centers of all lower dimensional faces, finally a vertex $(1, \dots, 1)$. Let X_0 be the set

$$X_0 = \{(1 = x_1 \ge x_2 \ge \cdots \ge x_n = 0) : x_i \in \{0, 1\}\}.$$

Note that all points in X_0 are on one face of the cube. For any $x \in X$ put $k = \sum_{i=1}^n x_i$ (the codimension of the face for which x is a center). Observe that

$$\begin{aligned}
&\sum_{x \in X_0} \frac{1}{(n - \sum x_i)!} \varphi_\lambda(x) \\
&\quad = \sum_{k=1}^{n-1} \sum_{\substack{\sum m_i^2 = \lambda \\ M = \{m_1 \le \cdots \le m_n\}}} a_M (-1)^{\sum m_i} \frac{1}{(n-k)!} \sum_{\sigma_n} \prod_{i=1}^{k} \cos(m_{\sigma_n(i)} \pi). \\
&\quad = \sum_{k=1}^{n-1} \sum_{\substack{\sum m_i^2 = \lambda \\ M = \{m_1 \le \cdots \le m_n\}}} a_M (-1)^{\sum m_i} \frac{1}{(n-k)!} \sum_{\sigma_n} (-1)^{\sum_{i=1}^k m_{\sigma_n(i)}}. \\
&\quad = \sum_{\substack{\sum m_i^2 = \lambda \\ M = \{m_1 \le \cdots \le m_n\}}} a_M (-1)^{\sum m_i} \sum_{k=1}^{n-1} \frac{1}{(n-k)!} \sum_{\sigma_n} (-1)^{\sum_{i=1}^k m_{\sigma_n(i)}}.
\end{aligned}$$

Note that in the innermost sum each term appears exactly $(n - k)!$ times, since we are using only the first k values of each σ_n. Hence we are adding (exactly once) all products of $(-1)^{m_i}$, except for the full and empty product, so we may rewrite this as

$$\sum_{x \in X_0} \frac{1}{(n - \sum x_i)!} \varphi_\lambda(x) = \sum_{\substack{\sum m_i^2 = \lambda \\ M = \{m_1 \leq \cdots \leq m_n\}}} a_M (-1)^{\sum m_i} \Big[\prod_{i=1}^{n} (1 + (-1)^{m_i}) - (-1)^{\sum m_i} - 1 \Big].$$

But at least one m_i is odd, hence the product in the bracket is 0. Furthermore, the sum of m_i is also odd, hence the whole bracket is 0. Thus

$$\sum_{x \in X_0} \frac{1}{(n - \sum x_i)!} \varphi_\lambda(x) = 0.$$

Therefore either φ_λ is 0 at the centers of faces of arbitrary dimension, or φ_λ must change sign. To prove the eigenfunction must change sign we will use a different method, similar to the one for equilateral triangles (Section 3.2). For the moment, we can show that the eigenfunction cannot be positive on the boundary for a few low-dimensional cases with an argument about the parity of the m_i.

Proposition 4.1. *In dimensions* 2, 3, *and* 4, *any positive Neumann eigenfunction on a cube must be constant.*

Remark. Dimension 2 was proved by Hoffmann-Ostenhof [11].

Proof. We only need to consider even m. For $0 < h < 1$ define

$$X_h = \{(1 = x_1 \geq x_2 \geq \cdots \geq x_n = h) : x_i \in \{h, 1\}\}.$$

As above we get

$$\sum_{x \in X_h} \frac{1}{(n - \sum 1_{x_i}(1))!} \varphi_\lambda(x) = \sum_{\substack{\sum m_i^2 = \lambda \\ M = \{m_1 \leq \cdots \leq m_n\}}} a_M \Big[\prod_{i=1}^{n} (\cos(m_i \pi h) + (-1)^{m_i}) - 1 - \prod_i \cos(m_i \pi h) \Big],$$

We now consider each dimension separately.

