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Spectrum theory of second-order difference equations with indefinite weight

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Abstract. In this paper, we study the spectrum structure of second-order difference operators with sign-changing weight. We apply the Sylvester inertia theorem to show that the spectrum consists of real and simple eigenvalues; the number of positive eigenvalues is equal to the number of positive elements in the weight function, and the number of negative eigenvalues is equal to the number of negative elements in the weight function. We also show that the eigenfunction corresponding to the j-th positive/negative eigenvalue changes its sign exactly $j-1$ times.

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1. Introduction

Let $T > 1$ be an integer, $\mathbb{T} = \{1, 2, \dots, T\}$. Let us consider the discrete linear second-order eigenvalue problem

$$\Delta[p(t-1)\Delta u(t-1)] - q(t)u(t) + \lambda m(t)u(t) = 0, \quad t \in \mathbb{T}, \tag{1.1$_\lambda$}$$

$$\alpha u(0) - \beta\Delta u(0) = 0, \quad \gamma u(T+1) + \delta\Delta u(T) = 0, \tag{1.2}$$

where Δ is a forward difference operator which is defined by

$$\Delta u(t) := u(t+1) - u(t),$$

λ is a spectrum parameter, $\alpha,\ \beta,\ \gamma,\ \delta \in \mathbb{R}$ satisfy $\alpha\beta \geq 0$, $\gamma\delta \geq 0$ with $\alpha^2+\beta^2 \neq 0$, $\gamma^2 + \delta^2 \neq 0$; $p\colon \{0, 1, \dots, T\} \to [0, \infty)$ with

$$p(j) > 0, \quad j \in \{0, 1, \dots, T\};$$

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$q: \mathbb{T} \to [0, \infty)$; the weight function $m: \mathbb{T} \to \mathbb{R}$ satisfies $m(t) \neq 0$ on $\mathbb{T}$ and m changes its sign on $\mathbb{T}$, i.e., there exists a proper subset $\mathbb{T}_+ \subset \mathbb{T}$, such that

$$m(t) > 0, \quad t \in \mathbb{T}_+; \qquad m(t) < 0, \quad t \in \mathbb{T} \setminus \mathbb{T}_+.$$

Let T^+ be the number of elements in $\mathbb{T}_+$ and let T^- be the number of elements in $\mathbb{T} \setminus \mathbb{T}_+$. Then

$$T^+ + T^- = T.$$

When the weight function $m(t)$ is of one sign, Atkinson [4] studied the linear eigenvalue problems

$$c(t)y(t+1) = (\lambda a(t) + b(t))y(t) - c(t-1)y(t-1), \quad t \in \mathbb{T}, \tag{1.3}$$

$$y(0) = 0, \quad y(T+1) + ly(T) = 0, \tag{1.4}$$

and obtained that (1.3), (1.4) has T real eigenvalues, which can be ordered as $\lambda_1 < \lambda_2 < \cdots < \lambda_T$. Here $a(t) > 0$, $c(t) > 0$ and l is some fixed real number. It can be seen that if we take $c(t) = p(t)$, $a(t) = -m(t)$, $b(t) = p(t) + p(t-1) + q(t)$, then (1.3) will convert to $(1.1)_\lambda$.

In 1995, Jirari [13] extended the results of Atkinson, under the more general boundary conditions

$$y(0) + hy(1) = 0, \quad y(T+1) + ly(T) = 0, \tag{1.5}$$

where $h, l \in \mathbb{R}$. He got that (1.3), (1.5) has T real eigenvalues, which can be ordered as $\lambda_1 < \lambda_2 < \cdots < \lambda_T$.

However, these two results do not give any information on the sign-changing of the eigenfunctions of (1.3), (1.4) or (1.3), (1.5).

In 1991, Kelley and Peterson [14] considered the linear eigenvalue problems

$$\Delta[p(t-1)\Delta y(t-1)] - q(t)y(t) + \lambda m(t)y(t) = 0, \quad t \in \mathbb{T}, \tag{1.6}$$

$$y(0) = y(T+1) = 0, \tag{1.7}$$

where $p(t) > 0$ on $\{0, 1, \ldots T\}$, $q(t)$ is defined and real-valued on $\mathbb{T}$ and $m(t) > 0$ on $\mathbb{T}$. They obtained the following result.

Theorem A. (1.6), (1.7) *has exactly T real and simple eigenvalues λ_k, $k \in \mathbb{T}$ which satisfy*

$$\lambda_1 < \lambda_2 < \cdots < \lambda_T$$

and the eigenfunction corresponding to λ_k changes its sign exactly $k-1$ times.

