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# Eigensystem bootstrap multiscale analysis for the Anderson model

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**Abstract.** We use a bootstrap argument to enhance the eigensystem multiscale analysis, introduced by Elgart and Klein for proving localization for the Anderson model at high disorder. The eigensystem multiscale analysis studies finite volume eigensystems, not finite volume Green's functions. It yields pure point spectrum with exponentially decaying eigenfunctions and dynamical localization. The starting hypothesis for the eigensystem bootstrap multiscale analysis only requires the verification of polynomial decay of the finite volume eigenfunctions, at some sufficiently large scale, with some minimal probability independent of the scale. It yields exponential localization of finite volume eigenfunctions in boxes of side $L$, with the eigenvalues and eigenfunctions labeled by the sites of the box, with probability higher than $1 - \mathrm{e}^{-L^{\xi}}$, for any desired $0 < \xi < 1$.

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## Contents



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## Introduction

The eigensystem multiscale analysis is a new approach for proving localization for the Anderson model introduced by Elgart and Klein [10]. The usual proofs of localization for random Schrödinger operators are based on the study of finite volume Green's functions [13, 14, 8, 9, 19, 7, 12, 15, 17, 5, 16, 4, 1, 2, 3]. In contrast to the usual strategy, the eigensystem multiscale analysis is based on finite volume eigensystems, not finite volume Green's functions. It treats all energies of the finite volume operator at the same time, establishing level spacing and localization of eigenfunctions in a fixed box with high probability. A new feature is the labeling of the eigenvalues and eigenfunctions by the sites of the box.

In this paper we use a bootstrap argument as in Germinet and Klein [15] to enhance the eigensystem multiscale analysis. It yields exponential localization of finite volume eigenfunctions in boxes of side $L$, with the eigenvalues and eigenfunctions labeled by the sites of the box, with probability higher than $1-\mathrm{e}^{-L^{\xi}}$, for any $0<\xi<1$. The starting hypothesis for the eigensystem bootstrap multiscale analysis only requires the verification of polynomial decay of the finite volume eigenfunctions, at some sufficiently large scale, with some minimal probability independent of the scale. The advantage of the bootstrap multiscale analysis is that from the same starting hypothesis we get conclusions that are valid for any $0<\xi<1$.

We consider the Anderson model $H_{\varepsilon,\omega}=-\varepsilon\Delta+V_{\omega}$ on $\ell^2(\mathbb{Z}^d)$ (see Definition 1.1; $\varepsilon>0$ is the inverse of the disorder parameter). Multiscale analyses study finite volume operators $H_{\varepsilon,\omega,\Lambda}$, the restrictions of $H_{\varepsilon,\omega}$ to finite boxes $\Lambda$. The objects of interest for the eigensystem multiscale analysis are finite volume eigensystems. An eigensystem $\{(\varphi_j,\lambda_j)\}_{j\in J}$ for $H_{\varepsilon,\omega,\Lambda}$ consists of eigenpairs $(\varphi_j,\lambda_j)$, where $\lambda_j$ is an eigenvalue for $H_{\varepsilon,\omega,\Lambda}$ and $\varphi_j$ is a corresponding normalized eigenfunction, such that $\{\varphi_j\}_{j\in J}$ is an orthonormal basis for the finite dimensional Hilbert space $\ell^2(\Lambda)$. Elgart and Klein [10] called *a box* $\Lambda$ *localizing for* $H_{\varepsilon,\omega}$ if the eigenvalues of $H_{\varepsilon,\omega,\Lambda}$ satisfy a level spacing condition, and there exists an eigensystem for $H_{\varepsilon,\omega,\Lambda}$ indexed by the sites in the box, $\{(\varphi_x,\lambda_x)\}_{x\in\Lambda}$, with the eigenfunctions $\{\varphi_x\}_{x\in\Lambda}$ exhibiting exponential localization around the label, i.e., $|\varphi_x(y)|\le \mathrm{e}^{-m\|x-y\|}$ for $y\in\Lambda$ distant from $x$. They showed [10, Theorem 1.6] that, fixing $\xi\in(0,1)$, at high disorder ($\varepsilon\ll 1$) boxes of (sufficiently large) side $L$ are localizing with probability $\ge 1-\mathrm{e}^{-L^{\xi}}$, yielding all the usual forms of localization [10, Theorem 1.7 and Corollary 1.8]. More precisely, it is shown in [10] that for $\xi\in(0,1)$ there exists $\varepsilon_{\xi}>0$, decreasing as $\xi$ increases, and for $\varepsilon>0$ a scale $L_{\varepsilon}$, increasing as $\varepsilon$ decreases, such that for $0<\varepsilon\le\varepsilon_{\xi}$ and $L\ge L_{\varepsilon_{\xi}}$ boxes of side $L$ are localizing for $H_{\varepsilon,\omega}$ with probability $\ge 1-\mathrm{e}^{-L^{\xi}}$.

We use the ideas of Germinet and Klein [15] to perform a bootstrap multiscale analysis for finite volume eigensystems (Theorem 1.6). To start the multiscale analysis, we only have to verify a statement of polynomial localization of the eigenfunctions with some minimal probability independent of the scale. We conclude that at high disorder boxes of side $L$ are localizing with probability $\geq 1 - e^{-L^{\xi}}$ for all $\xi \in (0,1)$. It follows (Theorem 1.7) that there exists $\varepsilon_0 > 0$, and for each $\xi \in (0,1)$ there exists a scale $L_{\varepsilon_0,\xi}$, such that for all $0 < \varepsilon \leq \varepsilon_0$ and $L \geq L_{\varepsilon_0,\xi}$ boxes of side $L$ are localizing for $H_{\varepsilon,\omega}$ with probability $\geq 1 - e^{-L^{\xi}}$. How large $L$ needs to be depends on $\xi$, but the required amount of disorder is independent of $\xi$. In addition, if we have the conclusions of [10, Theorem 1.6] for a fixed $\xi \in (0,1)$, it follows from Theorem 1.6 that for all $\xi' \in (0,1)$ there exists a scale $L_{\xi'}$, such that for all $0 < \varepsilon \leq \varepsilon_\xi$ and $L \geq L_{\xi'}$ boxes of side $L$ are localizing for $H_{\varepsilon,\omega}$ with probability $\geq 1 - e^{-L^{\xi'}}$. (Note that $\varepsilon_\xi$ depends on the fixed $\xi$ but does not depend on $\xi'$.)

Recently, Elgart and Klein [11] extended the eigensystem multiscale analysis to establish localization for the Anderson model in an energy interval. This extension yields localization at fixed disorder on an interval at the edge of the spectrum (or in the vicinity of a spectral gap), and at a fixed interval at the bottom of the spectrum for sufficiently high disorder. We expect that our bootstrap eigensystem multiscale analysis can also be extended to energy intervals.

Our main definitions and resuts are stated in Section 1. Theorem 1.6 is the bootstrap eigensystem multiscale analysis. Theorem 1.7 gives the high disorder result for the Anderson model, and yields Theorem 1.8, which encapsulates localization for the Anderson model at high disorder. Theorem 1.6 is proven in Section 4, and Theorem 1.7 is proven in Section 5. In Section 2 we provide notation, definitions and lemmas for the proof of the bootstrap eigensystem multiscale analysis. In Section 3 we state the probability estimates for level spacing used in the proof of the bootstrap eigensystem multiscale analysis.

## 1. Main definitions and results

We consider the Anderson model in the following form.

**Definition 1.1.** The Anderson model is the random Schrödinger operator

$$H_{\varepsilon,\omega} := -\varepsilon\Delta + V_\omega \quad \text{on } \ell^2(\mathbb{Z}^d),$$

where $\varepsilon > 0$; $\Delta$ is the (centered) discrete Laplacian:

$$(\Delta\varphi)(x) := \sum_{y\in\mathbb{Z}^d,|y-x|=1} \varphi(y) \quad \text{for } \varphi \in \ell^2(\mathbb{Z}^d);$$

$V_\omega(x) = \omega_x$ for $x \in \mathbb{Z}^d$, where $\omega = \{\omega_x\}_{x\in\mathbb{Z}^d}$ is a family of independent identically distributed random variables, with a non-degenerate probability distribution $\mu$ with bounded support and Hölder continuous of order $\alpha \in \left(\frac{1}{2}, 1\right]$:

$$S_\mu(t) \le K t^\alpha \quad \text{for all } t \in [0, 1],$$

with $S_\mu(t) := \sup_{a\in\mathbb{R}} \mu\{[a, a + t]\}$ the concentration function of the measure $\mu$ and $K$ a constant.

Given $\Theta \subset \mathbb{Z}^d$, we let $T_\Theta = \chi_\Theta T \chi_\Theta$ be the restriction of the bounded operator $T$ on $\ell^2(\mathbb{Z}^d)$ to $\ell^2(\Theta)$. If $\Phi \subset \Theta \subset \mathbb{Z}^d$, we identify $\ell^2(\Phi)$ with a subset of $\ell^2(\Theta)$ by extending functions on $\Phi$ to functions on $\Theta$ that are identically 0 on $\Theta \setminus \Phi$. We write $\varphi_\Phi = \chi_\Phi\varphi$ if $\varphi$ is a function on $\Theta$. We let $\|\varphi\| = \|\varphi\|_2$ and $\|\varphi\|_\infty = \max_{y\in\Theta} |\varphi(y)|$ for $\varphi \in \ell^2(\Theta)$.

For $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$ we set

$$\|x\| = |x|_\infty = \max_{j=1,2,\dots,d} |x_j|,$$

$$|x| = |x|_2 = \Big(\sum_{j=1}^d x_j^2\Big)^{\frac{1}{2}},$$

$$|x|_1 = \sum_{j=1}^d |x_j|.$$

Given $\Xi \subset \mathbb{R}^d$, we let $\operatorname{diam} \Xi = \sup_{x,y\in\Xi} \|y - x\|$ denote its diameter, and set

$$\operatorname{dist}(x, \Xi) = \inf_{y\in\Xi} \|y - x\| \quad \text{for } x \in \mathbb{R}^d.$$

We use boxes in $\mathbb{Z}^d$ centered at points in $\mathbb{R}^d$. The box in $\mathbb{Z}^d$ of side $L > 0$ centered at $x \in \mathbb{R}^d$ is given by

$$\Lambda_L(x) = \Lambda_L^{\mathbb{R}}(x) \cap \mathbb{Z}^d, \quad \text{where } \Lambda_L^{\mathbb{R}}(x) = \left\{y \in \mathbb{R}^d;\, \|y - x\| \le \tfrac{L}{2}\right\}.$$

We write $\Lambda_L$ to denote a box $\Lambda_L(x)$ for some $x \in \mathbb{R}^d$. We have $(L-2)^d < |\Lambda_L| \le (L + 1)^d$ for $L \ge 2$, where for a set $\Theta \subset \mathbb{Z}^d$ we let $|\Theta|$ denote its cardinality.

The following definitions are for a fixed discrete Schrödinger operator $H_\varepsilon$. We omit $\varepsilon$ from the notation (i.e., we write $H$ for $H_\varepsilon$, $H_\Theta$ for $H_{\varepsilon,\Theta}$) when it does not lead to confusion. We always consider scales $L \geq 200$, and, for $\tau \in (0,1)$, set

$$L' = \lfloor \tfrac{L}{20} \rfloor \quad \text{and} \quad L_\tau = \lfloor L^\tau \rfloor.$$

For fixed $q > 0$, $\beta, \tau \in (0,1)$, we have the following definitions.

**Definition 1.2.** Let $\Lambda_L$ be a box, $x \in \Lambda_L$, and $\varphi \in \ell^2(\Lambda_L)$ with $\|\varphi\| = 1$.

(i) Given $\tilde{\theta} > 0$, $\varphi$ is said $(x, \tilde{\theta})$-*polynomially localized* if

$$|\varphi(y)| \leq L^{-\tilde{\theta}} \quad \text{for all } y \in \Lambda_L \text{ with } \|y - x\| \geq L'. \tag{1.1}$$

(ii) Given $\tilde{s} \in (0,1)$, $\varphi$ is said $(x, \tilde{s})$-*subexponentially localized* if

$$|\varphi(y)| \leq \mathrm{e}^{-L^{\tilde{s}}} \quad \text{for all } y \in \Lambda_L \text{ with } \|y - x\| \geq L'. \tag{1.2}$$

(iii) Given $m > 0$, $\varphi$ is said $(x, m)$-*localized* if

$$|\varphi(y)| \leq \mathrm{e}^{-m\|y-x\|} \quad \text{for all } y \in \Lambda_L \text{ with } \|y - x\| \geq L_\tau. \tag{1.3}$$

**Definition 1.3.** Let $R > 0$, and $\Theta \subset \mathbb{Z}^d$ be a finite set such that all eigenvalues of $H_\Theta$ are simple (i.e., $|\sigma(H_\Theta)| = |\Theta|$). Then

(i) $\Theta$ is called *$R$-polynomially level spacing for $H_\Theta$* if $|\lambda - \lambda'| \geq R^{-q}$ for all $\lambda, \lambda' \in \sigma(H_\Theta), \lambda \neq \lambda'$;

(ii) $\Theta$ is called *$R$-level spacing for $H_\Theta$* if $|\lambda - \lambda'| \geq \mathrm{e}^{-R^\beta}$ for all $\lambda, \lambda' \in \sigma(H_\Theta), \lambda \neq \lambda'$.

When $\Theta = \Lambda_L$, a box, and $R = L$, we will just say that $\Lambda_L$ is *polynomially level spacing for $H_{\Lambda_L}$*, or $\Lambda_L$ *is level spacing for $H_{\Lambda_L}$*.

Note that $R$-polynomially level spacing implies $R$-level spacing for sufficiently large $R$.

Given $\Theta \subset \mathbb{Z}^d$, $(\varphi, \lambda)$ is called an *eigenpair for $H_\Theta$* if $\varphi \in \ell^2(\Theta)$, $\lambda \in \mathbb{R}$ with $\|\varphi\| = 1$, and $H_\Theta \varphi = \lambda\varphi$ (i.e., $\lambda$ is an eigenvalue for $H_\Theta$ with a corresponding normalized eigenfunction $\varphi$). A collection $\{(\varphi_j, \lambda_j)\}_{j \in J}$ of eigenpairs for $H_\Theta$ is called an *eigensystem for $H_\Theta$* if $\{\varphi_j\}_{j \in J}$ is an orthonormal basis for $\ell^2(\Theta)$. We may rewrite the eigensystem as $\{(\psi_\lambda, \lambda)\}_{\lambda \in \sigma(H_\Theta)}$ if all eigenvalues of $H_\Theta$ are simple.

**Definition 1.4.** Let $\Lambda_L$ be a box.

(i) Given $\tilde{\theta} > 0$, $\Lambda_L$ will be called *$\tilde{\theta}$-polynomially localizing* (*PL*) *for* $H$ if the following holds:

  (a) $\Lambda_L$ is polynomially level spacing for $H_{\Lambda_L}$;

  (b) there exists a $\tilde{\theta}$-polynomially localized eigensystem for $H_{\Lambda_L}$, that is, an eigensystem $\{(\varphi_x, \lambda_x)\}_{x\in\Lambda_L}$ for $H_{\Lambda_L}$ such that $\varphi_x$ is $(x, \tilde{\theta})$-polynomially localized for all $x \in \Lambda_L$.

(ii) Given $m^* > 0$, $\Lambda_L$ will be called *$m^*$-mix localizing* (*ML*) *for* $H$ if the following holds:

  (a) $\Lambda_L$ is polynomially level spacing for $H_{\Lambda_L}$;

  (b) there exists an $m^*$-localized eigensystem for $H_{\Lambda_L}$, that is, an eigensystem $\{(\varphi_x, \lambda_x)\}_{x\in\Lambda_L}$ for $H_{\Lambda_L}$ such that $\varphi_x$ is $(x, m^*)$-localized for all $x \in \Lambda_L$.

(iii) Given $\tilde{s} \in (0, 1)$, $\Lambda_L$ will be called *$\tilde{s}$-subexponentially localizing* (*SEL*) *for* $H$ if the following holds:

  (a) $\Lambda_L$ is level spacing for $H_{\Lambda_L}$;

  (b) there exists an $\tilde{s}$-subexponentially localized eigensystem for $H_{\Lambda_L}$, that is, an eigensystem $\{(\varphi_x, \lambda_x)\}_{x\in\Lambda_L}$ for $H_{\Lambda_L}$ such that $\varphi_x$ is $(x, \tilde{s})$-subexponentially localized for all $x \in \Lambda_L$.

(iv) Given $m > 0$, $\Lambda_L$ will be called *$m$-localizing* (*LOC*) *for* $H$ if the following holds:

  (a) $\Lambda_L$ is level spacing for $H_{\Lambda_L}$;

  (b) there exists an $m$-localized eigensystem for $H_{\Lambda_L}$.

**Remark 1.5.** It follows immediately from the definition that given $\tilde{s} \in (0, 1)$,

$$\Lambda_L \quad \text{is } m^*\text{-mix localizing} \implies \Lambda_L \text{ is } \Big(1 - \frac{\log \frac{40}{m^*}}{\log L}\Big)\text{-SEL} \implies \Lambda_L \text{ is } \tilde{s}\text{-SEL},$$

for sufficiently large $L$. (We consider $m^* < 40$.)

We now state the bootstrap multiscale analysis. We will use $C_{a,b,\ldots}$, $C'_{a,b,\ldots}$, $C(a, b, \ldots)$, etc., to denote a finite constant depending on the parameters $a, b, \ldots$. Note that $C_{a,b,\ldots}$ may denote different constants in different equations, and even in the same equation. We will omit the dependence on $d$ and $\mu$ from the notation.