- *Dimension* 2. The sum of the squares of two odd numbers is congruent to 2 modulo 4, hence it is not a square. Therefore, if m is even, then both m_i are even. Furthermore, by induction 2^s divides both m_i, but 2^{s+1} divides exactly one of them. Take $h = 1/2^s$. Then $\cos(m_i\pi/2^s)$ have both signs. But both m_i are even, hence the first product in the bracket is 0, and the second product equals -1. Hence the whole bracket equals 0.

- *Dimension* 3. The sum of three squares involving exactly two odd numbers is congruent to 2 modulo 4, hence it cannot be a square. Since m is even, we get that all m_i are even. By induction, there exists s such that 2^s divides all m_i, while 2^{s+1} divides none or two. In either case, the first product equals 0, and the second equals -1. Hence the bracket is again 0.

- *Dimension* 4. The sum of k odd squares is congruent to k modulo 8. Hence only 1 or 4 odd squares can give a square. Suppose some m_i are odd. Since m is even, all m_i must be odd and 4 does not divide m. Since all m_i are odd, $\cos(m_i\pi/2) = 0$ and the first product equals 1. The second product is obviously 0 and the bracket is again 0. If all m_i are even, but 4 does not divide m, then exactly one of the $m_i/2$ is odd. Therefore the first product is 0 and the second equals -1. Again the bracket is 0. Finally, suppose 4 divides m. Then 4 divides all m_i, and we can reduce the problem to $m' = m/4$ and apply the same argument recursively. □

Remark. In dimension 5 we have $36 = 6^2 = 4 \times 3^2$. The first decomposition does give 0 in the bracket. However the second gives 1.

In dimension 6 we have $36 = 6^2 = 2 \times 4^2 + 4 \times 1^2 = 5^2 + 2 \times 2^2 + 3 \times 1^2 = 5^2 + 3^2 + 2 \times 1^2$. Hence in dimensions 6 and higher, any integer smaller than m may appear in the decomposition for m^2. Therefore an argument based on divisibility will most likely fail.

4.3. Nonnegative eigenfunctions. To prove Theorem 1.4 we will generalize the approach used on equilateral triangles in Section 3.2. We will show that φ_λ integrates to 0 over a union of small cubes with codimension one contained in one of the faces. Since an eigenfunction cannot equal 0 on an open subset of the boundary (it already satisfies the Neumann condition there), it must change sign in the union of these cubes. Note also that it is irrelevant if these cubes are disjoint, but they must be subsets of the face.

Suppose that $\lambda = m^2$ and consider the following set of points uniformly distributed on $(-1, 1)$.

$$X = \left\{x_k = 1 - \frac{2k+1}{m} : k = 0, \ldots, m-1\right\}$$

By [9, Section 1.341, Formula 3]

$$\sum_{k=0}^{m-1} \cos(l\pi x_k) = \begin{cases} 0, & 0 < l < m, \\ m(-1)^{m+1}, & l = m. \end{cases}$$

Now take a lattice of cubes with centers on X^{n-1} and side length $2a$. That is

$$\mathcal{L} = \{C_x = \{y : y_n = 1, \max|x_i - y_i| \le a\} : x \in X^{n-1}\}$$

Note that all cubes C_x are on one face of $[-1, 1]^n$ if $a < 1/m$.

Consider one sequence m_i and one permutation in the definition of φ_λ. The integral over the lattice of the resulting function equals