Furthermore, when $m(t) \equiv 1$, Agarwal et al. [2] generalized the results of Theorem A to the dynamic equations with Sturm-Liouville boundary condition. Moreover, under the assumption that the weight functions are of one sign, further important results in linear Hamiltonian difference systems, including the oscillation properties of solutions, one can be found in Bohner [6], Shi and Chen [21], and the references therein.

However, there are only a few results on the spectrum of discrete second-order linear eigenvalue problems when $m(t)$ changes its sign on $\mathbb{T}$. In 2008, Shi and Yan [22] discussed the spectral theory of left definite difference operators when $m(t)$ changes its sign, however, they provided no information about the sign of the eigenvalues and no information on the sign-changing of the corresponding eigenfunctions. Recently, Ma et al. [17] obtained that $(1.1)_\lambda$, (1.2) has two principal eigenvalues $\lambda_{1,-} < 0 < \lambda_{1,+}$ when $p(t) \equiv 1$ and $q(t) \equiv 0$ and they used this result to deal with some discrete nonlinear problems.

Now, there are two questions: (a) how many eigenvalues do $(1.1)_\lambda$, (1.2) have? (b) how do these eigenvalues distribute?

In [12], Ince studied the linear eigenvalue problem of second-order ordinary differential equations

$$(k(t)u'(t))' + (\mu g(t) - l(t))u(t) = 0, \quad t \in (0, 1), \tag{1.8}$$

$$\alpha' u(0) - \alpha u'(0) = 0, \quad \beta' y(1) + \beta y'(1) = 0 \tag{1.9}$$

under the following assumptions:

(F1) $\alpha\alpha' \geq 0$, $\beta'\beta \geq 0$, $\alpha^2 + \alpha'^2 \neq 0$ and $\beta^2 + \beta'^2 \neq 0$;

(F2) $k \in C^1[0, 1]$, $l \in C[0, 1]$ are such that $k(t) > 0$, $l(t) \geq 0$;

(F3) $g\colon [0, 1] \to \mathbb{R}$ is continuous and changes its sign.

He obtained the following result.

Theorem B. *Suppose that* (F1)–(F3) *hold. If $l \not\equiv 0$ or $\alpha'^2 + \beta'^2 \neq 0$, then* (1.8), (1.9) *has an infinite sequence of simple eigenvalues*

$$-\infty \leftarrow \cdots < \mu_{k,-} < \cdots < \mu_{2,-} < \mu_{1,-} < 0 < \mu_{1,+} < \cdots < \mu_{k,+} < \cdots \to +\infty$$

and the eigenfunction corresponding to $\mu_{k,\pm}$ has exactly $k-1$ simple zeros in $(0, 1)$.

When $l \equiv 0$ and $\alpha'^2 + \beta'^2 = 0$, the corresponding results have been established by Bôcher [5]. Moreover, the result of Theorem B has been extended to the one-dimensional p−Laplacian operator by Anane, Chakrone and Moussa [3] and to the high-dimensional case by Hess and Kato [10], Ko and Brown [15], and Afrouzi and Brown [1]. These spectrum results have been used to deal with several nonlinear problems, see, for example [1], [15], [18], [19], and the references therein.

It is the purpose of this paper to establish the discrete analogue of Theorem B for the discrete problem $(1.1)_\lambda$, (1.2).

The main result of our paper is the following theorem.

Theorem 1. *Suppose that* $m\colon \mathbb{T} \to \mathbb{R}$ *satisfies* $m(t) \neq 0$ *on* $\mathbb{T}$*, and there exists a proper subset* $\mathbb{T}_+ \subset \mathbb{T}$*, such that* $m(t) > 0,\ \ t \in \mathbb{T}_+$ *and* $m(t) < 0,\ \ t \in \mathbb{T} \setminus \mathbb{T}_+$*,* $q(t) \not\equiv 0$ *on* $\mathbb{T}$ *or* $\alpha^2 + \gamma^2 \neq 0$ *and* $\nu \in \{+, -\}$*. Then*

(a) $(1.1)_\lambda$, (1.2) *has* T *real and simple eigenvalues, which can be ordered as follows*

$$\lambda^-_{T,T^-} < \lambda^-_{T,T^- -1} < \cdots < \lambda^-_{T,1} < 0 < \lambda^+_{T,1} < \lambda^+_{T,2} < \cdots < \lambda^+_{T,T^+};$$

(b) *every eigenfunction* $\psi^\nu_{T,k}$ *corresponding to the eigenvalue* $\lambda^\nu_{T,k}$ *changes its sign exactly* $k-1$ *times.*

Remark 1. From Theorem 1, it can be seen that if there are T^+ elements in $\mathbb{T}$ such that $m(t) > 0$ for $t \in T^+$, then $(1.1)_\lambda$, (1.2) has exactly T^+ positive eigenvalues and T^- negative eigenvalues. It is worth remarking that the times of sign-changing of the eigenfunction are given in Theorem 1. Thus, this result is not only the discrete analogue of Theorem B, but also the generalization of the result in Atkinson [4], Jirari [13], Kelly and Peterson [14], and Ma et al. [17].