Given $\theta > \left(\frac{6}{2\alpha-1} + \frac{9}{2}\right) d$ and $0 < \xi < 1$, we introduce the following parameters:

- We fix $q, p, \gamma_1$ such that

$$\tfrac{3d}{2\alpha-1} < q < \tfrac{1}{2}\big(\theta - \tfrac{9}{2}d\big), \tag{1.4a}$$

$$0 < p < (2\alpha-1)q - 3d, \tag{1.4b}$$

and

$$1 < \gamma_1 < \min\big\{1 + \tfrac{p}{p+2d}, \tfrac{2\theta-4d}{5d+4q}\big\}, \tag{1.4c}$$

and note that

$$\theta > 2d + \gamma_1\big(\tfrac{5d}{2} + 2q\big) > \tfrac{9d}{2} + 2q$$

- We fix $\zeta, \beta, \gamma, \tau$ such that

$$0 < \xi < \zeta < \beta < \tfrac{1}{\gamma} < 1 < \gamma < \sqrt{\tfrac{\zeta}{\xi}}, \tag{1.5a}$$

and

$$\max\big\{\tfrac{1+\gamma_1}{2\gamma_1}, \tfrac{1+\gamma\beta}{2}, \tfrac{(\gamma-1)\beta+1}{\gamma}\big\} < \tau < 1, \tag{1.5b}$$

and note that

$$\tfrac{1}{\gamma_1} < 1 - \tau + \tfrac{1}{\gamma_1} < \tau,$$

and

$$0 < \xi < \xi\gamma^2 < \zeta < \beta < \tfrac{\tau}{\gamma} < \tfrac{1}{\gamma} < \tau < 1 < \tfrac{1-\beta}{\tau-\beta} < \gamma < \tfrac{\tau}{\beta}.$$

- We fix $s$ such that

$$\max\big\{\gamma\beta, 1 - 2\gamma\big(\tau - \tfrac{1+\gamma\beta}{2}\big)\big\} < s < 1,$$

and note that

$$0 < \zeta < \beta < \gamma\beta < s < 1 \quad \text{and} \quad 1 - \tau + \tfrac{1-s}{\gamma} < \tau - \gamma\beta.$$

- We also let

$$\tilde{\zeta} = \tfrac{\zeta+\beta}{2} \in (\zeta, \beta), \quad \tilde{\tau} = \tfrac{1+\tau}{2} \in (\tau, 1), \quad L_{\tilde{\tau}} = \lfloor L^{\tilde{\tau}} \rfloor.$$

In what follows, given $\theta > \big(\tfrac{6}{2\alpha-1} + \tfrac{9}{2}\big)d$, we fix $q, p, \gamma_1$ as in (1.4), and then, given $0 < \xi < 1$, we fix $\zeta, \beta, \gamma, \tau$ as in (1.5). We use Definitions 1.2–1.4 with these fixed $q, \beta, \tau$, which we omit from the dependence of the constants.

**Theorem 1.6.** *Let* $\theta > \left(\frac{6}{2\alpha-1} + \frac{9}{2}\right) d$ *and* $\varepsilon_0 > 0$. *There exists a finite scale* $\mathcal{L}(\varepsilon_0, \theta)$ *with the following property: Suppose for some* $\varepsilon \in (0, \varepsilon_0]$, $L_0 \geq \mathcal{L}(\varepsilon_0, \theta)$, *and* $0 \leq P_0 < \frac{1}{2(800)^{2d}}$, *we have*

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_{L_0}(x) \textit{ is } \theta\textit{-polynomially localizing for } H_{\varepsilon,\omega}\} \geq 1 - P_0.$$

*Then, given* $0 < \xi < 1$, *we can find a finite scale* $\widetilde{L} = \widetilde{L}(\varepsilon_0, \theta, \xi, L_0)$ *and* $m_\xi = m(\xi, \widetilde{L}) > 0$ *such that*

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_L(x) \textit{ is } m_\xi\textit{-localizing for } H_{\varepsilon,\omega}\} \geq 1 - \mathrm{e}^{-L^\xi} \quad \textit{for all } L \geq \widetilde{L}. \tag{1.6}$$

The eigensystem bootstrap multiscale analysis, stated in Theorem 1.6, is proven in Section 4. It follows from a repeated use of a bootstrap argument, as in [15, Section 6], making successive use of Propositions 4.1, 4.3, 4.4, 4.6, 4.8, and 4.9. Propositions 4.1, 4.4, 4.6, and 4.9 are eigensystem multiscale analyses. But there is a difference in the procedure comparing with the Green's function bootstrap multiscale analysis of [15]. Unlike the definitions of good boxes for the Green's function multiscale analyses, the definitions of good (i.e., localizing) boxes for the eigensystem multiscale analyses, given in Definition 1.4, require intermediate scales, namely $\frac{L}{20}$ and $L^\tau$ in Definition 1.2. For this reason we only have the direct implications given in Remark 1.5. Thus the bootstrap between the eigensystem multiscale analyses requires some extra intermediate steps, given in Propositions 4.3 and 4.8.

In Section 5 we will prove that we can fulfill the hypotheses of Theorem 1.6, obtaining the following theorem.

**Theorem 1.7.** *There exists* $\varepsilon_0 > 0$ *such that, given* $0 < \xi < 1$, *we can find a finite scale* $\widetilde{L} = \widetilde{L}(\varepsilon_0, \xi)$ *and* $m_\xi = m(\xi, \widetilde{L}) > 0$ *such that for all* $0 < \varepsilon \leq \varepsilon_0$ *we have*

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_L(x) \textit{ is } m_\xi\textit{-localizing for } H_{\varepsilon,\omega}\} \geq 1 - \mathrm{e}^{-L^\xi} \quad \textit{for all } L \geq \widetilde{L}.$$

Theorem 1.7 yields all the usual forms of localization. To see this, we introduce some notation and definitions. We fix $\nu > \frac{d}{2}$, and set $\langle x \rangle = \sqrt{1 + \|x\|^2}$.

A function $\psi : \mathbb{Z}^d \to \mathbb{C}$ is called a *$\nu$-generalized eigenfunction for $H_\varepsilon$* if $\psi$ is a generalized eigenfunction (see (2.4)) and $0 < \|\langle x \rangle^{-\nu} \psi\| < \infty$. We let $\mathcal{V}_\varepsilon(\lambda)$ denote the collection of $\nu$-generalized eigenfunctions for $H_\varepsilon$ with generalized eigenvalue $\lambda \in \mathbb{R}$.

Given $\lambda \in \mathbb{R}$ and $a, b \in \mathbb{Z}^d$, we set

$$W_{\varepsilon,\lambda}^{(a)}(b) := \begin{cases} \sup_{\psi \in \mathcal{V}_\varepsilon(\lambda)} \frac{|\psi(b)|}{\|\langle x-a \rangle^{-\nu} \psi\|} & \text{if } \mathcal{V}_\varepsilon(\lambda) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1.7 yields the following theorem, from which one can derive Anderson localization (pure point spectrum with exponentially decaying eigenfunctions) dynamical localization, and more, as in [10, Corollary 1.8].

**Theorem 1.8.** *Let $H_{\varepsilon,\omega}$ be an Anderson model. There exists $\varepsilon_0 > 0$ such that, given $\xi \in (0,1)$, we can find a scale $\widehat{L} = \widehat{L}(\varepsilon_0, \xi)$ and $m_\xi = m(\xi, \widehat{L}) > 0$, such that for all $0 < \varepsilon \le \varepsilon_0$, $L \ge \widehat{L}$ with $L \in 2\mathbb{N}$, and $a \in \mathbb{Z}^d$ there exists an event $\mathcal{Y}_{\varepsilon,L,a}$ with the following properties*:

(i) *$\mathcal{Y}_{\varepsilon,L,a}$ depends only on the random variables $\{\omega_x\}_{x\in\Lambda_{5L}(a)}$, and*

$$\mathbb{P}\{\mathcal{Y}_{\varepsilon,L,a}\} \ge 1 - C_{\varepsilon_0} \mathrm{e}^{-L^\xi}.$$

(ii) *For all $\omega \in \mathcal{Y}_{\varepsilon,L,a}$ and $\lambda \in \mathbb{R}$ we have, with*

$$\max_{b\in\Lambda_{\frac{L}{3}}(a)} W^{(a)}_{\varepsilon,\omega,\lambda}(b) > \mathrm{e}^{-\frac{1}{4}m_\xi L} \implies \max_{y\in A_L(a)} W^{(a)}_{\varepsilon,\omega,\lambda}(y) \le \mathrm{e}^{-\frac{7}{132}m_\xi\|y-a\|},$$

*where*

$$A_L(a) := \left\{y \in \mathbb{Z}^d; \tfrac{8}{7}L \le \|y-a\| \le \tfrac{33}{14}L\right\}.$$

*In particular,*

$$W^{(a)}_{\varepsilon,\omega,\lambda}(a) W^{(a)}_{\varepsilon,\omega,\lambda}(y) \le \mathrm{e}^{-\frac{7}{132}m_\xi\|y-a\|} \quad \textit{for all } y \in A_L(a).$$

Theorem 1.8 is proved in the same way as [10, Theorem 1.7].

## 2. Preliminaries to the multiscale analysis

We consider a fixed discrete Schrödinger operator $H = -\varepsilon\Delta + V$ on $\ell^2(\mathbb{Z}^d)$, where $0 < \varepsilon \le \varepsilon_0$ for a fixed $\varepsilon_0$ and $V$ is a bounded potential.

**2.1. Some basic facts and definitions.** Let $\Phi \subset \Theta \subset \mathbb{Z}^d$. We define the boundary, exterior boundary, and interior boundary of $\Phi$ relative to $\Theta$, respectively, by

$$\boldsymbol{\partial}^\Theta \Phi = \{(u,v) \in \Phi \times (\Theta \setminus \Phi); |u-v| = 1\},$$

$$\partial^\Theta_{\mathrm{ex}} \Phi = \{v \in (\Theta \setminus \Phi); (u,v) \in \boldsymbol{\partial}^\Theta \Phi \text{ for some } u \in \Phi\},$$

$$\partial^\Theta_{\mathrm{in}} \Phi = \{u \in \Phi; (u,v) \in \boldsymbol{\partial}^\Theta \Phi \text{ for some } v \in \Theta \setminus \Phi\}.$$

We have

$$H_\Theta = H_\Phi \oplus H_{\Theta\setminus\Phi} + \varepsilon\Gamma_{\boldsymbol{\partial}^\Theta\Phi} \quad \text{on } \ell^2(\Theta) = \ell^2(\Phi) \oplus \ell^2(\Theta\setminus\Phi),$$

where

$$\Gamma_{\boldsymbol{\partial}^\Theta\Phi}(u,v) = \begin{cases} -1 & \text{if either } (u,v) \text{ or } (v,u) \in \boldsymbol{\partial}^\Theta\Phi, \\ 0 & \text{otherwise.} \end{cases}$$

For $t \geq 1$ we set

$$\Phi^{\Theta,t} = \{y \in \Phi; \Lambda_{2t}(y) \cap \Theta \subset \Phi\} = \{y \in \Phi; \operatorname{dist}(y, \Theta\setminus\Phi) > \lfloor t \rfloor\},$$

$$\partial_{\text{in}}^{\Theta,t}\Phi = \Phi \setminus \Phi^{\Theta,t} = \{y \in \Phi; \operatorname{dist}(y, \Theta\setminus\Phi) \leq \lfloor t \rfloor\},$$

$$\partial^{\Theta,t}\Phi = \partial_{\text{in}}^{\Theta,t}\Phi \cup \partial_{\text{ex}}^{\Theta}\Phi.$$

Given a box $\Lambda_L(x) \subset \Theta \subset \mathbb{Z}^d$ we write $\Lambda_L^{\Theta,t}(x)$ for $(\Lambda_L(x))^{\Theta,t}$.

For a box $\Lambda_L \subset \Theta \subset \mathbb{Z}^d$, there exists a unique $\hat{v} \in \partial_{\text{in}}^{\Lambda_L}\Theta$ for each $v \in \partial_{\text{ex}}^{\Lambda_L}\Theta$ such that $(\hat{v}, v) \in \partial_{\Lambda_L}\Theta$. Given $v \in \Theta$, we define $\hat{v}$ as above if $v \in \partial_{\text{ex}}^{\Lambda_L}\Theta$, and set $\hat{v} = v$ otherwise. Note that $|\partial_{\text{ex}}^{\Lambda_L}\Theta| = |\boldsymbol{\partial}_{\Lambda_L}\Theta|$. If $L \geq 2$, we have

$$|\partial_{\text{in}}^{\Theta}\Lambda_L| \leq |\partial_{\text{ex}}^{\Theta}\Lambda_L| = |\boldsymbol{\partial}^\Theta\Lambda_L| \leq s_d L^{d-1}, \quad \text{where } s_d = 2^d d.$$

To cover a box of side $L$ by boxes of side $\ell < L$, we will use suitable covers as in [10, Definition 3.10] (also see [16, Definition 3.12]).

**Definition 2.1.** Let $\Lambda_L = \Lambda_L(x_0)$, $x_0 \in \mathbb{R}^d$ be a box in $\mathbb{Z}^d$, and let $\ell < L$. A suitable $\ell$-cover of $\Lambda_L$ is the collection of boxes

$$\mathcal{C}_{L,\ell}(x_0) = \{\Lambda_\ell(a)\}_{a\in\Xi_{L,\ell}},$$

where

$$\Xi_{L,\ell} := \{x_0 + \rho\ell\mathbb{Z}^d\} \cap \Lambda_L^{\mathbb{R}} \quad \text{with } \rho \in \left[\tfrac{3}{5}, \tfrac{4}{5}\right] \cap \left\{\tfrac{L-\ell}{2\ell k}; k \in \mathbb{N}\right\}.$$

We call $\mathcal{C}_{L,\ell}(x_0)$ the suitable $\ell$-cover of $\Lambda_L$ if

$$\rho = \rho_{L,\ell} := \max\left\{\left[\tfrac{3}{5}, \tfrac{4}{5}\right] \cap \left\{\tfrac{L-\ell}{2\ell k}; k \in \mathbb{N}\right\}\right\}.$$

Note that $\left[\tfrac{3}{5}, \tfrac{4}{5}\right] \cap \left\{\tfrac{L-\ell}{2\ell k}; k \in \mathbb{N}\right\} \neq \emptyset$ if $\ell \leq \tfrac{L}{6}$. For a suitable $\ell$-cover $\mathcal{C}_{L,\ell}(x_0)$, we have (see [10, Lemma 3.11])

$$\Lambda_L = \bigcup_{a\in\Xi_{L,\ell}} \Lambda_\ell^{\Lambda_L, \frac{\ell}{10}}(a); \tag{2.1}$$

$$\left(\tfrac{L}{\ell}\right)^d \leq \#\Xi_{L,\ell} = \left(\tfrac{L-\ell}{\rho\ell} + 1\right)^d \leq \left(\tfrac{2L}{\ell}\right)^d. \tag{2.2}$$

**2.2. Lemmas about eigenpairs.** Given both $\Theta \subset \mathbb{Z}^d$ and an eigensystem $\{(\varphi_j, \lambda_j)\}_{j \in J}$ for $H_\Theta$. We have

$$\delta_y = \sum_{j \in J} \overline{\varphi_j(y)} \varphi_j \quad \text{for all } y \in \Theta, \tag{2.3a}$$

$$\psi(y) = \langle \delta_y, \psi \rangle = \sum_{j \in J} \varphi_j(y) \langle \varphi_j, \psi \rangle \quad \text{for all } \psi \in \ell^2(\Theta) \text{ and } y \in \Theta. \tag{2.3b}$$

Given $\Theta \subset \mathbb{Z}^d$, a function $\psi \colon \Theta \to \mathbb{C}$ is called a *generalized eigenfunction for $H_\Theta$ with generalized eigenvalue $\lambda \in \mathbb{R}$* if $\psi$ is not identically zero and

$$-\varepsilon \sum_{y \in \Theta, |y-x|=1} \psi(y) + (V(x) - \lambda)\psi(x) = 0 \quad \text{for all } x \in \Theta,$$

or, equivalently,

$$\langle (H_\Theta - \lambda)\varphi, \psi \rangle = 0 \quad \text{for all } \varphi \in \ell^2(\Theta) \text{ with finite support.} \tag{2.4}$$

If $\psi \in \ell^2(\Theta)$, $\psi$ is an eigenfunction for $H_\Theta$ with eigenvalue $\lambda$. We do not require generalized eigenfunctions to be in $\ell^2(\Theta)$, we only require the pointwise equality in (2.4). If $\Theta$ is finite there is no difference between generalized eigenfunctions and eigenfunctions.

**Lemma 2.2.** *Consider a box $\Lambda_L \subset \Theta \subset \mathbb{Z}^d$, and suppose $(\varphi, \lambda)$ is an eigenpair for $H_{\Lambda_L}$.*

(i) *Given $\tilde{\theta} > 0$, if $\varphi$ is $(x, \tilde{\theta})$-polynomially localized for some $x \in \Lambda_L^{\Theta, L'}$, we have*
$$\operatorname{dist}(\lambda, \sigma(H_\Theta)) \le \|(H_\Theta - \lambda)\varphi\| \le C_{d,\varepsilon_0} L^{-(\tilde{\theta} - \frac{d-1}{2})}.$$

(ii) *Given $\tilde{s} \in (0,1)$, if $\varphi$ is $(x, \tilde{s})$-subexponentially localized for some $x \in \Lambda_L^{\Theta, L'}$, we have*
$$\operatorname{dist}(\lambda, \sigma(H_\Theta)) \le \|(H_\Theta - \lambda)\varphi\| \le \mathrm{e}^{-c_1 L^{\tilde{s}}}, \tag{2.5}$$
*where $c_1 = c_1(L) \ge 1 - C_{d,\varepsilon_0} \frac{\log L}{L^{\tilde{s}}}$.*

(iii) *Given $m > 0$ and $\tau \in (0,1)$, if $\varphi$ is $(x, m)$ localized for some $x \in \Lambda_L^{\Theta, L_\tau}$, we have*
$$\operatorname{dist}(\lambda, \sigma(H_\Theta)) \le \|(H_\Theta - \lambda)\varphi\| \le \mathrm{e}^{-m_1 L_\tau}, \tag{2.6}$$
*where $m_1 = m_1(L) \ge m - C_{d,\varepsilon_0} \frac{\log L}{L_\tau}$.*

*Proof.* We prove part (i), the proofs of (ii) and (iii) are similar. If $x \in \Lambda_L^{\Theta,L'}$, we have $\operatorname{dist}(x, \partial_{\text{in}}^{\Theta}\Lambda_L) \geq L'$, thus it follows from [10, Lemma 3.2] that

$$\begin{aligned}\|(H_\Theta - \lambda)\varphi\| &\leq \varepsilon\sqrt{s_d}\, L^{\frac{d-1}{2}} \|\varphi_{\partial_{\text{in}}^{\Theta}\Lambda_L}\|_\infty \\ &\leq \varepsilon\sqrt{s_d}\, L^{\frac{d-1}{2}} L^{-\tilde{\theta}} \\ &\leq \varepsilon_0\sqrt{s_d}\, L^{-(\tilde{\theta}-\frac{d-1}{2})}. \qquad \square\end{aligned}$$

For the following lemmas in this and next subsections, we fix $\theta > \big(\frac{6}{2\alpha-1} + \frac{9}{2}\big) d$ and $0 < \xi < 1$ (so $q, p, \gamma_1, \zeta, \beta, \gamma, \tau, s$ are fixed). Also, when we consider $\Lambda_\ell$ to be a $\sharp$ box, where $\sharp$ stands for $\theta$-PL, $m^*$-ML, $s$-SEL or $m$-LOC, with $m^* \geq m_-^*(\ell) > 0$ and $m \geq m_-(\ell) > 0$, we let

$$L = L_\sharp = \begin{cases} Y\ell \text{ or } \ell^{\gamma_1} & \text{if } \sharp \text{ is } \theta\text{-PL}, \\ \ell^{\gamma_1} & \text{if } \sharp \text{ is } m^*\text{-ML}, \\ Y\ell \text{ or } \ell^{\gamma} & \text{if } \sharp \text{ is } s\text{-SEL}, \\ \ell^{\gamma} & \text{if } \sharp \text{ is } m\text{-LOC}, \end{cases} \tag{2.7a}$$

and

$$\ell_\sharp = \begin{cases} \ell' & \text{if } \sharp \text{ is } \theta\text{-PL or } s\text{-SEL}, \\ \ell_\tau & \text{if } \sharp \text{ is } m^*\text{-ML or } m\text{-LOC}, \end{cases} \tag{2.7b}$$

where $Y \geq 1$. We will omit the dependence on $\theta$, $\xi$ and $Y$ from the notation.