$$\begin{aligned} &\sum_{C_x \in \mathcal{L}} \int_{C_x} \cos(m_{\sigma(n)}\pi) \prod_{i=1}^{n-1} \cos(m_{\sigma(i)}\pi z_i) dz_1 \ldots dz_{n-1} \\ &\quad = \cos(m_{\sigma(n)}\pi) \prod_{i=1}^{n-1} \sum_{k=0}^{m-1} \int_{x_k - a}^{x_k + a} \cos(m_{\sigma(i)}\pi z_i) dz_i \\ &\quad = \cos(m_{\sigma(n)}\pi) \prod_{i=1}^{n-1} \sum_{k=0}^{m-1} \frac{2}{m_{\sigma(i)}\pi} \sin(m_{\sigma(i)}\pi a) \cos(m_{\sigma(i)}\pi x_k) \qquad (8) \\ &\quad = \begin{cases} 0, & (m_1, \ldots, m_n) \ne (0, \ldots, 0, m), \\ (-1)^m (2am)^{n-1}, & m_{\sigma(n)} = m, \\ \frac{1}{\pi} \sin(m\pi a)(-1)^{m+1}(2am)^{n-2}, & m_{\sigma(i)} = m \text{ for some } i < n. \end{cases} \qquad (9) \end{aligned}$$

Note that in (8) we mean $\frac{\sin x}{x} = 1$ if $x = 0$. The top case in (9) is equivalent to $k > 0$ in Section 3.2, while the other two cases correspond to the integral from (6).

Hence

$$\begin{aligned} \sum_{C_x \in \mathcal{L}} \int_{C_x} \varphi_\lambda(z) dz &= 2a_{0,\ldots,0,m}(2am)^{n-2} \Big(\sum_{\sigma(n)=n} am - \sum_{\sigma(n) \ne n} \frac{1}{\pi} \sin(m\pi a) \Big) \\ &= 2a_{0,\ldots,0,m}(2am)^{n-2}(n-1)! \Big(am - \frac{n-1}{\pi} \sin(m\pi a) \Big). \end{aligned}$$

The last expression equals 0 if we choose $0 < a < 1/m$ so that

$$\pi a m = (n-1)\sin(\pi a m).$$

The existence of such a is equivalent to the existence of $0 < x < \pi$ such that

$$x = (n-1)\sin x. \tag{10}$$

This equation has a positive solution when $n > 2$. This proves that in dimensions $n > 2$ any eigenfunction of a cube must change sign on the boundary. However this argument fails in dimension 2, and Proposition 4.1 (or the earlier result [11] by Hoffmann-Ostenhof) is the best we can expect. It is remarkable that (10) is exactly the same as the equation for a in the equilateral case (perhaps hinting at the fact that the equilateral triangle can be embedded in a cube as an intersection of that cube with a plane).

4.4. General boxes. Consider an n-dimensional box with sides of length $2a_i$ centered at the origin. The eigenvalues λ for this box can be indexed using a sequence L of n natural numbers l_i such that

$$\lambda = \frac{\pi^2}{4}\sum_{i=1}^{n}\frac{l_i^2}{a_i^2}. \tag{11}$$

The complete set of eigenfunctions is given by

$$\varphi(x) = \prod_{i=1}^{n} F(l_i \pi x_i / a_i),$$

where F is either sine or cosine. However, any eigenfunction that is nonnegative on the boundary can be axially symmetrized by summing over all sign changes for all coordinates. This procedure still gives a nonnegative boundary and eliminates all occurrences of sine. Therefore we can assume that $F(x) = \cos x$.

Any eigenfunction of a box, when restricted to a face, is also a sum of eigenfunctions on each face (put $x_i = a_i$ for some i). This lower dimensional combination of eigenfunctions consists of eigenfunctions that are orthogonal to a constant eigenfunction (that is, they integrate to 0 over the face), and/or a constant term. If the constant term is not present, the linear combination must change sign on the face. Therefore, an eigenfunction that is nonnegative on the boundary must have a constant term when restricted to any face. Hence, its eigenvalue must admit indexing sequences $L_j = \{l_i = \delta_j(i) l_j\}$. Taking $L = L_j$ in (11) thus yields

$$\lambda = \frac{\pi^2}{4}\frac{l_1^2}{a_1^2} = \frac{\pi^2}{4}\frac{l_2^2}{a_2^2} = \cdots = \frac{\pi^2}{4}\frac{l_n^2}{a_n^2}.$$

This immediately proves that if any ratio of two squares of the side lengths is not the square of a rational number, then nonnegative eigenfunctions do not exist.