The rest of the paper is devoted to proving Theorem 1. To do this, we make use of the Law of Inertia for Quadratic Forms and some techniques from oscillation matrices, see [9] and [7].

2. Proof of the main result

Let $c(t) = p(t-1) + p(t) + q(t)$ for $t = 2, \ldots, T-1$, $c(1) = \frac{\alpha}{\alpha+\beta} p(0) + p(1) + q(1)$, $c(T) = \frac{\gamma}{\gamma+\delta} p(T) + p(T-1) + q(T)$. Then $(1.1)_\lambda$, (1.2) can be written as a linear pencil problem

$$Ju = \lambda Du,$$

where

$$J = \begin{pmatrix} c(1) & -p(1) & 0 & \dots & 0 & 0 & 0 \\ -p(1) & c(2) & -p(2) & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -p(T-2) & c(T-1) & -p(T-1) \\ 0 & 0 & 0 & \dots & 0 & -p(T-1) & c(T) \end{pmatrix}$$

and $D = \operatorname{diag}(m(1), m(2), \dots, m(T))$.

Let J_j denote the j-th principal submatrix of J and D_j the j-th principal submatrix of D. Then J and J_i are positive definite. In fact, for any $\mathbf{x} = (x_1, x_2, \dots, x_T) \in \mathbb{R}^T$, it follows that

$$\mathbf{x}J\mathbf{x}^T = \alpha p(0)x_1^2 + \gamma p(T)x_T^2 + \sum_{i=1}^{T-1} p(i)(x_{i+1} - x_i)^2 + \sum_{i=1}^{T} q(i)x_i^2 \geq 0.$$

Moreover, $\mathbf{x}J\mathbf{x}^T = 0$ implies $\mathbf{x} = 0$. So, J is positive definite. By the same method, with obvious changes, we can conclude that J_i is also positive definite for $j = 1, 2, \dots, T$.

For $j = 1, 2, \dots, T$, let $Q_j(\lambda)$ denote the j-th principal subdeterminant of $J - \lambda D$ and suppose that $Q_0(\lambda) = 1$. Then $Q_T(\lambda) = \det(J - \lambda D)$, and

$$Q_0(\lambda) = 1; \tag{2.1a}$$

$$Q_1(\lambda) = c(1) - \lambda m(1); \tag{2.1b}$$

$$Q_j(\lambda) = (c(j) - \lambda m(j))Q_{j-1}(\lambda) - p^2(j-1)Q_{j-2}(\lambda), \quad j = 2, 3, \dots, T. \tag{2.1c}$$

As we know, finding the eigenvalues of $(1.1)_\lambda$, (1.2) is equivalent to finding the zeros of $Q_T(\lambda)$. Thus, it is necessary to discuss some properties of the sequence (2.1).

For $j \in \{1, \dots, T\}$, let j^+ be the number of the elements in $\{m(i) \mid m(i) > 0$ for some $i \in \{1, \dots, j\}\}$, and j^- the number of the elements in $\{m(i) \mid m(i) < 0$ for some $i \in \{1, \dots, j\}\}$.

Lemma 1. *For $j \in \{1, \dots, T\}$, we have*

$$\lim_{\lambda \to -\infty} (-1)^{j^-} Q_j(\lambda) = +\infty, \qquad \lim_{\lambda \to +\infty} (-1)^{j^+} Q_j(\lambda) = +\infty.$$

Proof. For $j \in \{1, \dots, T\}$, it is evident that $Q_j(\lambda)$ is a polynomial of degree precisely j, and

$$Q_j(\lambda) = m(1) \dots m(j)(-\lambda)^j + O(\lambda^{j-1}).$$

□

Lemma 2. *The roots of $Q_j(\lambda) = 0$ are real. Moreover, $Q_j(\lambda) = 0$ has j^+ positive roots and j^- negative roots.*

Proof. For the positive definite matrix J_j, there exists a unique lower triangular real matrix L such that

$$LL^T = J_j$$

(this is the well-known Cholesky decomposition, see [11, Corollary 7.2.9]). It is easy to check that the matrix $L^{-1}D_j(L^T)^{-1}$ is real and symmetric, and λ is a zero of $Q_j(\lambda)$ if and only if $1/\lambda$ is an eigenvalue of $L^{-1}D_j(L^T)^{-1}$.