We prove most of the lemmas only for $\sharp$ being $\theta$-PL. The proofs of other cases are similar.

**Lemma 2.3.** *Given $\Theta \subset \mathbb{Z}^d$, let $\psi\colon \Theta \to \mathbb{C}$ be a generalized eigenfunction for $H_\Theta$ with generalized eigenvalue $\lambda \in \mathbb{R}$. Consider a $\sharp$ box $\Lambda_\ell \subset \Theta$ with a corresponding eigensystem $\{(\varphi_u, \nu_u)\}_{u\in\Lambda_\ell}$, and suppose for all $u \in \Lambda_\ell^{\Theta,\ell_\sharp}$ we have*

$$|\lambda - \nu_u| \geq \begin{cases} \frac{1}{2}L^{-q} & \textit{if } \sharp \textit{ is } \theta\textit{-PL or } m^*\textit{-ML} \\ \frac{1}{2}\mathrm{e}^{-L^{\beta}} & \textit{if } \sharp \textit{ is } s\textit{-SEL or } m\textit{-LOC} \end{cases}. \tag{2.8}$$

*Then the following holds for sufficiently large $\ell$:*

(i) *Let $y \in \Lambda_\ell^{\Theta,2\ell_\sharp}$. Then*

(a) *if $\sharp$ is $\theta$-PL,*

$$|\psi(y)| \leq C_{d,\varepsilon_0} L^q \ell^{-(\theta-2d)} |\psi(y_1)| \quad \textit{for some } y_1 \in \partial^{\Theta,2\ell'}\Lambda_\ell; \tag{2.9}$$

(b) *if* $\sharp$ *is* $s$*-SEL,*

$$|\psi(y)| \leq \mathrm{e}^{-c_2 \ell^s} |\psi(y_1)| \quad \textit{for some } y_1 \in \partial^{\Theta,2\ell'} \Lambda_\ell, \tag{2.10}$$

*where* $c_2 = c_2(\ell) \geq 1 - C_{d,\varepsilon_0} L^\beta \ell^{-s}$*;*

(c) *if* $\sharp$ *is* $m^*$*-ML,*

$$|\psi(y)| \leq \mathrm{e}^{-m_2^* \ell_\tau} |\psi(y_1)| \quad \textit{for some } y_1 \in \partial^{\Theta,2\ell_\tau} \Lambda_\ell, \tag{2.11}$$

*where* $m_2^* = m_2^*(\ell) \geq m^* - C_{d,\varepsilon_0} \gamma_1 q \frac{\log \ell}{\ell_\tau}$*;*

(d) *If* $\sharp$ *is* $m$*-LOC,*

$$|\psi(y)| \leq \mathrm{e}^{-m_2 \ell_\tau} |\psi(y_1)| \quad \textit{for some } y_1 \in \partial^{\Theta,2\ell_\tau} \Lambda_\ell, \tag{2.12}$$

*where* $m_2 = m_2(\ell) \geq m - C_{d,\varepsilon_0} \ell^{\gamma\beta - \tau}$*.*

(ii) *Let* $y \in \Lambda_\ell^{\Theta,2\ell_{\tilde\tau}}$*. Then*

(a) *if* $\sharp$ *is* $m^*$*-ML,*

$$|\psi(y)| \leq \mathrm{e}^{-m_3^* \|y_2 - y\|} |\psi(y_2)| \quad \textit{for some } y_2 \in \partial^{\Theta,\ell_{\tilde\tau}} \Lambda_\ell, \tag{2.13}$$

*where* $m_3^* = m_3^*(\ell) \geq m^*(1 - 4\ell^{\frac{\tau-1}{2}}) - C_{d,\varepsilon_0} \gamma_1 q \frac{\log \ell}{\ell_{\tilde\tau}}$*;*

(b) *if* $\sharp$ *is* $m$*-LOC,*

$$|\psi(y)| \leq \mathrm{e}^{-m_3 \|y_2 - y\|} |\psi(y_2)| \quad \textit{for some } y_2 \in \partial^{\Theta,\ell_{\tilde\tau}} \Lambda_\ell, \tag{2.14}$$

*where* $m_3 = m_3(\ell) \geq m(1 - 4\ell^{\frac{\tau-1}{2}}) - C_{d,\varepsilon_0} \ell^{\gamma\beta - \tilde\tau}$*.*

*Proof.* Let $y \in \Lambda_\ell$, we have (see (2.3))

$$\psi(y) = \sum_{u \in \Lambda_\ell} \varphi_u(y) \langle \varphi_u, \psi \rangle = \sum_{u \in \Lambda_\ell^{\Theta,\ell'}} \varphi_u(y) \langle \varphi_u, \psi \rangle + \sum_{u \in \partial_{\mathrm{in}}^{\Theta,\ell'} \Lambda_\ell} \varphi_u(y) \langle \varphi_u, \psi \rangle. \tag{2.15}$$

If $u \in \Lambda_\ell^{\Theta,\ell'}$, we have $|\lambda - \nu_u| \geq \frac{1}{2} L^{-q}$ by (2.8). Using (2.4), we get

$$\langle \varphi_u, \psi \rangle = (\lambda - \nu_u)^{-1} \langle \varphi_u, (H_\Theta - \nu_u)\psi \rangle = (\lambda - \nu_u)^{-1} \langle (H_\Theta - \nu_u)\varphi_u, \psi \rangle.$$

It follows from [10, Lemma 3.2] that

$$|\varphi_u(y) \langle \varphi_u, \psi \rangle| \leq 2L^q \varepsilon \sum_{v \in \partial_{\mathrm{ex}}^{\Theta} \Lambda_\ell} |\varphi_u(y) \varphi_u(\hat v)| |\psi(v)|. \tag{2.16}$$

If $v' \in \partial_{\text{in}}^{\Theta}\Lambda_\ell$, we have $\|v'-u\| \geq \ell'$, so (1.1) gives $|\varphi_u(v')| \leq \ell^{-\theta}$. It follows from (2.16) and $\|\varphi_u\| = 1$ that

$$|\varphi_u(y)\langle\varphi_u,\psi\rangle| \leq 2\varepsilon L^q \ell^{-\theta} \sum_{v\in\partial_{\text{ex}}^{\Theta}\Lambda_\ell} |\psi(v)| \leq 2\varepsilon s_d L^q \ell^{-(\theta-d+1)}|\psi(v_1)|$$

for some $v_1 \in \partial_{\text{ex}}^{\Theta}\Lambda_\ell$. Therefore

$$\left|\sum_{u\in\Lambda_\ell^{\Theta,\ell'}} \varphi_u(y)\langle\varphi_u,\psi\rangle\right| \leq 2\varepsilon s_d L^q \ell^{-(\theta-2d+1)}|\psi(v_2)| \tag{2.17}$$

for some $v_2 \in \partial_{\text{ex}}^{\Theta}\Lambda_\ell$.

Let $y \in \Lambda_\ell^{\Theta,2\ell'}$. If $u \in \partial_{\text{in}}^{\Theta,\ell'}\Lambda_\ell$, we have $\|u-y\| \geq 2\ell' - \ell' = \ell'$, thus (1.1) gives $|\varphi_u(y)| \leq \ell^{-\theta}$, and hence

$$\left|\sum_{u\in\partial_{\text{in}}^{\Theta,\ell'}\Lambda_\ell} \varphi_u(y)\langle\varphi_u,\psi\rangle\right| \leq \ell^{-(\theta-d)}\|\psi\chi_{\Lambda_\ell}\| \leq \ell^{-(\theta-\frac{3d}{2})}|\psi(v_3)| \tag{2.18}$$

for some $v_3 \in \Lambda_\ell$. Combining (2.15), (2.17), and (2.18), we conclude that

$$|\psi(y)| \leq (1+2\varepsilon_0 s_d)L^q \ell^{-(\theta-2d)}|\psi(y_1)| \tag{2.19}$$

for some $y_1 \in \Lambda_\ell \cup \partial_{\text{ex}}^{\Theta}\Lambda_\ell$. If $y_1 \notin \partial^{\Theta,2\ell'}\Lambda_\ell$ we repeat the procedure to estimate $|\psi(y_1)|$. Since we can suppose $\psi(y) \neq 0$ without loss of generality, the procedure must stop after finitely many times, and at that time we must have (2.9).

We prove part (ii) only for $\sharp$ being $m^*$-ML. The proof for $\sharp$ being $m$-LOC is similar. Let $y \in \Lambda_\ell^{\Theta,\ell_{\tilde\tau}}$, then $\|y-v'\| \geq \ell_{\tilde\tau}$ for $v' \in \partial_{\text{in}}^{\Theta}\Lambda_\ell$. Thus for $u \in \Lambda_\ell^{\Theta,\ell_\tau}$ and $v' \in \partial_{\text{in}}^{\Theta}\Lambda_\ell$ we have

$$|\varphi_u(y)\varphi_u(v')| \leq \begin{cases} \mathrm{e}^{-m^*(\|y-u\|+\|v'-u\|)} \leq \mathrm{e}^{-m^*\|v'-y\|} & \text{if } \|y-u\| \geq \ell_\tau, \\ \mathrm{e}^{-m^*\|v'-u\|} \leq \mathrm{e}^{-m_1'\|v'-y\|} & \text{if } \|y-u\| < \ell_\tau, \end{cases} \tag{2.20}$$

where

$$m_1' \geq m^*(1-2\ell^{\tau-\tilde\tau}) = m^*(1-2\ell^{\frac{\tau-1}{2}}),$$

since for $\|y-u\| < \ell_\tau$,

$$\|v'-u\| \geq \|v'-y\| - \|y-u\| \geq \|v'-y\| - \ell_\tau \geq \|v'-y\|\big(1-\tfrac{\ell_\tau}{\ell_{\tilde\tau}}\big).$$

Combining (2.16) and (2.20), we conclude that

$$\begin{aligned} |\varphi_u(y)\langle\varphi_u,\psi\rangle| &\leq 2\varepsilon L^q \sum_{v\in\partial_{\text{ex}}^{\Theta}\Lambda_\ell} \mathrm{e}^{-m_1'(\|v-y\|-1)}|\psi(v)| \\ &\leq 2\varepsilon s_d \ell^{\gamma_1 q+d-1}\mathrm{e}^{-m_1'(\|v_1-y\|-1)}|\psi(v_1)| \\ &\leq \mathrm{e}^{-m_2'\|v_1-y\|}|\psi(v_1)| \end{aligned} \tag{2.21}$$

for some $v_1 \in \partial_{\text{ex}}^{\Theta}\Lambda_\ell$, where we used $\|v_1 - y\| \geq \ell_{\tilde{\tau}}$ and took

$$m_2' \geq m_1'(1 - 2\ell^{\tilde{\tau}}) - C_{d,\varepsilon_0}\gamma_1 q \frac{\log \ell}{\ell_{\tilde{\tau}}} \geq m^*(1 - 4\ell^{\frac{\tau-1}{2}}) - C_{d,\varepsilon_0}\gamma_1 q \frac{\log \ell}{\ell_{\tilde{\tau}}}.$$

Therefore

$$\Bigg|\sum_{u \in \Lambda_\ell^{\Theta,\ell_\tau}} \varphi_u(y)\langle\varphi_u, \psi\rangle\Bigg| \leq \ell^d \mathrm{e}^{-m_2'\|v_2 - y\|}|\psi(v_2)| \leq \mathrm{e}^{-m_3'\|v_2 - y\|}|\psi(v_2)| \tag{2.22}$$

for some $v_2 \in \partial_{\text{ex}}^{\Theta}\Lambda_\ell$, where

$$m_3' \geq m_2' - C_d \frac{\log \ell}{\ell_{\tilde{\tau}}} \geq m^*(1 - 4\ell^{\frac{\tau-1}{2}}) - C_{d,\varepsilon_0}\gamma_1 q \frac{\log \ell}{\ell_{\tilde{\tau}}}.$$

If $u \in \partial_{\text{in}}^{\Theta,\ell_\tau}\Lambda_\ell$, then

$$\|u - y\| \geq \ell_{\tilde{\tau}} - \ell_\tau > \tfrac{1}{2}\ell_{\tilde{\tau}},$$

thus (1.3) gives $|\varphi_u(y)| \leq \mathrm{e}^{-m^*\|u-y\|}$. Also, (1.3) implies

$$|\varphi_u(v)| \leq \mathrm{e}^{m^*\ell_\tau}\mathrm{e}^{-m^*\|v-u\|} \quad \text{for all } v \in \Lambda_\ell.$$

Therefore

$$|\langle\varphi_u, \psi\rangle| = \Bigg|\sum_{v \in \Lambda_\ell} \varphi_u(v)\psi(v)\Bigg| \leq \sum_{v \in \Lambda_\ell} \mathrm{e}^{-m^*(\|v-u\| - \ell_\tau)}|\psi(v)|,$$

so we get

$$\begin{aligned}
|\varphi_u(y)\langle\varphi_u, \psi\rangle| &\leq \sum_{v \in \Lambda_\ell} \mathrm{e}^{-m^*(\|u-y\| - \ell_\tau + \|v-u\|)}|\psi(v)| \\
&\leq (\ell + 1)^d \mathrm{e}^{-m^*(\|u-y\| - \ell_\tau) - m^*\|v_3 - u\|}|\psi(v_3)| \\
&\leq \mathrm{e}^{-m_4'\|u-y\| - m^*\|v_3-u\|}|\psi(v_3)| \\
&\leq \mathrm{e}^{-m_4'\max\{\|v_3-y\|, \|u-y\|\}}|\psi(v_3)| \\
&\leq \mathrm{e}^{-m_4'\max\{\|v_3-y\|, \frac{1}{2}\ell_{\tilde{\tau}}\}}|\psi(v_3)|
\end{aligned}$$

for some $v_3 \in \Lambda_\ell$, where we used $\|u - y\| \geq \frac{1}{2}\ell_{\tilde{\tau}}$ and took

$$m_4' \geq m^*(1 - 4\ell^{\frac{\tau-1}{2}}) - C_d \frac{\log \ell}{\ell_{\tilde{\tau}}}. \tag{2.23}$$

Therefore

$$\begin{aligned}
\Bigg|\sum_{u \in \partial_{\text{in}}^{\Theta,\ell_\tau}\Lambda_\ell} \varphi_u(y)\langle\varphi_u, \psi\rangle\Bigg| &\leq \ell^d \mathrm{e}^{-m_4'\max\{\|v_3-y\|, \frac{1}{2}\ell_{\tilde{\tau}}\}}|\psi(v_3)| \\
&\leq \mathrm{e}^{-m_5'\max\{\|v_3-y\|, \frac{1}{2}\ell_{\tilde{\tau}}\}}|\psi(v_3)|
\end{aligned} \tag{2.24}$$

for some $v_3 \in \Lambda_\ell$, where

$$m_5' \geq m_4^{*\prime} - C_d \frac{\log \ell}{\ell_{\tilde{\tau}}} \geq m^*(1 - 4\ell^{\frac{\tau-1}{2}}) - C_d \frac{\log \ell}{\ell_{\tilde{\tau}}}.$$

Combining (2.15), (2.22), and (2.24), we conclude that

$$|\psi(y)| \leq \mathrm{e}^{-m_3^* \max\{\|y_1 - y\|, \frac{1}{2}\ell_{\tilde{\tau}}\}} |\psi(y_1)| \quad \text{for some } y_1 \in \Lambda_\ell \cup \partial_{\mathrm{ex}}^{\Theta} \Lambda_\ell,$$

where $m_3^*$ is given in (2.13). If $y_1 \notin \partial^{\Theta, \ell_{\tilde{\tau}}} \Lambda_\ell$ we repeat the procedure to estimate $|\psi(y_1)|$. Since we can suppose $\psi(y) \neq 0$ without loss of generality, the procedure must stop after finitely many times, and at that time we must have

$$|\psi(y)| \leq \mathrm{e}^{-m_3^* \max\{\|\tilde{y} - y\|, \frac{1}{2}\ell_{\tilde{\tau}}\}} |\psi(\tilde{y})| \quad \text{for some } \tilde{y} \in \partial^{\Theta, \ell_{\tilde{\tau}}} \Lambda_\ell. \tag{2.25}$$

If $y \in \Lambda_\ell^{\Theta, 2\ell_{\tilde{\tau}}}$, (2.13) follows immediately from (2.25). □

**Lemma 2.4.** *Given a finite set $\Theta \subset \mathbb{Z}^d$, let $\{(\psi_\lambda, \lambda)\}_{\lambda \in \sigma(H_\Theta)}$ be an eigensystem for $H_\Theta$.*

*Then the following holds for sufficiently large $\ell$.*

(i) *Let $\Lambda_\ell(a) \subset \Theta$, where $a \in \mathbb{R}^d$, be a $\sharp$-localizing box with a corresponding eigensystem $\{(\varphi_x^{(a)}, \lambda_x^{(a)})\}_{x \in \Lambda_\ell(a)}$, and let $\Theta$ be $L$-polynomially level spacing for $H$ if $\sharp$ is $\theta$-PL or $m^*$-ML, $L$-level spacing for $H$ if $\sharp$ is $s$-SEL or $m$-LOC.*

(a) *There exists an injection*

$$\Lambda_\ell^{\Theta, \ell_\sharp}(a) \ni x \longmapsto \tilde{\lambda}_x^{(a)} \in \sigma(H_\Theta),$$