For any $i \neq j$ we have

$$\frac{a_i^2}{a_j^2} = \frac{l_i^2}{l_j^2}.$$

Hence a_i/a_j is also rational for any $i \neq j$. Therefore $a_i = r_i \alpha$ for some rational r_i and real α. Rewriting $r_i = \frac{k_i}{n}$ (with k_i, n integers), we see that copies of the box can be used to build a cube with side length $\frac{\alpha}{n} \prod k_i$. Hence any eigenfunction positive on the boundary of the box would produce an eigenfunction on the large cube with the same property (thanks to Neumann boundary matching in the tiling). But we proved these do not exist. Therefore Theorem 1.4 also holds for arbitrary boxes.

5. Proof of Theorem 1.3 for $n \geq 6$

For a regular hexagon we can simply take the symmetric mode $\varphi_{0,1}$ of the equilateral triangle (defined in (3)) and cover the hexagon with its reflections to get an eigenfunction which is positive on the boundary.

Now consider a regular polygon with n sides, where $n > 6$. Such a polygon can be decomposed into n subequilateral triangles (ABD on Figure 1). The second Neumann eigenvalue of a subequilateral triangle is simple (Lemma 2.1) and the second Neumann eigenfunction is symmetric [13, Theorem 3.1]. Hence it is also the second eigenfunction on the right triangle formed by cutting the isosceles triangle in half (ABO and ADO on Figure 1).

The second Neumann eigenfunction must have exactly 2 nodal domains, by Courant's nodal domain theorem (see e.g. [7, Sec. V.5, VI.6]). By symmetry, the nodal line must either connect the two long sides (AB and AD) of the subequilateral triangle, or start and end on the short side (BD). From Lemma 2.3, the second case is not possible, regardless of the shape of the triangle. Hence this eigenfunction is positive on the short side, and it can be reflected n times inside of the regular polygon to cover the whole regular polygon. We obtain an eigenfunction on the regular polygon that is positive on the boundary. Therefore Theorem 1.3 holds for $n > 6$.

As a corollary from the above proof we also get a partial result for Conjecture 2.4

Corollary 5.1. *The nodal line for the second Neumann eigenfunction on right triangles with smallest angle $\alpha < \pi/6$ connects the interiors of the two longest sides.*

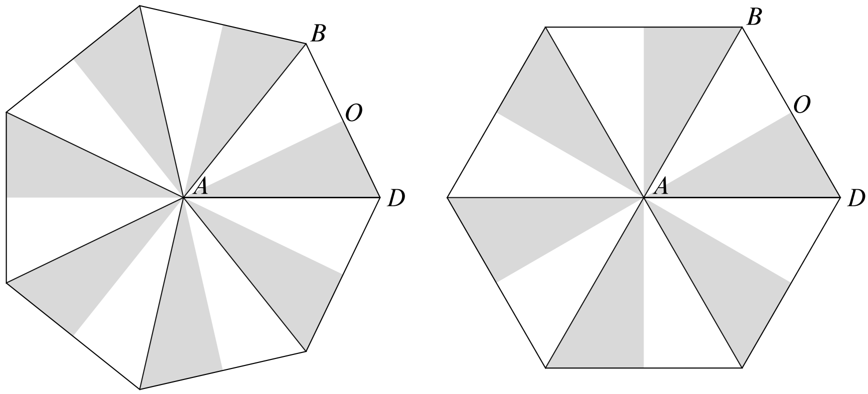

Figure 1. Regular heptagon decomposed into *subequilateral* triangles, and regular hexagon decomposed into equilateral triangles.

Appendix A. Towards a proof for regular pentagons.

A regular pentagon decomposes into acute superequilateral triangles instead of subequilateral triangles (as was the case of $n \geq 6$ sides). This invalidates the approach taken in Section 5. In this appendix we present some preliminary results needed to prove the pentagonal case of Theorem 1.3. However, due to very computational nature of the proof, it is postponed to a companion paper coauthored with Nilima Nigam and Benjamin Young [18].