The fact that $L^{-1}D_j(L^T)^{-1}$ is real and symmetric implies that there exists an orthogonal matrix Q such that

$$Q^T L^{-1} D_j (L^T)^{-1} Q = \operatorname{diag}(a_1, \dots, a_j), \tag{2.2}$$

where $a_1 \geq a_2 \geq \cdots \geq a_j$ are all eigenvalues of $L^{-1}D_j(L^T)^{-1}$. Let $\mathbf{x}^T = (L^T)^{-1}Q\mathbf{z}^T$. It is seen from (2.2) that

$$\sum_{i=1}^{j} a_i z_i^2 = \sum_{i=1}^{j} m(i) x_i^2$$

are two representations of the real quadratic form $\mathbf{x}D_j\mathbf{x}^T$. In view of the Law of Inertia for Quadratic Forms [8, Theorem 1, p. 297], we immediately deduce that the number of positive and the number of negative elements in the set $\{a_1, \dots, a_j\}$ are j^+ and j^-, respectively. □

Lemma 3. *Two consecutive polynomials $Q_{i-1}(\lambda)$, $Q_i(\lambda)$ have no common zeros for $i = 1, \dots, T$.*

Proof. Suppose on the contrary there exists $\lambda = \lambda_0$ such that $Q_{i-1}(\lambda_0) = Q_i(\lambda_0) = 0$. Then by the recurrence relation (2.1), we get $Q_{i-2}(\lambda_0) = 0$. Furthermore, we can get $Q_{i-3}(\lambda_0) = \cdots = Q_1(\lambda_0) = Q_0(\lambda_0) = 0$. However, this contradicts $Q_0(\lambda_0) = 1$. □

Lemma 4. *Suppose that $\lambda = \lambda_0$ is a zero of $Q_i(\lambda)$. Then $Q_{i-1}(\lambda_0)Q_{i+1}(\lambda_0) < 0$ for $i = 1, \dots, T-1$.*

Proof. Since $Q_i(\lambda_0) = 0$, by the Lemma 3, we have $Q_{i-1}(\lambda_0) \neq 0$. By the recurrence relation (2.1), $Q_{i+1}(\lambda_0) = -p^2(i)Q_{i-1}(\lambda_0)$, which implies that $Q_{i+1}(\lambda_0)Q_{i-1}(\lambda_0) = -p^2(i)Q_{i-1}^2(\lambda_0) < 0$. This completes the proof. □

Lemma 5. *For $j = 1, \dots, T$, the roots of $Q_j(\lambda) = 0$ are simple. Moreover,*

(i) *the largest negative root $\lambda^-_{j,1}$ and the smallest positive root $\lambda^+_{j,1}$ of $Q_j(\lambda) = 0$ and the largest negative root $\lambda^-_{j+1,1}$ and the smallest positive root $\lambda^+_{j+1,1}$ of $Q_{j+1}(\lambda) = 0$ satisfy*

$$(\lambda^-_{j,1}, \lambda^+_{j,1}) \supset (\lambda^-_{j+1,1}, \lambda^+_{j+1,1}), \quad j = 1, \dots, T-1;$$

(ii) *for $j = 1, \dots, T-1$, the positive roots of $Q_j(\lambda) = 0$ and $Q_{j+1}(\lambda) = 0$ separate one another; and the negative roots of $Q_j(\lambda) = 0$ and $Q_{j+1}(\lambda) = 0$ separate one another.*

Proof. First, we deal with the case $j = 1$.

Obviously, $Q_1(\lambda) = c(1) - \lambda m(1)$. If $m(1) > 0$, then $j = 1,\ j^+ = 1,\ j^- = 0$, and $\lambda^+_{1,1} = \frac{c(1)}{m(1)} > 0$. If $m(1) < 0$, then $j = 1,\ j^+ = 0,\ j^- = 1$, and $\lambda^-_{1,1} = \frac{c(1)}{m(1)} < 0$.

Recall that $Q_2(\lambda) = (c(2) - \lambda m(2))(c(1) - \lambda m(1)) - p^2(1)$. Then $Q_2(\lambda) = 0$ has two different roots as follows:

$$\lambda_1 = \frac{c(1)m(2) + c(2)m(1) - \sqrt{(c(1)m(2) - c(2)m(1))^2 + 4p^2(1)m(1)m(2)}}{2m(1)m(2)},$$

$$\lambda_2 = \frac{c(1)m(2) + c(2)m(1) + \sqrt{(c(1)m(2) - c(2)m(1))^2 + 4p^2(1)m(1)m(2)}}{2m(1)m(2)}.$$

If $m(1) > 0$, $m(2) > 0$, then $j = 2,\ j^+ = 2,\ j^- = 0$. By direct computation, we get $0 < \lambda_1 < \lambda_2$. Now, let $\lambda^+_{2,1} = \lambda_1$ and $\lambda^+_{2,2} = \lambda_2$. Then we have $0 < \lambda^+_{2,1} < \lambda^+_{1,1} < \lambda^+_{2,2}$.