*such that, for all $x \in \Lambda_\ell^{\Theta, \ell_\sharp}(a)$,*

(i) *if $\sharp$ is $\theta$-PL,*

$$|\tilde{\lambda}_x^{(a)} - \lambda_x^{(a)}| \leq C_{d,\varepsilon_0} \ell^{-(\theta - \frac{d-1}{2})}, \tag{2.26}$$

*and, multiplying each $\varphi_x^{(a)}$ by a suitable phase factor,*

$$\|\psi_{\tilde{\lambda}_x^{(a)}} - \varphi_x^{(a)}\| \leq 2C_{d,\varepsilon_0} L^q \ell^{-(\theta - \frac{d-1}{2})}; \tag{2.27}$$

(ii) *if $\sharp$ is $s$-SEL,*

$$|\tilde{\lambda}_x^{(a)} - \lambda_x^{(a)}| \leq \mathrm{e}^{-c_1 \ell^s}, \quad \textit{with } c_1 = c_1(\ell) \textit{ as in } (2.5),$$

*and, multiplying each $\varphi_x^{(a)}$ by a suitable phase factor*

$$\|\psi_{\tilde{\lambda}_x^{(a)}} - \varphi_x^{(a)}\| \leq 2\mathrm{e}^{-c_1 \ell^s} \mathrm{e}^{L^\beta}; \tag{2.28}$$

(iii) *if* $\sharp$ *is* $m^*$*-ML,*

$$|\tilde{\lambda}_x^{(a)} - \lambda_x^{(a)}| \le \mathrm{e}^{-m_1^* \ell_\tau}, \quad \text{with } m_1^* = m_1^*(\ell) \text{ as in (2.6)},$$

*and, multiplying each* $\varphi_x^{(a)}$ *by a suitable phase factor*

$$\|\psi_{\tilde{\lambda}_x^{(a)}} - \varphi_x^{(a)}\| \le 2\mathrm{e}^{-m_1^* \ell_\tau} L^q; \tag{2.29}$$

(iv) *if* $\sharp$ *is* $m$*-LOC,*

$$|\tilde{\lambda}_x^{(a)} - \lambda_x^{(a)}| \le \mathrm{e}^{-m_1 \ell_\tau}, \quad \text{with } m_1 = m_1(\ell) \text{ as in (2.6)},$$

*and, multiplying each* $\varphi_x^{(a)}$ *by a suitable phase factor,*

$$\|\psi_{\tilde{\lambda}_x^{(a)}} - \varphi_x^{(a)}\| \le 2\mathrm{e}^{-m_1 \ell_\tau} \mathrm{e}^{L^\beta}.$$

(b) *Set*

$$\sigma_{\{a\}}(H_\Theta) := \{\tilde{\lambda}_x^{(a)}; x \in \Lambda_\ell^{\Theta,\ell_\sharp}(a)\}.$$

*If* $\lambda \in \sigma_{\{a\}}(H_\Theta)$*, for all* $y \in \Theta \setminus \Lambda_\ell(a)$*, then*

$$|\psi_\lambda(y)| \le \begin{cases} 2C_{d,\varepsilon_0} L^q \ell^{-(\theta - \frac{d-1}{2})} & \text{if } \sharp \text{ is } \theta\text{-PL}, \\ 2\mathrm{e}^{-c_1 \ell^s} \mathrm{e}^{L^\beta} & \text{if } \sharp \text{ is } s\text{-SEL}, \\ 2\mathrm{e}^{-m_1^* \ell_\tau} L^q & \text{if } \sharp \text{ is } m^*\text{-ML}, \\ 2\mathrm{e}^{-m_1 \ell_\tau} \mathrm{e}^{L^\beta} & \text{if } \sharp \text{ is } m\text{-LOC}. \end{cases} \tag{2.30}$$

(c) *If* $\lambda \in \sigma(H_\Theta) \setminus \sigma_{\{a\}}(H_\Theta)$*, then for all* $x \in \Lambda_\ell^{\Theta,\ell_\sharp}(a)$

$$|\lambda - \lambda_x^{(a)}| \ge \begin{cases} \frac{1}{2} L^{-q} & \text{if } \sharp \text{ is } \theta\text{-PL or } m^*\text{-ML}, \\ \frac{1}{2}\mathrm{e}^{-L^\beta} & \text{if } \sharp \text{ is } s\text{-SEL or } m\text{-LOC}, \end{cases} \tag{2.31}$$

*and for all* $y \in \Lambda_\ell^{\Theta,2\ell_\sharp}(a)$

$$|\psi_\lambda(y)| \le \begin{cases} C_{d,\varepsilon_0} L^q \ell^{-(\theta - 2d)} |\psi_\lambda(y_1)| & \text{if } \sharp \text{ is } \theta\text{-PL}, \\ \mathrm{e}^{-c_2 \ell^s} |\psi_\lambda(y_1)| & \text{if } \sharp \text{ is } s\text{-SEL}, \\ \mathrm{e}^{-m_2^* \ell_\tau} |\psi_\lambda(y_1)| & \text{if } \sharp \text{ is } m^*\text{-ML}, \\ \mathrm{e}^{-m_2 \ell_\tau} |\psi_\lambda(y_1)| & \text{if } \sharp \text{ is } m\text{-LOC}, \end{cases} \tag{2.32}$$

*for some* $y_1 \in \partial^{\Theta,2\ell_\sharp} \Lambda_\ell(a)$*, where* $c_2 = c_2(\ell)$ *as in (2.10),* $m_2^* = m_2^*(\ell)$ *as in (2.11),* $m_2 = m_2(\ell)$ *as in (2.12). Moreover, for all* $y \in \Lambda_\ell^{\Theta,2\ell_{\tilde{\tau}}}(a)$*,*

$$|\psi_\lambda(y)| \le \begin{cases} \mathrm{e}^{-m_3^* \|y_2 - y\|} |\psi_\lambda(y_2)| & \text{if } \sharp \text{ is } m^*\text{-ML}, \\ \mathrm{e}^{-m_3 \|y_2 - y\|} |\psi_\lambda(y_2)| & \text{if } \sharp \text{ is } m\text{-LOC}, \end{cases} \tag{2.33}$$

*for some* $y_2 \in \partial^{\Theta,\ell_{\tilde{\tau}}} \Lambda_\ell(a)$*, where* $m_3^* = m_3^*(\ell)$ *as in (2.13),* $m_3 = m_3(\ell)$ *as in (2.14).*

(ii) *Let* $\{\Lambda_\ell(a)\}_{a\in\mathcal{G}}$*, where* $\mathcal{G} \subset \mathbb{R}^d$ *such that* $\Lambda_\ell(a) \subset \Theta$ *for all* $a \in \mathcal{G}$*, be a collection of* $\sharp$ *boxes with corresponding eigensystems* $\{(\varphi_x^{(a)}, \lambda_x^{(a)})\}_{x\in\Lambda_\ell(a)}$ *and let* $\Theta$ *be L-polynomially level spacing for H if* $\sharp$ *is* $\theta$*-PL or* $m^*$*-ML, L-level spacing for H if* $\sharp$ *is s-SEL or m-LOC. Set*

$$\mathcal{E}_{\mathcal{G}}^{\Theta}(\lambda) = \{\lambda_x^{(a)}; a \in \mathcal{G}, x \in \Lambda_\ell^{\Theta,\ell_\sharp}(a), \tilde{\lambda}_x^{(a)} = \lambda\} \quad \text{for } \lambda \in \sigma(H_\Theta), \tag{2.34a}$$

$$\sigma_{\mathcal{G}}(H_\Theta) = \left\{\lambda \in \sigma(H_\Theta); \mathcal{E}_{\mathcal{G}}^{\Theta}(\lambda) \neq \emptyset\right\} = \bigcup_{;a\in\mathcal{G}} \sigma_{\{a\}}(H_\Theta). \tag{2.34b}$$

(a) *For* $a, b \in \mathcal{G}$*,* $a \neq b$*, if* $x \in \Lambda_\ell^{\Theta,\ell_\sharp}(a)$ *and* $y \in \Lambda_\ell^{\Theta,\ell_\sharp}(b)$*,*

$$\lambda_x^{(a)}, \lambda_x^{(b)} \in \mathcal{E}_{\mathcal{G}}^{\Theta}(\lambda) \implies \|x - y\| < 2\ell_\sharp. \tag{2.35}$$

*As a consequence,*

$$\Lambda_\ell(a) \cap \Lambda_\ell(b) = \emptyset \implies \sigma_{\{a\}}(H_\Theta) \cap \sigma_{\{b\}}(H_\Theta) = \emptyset. \tag{2.36}$$

(b) *If* $\lambda \in \sigma_{\mathcal{G}}(H_\Theta)$*, then for all* $y \in \Theta \setminus \Theta_{\mathcal{G}}$*, where* $\Theta_{\mathcal{G}} := \bigcup_{a\in\mathcal{G}} \Lambda_\ell(a)$*,*

$$|\psi_\lambda(y)| \le \begin{cases} 2C_{d,\varepsilon_0} L^q \ell^{-(\theta - \frac{d-1}{2})} & \text{if } \sharp \text{ is } \theta\text{-PL},\\ 2\mathrm{e}^{-c_1\ell^s} \mathrm{e}^{L^\beta} & \text{if } \sharp \text{ is } s\text{-SEL},\\ 2\mathrm{e}^{-m_1^*\ell_\tau} L^q & \text{if } \sharp \text{ is } m^*\text{-ML},\\ 2\mathrm{e}^{-m_1\ell_\tau} \mathrm{e}^{L^\beta} & \text{if } \sharp \text{ is } m\text{-LOC}. \end{cases} \tag{2.37}$$

(c) *If* $\lambda \in \sigma(H_\Theta) \setminus \sigma_{\mathcal{G}}(H_\Theta)$*, then for all* $y \in \Theta'_{\mathcal{G}} := \bigcup_{a\in\mathcal{G}} \Lambda_\ell^{\Theta,2\ell_\sharp}(a)$*,*

$$|\psi_\lambda(y)| \le \begin{cases} C_{d,\varepsilon_0} L^q \ell^{-(\theta - 2d)} & \text{if } \sharp \text{ is } \theta\text{-PL},\\ \mathrm{e}^{-c_2\ell^s} & \text{if } \sharp \text{ is } s\text{-SEL},\\ \mathrm{e}^{-m_2^*\ell_\tau} & \text{if } \sharp \text{ is } m^*\text{-ML},\\ \mathrm{e}^{-m_2\ell_\tau} & \text{if } \sharp \text{ is } m\text{-LOC}. \end{cases} \tag{2.38}$$

(d) *If* $|\Theta| \le (L+1)^d$*, we have*

$$|\Theta'_{\mathcal{G}}| \le |\sigma_{\mathcal{G}}(H_\Theta)| \le |\Theta_{\mathcal{G}}|.$$

*Proof.* Let $\Lambda_\ell(a) \subset \Theta$, where $a \in \mathbb{R}^d$, be a $\theta$-polynomially localizing box with a corresponding eigensystem $\{(\varphi_x^{(a)}, \lambda_x^{(a)})\}_{x\in\Lambda_\ell(a)}$. It follows from Lemma 2.2 that there exists $\tilde{\lambda}_x^{(a)} \in \sigma(H_\Theta)$ satisfying (2.26) for $x \in \Lambda_\ell^{\Theta,\ell'}(a)$. $\tilde{\lambda}_x^{(a)}$ is unique since $\Theta$ is $L$-polynomially level spacing for $H_\Theta$ and $q < \gamma_1 q < \theta - \frac{d-1}{2}$. Moreover, we have $\tilde{\lambda}_x^{(a)} \neq \tilde{\lambda}_y^{(a)}$ if $x, y \in \Lambda_\ell^{\Theta,\ell'}(a)$, $x \neq y$, since

$$\begin{aligned}|\tilde{\lambda}_x^{(a)} - \tilde{\lambda}_y^{(a)}| &\geq |\lambda_x^{(a)} - \lambda_y^{(a)}| - |\tilde{\lambda}_x^{(a)} - \lambda_x^{(a)}| - |\tilde{\lambda}_y^{(a)} - \lambda_y^{(a)}| \\ &\geq \ell^{-q} - 2C_{d,\varepsilon_0}\ell^{-(\theta-\frac{d-1}{2})} \\ &\geq \tfrac{1}{2}\ell^{-q},\end{aligned}$$

$\Lambda_\ell(a)$ is polynomially level spacing for $H_{\Lambda_\ell(a)}$, and $q < \theta - \frac{d-1}{2}$. (2.27) follows from [10, Lemma 3.3].

If $\lambda \in \sigma_{\{a\}}(H_\Theta)$, we have $\lambda = \tilde{\lambda}_x^{(a)}$ for some $x \in \Lambda_\ell^{\Theta,\ell'}(a)$, thus (2.30) follows from (2.27) as $\varphi_x^{(a)}(y) = 0$ for all $y \in \Theta \setminus \Lambda_\ell(a)$.

If $\lambda \in \sigma(H_\Theta) \setminus \sigma_{\{a\}}(H_\Theta)$, then for all $x \in \Lambda_\ell^{\Theta,\ell'}(a)$

$$|\lambda - \lambda_x^{(a)}| \geq |\lambda - \tilde{\lambda}_x^{(a)}| - |\tilde{\lambda}_x^{(a)} - \lambda_x^{(a)}| \geq L^{-q} - C_{d,\varepsilon_0}\ell^{-(\theta-\frac{d-1}{2})} \geq \tfrac{1}{2}L^{-q}, \quad (2.39)$$

since $\Theta$ is $L$-polynomially level spacing for $H_\Theta$, we have (2.26), and $q < \gamma_1 q < \theta - \frac{d-1}{2}$. Therefore (2.32) follows from Lemma 2.3(i). (Note that (2.33) follows from Lemma 2.3(ii).)

Now let $\{\Lambda_\ell(a)\}_{a\in\mathcal{G}}$, where $\mathcal{G} \subset \mathbb{R}^d$ such that $\Lambda_\ell(a) \subset \Theta$ for all $a \in \mathcal{G}$, be a collection of $\theta$-polynomially localizing boxes with corresponding eigensystems $\{(\varphi_x^{(a)}, \lambda_x^{(a)})\}_{x\in\Lambda_\ell(a)}$. Let $\lambda \in \sigma(H_\Theta)$, $a, b \in \mathcal{G}$, $a \neq b$, $x \in \Lambda_\ell^{\Theta,\ell'}(a)$ and $y \in \Lambda_\ell^{\Theta,\ell'}(b)$. Assume $\lambda_x^{(a)}, \lambda_x^{(b)} \in \mathcal{E}_{\mathcal{G}}^{\Theta}(\lambda)$, then it follows from (2.27) that

$$\|\varphi_x^{(a)} - \varphi_y^{(b)}\| \leq 4C_{d,\varepsilon_0}L^q\ell^{-(\theta-\frac{d-1}{2})},$$

thus

$$|\langle\varphi_x^{(a)}, \varphi_y^{(b)}\rangle| \geq \Re\langle\varphi_x^{(a)}, \varphi_y^{(b)}\rangle \geq 1 - 8C_{d,\varepsilon_0}^2 L^{2q}\ell^{-2(\theta-\frac{d-1}{2})}. \quad (2.40)$$

On the other hand, (1.1) gives

$$\|x - y\| \geq 2\ell' \implies |\langle\varphi_x^{(a)}, \varphi_y^{(b)}\rangle| \leq (\ell+1)^d\ell^{-\theta}. \quad (2.41)$$

Combining (2.40) and (2.41), we conclude that

$$\lambda_x^{(a)}, \lambda_x^{(b)} \in \mathcal{E}_{\mathcal{G}}^{\Theta}(\lambda) \implies \|x - y\| < 2\ell'.$$

To prove (2.36), let $a, b \in \mathcal{G}$, $a \neq b$. Assume $\Lambda_\ell(a) \cap \Lambda_\ell(b) = \emptyset$, then

$$(x \in \Lambda_\ell^{\Theta,\ell'}(a) \text{ and } y \in \Lambda_\ell^{\Theta,\ell'}(b)) \implies \|x - y\| \geq 2\ell',$$

thus it follows from (2.35) that $\sigma_{\{a\}}(H_\Theta) \cap \sigma_{\{b\}}(H_\Theta) = \emptyset$.

Parts (ii)(b) and (ii)(c) follow immediately from parts (i)(b) and (i)(c) respectively. To prove part (ii)(d), we let $P_{\mathcal{G}}$ be the orthogonal projection onto the span of $\{\psi_\lambda; \lambda \in \sigma_{\mathcal{G}}(H_\Theta)\}$. (2.38) gives

$$\|(1 - P_{\mathcal{G}})\delta_y\| \leq C_{d,\varepsilon_0} L^q \ell^{-(\theta-2d)} |\Theta|^{\frac{1}{2}} \quad \text{for all } y \in \Theta'_{\mathcal{G}},$$

thus

$$\|(1 - P_{\mathcal{G}})\chi_{\Theta'_{\mathcal{G}}}\| \leq |\Theta'_{\mathcal{G}}|^{\frac{1}{2}} |\Theta|^{\frac{1}{2}} C_{d,\varepsilon_0} L^q \ell^{-(\theta-2d)} \leq |\Theta| C_{d,\varepsilon_0} L^q \ell^{-(\theta-2d)}.$$

If $|\Theta| \leq (L+1)^d$, then

$$\|(1 - P_{\mathcal{G}})\chi_{\Theta'_{\mathcal{G}}}\| \leq (L+1)^d C_{d,\varepsilon_0} L^q \ell^{-(\theta-2d)} < 1$$

since $d + q < \gamma_1(d + q) < \theta - 2d$, so it follows from [10, Lemma A.1] that

$$|\Theta'_{\mathcal{G}}| = \operatorname{tr} \chi_{\Theta'_{\mathcal{G}}} \leq \operatorname{tr} P_{\mathcal{G}} = |\sigma_{\mathcal{G}}(H_\Theta)|.$$

Using a similar argument and (2.37), we can prove $|\sigma_{\mathcal{G}}(H_\Theta)| \leq |\Theta_{\mathcal{G}}|$. □

**2.3. Buffered subsets.** For boxes $\Lambda_\ell \subset \Lambda_L$ that are not ♯ for $H$, we will surround them with a buffer of ♯ boxes and study eigensystems for the augmented subset.

**Definition 2.5.** Let $\Lambda_L = \Lambda_L(x_0)$ and $x_0 \in \mathbb{R}^d$. $\Upsilon \subset \Lambda_L$ is called a *♯-buffered subset of* $\Lambda_L$, where ♯ stands for $\theta$-PL, $s$-SEL, $m^*$-ML or $m$-LOC, if the following holds.