First we look more closely at the differences between $n \geq 6$, and the pentagon. The second Neumann eigenvalue μ_2 of a superequilateral triangle (ABD on Figure 2) is simple but the second eigenfunction is antisymmetric [13, Theorem 3.2] (as opposed to symmetric for subequilateral triangles). By Lemma 2.2 all eigenfunctions for μ_3 are therefore symmetric. But all these eigenfunctions belong to the second (simple) eigenvalue of the right triangle OAB obtained by cutting the isosceles triangle ABD in half (shaded on Figure 2). Therefore μ_3 of a superequilateral triangle ABD is simple, with the eigenfunction symmetric with respect to OA. Unfortunately, Lemma 2.3 applies only to eigenfunctions for μ_2. Moreover, we need to exclude a possibility of having 3 nodal domains (allowed for μ_3).

These two problems make the pentagonal case much harder than regular polygons with $n \geq 6$ sides. Moreover, there is essentially no hope of finding explicit trigonometric formulae for its eigenfunctions. A completely different approach is required.

Consider the rhombus R ($ABCD$ on Figure 2) built using right triangle T (ABD on the same figure) with the smallest angle at least $\pi/6$ (equal to $\pi/5$ for our regular pentagon). Then [20, Corollary 1.3] gives

$$\mu_4(R) < \lambda_1(R).$$

Note that the classical Levine–Weinberger inequality [14] only gives $\mu_3 \leq \lambda_1$. Furthermore, the eigenfunction u_2 that belongs to $\mu_2(T)$ extends by symmetry to a doubly symmetric eigenfunction $\tilde{u}$ on R. Then $\tilde{u}$ must belong to the lowest eigenvalue of R which possesses a doubly symmetric mode. Otherwise, a doubly symmetric eigenfunction of the lower eigenvalue of R would be an eigenfunction for T. Therefore

$$\mu_2(T) = \mu_4(R).$$

For the triangle $T = ABD$ we have the following lemma.

Lemma A.1. *The partial derivatives u_x and u_y of the second Neumann eigenfunction u of T are never zero and have opposite signs.*

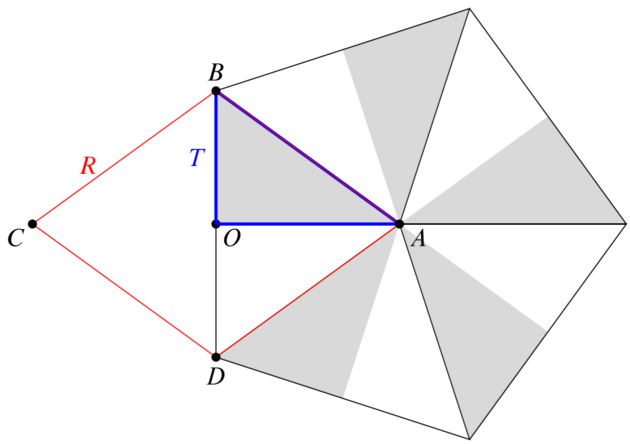

Figure 2. Regular pentagon decomposed into acute *superequilateral* triangles, blue triangle T and red rhombus R.

Remark. Note that this result can be deduced from the last paragraph on p. 244 of Atar and Burdzy [2]. Nevertheless we present a simpler proof.

Proof. We will follow the proofs of [16, Lemmas 3.2,3.4] and [19, Theorem 2]. First note that [19, Lemma 2] applies to T, hence its second Neumann eigenfunction u is strictly monotonic on AB. We can assume that $u_x > 0$ and $u_y < 0$ on AB.

Now we consider the doubly symmetric extension of u to the rhombus R. On CB we also have $u_y < 0$ due to double symmetry of u, while on CD and DA we have $u_y > 0$. Similarly, $u_x > 0$ on DA, and $u_x < 0$ on CB and CD. Furthermore, u_x is antisymmetric with respect to y-axis and symmetric with respect to x-axis (again by double symmetry of u), while u_y has reversed symmetries.