If $m(1) < 0$, $m(2) > 0$, then $j = 2,\ j^+ = 1,\ j^- = 1$ and $\lambda_1 > 0 > \lambda_2$. Let $\lambda^+_{2,1} = \lambda_1$ and $\lambda^-_{2,1} = \lambda_2$. Then we have $\lambda^-_{1,1} < \lambda^-_{2,1} < 0 < \lambda^+_{2,1}$.

If $m(1) > 0$, $m(2) < 0$, then $j = 2,\ j^+ = 1,\ j^- = 1$ and $\lambda_1 > 0 > \lambda_2$. Let $\lambda^+_{2,1} = \lambda_1$ and $\lambda^-_{2,1} = \lambda_2$. Then we have $\lambda^-_{2,1} < 0 < \lambda^+_{1,1} < \lambda^+_{2,1}$.

If $m(1) < 0$, $m(2) < 0$, then $j = 2,\ j^+ = 0,\ j^- = 2$ and $\lambda_2 < \lambda_1 < 0$. Let $\lambda^-_{2,1} = \lambda_1$ and $\lambda^-_{2,2} = \lambda_2$. Then $\lambda^-_{2,2} < \lambda^-_{1,1} < \lambda^-_{2,1} < 0$. Thus, the assertion is true for $j = 1$.

Second, suppose that for $j = k$, the relations of $Q_k(\lambda) = 0$ and $Q_{k+1}(\lambda) = 0$ are true, i.e., the following two assertions hold.

- If $m(k+1) > 0$, then $(k+1)^+ = k^+ + 1$, $(k+1)^- = k^-$ and accordingly,

$$\lambda^-_{k,k^-} < \lambda^-_{k+1,k^-} < \cdots < \lambda^-_{k,1} < \lambda^-_{k+1,1} < 0 \tag{2.3}$$

and

$$0 < \lambda^+_{k+1,1} < \lambda^+_{k,1} < \cdots < \lambda^+_{k+1,k^+} < \lambda^+_{k,k^+} < \lambda^+_{k+1,(k+1)^+}. \tag{2.4}$$

- If $m(k+1) < 0$, then $(k+1)^+ = k^+$, $(k+1)^- = k^- + 1$ and accordingly,

$$\lambda^-_{k+1,k^-+1} < \lambda^-_{k,k^-} < \lambda^-_{k+1,k^-} < \cdots < \lambda^-_{k,1} < \lambda^-_{k+1,1} < 0$$

  and

$$0 < \lambda^+_{k+1,1} < \lambda^+_{k,1} < \cdots < \lambda^+_{k+1,k^+} < \lambda^+_{k,k^+}.$$

Now, we consider the case $j = k+1$. It is enough to consider the following four cases.

CASE 1: $m(k+1) > 0$ AND $m(k+2) > 0$. In this case, $(k+2)^+ = k^+ + 2$, $(k+2)^- = k^-$, we need to prove that

$$\lambda^-_{k+1,k^-} < \lambda^-_{k+2,k^-} < \cdots < \lambda^-_{k+1,1} < \lambda^-_{k+2,1} < 0, \tag{2.5}$$

and

$$0 < \lambda^+_{k+2,1} < \lambda^+_{k+1,1} < \cdots < \lambda^+_{k+2,k^++1} < \lambda^+_{k+1,k^++1} < \lambda^+_{k+2,k^++2}. \tag{2.6}$$

CASE 2: $m(k+1) > 0$ AND $m(k+2) < 0$. In this case, $(k+2)^+ = k^+ + 1$, $(k+2)^- = k^- + 1$, we need to prove that

$$\lambda^-_{k+2,k^-+1} < \lambda^-_{k+1,k^-} < \lambda^-_{k+2,k^-} < \cdots < \lambda^-_{k+1,1} < \lambda^-_{k+2,1} < 0$$

and

$$0 < \lambda^+_{k+2,1} < \lambda^+_{k+1,1} < \cdots < \lambda^+_{k+2,k^++1} < \lambda^+_{k+1,k^++1}.$$

CASE 3: $m(k+1) < 0$ AND $m(k+2) < 0$. In this case, $(k+2)^+ = k^+$, $(k+2)^- = k^- + 2$, we need to prove that

$$\lambda^-_{k+2,k^-+2} < \lambda^-_{k+1,k^-+1} < \lambda^-_{k+2,k^-+1} < \cdots < \lambda^-_{k+1,1} < \lambda^-_{k+2,1} < 0$$

and

$$0 < \lambda^+_{k+2,1} < \lambda^+_{k+1,1} < \cdots < \lambda^+_{k+2,k^+} < \lambda^+_{k+1,k^+}.$$