(i) $\Upsilon$ is a connected set in $\mathbb{Z}^d$ of the form

$$\Upsilon = \bigcup_{j=1}^{J} \Lambda_{R_j}(a_j) \cap \Lambda_L,$$

where $J \in \mathbb{N}$, $a_1, a_2, \ldots, a_J \in \Lambda_L^{\mathbb{R}}$, and $\ell \leq R_j \leq L$ for $j = 1, 2, \ldots, J$.

(ii) $\Upsilon$ is $L$-polynomially level spacing for $H$ if ♯ is $\theta$-PL or $m^*$-ML, $L$-level spacing for $H$ if ♯ is $s$-SEL or $m$-LOC.

(iii) There exists $\mathcal{G}_\Upsilon \subset \Lambda_L^{\mathbb{R}}$ such that

(a) for all $a \in \mathcal{G}_\Upsilon$ we have $\Lambda_\ell(a) \subset \Upsilon$, $\Lambda_\ell(a)$ is a $\sharp$ box for $H$;

(b) for all $y \in \partial_{\text{in}}^{\Lambda_L}\Upsilon$ there exists $a_y \in \mathcal{G}_\Upsilon$ such that $y \in \Lambda_\ell^{\Upsilon,2\ell_\sharp}(a_y)$.

In this case we set

$$\check{\Upsilon} = \bigcup_{a\in\mathcal{G}_\Upsilon} \Lambda_\ell(a), \quad \check{\Upsilon}' = \bigcup_{a\in\mathcal{G}_\Upsilon} \Lambda_\ell^{\Upsilon,2\ell_\sharp}(a), \quad \hat{\Upsilon} = \Upsilon\backslash\check{\Upsilon}, \quad \hat{\Upsilon}' = \Upsilon\backslash\check{\Upsilon}'. \tag{2.42}$$

($\check{\Upsilon} = \Upsilon_{\mathcal{G}_\Upsilon}$ and $\check{\Upsilon}' = \Upsilon'_{\mathcal{G}_\Upsilon}$ in the notation of Lemma 2.4.)

**Lemma 2.6.** *Given a $\sharp$-buffered subset $\Upsilon$ of $\Lambda_L$, let $\{(\psi_\nu,\nu)\}_{\nu\in\sigma(H_\Upsilon)}$ be an eigensystem for $H_\Upsilon$. Let $\mathcal{G} = \mathcal{G}_\Upsilon$ and set*

$$\sigma_{\mathcal{B}}(H_\Upsilon) = \sigma(H_\Upsilon) \setminus \sigma_{\mathcal{G}}(H_\Upsilon),$$

*where $\sigma_{\mathcal{G}}(H_\Upsilon)$ is as in (2.34). Then the following holds for sufficiently large $\ell$:*

(i) *If $\nu \in \sigma_{\mathcal{B}}(H_\Upsilon)$, then for all $y \in \check{\Upsilon}'$*

$$|\psi_\lambda(y)| \le \begin{cases} C_{d,\varepsilon_0} L^q \ell^{-(\theta-2d)} & \text{if } \sharp \text{ is } \theta\text{-PL},\\ \mathrm{e}^{-c_2\ell^s}, \text{ with } c_2 = c_2(\ell) \text{ as in (2.10)} & \text{if } \sharp \text{ is } s\text{-SEL},\\ \mathrm{e}^{-m_2^*\ell_\tau}, \text{ with } m_2^* = m_2^*(\ell) \text{ as in (2.11)} & \text{if } \sharp \text{ is } m^*\text{-ML},\\ \mathrm{e}^{-m_2\ell_\tau}, \text{ with } m_2 = m_2(\ell) \text{ as in (2.12)} & \text{if } \sharp \text{ is } m\text{-LOC}, \end{cases} \tag{2.43}$$

*and*

$$|\hat{\Upsilon}| \le |\sigma_{\mathcal{B}}(H_\Upsilon)| \le |\hat{\Upsilon}'|.$$

(ii) *Let $\Lambda_L$ be polynomially level spacing for $H$ if $\sharp$ is $\theta$-PL or $m^*$-ML, level spacing for $H$ if $\sharp$ is $s$-SEL or $m$-LOC, and let $\{(\phi_\lambda,\lambda)\}_{\lambda\in\sigma(H_{\Lambda_L})}$ be an eigensystem for $H_{\Lambda_L}$. There exists an injection*

$$\sigma_{\mathcal{B}}(H_\Upsilon) \ni \nu \longmapsto \tilde{\nu} \in \sigma(H_{\Lambda_L}) \setminus \sigma_{\mathcal{G}}(H_{\Lambda_L}), \tag{2.44}$$

*such that for all $\nu \in \sigma_{\mathcal{B}}(H_\Upsilon)$*

(a) *if $\sharp$ is $\theta$-PL, then*

$$|\tilde{\nu} - \nu| \le C_{d,\varepsilon_0} L^{\frac{d}{2}+q} \ell^{-(\theta-2d)}, \tag{2.45}$$

*and, multiplying each $\psi_\nu$ by a suitable phase factor,*

$$\|\phi_{\tilde{\nu}} - \psi_\nu\| \le 2C_{d,\varepsilon_0} L^{\frac{d}{2}+2q} \ell^{-(\theta-2d)}; \tag{2.46}$$

(b) *if $\sharp$ is s-SEL, then*

$$|\tilde{\nu}-\nu| \le \mathrm{e}^{-c_3\ell^s}, \quad \textit{where } c_3 = c_3(\ell) \ge 1 - C_{d,\varepsilon_0} L^\beta \ell^{-s},$$

*and, multiplying each $\psi_\nu$ by a suitable phase factor,*

$$\|\phi_{\tilde{\nu}} - \psi_\nu\| \le 2\mathrm{e}^{-c_3\ell^s}\mathrm{e}^{L^\beta}; \tag{2.47}$$

(c) *if $\sharp$ is $m^*$-ML, then*

$$|\tilde{\nu}-\nu| \le \mathrm{e}^{-m_4^*\ell_\tau}, \quad \textit{where } m_4^* = m_4^*(\ell) \ge m^* - C_{d,\varepsilon_0}\gamma_1 q \tfrac{\log \ell}{\ell_\tau},$$

*and, multiplying each $\psi_\nu$ by a suitable phase factor,*

$$\|\phi_{\tilde{\nu}} - \psi_\nu\| \le 2\mathrm{e}^{-m_4^*\ell_\tau} L^q; \tag{2.48}$$

(d) *if $\sharp$ is m-LOC, then*

$$|\tilde{\nu}-\nu| \le \mathrm{e}^{-m_4\ell_\tau}, \quad \textit{where } m_4 = m_4(\ell) \ge m - C_{d,\varepsilon_0}\ell^{\gamma\beta-\tau},$$

*and, multiplying each $\psi_\nu$ by a suitable phase factor,*

$$\|\phi_{\tilde{\nu}} - \psi_\nu\| \le 2\mathrm{e}^{-m_4\ell_\tau}\mathrm{e}^{L^\beta}.$$

*Proof.* Part (i) follows immediately from Lemma 2.4(ii)(c) and (ii)(d).

Let $\Lambda_L$ be polynomially level spacing, and let $\{(\phi_\lambda,\lambda)\}_{\lambda\in\sigma(H_{\Lambda_L})}$ be an eigensystem for $H_{\Lambda_L}$. It follows from [10, Lemma 3.2] that for $\nu \in \sigma_{\mathcal{B}}(H_\Upsilon)$, then

$$\begin{aligned}\|(H_{\Lambda_L}-\nu)\psi_\nu\| &\le (2d-1)\varepsilon|\partial_{\mathrm{ex}}^{\Lambda_L}\Upsilon|^{\frac{1}{2}}\|\varphi_{\partial_{\mathrm{in}}^{\Lambda_L}\Upsilon}\|_\infty \\ &\le (2d-1)\varepsilon L^{\frac{d}{2}} C_{d,\varepsilon_0} L^q \ell^{-(\theta-2d)} \\ &\le C_{d,\varepsilon_0} L^{\frac{d}{2}+q}\ell^{-(\theta-2d)},\end{aligned}$$

where we used $\partial_{\mathrm{in}}^{\Lambda_L}\Upsilon \subset \check{\Upsilon}'$ and (2.43). The map in (2.44) is a well defined injection into $\sigma(H_{\Lambda_L})$ since $\Lambda_L$ and $\Upsilon$ are $L$-polynomially level spacing for $H$, and (2.46) follows from (2.45) and [10, Lemma 3.3].

To show $\tilde{\nu} \notin \sigma_{\mathcal{G}}(H_{\Lambda_L})$ for all $\nu \in \sigma_{\mathcal{B}}(H_\Upsilon)$, we assume $\tilde{\nu}_1 \in \sigma_{\mathcal{G}}(H_{\Lambda_L})$ for some $\nu_1 \in \sigma_{\mathcal{B}}(H_\Upsilon)$. Then there is $a \in \mathcal{G}$ and $x \in \Lambda_\ell^{\Lambda_L,\ell'}(a)$ such that $\lambda_x^{(a)} \in \mathcal{E}_{\mathcal{G}}^{\Lambda_L}(\tilde{\nu}_1)$. On the other hand, $\lambda_x^{(a)} \in \mathcal{E}_{\mathcal{G}}^{\Upsilon}(\lambda_1)$ for some $\lambda_1 \in \sigma_{\mathcal{G}}(H_\Upsilon)$ by Lemma 2.4(i)(a). We conclude from (2.27) and (2.46) that

$$\begin{aligned}\sqrt{2} &= \|\psi_{\lambda_1} - \psi_{\nu_1}\| \\ &\le \|\psi_{\lambda_1} - \varphi_x^{(a)}\| + \|\varphi_x^{(a)} - \phi_{\tilde{\nu}_1}\| + \|\phi_{\tilde{\nu}_1} - \psi_{\nu_1}\| \\ &\le 4C_{d,\varepsilon_0}L^q\ell^{-(\theta-\frac{d-1}{2})} + 2C_{d,\varepsilon_0}L^{\frac{d}{2}+2q}\ell^{-(\theta-2d)} \\ &< 1,\end{aligned}$$

a contradiction. □

**Lemma 2.7.** *Given $\Lambda_L = \Lambda_L(x_0)$, $x_0 \in \mathbb{R}^d$, let $\Upsilon$ be a $\sharp$-buffered subset of $\Lambda_L$. Let $\mathcal{G} = \mathcal{G}_\Upsilon$ and set*

$$\mathcal{E}_{\mathcal{G}}^{\Lambda_L}(\nu) = \{\lambda_x^{(a)}; a \in \mathcal{G}, x \in \Lambda_\ell^{\Lambda_L,\ell_\sharp}(a), \tilde{\lambda}_x^{(a)} = \nu\} \subset \mathcal{E}_{\mathcal{G}}^{\Upsilon}(\nu) \quad \text{for } \nu \in \sigma(H_\Upsilon),$$

$$\sigma_{\mathcal{G}}^{\Lambda_L}(H_\Upsilon) = \{\nu \in \sigma(H_\Upsilon); \mathcal{E}_{\mathcal{G}}^{\Lambda_L}(\lambda) \neq \emptyset\} \subset \sigma_{\mathcal{G}}(H_\Upsilon).$$

*The following holds for sufficiently large $\ell$.*

(i) *Let $(\psi, \lambda)$ be an eigenpair for $H_{\Lambda_L}$ such that for all $\nu \in \sigma_{\mathcal{G}}^{\Lambda_L}(H_\Upsilon) \cup \sigma_{\mathcal{B}}(H_\Upsilon)$,*

$$|\lambda - \nu| \geq \begin{cases} \frac{1}{2}L^{-q} & \text{if } \sharp \text{ is } \theta\text{-PL or } m^*\text{-ML}, \\ \frac{1}{2}\mathrm{e}^{-L^\beta} & \text{if } \sharp \text{ is } s\text{-SEL or } m\text{-LOC}. \end{cases} \tag{2.49}$$

*For all $y \in \Upsilon^{\Lambda_L, 2\ell_\sharp}$,*

(a) *if $\sharp$ is $\theta$-PL, then*

$$|\psi(y)| \leq C_{d,\varepsilon_0} L^{2d+2q} \ell^{-(\theta-2d)} |\psi(v)| \quad \text{for some } v \in \partial^{\Lambda_L, 2\ell'}\Upsilon; \tag{2.50}$$

(b) *if $\sharp$ is $s$-SEL, then*

$$|\psi(y)| \leq \mathrm{e}^{-c_4 \ell^s} |\psi(v)| \quad \text{for some } v \in \partial^{\Lambda_L, 2\ell'}\Upsilon,$$

*where $c_4 = c_4(\ell) \geq 1 - C_{d,\varepsilon_0} L^\beta \ell^{-s}$;*

(c) *if $\sharp$ is $m^*$-ML, then*

$$|\psi(y)| \leq \mathrm{e}^{-m_5^* \ell_\tau} |\psi(v)| \quad \text{for some } v \in \partial^{\Lambda_L, 2\ell_\tau}\Upsilon,$$

*where $m_5^* = m_5^*(\ell) \geq m^* - C_{d,\varepsilon_0} \gamma_1 q \frac{\log \ell}{\ell_\tau}$;*

(d) *If $\sharp$ is $m$-LOC, then*

$$|\psi(y)| \leq \mathrm{e}^{-m_5 \ell_\tau} |\psi(v)| \quad \text{for some } v \in \partial^{\Lambda_L, 2\ell_\tau}\Upsilon,$$

*where $m_5 = m_5(\ell) \geq m - C_{d,\varepsilon_0} \ell^{\gamma\beta - \tau}$.*

(ii) *Let $\Lambda_L$ be polynomially level spacing for $H$ if $\sharp$ is $\theta$-PL or $m^*$-ML, level spacing for $H$ if $\sharp$ is $s$-SEL or $m$-LOC. Let $\{(\psi_\lambda, \lambda)\}_{\lambda \in \sigma(H_{\Lambda_L})}$ be an eigensystem for $H_{\Lambda_L}$, and set (recalling (2.44))*

$$\sigma_\Upsilon(H_{\Lambda_L}) = \{\tilde{\nu}; \nu \in \sigma_{\mathcal{B}}(H_\Upsilon)\} \subset \sigma(H_{\Lambda_L}) \setminus \sigma_{\mathcal{G}}(H_{\Lambda_L}).$$

*Then condition* (2.49) *is satisfied for all* $\lambda \in \sigma(H_{\Lambda_L}) \setminus (\sigma_{\mathcal{G}}(H_{\Lambda_L}) \cup \sigma_{\Upsilon}(H_{\Lambda_L}))$*, so for all* $y \in \Upsilon^{\Lambda_L, 2\ell_\sharp}$

$$|\psi_\lambda(y)| \le \begin{cases} C_{d,\varepsilon_0} L^{2d+2q} \ell^{-(\theta-2d)} |\psi(v)| & \text{if } \sharp \text{ is } \theta\text{-PL}, \\ \mathrm{e}^{-c_4 \ell^s} |\psi(v)| & \text{if } \sharp \text{ is } s\text{-SEL}, \\ \mathrm{e}^{-m_5^* \ell_\tau} |\psi(v)| & \text{if } \sharp \text{ is } m^*\text{-ML}, \\ \mathrm{e}^{-m_5 \ell_\tau} |\psi(v)| & \text{if } \sharp \text{ is } m\text{-LOC}, \end{cases}$$

*for some* $v \in \partial^{\Lambda_L, 2\ell_\sharp} \Upsilon$.

*Proof.* Let $\{(\vartheta_\nu, \nu)\}_{\nu \in \sigma(H_\Upsilon)}$ be an eigensystem for $H_\Upsilon$. For $\nu \in \sigma_{\mathcal{G}}(H_\Upsilon)$ we fix $\lambda_{x_\nu}^{(a_\nu)} \in \mathcal{E}_{\mathcal{G}}^{\Upsilon}(\nu)$, where $a_\nu \in \mathcal{G}$, $x_\nu \in \Lambda_\ell^{\Upsilon, \ell'}(a_\nu)$. If $\nu \in \sigma_{\mathcal{G}}^{\Lambda_L}(H_\Upsilon)$, we choose $\lambda_{x_\nu}^{(a_\nu)} \in \mathcal{E}_{\mathcal{G}}^{\Lambda_L}(\nu)$, thus $x_\nu \in \Lambda_\ell^{\Lambda_L, \ell'}(a_\nu)$. If $\nu \in \sigma_{\mathcal{G}}(H_\Upsilon) \setminus \sigma_{\mathcal{G}}^{\Lambda_L}(H_\Upsilon)$ we have $x_\nu \in \Lambda_\ell^{\Upsilon, \ell'}(a_\nu) \setminus \Lambda_\ell^{\Lambda_L, \ell'}(a_\nu)$.