Suppose u_y is zero somewhere in R, then by [16, Proposition 2.1(i)] it must change sign inside R. By antisymmetry, it must be positive somewhere in ABC. But $u_y < 0$ on AB and CB and $u_y = 0$ on AC. Hence a nodal domain of u_y is a subset of ABC (part of the nodal line might be a subset of AC). We already noticed that u belongs to $\mu_4(R)$, hence

$$\mu_4(R) = \lambda_1(N) > \lambda_1(R) > \mu_4(R),$$

a contradiction. Hence $u_y < 0$ on ABC (hence also on T). Similarly we can prove that $u_x > 0$ on T. □

We need a domain monotonicity result for the eigenvalues of the domains with mixed boundary conditions. This is a special case of a more general partial domain monotonicity principle proved by Harrell. We present this special case due to its rather simple proof.

Lemma A.2 (special case of Harrell [10, Corollary II.2]). *Suppose $D_1 \subset D_2$ are open and the Neumann boundary $\partial_N D_1$ of D_1 is contained in the Neumann boundary $\partial_N D_2$ of D_2 (see Figure 3). Then the lowest eigenvalue on D_2 for the mixed Dirichlet–Neumann problem is smaller than the lowest mixed eigenvalue on D_1, unless $D_1 = D_2$ and $\partial_N D_1 = \partial_N D_2$.*

Proof. Suppose φ is the eigenfunction for D_1. Extend it with 0 to the whole set D_2. Note that the extension satisfies the Dirichlet boundary condition on $\partial D_2 \setminus \partial_N D_1$, hence on the Dirichlet boundary $\partial_D D_2$. Note also that $\partial_N D_1$ does not intersect $D_2 \setminus D_1$, hence the extension is continuous. Therefore it is a valid trial function for the Rayleigh–Ritz quotient on D_2. But it also equals 0 on an open set (if D_1 is strictly included in D_2), or equals 0 on a piece of boundary $\partial_N D_2 \setminus \partial_N D_1$ (satisfies both Dirichlet and Neumann condition). In either case it must be 0 everywhere. □

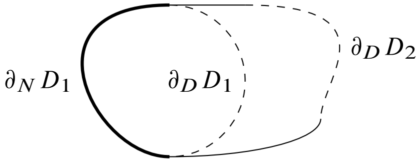

Figure 3. Domain monotonicity: $D_1 \subset D_2$ and Neumann boundary condition on D_1 is specified only on a portion of the Neumann boundary $\partial_N D_2$. Solid lines indicate a Neumann boundary while dashed lines indicate Dirichlet boundarh.

Corollary A.3. *Let $D_1 \subset D$ be a nodal domain for the eigenfunction for $\mu_2(D)$. Suppose we find $D_2 \subset D$ such that $\partial_N D_1 \subset \partial_N D_2 \subset \partial D$, and the mixed eigenvalue $\lambda_1(D_2) > \mu_2(D)$. Then the nodal line for $\mu_2(D)$ must intersect the Dirichlet boundary $\partial_D D_2$.*

Proof. The mixed eigenvalue of D_1 equals $\mu_2(D)$ and is smaller than $\lambda_1(D_2)$. Note also that for a nodal domain we always have $\partial_N D_1 \subset \partial D$ and $\partial_D D_1$ is the nodal line. If $D_1 \subset D_2$, then the above lemma gives $\lambda_1(D_1) > \lambda_1(D_2)$, leading to a contradiction. Hence $D_1 \not\subset D_2$, and the nodal line $\partial_D D_1$ must have a nonempty intersection with $D \setminus D_2$. Hence it must intersect $\partial_D D_2$. □

To summarize, all the presented results can be useful in deciding whether the nodal line in a right triangle connects the longest side to the shortest side, or to the midium side. In our companion paper [18] we show that the latter is true, hence the proof for $n \geq 6$ can be adapted to the pentagonal case.

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