CASE 4: $m(k+1) < 0$ AND $m(k+2) > 0$. In this case, $(k+2)^+ = k^+ + 1$, $(k+2)^- = k^- + 1$, we need to prove that

$$\lambda^-_{k+1,k^-+1} < \lambda^-_{k+2,k^-+1} < \lambda^-_{k+1,k^-} < \lambda^-_{k+2,k^-} < \cdots < \lambda^-_{k+1,1} < \lambda^-_{k+2,1} < 0$$

and

$$0 < \lambda^+_{k+2,1} < \lambda^+_{k+1,1} < \cdots < \lambda^+_{k+2,k^+} < \lambda^+_{k+1,k^+} < \lambda^+_{k+2,k^++1}.$$

We only deal with the Case 1. The other cases can be disposed of via the similar method. First, let us show that (2.5) holds. Since $(k+2)^- = (k+1)^- = k^-$, it follows from Lemma 1 that

$$(-1)^{k^-} Q_k(-\infty) > 0, \quad (-1)^{(k+1)^-} Q_{k+1}(-\infty) = (-1)^{k^-} Q_{k+1}(-\infty) > 0. \tag{2.7}$$

To wit, we only deal with the case that k^- is even. The case k^- is odd can be treated by the similar way. In this case, (2.7) reduces to

$$Q_k(-\infty) > 0, \quad Q_{k+1}(-\infty) > 0. \tag{2.8}$$

Then, from (2.7), (2.8), and (2.3), we get $(-1)^j Q_k(\lambda^-_{k+1,k^--j}) < 0$, $j = 0, \ldots, k^- - 1$. Meanwhile, $Q_{k+1}(\lambda^-_{k+1,k^--j}) = 0$. From Lemma 4, we get that

$$(-1)^j Q_{k+2}(\lambda^-_{k+1,k^--j}) > 0, \quad j = 0, \ldots, k^- - 1. \tag{2.9}$$

From Lemma 2, we know that $Q_{k+2}(\lambda) = 0$ has exactly k^- zeros in $(-\infty, 0)$. This together with (2.9) and the fact that $Q_k(0) > 0$, $Q_{k+1}(0) > 0$, $Q_{k+2}(0) > 0$ implies that there exist $\lambda^-_{k+2,k^--j} \in (\lambda^-_{k+1,k^--j}, \lambda^-_{k+1,k^--j-1})$, $j = 0, \ldots, k^- - 2$, and $\lambda^-_{k+2,1} \in (\lambda^-_{k+1,1}, 0)$, such that

$$Q_{k+2}(\lambda^-_{k+2,k^--j}) = 0, \quad j = 0, \ldots, k^- - 1.$$

Therefore, (2.5) holds.

We show (2.6) is true. Recall that we are working with Case 1: $m(k+1) > 0$, $m(k+2) > 0$. So, $(k+2)^+ = (k+1)^+ + 1 = k^+ + 2$. Here, we also only deal with the case that k^+ is even. From Lemma 1, we have that

$$Q_k(+\infty) > 0, \quad Q_{k+1}(+\infty) < 0, \quad Q_{k+2}(+\infty) > 0.$$

Combining this with (2.4) and the fact $Q_{k+1}(\lambda^+_{k+1,(k+1)^+-j}) = 0$, $j = 0, \ldots, (k+1)^+ - 1$, we get that $(-1)^j Q_k(\lambda^+_{k+1,(k+1)^+-j}) > 0$, $j = 0, \ldots, (k+1)^+ - 1$. This together with Lemma 4 implies that

$$(-1)^j Q_{k+2}(\lambda^+_{k+1,(k+1)^+-j}) < 0, \quad j = 0, \ldots, (k+1)^+ - 1. \tag{2.10}$$

In particular, for $j = 0$, $Q_{k+2}(\lambda^+_{k+1,(k+1)^+}) < 0$. This together with the fact $Q_{k+1}(+\infty) > 0$ gives

$$Q_{k+2}(\lambda^+_{k+2,(k+1)^++1}) = 0$$

for some $\lambda^+_{k+2,(k+1)^++1} \in (\lambda^+_{k+1,(k+1)^+}, \infty)$.

Using (2.10) with $j = (k+1)^+ - 1$, we get $Q_{k+2}(\lambda^+_{k+1,1}) < 0$. Combining this with the fact $Q_{k+2}(0) > 0$, we get that

$$Q_{k+2}(\lambda^+_{k+2,1}) = 0$$

for some $\lambda^+_{k+2,1} \in (0, \lambda^+_{k+1,1})$.