Given $y \in \Upsilon$, we have (see (2.3))

$$\begin{aligned} \psi(y) &= \sum_{\nu \in \sigma(\Upsilon)} \vartheta_\nu(y) \langle \vartheta_\nu, \psi \rangle \\ &= \sum_{\nu \in \sigma_{\mathcal{G}}^{\Lambda_L}(H_\Upsilon) \cup \sigma_{\mathcal{B}}(H_\Upsilon)} \vartheta_\nu(y) \langle \vartheta_\nu, \psi \rangle + \sum_{\nu \in \sigma_{\mathcal{G}}(H_\Upsilon) \setminus \sigma_{\mathcal{G}}^{\Lambda_L}(H_\Upsilon)} \vartheta_\nu(y) \langle \vartheta_\nu, \psi \rangle. \end{aligned} \tag{2.51}$$

Let $(\psi, \lambda)$ be an eigenpair for $H_{\Lambda_L}$ satisfying (2.49). If $\nu \in \sigma_{\mathcal{G}}^{\Lambda_L}(H_\Upsilon) \cup \sigma_{\mathcal{B}}(H_\Upsilon)$, then

$$\langle \vartheta_\nu, \psi \rangle = (\lambda - \nu)^{-1} \langle \vartheta_\nu, (H_{\Lambda_L} - \nu) \psi \rangle = (\lambda - \nu)^{-1} \langle (H_{\Lambda_L} - \nu) \vartheta_u, \psi \rangle.$$

It follows from (2.49) and [10, Lemma 3.2] that

$$\begin{aligned} |\vartheta_\nu(y) \langle \vartheta_\nu, \psi \rangle| &\le 2 L^q \varepsilon |\vartheta_\nu(y)| \sum_{v \in \partial_{\mathrm{ex}}^{\Lambda_L} \Upsilon} \Big( \sum_{v' \in \partial_{\mathrm{in}}^{\Lambda_L} \Upsilon, |v'-v|=1} |\vartheta_\nu(v')| \Big) |\psi(v)| \\ &\le 2 \varepsilon L^{q+d} (2d \max_{u \in \partial_{\mathrm{in}}^{\Lambda_L} \Upsilon} |\vartheta_\nu(u)|) |\psi(v_1)| \quad \text{for some } v_1 \in \partial_{\mathrm{ex}}^{\Lambda_L} \Upsilon. \end{aligned}$$

If $\nu \in \sigma_{\mathcal{B}}(H_\Upsilon)$, (2.43) gives

$$\max_{u \in \partial_{\mathrm{in}}^{\Lambda_L} \Upsilon} |\vartheta_\nu(u)| \le C_{d,\varepsilon_0} L^q \ell^{-(\theta-2d)}.$$

If $\nu \in \sigma_{\mathcal{G}}^{\Lambda_L}(H_{\Upsilon})$, it follows from (2.27) and (1.1), that

$$\begin{aligned}
&\max_{u \in \partial_{\text{in}}^{\Lambda_L}\Upsilon} |\vartheta_\nu(u)| \\
&\leq \max_{u \in \partial_{\text{in}}^{\Lambda_L}} (|\vartheta_\nu(u) - \varphi_{x_\nu}^{(a_\nu)}| + |\varphi_{x_\nu}^{(a_\nu)}|) \\
&\leq 2C_{d,\varepsilon_0} L^q \ell^{-(\theta - \frac{d-1}{2})} + \ell^{-\theta} \\
&\leq 3C_{d,\varepsilon_0} L^q \ell^{-(\theta - \frac{d-1}{2})} \\
&\leq C_{d,\varepsilon_0} L^q \ell^{-(\theta - 2d)}.
\end{aligned}$$

Therefore (recalling (2.19)),

$$\begin{aligned}
\left| \sum_{\nu \in \sigma_{\mathcal{G}}^{\Lambda_L}(H_\Upsilon) \cup \sigma_{\mathcal{B}}(H_\Upsilon)} \vartheta_\nu(y) \langle \vartheta_\nu, \psi \rangle \right| &\leq 4d\varepsilon L^{2d+q} (C_{d,\varepsilon_0} L^q \ell^{-(\theta-2d)}) |\psi(v_2)| \\
&\leq C_{d,\varepsilon_0} L^{2d+2q} \ell^{-(\theta-2d)} |\psi(v_2)|,
\end{aligned} \tag{2.52}$$

for some $v_2 \in \partial_{\text{ex}}^{\Lambda_L}\Upsilon$.

If $\nu \in \sigma_{\mathcal{G}}(H_\Upsilon) \setminus \sigma_{\mathcal{G}}^{\Lambda_L}(H_\Upsilon)$, we have $x_\nu \in \Lambda_\ell^{\Upsilon,\ell'}(a_\nu) \setminus \Lambda_\ell^{\Lambda_L,\ell'}(a_\nu)$, thus

$$\operatorname{dist}(x_\nu, \Upsilon \setminus \Lambda_\ell(a_\nu)) > \ell' \quad \text{and} \quad \operatorname{dist}(x_\nu, \Lambda_L \setminus \Lambda_\ell(a_\nu)) \leq \ell',$$

and hence there is $u_0 \in \Lambda_L \setminus \Upsilon$ such that $\|x_\nu - u_0\| \leq \ell'$. We suppose $y \in \Upsilon^{\Lambda_L, 2\ell'}$, then $\|y - u_0\| > 2\ell'$. Therefore

$$\|x_\nu - y\| \geq \|y - u_0\| - \|x_\nu - u_0\| > 2\ell' - \ell' = \ell'.$$

Thus it follows from (2.27) and (1.1) that

$$\begin{aligned}
|\vartheta_\nu(u)| &\leq |\vartheta_\nu(u) - \varphi_{x_\nu}^{(a_\nu)}| + |\varphi_{x_\nu}^{(a_\nu)}| \\
&\leq 2C_{d,\varepsilon_0} L^q \ell^{-(\theta - \frac{d-1}{2})} + \ell^{-\theta} \\
&\leq 3C_{d,\varepsilon_0} L^q \ell^{-(\theta - \frac{d-1}{2})}.
\end{aligned}$$

Therefore

$$\left| \sum_{\nu \in \sigma_{\mathcal{G}}(H_\Upsilon) \setminus \sigma_{\mathcal{G}}^{\Lambda_L}(H_\Upsilon)} \vartheta_\nu(y) \langle \vartheta_\nu, \psi \rangle \right| \leq 3C_{d,\varepsilon_0} L^q (L+1)^{\frac{3d}{2}} \ell^{-(\theta - \frac{d-1}{2})} |\psi(v_3)|, \tag{2.53}$$

for some $v_3 \in \Upsilon$.

Combining (2.51), (2.52), and (2.53), we conclude that for all $y \in \Upsilon^{\Lambda_L, 2\ell'}$,

$$|\psi(y)| \leq C_{d,\varepsilon_0} L^{2d+2q} \ell^{-(\theta-2d)} |\psi(v_4)|,$$

for some $v_4 \in \Upsilon \cup \partial_{\mathrm{ex}}^{\Lambda_L} \Upsilon$. If $v_4 \in \Upsilon^{\Lambda_L, 2\ell'}$ we repeat the procedure to estimate $|\psi(v_4)|$. Since we can suppose $\psi(y) \neq 0$ without loss of generality, the procedure must stop after finitely many times, and at that time we must have (2.50).

Now let $\Lambda_L$ be polynomially level spacing. If $\lambda \notin \sigma_{\mathcal{G}}(H_{\Lambda_L})$, it follows from Lemma 2.4(i)(c) that (2.31) holds for all $a \in \mathcal{G}$. If $\lambda \notin \sigma_{\Upsilon}(H_{\Lambda_L})$, using the argument in (2.39), with (2.45) instead of (2.26), we get $|\lambda - \nu| \geq \frac{1}{2} L^{-q}$ for all $\nu \in \sigma_{\mathcal{B}}(H_{\Upsilon})$. Therefore we have (2.49), which implies (2.50). □

## 3. Probability estimates

The following lemma gives the probability estimates for polynomially level spacing and level spacing.

**Lemma 3.1.** *Let $H_{\varepsilon,\omega}$ be the Anderson model. Let $\Theta \subset \mathbb{Z}^d$ and $L > 1$. Then, for all $\varepsilon \leq \varepsilon_0$,*

$$\mathbb{P}\{\Theta \text{ is } L\text{-polynomially level spacing for } H\} \geq 1 - Y_{\varepsilon_0} L^{-(2\alpha-1)q} |\Theta|^2,$$

*and*

$$\mathbb{P}\{\Theta \text{ is } L\text{-level spacing for } H\} \geq 1 - Y_{\varepsilon_0} \mathrm{e}^{-(2\alpha-1)L^{\beta}} |\Theta|^2,$$

*where*

$$Y_{\varepsilon_0} = 2^{2\alpha-1} \widetilde{K}^2 (\operatorname{diam} \operatorname{supp} \mu + 2d\varepsilon_0 + 1),$$

*with $\widetilde{K} = K$ if $\alpha = 1$ and $\widetilde{K} = 8K$ if $\alpha \in \left(\frac{1}{2}, 1\right)$.*

Lemma 3.1 follows from [10, Lemma 2.1] and its proof. (Also see [18, Lemma 2].)

## 4. Bootstrap multiscale analysis

In this section, we fix $\theta > \left(\frac{6}{2\alpha-1} + \frac{9}{2}\right) d$ and $0 < \xi < 1$. (Note that Proposition 4.1 is independent of $\xi$.) We will omit the dependence on $\theta$ and $\xi$ from the notation. We denote the complementary event of an event $\mathcal{E}$ by $\mathcal{E}^c$.

### 4.1. The first multiscale analysis

**Proposition 4.1.** *Fix $\varepsilon_0 > 0$, $Y \geq 400$, and $P_0 < \frac{1}{2}(2Y)^{-2d}$. There exists a finite scale $\mathcal{L}(\varepsilon_0, Y)$ with the following property: Suppose for some scale $L_0 \geq \mathcal{L}(\varepsilon_0, Y)$, and $0 < \varepsilon \leq \varepsilon_0$ we have*

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_{L_0}(x) \text{ is } \theta\text{-polynomially localizing for } H_{\varepsilon,\omega}\} \geq 1 - P_0. \tag{4.1}$$

*Then, setting $L_{k+1} = Y L_k$ for $k = 0, 1, \ldots$, there exists $K_0 = K_0(Y, L_0, P_0) \in \mathbb{N}$ such that*

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_{L_k}(x) \text{ is } \theta\text{-polynomially localizing for } H_{\varepsilon,\omega}\} \geq 1 - L_k^{-p} \quad \text{for } k \geq K_0. \tag{4.2}$$

Proposition 4.1 follows from the following induction step for the multiscale analysis.

**Lemma 4.2.** *Fix $\varepsilon_0 > 0$, $Y \geq 400$, and $P \leq 1$. Suppose for some scale $\ell$ and $0 < \varepsilon \leq \varepsilon_0$ we have*

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_{\ell}(x) \text{ is } \theta\text{-polynomially localizing for } H_{\varepsilon,\omega}\} \geq 1 - P. \tag{4.3}$$

*If $\ell$ is sufficiently large, for $L = Y\ell$, then*

$$\begin{aligned}\inf_{x \in \mathbb{R}^d} & \mathbb{P}\{\Lambda_{L}(x) \text{ is } \theta\text{-polynomially localizing for } H_{\varepsilon,\omega}\} \\ & \geq 1 - \left((2Y)^{2d} P^2 + \tfrac{1}{2} L^{-p}\right).\end{aligned}$$

*Proof.* We fix $0 < \varepsilon \leq \varepsilon_0$ and suppose (4.3) for some scale $\ell$. Let $\Lambda_L = \Lambda_L(x_0)$, where $x_0 \in \mathbb{R}^d$, and let $\mathcal{C}_{L,\ell} = \mathcal{C}_{L,\ell}(x_0)$ be the suitable $\ell$-cover of $\Lambda_L$. For $N \in \mathbb{N}$, let $\mathcal{B}_N$ denote the event that there exist at most $N$ disjoint boxes in $\mathcal{C}_{L,\ell}$ that are not $\theta$-PL for $H_{\varepsilon,\omega}$. Using (4.3), (2.2) and the fact that events on disjoint boxes are independent, if $N = 1$, then

$$\mathbb{P}\{\mathcal{B}_N^c\} \leq \left(\tfrac{2L}{\ell}\right)^{(N+1)d} P^{N+1} = (2Y)^{(N+1)d} P^{N+1} = (2Y)^{2d} P^2. \tag{4.4}$$

We now fix $\omega \in \mathcal{B}_N$. There exists $\mathcal{A}_N = \mathcal{A}_N(\omega) \in \Xi_{L,\ell} = \Xi_{L,\ell}(x_0)$, with $|\mathcal{A}_N| \leq N$ and $\|a - b\| \geq 2\rho\ell$ (i.e., $\Lambda_\ell(a) \cap \Lambda_\ell(b) = \emptyset$) if $a, b \in \mathcal{A}_N$, $a \neq b$, such that for all $a \in \Xi_{L,\ell}$ with $\operatorname{dist}(a, \mathcal{A}_N) \geq 2\rho\ell$ (i.e., $\Lambda_\ell(a) \cap \Lambda_\ell(b) = \emptyset$ for all $b \in \mathcal{A}_N$), $\Lambda_\ell(a)$ is a $\sharp$ box for $H_{\varepsilon,\omega}$ ($\sharp$ stands for $\theta$-PL). In other words,

$$a \in \Xi_{L,\ell} \setminus \bigcup_{b \in \mathcal{A}_N} \Lambda^{\mathbb{R}}_{(2\rho+1)\ell}(a_0) \implies \Lambda_\ell(a) \text{ is a } \sharp \text{ box for } H_{\varepsilon,\omega}. \tag{4.5}$$

To embed the box $\{\Lambda_\ell(b)\}_{b\in\mathcal{A}_N}$ into $\sharp$-buffered subsets of $\Lambda_L$, we consider graphs $\mathbb{G}_i = (\Xi_{L,\ell}, \mathbb{E}_i)$, $i = 1, 2$, both having $\Xi_{L,\ell}$ as the set of vertices, with sets of edges given by

$$\begin{aligned}\mathbb{E}_1 &= \{\{a,b\} \in \Xi_{L,\ell}^2; \|a-b\| = \rho\ell\}\\ &= \{\{a,b\} \in \Xi_{L,\ell}^2; a \neq b \text{ and } \Lambda_\ell(a) \cap \Lambda_\ell(b) \neq \emptyset\},\end{aligned}$$

$$\begin{aligned}\mathbb{E}_2 &= \{\{a,b\} \in \Xi_{L,\ell}^2; \text{ either } \|a-b\| = 2\rho\ell \text{ or } \|a-b\| = 3\rho\ell\}\\ &= \{\{a,b\} \in \Xi_{L,\ell}^2; \Lambda_\ell(a) \cap \Lambda_\ell(b) = \emptyset \text{ and } \Lambda_{(2\rho+1)\ell}(a) \cap \Lambda_{(2\rho+1)\ell}(b) \neq \emptyset\}.\end{aligned}$$

Let $\{\Phi_r\}_{r=1}^R = \{\Phi_r(\omega)\}_{r=1}^R$ denote the $\mathbb{G}_2$-connected components of $\mathcal{A}_N$ (i.e., connected in the graph $\mathcal{G}_2$). Note that

$$R \in \{1, 2, \ldots, N\}, \quad \sum_{r=1}^R |\Phi_r| = |\mathcal{A}_N| \le N, \quad \text{and} \quad \operatorname{diam} \Phi_r \le 3\rho\ell(|\Phi_r| - 1).$$

Set

$$\widetilde{\Phi}_r = \Xi_{L,\ell} \cap \bigcup_{a\in\Phi_r} \Lambda_{(2\rho+1)\ell}^{\mathbb{R}}(a) = \{a \in \Xi_{L,\ell}; \operatorname{dist}(a, \Phi_r) \le \rho\ell\},$$

and note that $\{\widetilde{\Phi}_r\}_{r=1}^R$ is a collection of disjoint, $\mathbb{G}_1$-connected subsets of $\Xi_{L,\ell}$, such that

$$\operatorname{diam} \widetilde{\Phi}_r \le \operatorname{diam} \Phi_r + 2\rho\ell \le \rho\ell(3|\Phi_r| - 1) \text{ and } \operatorname{dist}(\widetilde{\Phi}_r, \widetilde{\Phi}_{\tilde{r}}) \ge 2\rho\ell,\ r \neq \tilde{r}.$$

Moreover, (4.5) gives

$$a \in \mathcal{G} = \mathcal{G}(\omega) = \Xi_{L,\ell} \setminus \bigcup_{r=1}^R \widetilde{\Phi}_r \implies \Lambda_\ell(a) \text{ is a } \sharp \text{ box for } H_{\varepsilon,\omega}. \tag{4.6}$$

For $\Psi \subset \Xi_{L,\ell}$, we define the exterior boundary of $\Psi$ in the graph $\mathbb{G}_1$ by

$$\partial_{\text{ex}}^{\mathbb{G}_1} \Psi = \{a \in \Xi_{L,\ell}; \operatorname{dist}(a, \Psi) = \rho\ell\}.$$

It follows from (4.6) that $\Lambda_\ell(a)$ is $\sharp$ for $H_{\varepsilon,\omega}$ for all $a \in \partial_{\text{ex}}^{\mathbb{G}_1} \widetilde{\Phi}_r$, $r = 1, 2, \ldots, R$. Set $\overline{\Psi} = \Psi \cup \partial_{\text{ex}}^{\mathbb{G}_1} \Psi$, and set, for $r = 1, 2, \ldots, R$,

$$\Upsilon_r^{(0)} = \Upsilon_r^{(0)}(\omega) = \bigcup_{a\in\widetilde{\Phi}_r} \Lambda_\ell(a), \tag{4.7a}$$

$$\Upsilon_r = \Upsilon_r(\omega) = \Upsilon_r^{(0)} \cup \bigcup_{a\in\partial_{\text{ex}}^{\mathbb{G}_1}\widetilde{\Phi}_r} \Lambda_\ell(a) = \bigcup_{a\in\overline{\widetilde{\Phi}}_r} \Lambda_\ell(a). \tag{4.7b}$$

Each $\Upsilon_r$, $r = 1, 2, \ldots, R$, satisfies all the requirements to be a $\theta$-PL-buffered subset of $\Lambda_L$ with $\mathcal{G}_{\Upsilon_r} = \partial_{\mathrm{ex}}^{\mathbb{G}_1} \widetilde{\Phi}_r$ (see Definition 2.5), except that we do not know if $\Upsilon_r$ is $L$-polynomially level spacing for $H_{\varepsilon,\omega}$. (Note that the sets $\{\Upsilon_r^{(0)}\}_{r=1}^R$ are disjoint, but the sets $\{\Upsilon_r\}_{r=1}^R$ are not necessarily disjoint.) Note also that

$$\operatorname{diam} \overline{\overline{\widetilde{\Phi}}}_r \le \operatorname{diam} \widetilde{\Phi}_r + 2\rho\ell \le \rho\ell(3|\Phi_r| + 1),$$

and hence

$$\operatorname{diam} \Upsilon_r \le \operatorname{diam} \overline{\overline{\widetilde{\Phi}}}_r + \ell \le \rho\ell(3|\Phi_r| + 1) + \ell \le 5\ell|\Phi_r|,$$

thus

$$\sum_{r=1}^{R} \operatorname{diam} \Upsilon_r \le 5\ell N. \tag{4.8}$$

We can arrange for $\{\Upsilon_r\}_{r=1}^R$ to be a collection of $\theta$-PL-buffered subsets of $\Lambda_L$ as follows. It follows from Lemma 3.1 that for any $\Theta \subset \Lambda_L$ we have

$$\mathbb{P}\{\Theta \text{ is } L\text{-polynomially level spacing for } H_{\varepsilon,\omega}\} \ge 1 - Y_{\varepsilon_0} \mathrm{e}^{-(2\alpha-1)L^\beta} (L+1)^{2d}. \tag{4.9}$$

Given a $\mathbb{G}_2$-connected subset $\Phi$ of $\Xi_{L,\ell}$, let $\Upsilon(\Phi) \subset \Lambda_L$ be constructed from $\Phi$ as in (4.7). Set

$$\mathcal{F}_N = \bigcup_{r=1}^{N} \mathcal{F}(r), \quad \text{where } \mathcal{F}(r) = \{\Phi \subset \Xi_{L,\ell}; \Phi \text{ is } \mathbb{G}_2\text{-connected and } |\Phi| = r\}.$$

Let $\mathcal{F}(r,a) = \{\Phi \in \mathcal{F}_r; a \in \Phi\}$ for $a \in \Xi_{L,\ell}$, and note that each vertex in the graph $\mathbb{G}_2$ has less than $d(3^{d-1} + 4^{d-1}) \le d4^d$ nearest neighbors , we have

$$\begin{aligned} |\mathcal{F}(r,a)| \le (r-1)!(d4^d)^{r-1} &\implies |\mathcal{F}(r)| \le (L+1)^d (r-1)!(d4^d)^{r-1} \\ &\implies |\mathcal{F}_N| \le (L+1)^d N!(d4^d)^{N-1}. \end{aligned} \tag{4.10}$$

Let $\mathcal{S}_N$ denote the event that the box $\Lambda_L$ and the subsets $\{\Upsilon(\Phi)\}_{\Phi \in \mathcal{F}_N}$ are all $L$-polynomially level spacing for $H_{\varepsilon,\omega}$, using (4.9) and (4.10), if $N = 1$, then

$$\mathbb{P}\{\mathcal{S}_N^c\} \le Y_{\varepsilon_0}(1 + (L+1)^d N!(d4^d)^{N-1})(L+1)^{2d}(L+1)^{2d} L^{-(2\alpha-1)q} < \tfrac{1}{2} L^{-p} \tag{4.11}$$

for sufficiently large $L$ since $p < (2\alpha - 1)q - 3d$.

Let $\mathcal{E}_N = \mathcal{B}_N \cap \mathcal{S}_N$. Combining (4.4) and (4.11), we conclude that if $N = 1$,

$$\mathbb{P}\{\mathcal{E}_N\} > 1 - ((2Y)^{2d} P^2 + \tfrac{1}{2} L^{-p}).$$

To finish the proof we need to show that for all $\omega \in \mathcal{E}_N$ the box $\Lambda_L$ is $\theta$-PL for $H_{\varepsilon,\omega}$.

We fix $\omega \in \mathcal{E}_N$. Then we have (4.6), $\Lambda_L$ is polynomially level spacing for $H_{\varepsilon,\omega}$, and the subsets $\{\Upsilon_r\}_{r=1}^R$ constructed in (4.7) are $\theta$-PL-buffered subsets of $\Lambda_L$ for $H_{\varepsilon,\omega}$. It follows from (2.1) and Definition 2.5(iii) that

$$\Lambda_L = \Big\{ \bigcup_{a \in \mathcal{G}} \Lambda_\ell^{\Lambda_L, \frac{\ell}{10}}(a) \Big\} \cup \Big\{ \bigcup_{r=1}^{R} \Upsilon_r^{\Lambda_L, \frac{\ell}{10}} \Big\}. \tag{4.12}$$

We omit both $\varepsilon$ and $\omega$ from the notation since they are now fixed. Let $\{(\psi_\lambda, \lambda)\}_{\lambda \in \sigma(H_{\Lambda_L})}$ be an eigensystem for $H_{\Lambda_L}$. For $a \in \mathcal{G}$, let $\{(\varphi_x^{(a)}, \lambda_x^{(a)})\}_{x \in \Lambda_\ell(a)}$ be a $\theta$-polynomially localized eigensystem for $\Lambda_\ell(a)$. For $r = 1, 2, \ldots, R$, let $\big\{(\phi_{\nu^{(r)}}, \nu^{(r)})\big\}_{\nu^{(r)} \in \sigma(H_{\Upsilon_r})}$ be an eigensystem for $H_{\Upsilon_r}$, and set

$$\sigma_{\Upsilon_r} = \{\tilde{\nu}^{(r)}; \nu^{(r)} \in \sigma_{\mathcal{B}}(H_{\Upsilon_r})\} \subset \sigma(H_{\Lambda_L}) \setminus \sigma_{\mathcal{G}}(H_{\Lambda_L}), \tag{4.13}$$

where $\tilde{\nu}^{(r)}$ is given in (2.44), which also gives $\sigma_{\Upsilon_r}(H_{\Lambda_L}) \subset \sigma(H_{\Lambda_L}) \setminus \sigma_{\mathcal{G}_{\Upsilon_r}}(H_{\Lambda_L})$, but the argument actually shows $\sigma_{\Upsilon_r}(H_{\Lambda_L}) \subset \sigma(H_{\Lambda_L}) \setminus \sigma_{\mathcal{G}}(H_{\Lambda_L})$. We also set

$$\sigma_{\mathcal{B}}(H_{\Lambda_L}) = \bigcup_{r=1}^{R} \sigma_{\Upsilon_r}(H_{\Lambda_L}) \subset \sigma(H_{\Lambda_L}) \setminus \sigma_{\mathcal{G}}(H_{\Lambda_L}).$$

We claim

$$\sigma(H_{\Lambda_L}) = \sigma_{\mathcal{G}}(H_{\Lambda_L}) \cup \sigma_{\mathcal{B}}(H_{\Lambda_L}). \tag{4.14}$$

To do this, we assume $\lambda \in \sigma_{\mathcal{G}} \setminus (\sigma_{\mathcal{G}}(H_{\Lambda_L}) \cup \sigma_{\mathcal{B}}(H_{\Lambda_L}))$. Since $\Lambda_L$ is polynomially level spacing for $H$, Lemma 2.4(ii)(c) gives

$$|\psi_\lambda(y)| \le C_{d,\varepsilon_0} L^q \ell^{-(\theta - 2d)} \quad \text{for all } y \in \bigcup_{a \in \mathcal{G}} \Lambda_\ell^{\Lambda_L, 2\ell'}(a),$$

and Lemma 2.7(ii) gives

$$|\psi_\lambda(y)| \le C_{d,\varepsilon_0} L^{2d+2q} \ell^{-(\theta - 2d)} \quad \text{for all } y \in \bigcup_{r=1}^{R} \Upsilon_r^{\Lambda_L, 2\ell'}.$$

Using (4.12) and $\theta - 2d > \gamma_1\big(\frac{5d}{2} + 2q\big) > \frac{5d}{2} + 2q$, we conclude that

$$1 = \|\psi_\lambda(y)\| \le C_{d,\varepsilon_0} L^{2d+2q} \ell^{-(\theta-2d)} (L+1)^{\frac{d}{2}} < 1$$

for sufficiently large $\ell$, a contradiction. This establishes the claim.

We now index the eigenvalues and eigenvectors of $H_{\Lambda_L}$ by sites in $\Lambda_L$ using Hall's Marriage Theorem, which states a necessary and sufficient condition for the existence of a perfect matching in a bipartite graph. (See [10, Appendix C] and [6, Chapter 2].) We consider the bipartite graph $\mathbb{G} = (\Lambda_L, \sigma(H_{\Lambda_L}); \mathbb{E})$, where the edge set $\mathbb{E} \subset \Lambda_L \times \sigma(H_{\Lambda_L})$ is defined as follows. For each $\lambda \in \sigma_{\mathcal{G}}(H_{\Lambda_L})$ we fix $\lambda_{x_\lambda}^{(a_\lambda)} \in \mathcal{E}_{\mathcal{G}}^{\Lambda_L}(\lambda)$, and set (recall (2.42) and (2.7))

$$\mathcal{N}_0(x) = \begin{cases} \{\lambda \in \sigma_{\mathcal{G}}(H_{\Lambda_L}); \|x_\lambda - x\| < \ell_\sharp\} & \text{for } x \in \Lambda_L \setminus \bigcup_{r=1}^R \widehat{\Upsilon}_r, \\ \emptyset & \text{for } x \in \bigcup_{r=1}^R \widehat{\Upsilon}_r. \end{cases}$$

We define

$$\mathcal{N}(x) = \begin{cases} \mathcal{N}_0(x) & \text{for } x \in \Lambda_L \setminus \bigcup_{r=1}^R \widehat{\Upsilon}'_r, \\ \sigma_\Upsilon(H_{\Lambda_L}) & \text{for } x \in \widehat{\Upsilon}_r,\ r = 1, 2, \ldots, R, \\ \mathcal{N}_0(x) \cup \sigma_\Upsilon(H_{\Lambda_L}) & \text{for } x \in \widehat{\Upsilon}'_r, \setminus \widehat{\Upsilon}_r,\ r = 1, 2, \ldots, R, \end{cases} \tag{4.15}$$

and let $\mathbb{E} = \{(x, \lambda) \in \Lambda_L \times \sigma(H_{\Lambda_L}); \lambda \in \mathcal{N}(x)\}$.

$\mathcal{N}(x)$ was defined to ensure $|\psi_\lambda(x)| \ll 1$ for $\lambda \notin \mathcal{N}(x)$. This can be seen as follows.

- If $x \in \Lambda_L$ and $\lambda \in \sigma_{\mathcal{G}}(H_{\Lambda_L}) \setminus \mathcal{N}_0(x)$, we have $\lambda = \tilde{\lambda}_{x_\lambda}^{(a_\lambda)}$ with $\|x_\lambda - x\| \geq \ell'$, so, using (1.1) and (2.27),

$$\begin{aligned} |\psi_\lambda(x)| &\leq |\varphi_{x_\lambda}^{(a_\lambda)}(x)| + \|\varphi_{x_\lambda}^{(a_\lambda)} - \psi_\lambda\| \\ &\leq \ell^{-\Theta} + 2C_{d,\varepsilon_0} L^q \ell^{-(\theta - \frac{d-1}{2})} \\ &\leq 3C_{d,\varepsilon_0} L^q \ell^{-(\theta - \frac{d-1}{2})}. \end{aligned}$$

- If $x \in \Lambda_L \setminus \widehat{\Upsilon}'_r$ and $\lambda \in \sigma_{\Upsilon_r}(H_{\Lambda_L})$, then $\lambda = \tilde{\nu}^{(r)}$ for some $\nu^{(r)} \in \sigma_{\mathcal{B}}(H_\Upsilon)$, and, using (2.43) and (2.46) (note $\phi_{\nu^{(r)}}(x) = 0$ if $x \notin \Upsilon_r$),

$$\begin{aligned} |\psi_\lambda(x)| &\leq |\phi_{\nu^{(r)}}(x)| + \|\phi_{\nu^{(r)}}(x) - \psi_\lambda\| \\ &\leq C_{d,\varepsilon_0} L^q \ell^{-(\theta - 2d)} + 2C_{d,\varepsilon_0} L^{\frac{d}{2} + 2q} \ell^{-(\theta - 2d)} \\ &\leq 3C_{d,\varepsilon_0} L^{\frac{d}{2} + 2q} \ell^{-(\theta - 2d)}. \end{aligned}$$

Therefore for all $x \in \Lambda_L$ and $\lambda \in \sigma(H_{\Lambda_L}) \setminus \mathcal{N}(x)$ we have

$$|\psi_\lambda(x)| \leq C_{d,\varepsilon_0} L^{\frac{d}{2} + 2q} \ell^{-(\theta - 2d)}. \tag{4.16}$$

Since $|\Lambda_L| = |\sigma(H_{\Lambda_L})|$, to apply Hall's Marriage Theorem we only need to verify $|\Theta| \le |\mathcal{N}(\Theta)|$, where $\mathcal{N}(\Theta) = \bigcup_{x\in\Theta} \mathcal{N}(x)$ for $\Theta \subset \Lambda_L$. For $\Theta \subset \Lambda_L$, let $Q_\Theta$ be the orthogonal projection onto the span of $\{\psi_\lambda; \lambda \in \mathcal{N}(\Theta)\}$. If $\lambda \notin \mathcal{N}(\Theta)$, for all $x \in \Theta$ we have (4.16), thus

$$\begin{aligned}\|(1-Q_\Theta)\chi_\Theta\| &\le |\Lambda_L|^{\frac{1}{2}}|\Theta|^{\frac{1}{2}} C_{d,\varepsilon_0} L^{\frac{d}{2}+2q}\ell^{-(\theta-2d)} \\ &\le (L+1)^d C_{d,\varepsilon_0} L^{\frac{d}{2}+2q}\ell^{-(\theta-2d)} \\ &< 1,\end{aligned}$$

for sufficiently large $\ell$ since $\theta - 2d > \gamma_1\left(\frac{5d}{2} + 2q\right) > \frac{5}{2}d + 2q$, so it follows from [10, Lemma A.1] that

$$|\Theta| = \operatorname{tr}\chi_\Theta \le \operatorname{tr} Q_\Theta = |\mathcal{N}(\Theta)|.$$

Using Hall's Marriage Theorem, we conclude that there exists a bijection

$$x \in \Lambda_L \mapsto \lambda_x \in \sigma(H_{\Lambda_L}), \quad \text{where } \lambda_x \in \mathcal{N}(x).$$

We set $\psi_x = \psi_{\lambda_x}$ for all $x \in \Lambda_L$.

To finish the proof we need to show that $\{(\psi_x, \lambda_x)\}_{x\in\Lambda_L}$ is a $\theta$-polynomially localized eigensystem for $\Lambda_L$. We fix $N = 1$, $x \in \Lambda_L$, take $y \in \Lambda_L$, and consider several cases.

(i) Suppose $\lambda_x \in \sigma_{\mathcal{G}}(\Lambda_L)$. Then $x \in \Lambda_\ell(a_{\lambda_x})$ with $a_{\lambda_x} \in \mathcal{G}$, and $\lambda_x \in \sigma_{\{a_{\lambda_x}\}}(H_{\Lambda_L})$. In view of (4.12) we consider two cases.

(a) If $y \in \Lambda_\ell^{\Lambda_L,\frac{\ell}{10}}(a)$ for some $a \in \mathcal{G}$ and $\|y - x\| \ge 2\ell$, we must have $\Lambda_\ell(a_{\lambda_x}) \cap \Lambda_\ell(a) = \emptyset$, so it follows from (2.36) that $\lambda_x \notin \sigma_{\{a\}}(H_{\Lambda_L})$, and (2.32) gives

$$|\psi_x| \le C_{d,\varepsilon_0} L^q \ell^{-(\theta-2d)} |\psi_x(y_1)| \quad \text{for some } y_1 \in \partial^{\Theta,2\ell'}\Lambda_\ell(a). \tag{4.17}$$

(b) If $y \in \Upsilon_1^{\Lambda_L,\frac{\ell}{10}}$, and $\|y - x\| \ge \ell + \operatorname{diam}\Upsilon_1$, we must have $\Lambda_\ell(a_{\lambda_x}) \cap \Upsilon_1 = \emptyset$, so it follows from (2.36) that $\lambda_x \notin \sigma_{\mathcal{G}_{\Upsilon_1}}(H_{\Lambda_L})$, and clearly $\lambda_x \notin \sigma_{\Upsilon_1}(H_{\Lambda_L})$ in view of (4.13). Thus Lemma 2.7(ii) gives

$$|\psi_x(y)| \le C_{d,\varepsilon_0} L^{2d+2q}\ell^{-(\theta-2d)}|\psi_x(v)| \quad \text{for some } v \in \partial^{\Lambda_L,2\ell'}\Upsilon_1. \tag{4.18}$$

(ii) Suppose $\lambda_x \notin \sigma_{\mathcal{G}}(\Lambda_L)$. Then it follows from (4.14) that we must have $\lambda_x \in \sigma_{\Upsilon_1}(H_{\Lambda_L})$. If $y \in \Lambda_\ell^{\Lambda_L,\frac{\ell}{10}}(a)$ for some $a \in \mathcal{G}$, and $\|y - x\| \ge \ell + \operatorname{diam}\Upsilon_1$, we must have $\Lambda_\ell(a) \cap \Upsilon_1 = \emptyset$, and (2.32) gives (4.17).

Now we fix $x \in \Lambda_L$, and take $y \in \Lambda_L$ such that $\|y - x\| \geq L'$. Suppose $|\psi_x(y)| > 0$ without loss of generality. We estimate $|\psi_x(y)|$ using either (4.17) or (4.18) repeatedly, as appropriate, stopping when we get too close to $x$ so we are not in any case described above. (Note that this must happen since $|\psi_x(y)| > 0$.) We accumulate decay only when using (4.17), and just use $C_{d,\varepsilon_0} L^{2d+2q} \ell^{-(\theta-2d)} < 1$ when using (4.18), then recalling $L = Y\ell$, we get

$$|\psi_x(y)| \leq (C_{d,\varepsilon_0} L^q \ell^{-(\theta-2d)})^{n(Y)}, \tag{4.19}$$

where $n(Y)$ is the number of times we used (4.17). We have

$$n(Y)(\ell + 1) + \operatorname{diam} \Upsilon_1 + 2\ell \geq L'.$$

Thus, using (4.8),

$$n(Y) \geq \tfrac{1}{\ell+1}(L' - 5\ell - 2\ell) \geq \tfrac{\ell}{\ell+1}\big(\tfrac{Y}{40} - 7\big) \geq 2.$$

for sufficiently large $\ell$ since $Y \geq 400$. It follows from (4.19),

$$|\psi_x(y)| \leq (C_{d,\varepsilon_0} Y^q \ell^{-(\theta-2d-q)})^2 \leq L^{-\theta},$$

for sufficiently large $\ell$ since $2(\theta - 2d - q) = \theta + (\theta - 4d - 2q) > \theta$.