Now, for $j = 1, \dots, (k+1)^+ - 1$, there exist

$$\lambda^+_{k+2,(k+2)^+-j} \in (\lambda^+_{k+1,(k+1)^+-j}, \lambda^+_{k+1,(k+1)^+-j+1}),$$

such that

$$Q_{k+2}(\lambda^+_{k+2,(k+2)^+-j}) = 0, \quad j = 0, \dots, (k+1)^+.$$

Therefore, (2.6) is valid. □

Lemma 6. *Let $w(\lambda)$ be the sign-changing number of the sequence (2.1). Then for $i \in \{1, \dots, T^+\}$,*

$$\lim_{\lambda \to \lambda^+_{T,i} - 0} w(\lambda) = i - 1, \qquad \lim_{\lambda \to \lambda^+_{T,i} + 0} w(\lambda) = i,$$

where $\lambda \to C - 0$ means that $\lambda \to C$ from left hand side of C, and $\lambda \to C + 0$ means that $\lambda \to C$ from right hand side of C.

Proof. Our proof is motivated by the proof of Sturm theorem, see [20, Theorem 1.4.3] and its proof.

The idea of the proof is to follow the changes in w as λ passes through the interval $[a, b]$. In particular, we will show that w is a monotonically increasing function and that each zero of Q_T and only one zero of Q_T makes w jump by 1.

Suppose $Q_j(\hat{\lambda}) = 0$ for some $j \in \{1, \dots, T-1\}$. Then for Q_{j-1}, Q_j, Q_{j+1} we have from Lemma 4 that Q_{j-1} and Q_{j+1} have opposite, but constant signs, since Q_{j-1} and Q_{j+1} cannot be zero in a sufficiently small neighborhood $U(\hat{\lambda})$ and thus cannot change sign. Hence, whatever the sign of Q_j in $U(\hat{\lambda})$ is , it does not change the overall sign change count (To see this, note that Q_{j-1} and Q_{j+1} have opposite signs, hence if the sign sequence before is $+ - -$, it is $+ + -$ afterwards and the number of sign changes remains the same. Similarly for the other cases). In other words, $w(\lambda)$ stays constant when λ passes through a zero of Q_j from some $j \in \{1, \dots, T-1\}$.

It is easy to see from Lemma 5 that

$$\operatorname{sgn} Q_{T-1}(\lambda^+_{T,i}) = (-1)^{i-1}, \quad i \in \{1, \dots, T^+\}.$$

Next, we show that each zero of Q_T and only one zero of Q_T makes w jump by 1.

In fact, for $i = 1$, $Q_{T-1}(\lambda^+_{T,1}) > 0$, which implies that there exists a neighborhood $U(\lambda^+_{T,1})$ of $\lambda^+_{T,1}$, such that

$$Q_{T-1}(\lambda) > 0, \quad \lambda \in U(\lambda^+_{T,1}).$$

From the definition of $\lambda^+_{T,1}$,

$$Q_T(\lambda) > 0, \quad \lambda \in [0, \lambda^+_{T,1}).$$

The chain of signs switches from "$\cdots + +$" to "$\cdots + -$" when passing through $\lambda^+_{T,1}$, so w increases by 1.

For $i = 2$, $Q_{T-1}(\lambda^+_{T,2}) < 0$ and $Q_T(\lambda) < 0$, $\lambda \in (\lambda^+_{T,1}, \lambda^+_{T,2})$. The chain of signs switches from "$\cdots - -$" to "$\cdots - +$" when passing through $\lambda^+_{T,2}$, so w increases by 1.

Repeating the above argument, we may deduce that

$$\lim_{\lambda \to \lambda^+_{T,i} - 0} w(\lambda) = i - 1, \qquad \lim_{\lambda \to \lambda^+_{T,i} + 0} w(\lambda) = i.$$

This completes the proof. □

Lemma 7. *If $u(\cdot, \lambda)$ satisfies* (1.1)$_\lambda$, (1.2) *with $u(1, \lambda) = 1$, then*

$$Q_k(\lambda) = p(1) \ldots p(k) u(k+1, \lambda), \quad k = 1, \ldots, T. \tag{2.11}$$

Proof. Let $u = (u_1, u_2, \ldots, u_T)^T$. Then

$$\begin{cases} \;\;(c(1) - \lambda m(1))u_1 - p(1)u_2 = 0, \\ -p(1)u_1 + (c(2) - \lambda m(2))u_2 - p(2)u_3 = 0, \\ \;\;\vdots \\ -p(T-2)u_{T-2} + (c(T-1) - \lambda m(T-1))u_{T-1} - p(T-1)u_T = 0, \\ -p(T-1)u_{T-1} + (c(T) - \lambda m(T))u_T = 0. \end{cases} \tag{2.12}$$

Since $p(0) = p(T) = 0$, (2.12) is equivalent to

$$-p(k-1)u_{k-1} + (c(k) - \lambda m(k))u_k - p(k)u_{k+1} = 0, \quad k = 1, \ldots, T,$$

where u_0 and u_{T+1} are determined by (1.2).

Let

$$v_0 = u_0, \;\; v_1 = u_1, \;\; v_k = p(1)p(2) \ldots p(k-1)u_k, \quad k = 2, \ldots T + 1.$$

Then

$$v_{k+1} = (c(k) - \lambda m(k))v_k - p^2(k-1)v_{k-1}, \quad k = 1, \dots T,$$
$$v_1 = u(1,\lambda) = 1 = Q_0(\lambda), \quad v_2 = Q_1(\lambda).$$

Obviously, since v_{k+1} and $Q_k(\lambda)$ satisfy the same recurrence formula (2.1), it follows that

$$v_{k+1} = Q_k(\lambda), \quad k = 1, \dots, T,$$

and accordingly, (2.11) holds. □

Proof of Theorem 1. (a) is a direct consequence of Lemma 2 and Lemma 5.

(b) From Lemma 7, we may determine that the number of sign changes of $\{u(1), \dots, u(T), u(T+1)\}$ via that of

$$\{Q_0(\lambda^+_{T,i}), Q_1(\lambda^+_{T,i}), \dots, Q_{T-1}(\lambda^+_{T,i}), Q_T(\lambda^+_{T,i})\}. \tag{2.13}$$

Notice that (1.2) implies that $u(T)u(T+1) \geq 0$. Then, the number of sign changes of (2.13) equals the number of sign changes in the sequence

$$\{Q_0(\lambda^+_{T,i}), Q_1(\lambda^+_{T,i}), \dots, Q_{T-1}(\lambda^+_{T,i})\}.$$

Let $\hat{w}(\lambda)$ be the sign-change time of the sequence

$$\{Q_0(\lambda),\ Q_1(\lambda), \dots, Q_{T-1}(\lambda)\}.$$

Using the same method to prove Lemma 6, with obvious changes, we may obtain that for $i \in \{2, \dots, (T-1)^+\}$,

$$\lim_{\lambda \to \lambda^+_{T-1,i-1}+0} \hat{w}(\lambda) = i-1, \qquad \lim_{\lambda \to \lambda^+_{T-1,i}-0} \hat{w}(\lambda) = i-1. \tag{2.14}$$

Thus, for $i \in \{2, \dots, (T-1)^+\}$, Lemma 5 yields $\lambda^+_{T-1,i-1} < \lambda^+_{T,i} < \lambda^+_{T-1,i}$. Combining this with (2.14) and the fact that $\hat{w}(\lambda)$ is nondecreasing in $(0,\infty)$, we obtain that

$$\hat{w}(\lambda^+_{T,i}) = \hat{w}\Big(\frac{\lambda^+_{T-1,i-1} + \lambda^+_{T-1,i}}{2}\Big) = i-1,$$

and accordingly

$$w(\lambda^+_{T,i}) = \hat{w}(\lambda^+_{T,i}) = i-1.$$

For the case that $i = 1$, since $Q_j(0) > 0$ for $j \in \{0, 1, \dots, T-1\}$, we get that $\lim_{\lambda \to 0+0} \hat{w}(\lambda) = 0$. Combining this with the facts $0 < \lambda^+_{T,1} < \lambda^+_{T-1,1}$ and $\lim_{\lambda \to \lambda^+_{T-1,1}-0} \hat{w}(\lambda) = 0$, we get that

$$w(\lambda^+_{T,1}) = \hat{w}(\lambda^+_{T,1}) = 0.$$

If $T^+ = (T-1)^+$, then it has been done! If $T^+ = (T-1)^+ + 1$, then by the same method with obvious changes, we get

$$w(\lambda^+_{T,T^+}) = \hat{w}(\lambda^+_{T,T^+}) = T^+ - 1.$$

Finally, by using the above method with obvious changes, we could prove that the number of sign changes $\psi^-_{T,i}$ is $i-1$. □

Remark 2. Applying the spectrum theory established in Theorem 1 and the similar methods developed in [16] and [17], we may prove the existence and multiplicity of sign-changing solutions of the corresponding nonlinear analogous

$$\Delta[p(t-1)\Delta u(t-1)] - q(t)u(t) + \lambda m(t) f(u(t)) = 0, \quad t \in \mathbb{T},$$

$$\alpha u(0) - \beta \Delta u(0) = 0, \quad \gamma u(T+1) + \delta \Delta u(T) = 0.$$

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