We conclude that $\{(\psi_x, \lambda_x)\}_{x\in\Lambda_L}$ is a $\theta$-polynomially localized eigensystem for $\Lambda_L$, so the box $\Lambda_L$ is $\theta$-polynomially localizing for $H_{\varepsilon,\omega}$. □

*Proof of Proposition* 4.1. We assume (4.1) and set $L_{k+1} = Y L_k$ for $k = 0, 1, \ldots$. For $k = 1, 2, \ldots$ we set

$$P_k = \sup_{x\in\mathbb{R}^d} \mathbb{P}\{\Lambda_{L_k}(x) \text{ is not } \theta\text{-polynomially localizing for } H_{\varepsilon,\omega}\}.$$

Then by Lemma 4.2, we have

$$P_{k+1} \leq (2Y)^{2d} P_k^2 + \tfrac{1}{2} L_{k+1}^{-p} \quad \text{for} \quad k = 0, 1, \ldots \tag{4.20}$$

If $P_k \leq L_k^{-p}$ for some $k \geq 0$, we have

$$P_{k+1} \leq (2Y)^{2d} L_k^{-2p} + \tfrac{1}{2} L_{k+1}^{-p} \leq (2Y)^{2d+2p} L_{k+1}^{-2p} + \tfrac{1}{2} L_{k+1}^{-p} \leq L_{k+1}^{-p}$$

for $L_0$ sufficiently large. Therefore to finish the proof, we need to show that

$$K_0 = \inf\{k \in \mathbb{N}; P_k \leq L_k^{-p}\} < \infty.$$

It follows from (4.20) that for any $1 \le k < K_0$,

$$P_k \le (2Y)^{2d} P_{k-1}^2 + \tfrac{1}{2} L_k^{-p} < (2Y)^{2d} P_{k-1}^2 + \tfrac{1}{2} P_k,$$

so

$$2(2Y)^{2d} P_k < (2(2Y)^{2d} P_{k-1})^2.$$

Therefore for $1 \le k < K_0$, we have

$$2^{2d+1} Y^{-(kp-2d)} L_0^{-p} = 2(2Y)^{2d} L_k^{-p} < 2(2Y)^{2d} P_k < (2(2Y)^{2d} P_0)^{2^k}. \quad (4.21)$$

Since $2(2Y)^{2d} P_0 < 1$, (4.21) cannot be satisfied for large $k$. We conclude that $K_0 < \infty$. □

## 4.2. The first intermediate step

**Proposition 4.3.** *Fix $\varepsilon_0 > 0$. Suppose that for some scale $\ell$ and $0 < \varepsilon \le \varepsilon_0$*

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_\ell(x) \textit{ is } \theta\textit{-polynomially localizing for } H_{\varepsilon,\omega}\} \ge 1 - \ell^{-p}. \quad (4.22)$$

*If $\ell$ is sufficiently large, for $L = \ell^{\gamma_1}$, then*

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_L(x) \textit{ is } m_0^*\textit{-mix localizing for } H_{\varepsilon,\omega}\} \ge 1 - L^{-p}, \quad (4.23)$$

*where*

$$m_0^* \ge \tfrac{1}{8}\big(\tfrac{5d}{2} + q\big) L^{-(1-\tau+\frac{1}{\gamma_1})} \log L. \quad (4.24)$$

*Proof.* We follow the proof of Lemma 4.2. For $N \in \mathbb{N}$, let $\mathcal{B}_N$, $\mathcal{S}_N$ and $\mathcal{E}_N$ as in the proof of Lemma 4.2. Using (4.22), (2.2) and the fact that events on disjoint boxes are independent, if $N = 1$, then

$$\mathbb{P}\{\mathcal{B}_N^c\} \le \big(\tfrac{2L}{\ell}\big)^{2d} \ell^{-2p} = 2^{2d} \ell^{-2p-2d(\gamma_1-1)} < \tfrac{1}{2} \ell^{-\gamma_1 p} = \tfrac{1}{2} L^{-p} \quad (4.25)$$

for all $\ell$ sufficiently large since $1 < \gamma_1 < 1 + \frac{p}{p+2d}$. Also, using (4.9) and (4.10), if $N = 1$, then

$$\mathbb{P}\{\mathcal{S}_N^c\} \le (1 + (L+1)^d) Y_{\varepsilon_0} (L+1)^{2d} L^{-(2\alpha-1)q} < \tfrac{1}{2} L^{-p} \quad (4.26)$$

for sufficiently large $L$, since $p < (2\alpha - 1)q - 3d$. Combining (4.25) and (4.26), we conclude that

$$\mathbb{P}\{\mathcal{E}_N\} > 1 - L^{-p}.$$

To finish the proof we need to show that for all $\omega \in \mathcal{E}_N$ the box $\Lambda_L$ is $m_0^*$-mix localizing for $H_{\varepsilon,\omega}$, where $m_0^*$ is given in (4.24). Following the proof of Lemma 4.2, we get (4.14) and obtain an eigensystem $\{(\psi_x, \lambda_x)\}_{x\in\Lambda_L}$ for $H_{\Lambda_L}$ using Hall's Marriage Theorem. To finish the proof we need to show that $\{(\psi_x, \lambda_x)\}_{x\in\Lambda_L}$ is an $m_0^*$-localized eigensystem for $\Lambda_L$. We proceed as in the proof of Lemma 4.2. We fix $N = 1$, $x \in \Lambda_L$, and take $y \in \Lambda_L$ such that $\|y - x\| \geq L_\tau$, we have

$$n(\ell)(\ell + 1) + \operatorname{diam} \Upsilon_1 + 2\ell \geq L_\tau. \tag{4.27}$$

where $n(\ell)$ is the number of times we used (4.17). Thus, using (4.8), we have

$$n(\ell) \geq \tfrac{1}{\ell+1}(L_\tau - 5\ell - 2\ell) \geq \tfrac{\ell}{\ell+1}\left(\tfrac{1}{2}\ell^{\gamma_1\tau-1} - 7\right) \geq \tfrac{1}{4}\ell^{\gamma_1\tau-1}. \tag{4.28}$$

for sufficiently large $\ell$. It follows from (4.19),

$$\begin{aligned} |\psi_x(y)| &\leq \left(C_{d,\varepsilon_0}\ell^{-(\theta-2d-\gamma_1 q)}\right)^{\frac{1}{4}\ell^{\gamma_1\tau-1}} \\ &\leq \mathrm{e}^{-\frac{1}{8}\left(\frac{5d}{2}+q\right)L^{-(1-\tau+\frac{1}{\gamma_1})}(\log L)\|y-x\|}, \end{aligned}$$

for sufficiently large $\ell$.

We conclude that $\{(\psi_x, \lambda_x)\}_{x\in\Lambda_L}$ is an $m_0^*$-localized eigensystem for $\Lambda_L$, where $m_0^*$ is given in (4.24), so the box $\Lambda_L$ is $m_0^*$-mix localizing for $H_{\varepsilon,\omega}$. □

### 4.3. The second multiscale analysis

**Proposition 4.4.** *Fix $\varepsilon_0 > 0$. There exists a finite scale $\mathcal{L}(\varepsilon_0)$ with the following property: Suppose for some scale $L_0 \geq \mathcal{L}(\varepsilon_0)$, $0 < \varepsilon \leq \varepsilon_0$, and $m_0^* \geq L_0^{-\kappa}$ where $0 < \kappa < \tau$, we have*

$$\inf_{x\in\mathbb{R}^d} \mathbb{P}\{\Lambda_{L_0}(x) \text{ is } m_0^*\text{-mix localizing for } H_{\varepsilon,\omega}\} \geq 1 - L_0^{-p}. \tag{4.29}$$

*Then, setting $L_{k+1} = L_k^{\gamma_1}$ for $k = 0, 1, \ldots$, we have*

$$\inf_{x\in\mathbb{R}^d} \mathbb{P}\{\Lambda_{L_k}(x) \text{ is } \tfrac{m_0^*}{2}\text{-mix localizing for } H_{\varepsilon,\omega}\} \geq 1 - L_k^{-p} \quad \text{for } k = 0, 1, \ldots. \tag{4.30}$$

Proposition 4.4 follows from the following induction step for the multiscale analysis.

**Lemma 4.5.** *Fix $\varepsilon_0 > 0$. Suppose that for some scale $\ell$, $0 < \varepsilon \leq \varepsilon_0$, and $m^* \geq \ell^{-\kappa}$, where $0 < \kappa < \tau$,*

$$\inf_{x\in\mathbb{R}^d} \mathbb{P}\{\Lambda_\ell(x) \text{ is } m^*\text{-mix localizing for } H_{\varepsilon,\omega}\} \geq 1 - \ell^{-p}. \tag{4.31}$$

*If $\ell$ is sufficiently large, for $L = \ell^{\gamma_1}$, then*

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_L(x) \text{ is } M^*\text{-mix localizing for } H_{\varepsilon,\omega}\} \geq 1 - L^{-p},$$

*where*

$$M^* \geq m^*(1 - C_{d,\varepsilon_0}\gamma_1 q \ell^{-\min\{\frac{1-\tau}{2}, \gamma_1\tau - 1, \tau - \kappa\}}) \geq L^{-\kappa}. \tag{4.32}$$

*Proof.* We follow the proof of Lemma 4.2. For $N \in \mathbb{N}$, let $\mathcal{B}_N$ denote the event that there do not exist two disjoint boxes in $\mathcal{C}_{L,\ell}$ that are not $m^*$-mix localizing for $H_{\varepsilon,\omega}$. Using (4.31), (2.2), and the fact that events on disjoint boxes are independent, if $N = 1$, then

$$\mathbb{P}\{\mathcal{B}_N^c\} \leq \left(\tfrac{2L}{\ell}\right)^{(N+1)d} \ell^{-(N+1)p} = 2^{2d} \ell^{-(2p - 2d(\gamma_1 - 1))} < \tfrac{1}{2}\ell^{-\gamma_1 p} = \tfrac{1}{2}L^{-p} \tag{4.33}$$

for all $\ell$ sufficiently large since $1 < \gamma_1 < 1 + \frac{p}{p+2d}$.

We now fix $\omega \in \mathcal{B}_N$, and proceed as in the proof of Lemma 4.2 with $\sharp$ being $m^*$-ML. Then we have $\Upsilon_r$, $r = 1, 2, \ldots, R$ such that each $\Upsilon_r$ satisfies all the requirements to be an $m^*$-ML-buffered subset of $\Lambda_L$ with $\mathcal{G}_{\Upsilon_r} = \partial_{\text{ex}}^{\mathbb{G}_1} \widetilde{\Phi}_r$, except we do not know if $\Upsilon_r$ is $L$-polynomially level spacing for $H_{\varepsilon,\omega}$.

Given a $\mathbb{G}_2$-connected subset $\Phi$ of $\Xi_{L,\ell}$, let $\Upsilon(\Phi) \subset \Lambda_L$ be constructed from $\Phi$ as in (4.7) with $\sharp$ being $m^*$-ML. Let $\mathcal{S}_N$ denote the event that the box $\Lambda_L$ and the subsets $\{\Upsilon(\Phi)\}_{\Phi \in \mathcal{F}_N}$ are all $L$-polynomially level spacing for $H_{\varepsilon,\omega}$. Using (4.9) and (4.10), if $N = 1$ we have

$$\mathbb{P}\{\mathcal{S}^c\} \leq \left(1 + \left(\tfrac{2L}{\ell}\right)^d\right) Y_{\varepsilon_0} (L+1)^{2d} L^{-(2\alpha - 1)q} < \tfrac{1}{2} L^{-p} \tag{4.34}$$

for sufficiently large $L$, since $p < (2\alpha - 1)q - 3d$.

Let $\mathcal{E}_N = \mathcal{B}_N \cap \mathcal{S}_N$. Combining (4.33) and (4.34), we conclude that if $N = 1$,

$$\mathbb{P}\{\mathcal{E}_N\} > 1 - L^{-p}.$$

To finish the proof we need to show that for all $\omega \in \mathcal{E}_N$ the box $\Lambda_L$ is $M^*$-mix localizing for $H_{\varepsilon,\omega}$, where $M^*$ is given in (4.32).

We fix $\omega \in \mathcal{E}_N$. Then we have (4.6), $\Lambda_L$ is polynomially level spacing for $H_{\varepsilon,\omega}$, and the subsets $\{\Upsilon_r\}_{r=1}^R$ constructed in (4.7) are $m^*$-ML-buffered subset of $\Lambda_L$ for $H_{\varepsilon,\omega}$. We proceed as in the proof of Lemma 4.2. To claim (4.14), we assume $\lambda \in \sigma_{\mathcal{G}} \setminus (\sigma_{\mathcal{G}}(H_{\Lambda_L}) \cup \sigma_{\mathcal{B}}(H_{\Lambda_L}))$. Since $\Lambda_L$ is polynomially level spacing for $H$, Lemma 2.4(ii)(c) gives

$$|\psi_\lambda(y)| \leq \mathrm{e}^{-m_2^* \ell_\tau} \quad \text{for all } y \in \bigcup_{a \in \mathcal{G}} \Lambda_\ell^{\Lambda_L, 2\ell_\tau}(a),$$

and Lemma 2.7(ii) gives

$$|\psi_\lambda(y)| \le \mathrm{e}^{-m_5^* \ell_\tau} \quad \text{for all } y \in \bigcup_{r=1}^{R} \Upsilon_r^{\Lambda_L, 2\ell_\tau}.$$

Using (4.12), we conclude that (note $m_5^* \le m_2^*$)

$$1 = \|\psi_\lambda(y)\| \le \mathrm{e}^{-m_5^* \ell_\tau}(L+1)^{\frac{d}{2}} < 1, \tag{4.35}$$

a contradiction. This establishes the claim.

To index the eigenvalues and eigenvectors of $H_{\Lambda_L}$ by sites in $\Lambda_L$, we define $\mathcal{N}(x)$ as in (4.15) and proceed as in the proof of Lemma 4.2.

- If $x \in \Lambda_L$ and $\lambda \in \sigma_{\mathcal{G}}(H_{\Lambda_L}) \setminus \mathcal{N}_0(x)$, we have $\lambda = \tilde{\lambda}_{x_\lambda}^{(a_\lambda)}$ with $\|x_\lambda - x\| \ge \ell_\tau$, so, using (1.3) and (2.29),

  $$|\psi_\lambda(x)| \le |\varphi_{x_\lambda}^{(a_\lambda)}(x)| + \|\varphi_{x_\lambda}^{(a_\lambda)} - \psi_\lambda\| \le \mathrm{e}^{-m^* \ell_\tau} + 2\mathrm{e}^{-m_1^* \ell_\tau} L^q \le 3\mathrm{e}^{-m_1 \ell_\tau} L^q.$$

- If $x \in \Lambda_L \setminus \widehat{\Upsilon}_r'$ and $\lambda \in \sigma_{\Upsilon_r}(H_{\Lambda_L})$, then $\lambda = \tilde{\nu}^{(r)}$ for some $\nu^{(r)} \in \sigma_{\mathcal{B}}(H_{\Upsilon_r})$, and, using (2.43) and (2.48), (Note $\phi_{\nu^{(r)}}(x) = 0$ if $x \notin \Upsilon_r$.)

  $$|\psi_\lambda(x)| \le |\phi_{\nu^{(r)}}(x)| + \|\phi_{\nu^{(r)}}(x) - \psi_\lambda\| \le \mathrm{e}^{-m_2^* \ell_\tau} + 2\mathrm{e}^{-m_4^* \ell_\tau} L^q \le 3\mathrm{e}^{-m_4^* \ell_\tau} L^q.$$

Therefore, for all $x \in \Lambda_L$ and $\lambda \in \sigma(H_{\Lambda_L}) \setminus \mathcal{N}(x)$

$$|\psi_\lambda(x)| \le 3\mathrm{e}^{-m_4^* \ell_\tau} L^q \le \mathrm{e}^{-\frac{1}{2} m_4^* \ell_\tau}. \tag{4.36}$$

If $\lambda \notin \mathcal{N}(\Theta)$, for all $x \in \Theta$ we have (4.36); thus

$$\|(1 - Q_\Theta)\chi_\Theta\| \le |\Lambda_L|^{\frac{1}{2}} |\Theta|^{\frac{1}{2}} \mathrm{e}^{-\frac{1}{2} m_4^* \ell_\tau} \le (L+1)^d \mathrm{e}^{-\frac{1}{2} m_4^* \ell_\tau} < 1.$$

Following the proof of Lemma 4.2, we can apply Hall's Marriage Theorem to obtain an eigensystem $\{(\psi_x, \lambda_x)\}_{x \in \Lambda_L}$ for $H_{\Lambda_L}$.

To finish the proof we need to show that $\{(\psi_x, \lambda_x)\}_{x \in \Lambda_L}$ is an $M^*$-localized eigensystem for $\Lambda_L$, where $M^*$ is given in (4.32). We fix $N = 1$, $x \in \Lambda_L$, take $y \in \Lambda_L$, and consider several cases.

(i) Suppose $\lambda_x \in \sigma_{\mathcal{G}}(\Lambda_L)$. Then $x \in \Lambda_\ell(a_{\lambda_x})$ with $a_{\lambda_x} \in \mathcal{G}$, and $\lambda_x \in \sigma_{\{a_{\lambda_x}\}}(H_{\Lambda_L})$. In view of (4.12) we consider two cases.

  (a) If $y \in \Lambda_\ell^{\Lambda_L, \frac{\ell}{10}}(a)$ for some $a \in \mathcal{G}$ and $\|y - x\| \ge 2\ell$, we must have $\Lambda_\ell(a_{\lambda_x}) \cap \Lambda_\ell(a) = \emptyset$, so it follows from (2.36) that $\lambda_x \notin \sigma_{\{a\}}(H_{\Lambda_L})$, and (2.33) gives

  $$|\psi_x| \le \mathrm{e}^{-m_3^* \|y_1 - y\|} |\psi_x(y_1)| \quad \text{for some } y_1 \in \partial^{\Theta, \ell_{\tilde{\tau}}} \Lambda_\ell(a). \tag{4.37}$$

(b) If $y \in \Upsilon_1^{\Lambda_L, \frac{\ell}{10}}$, and $\|y-x\| \geq \ell + \operatorname{diam} \Upsilon_1$, we must have $\Lambda_\ell(a_{\lambda_x}) \cap \Upsilon_1 = \emptyset$, so it follows from (2.36) that $\lambda_x \notin \sigma_{\mathcal{G}_{\Upsilon_1}}(H_{\Lambda_L})$, and clearly $\lambda_x \notin \sigma_{\Upsilon_1}(H_{\Lambda_L})$ in view of (4.13). Thus Lemma 2.7(ii) gives

$$|\psi_x(y)| \leq \mathrm{e}^{-m_5^* \ell_\tau} |\psi_x(v)| \quad \text{for some } v \in \partial^{\Lambda_L, 2\ell_\tau} \Upsilon_1. \tag{4.38}$$

(i) Suppose $\lambda_x \notin \sigma_{\mathcal{G}}(\Lambda_L)$. Then it follows from (4.14) that we must have $\lambda_x \in \sigma_{\Upsilon_1}(H_{\Lambda_L})$. If $y \in \Lambda_\ell^{\Lambda_L, \frac{\ell}{10}}(a)$ for some $a \in \mathcal{G}$, and $\|y-x\| \geq \ell + \operatorname{diam} \Upsilon_1$, we must have $\Lambda_\ell(a) \cap \Upsilon_1 = \emptyset$, and (2.33) gives (4.37).

Now we fix $x \in \Lambda_L$, and take $y \in \Lambda_L$ such that $\|y-x\| \geq L_\tau$. Suppose $|\psi_x(y)| > 0$ without loss of generality. We estimate $|\psi_x(y)|$ using either (4.37) or (4.38) repeatedly, as appropriate, stopping when we get too close to $x$ so we are not in any case described above. (Note that this must happen since $|\psi_x(y)| > 0$.) We accumulate decay only when using (4.37), and just use $\mathrm{e}^{-m_5^* \ell_\tau} < 1$ when using (4.38), then we get

$$\begin{aligned} |\psi_x(y)| &\leq \mathrm{e}^{-m_3^*(\|y-x\| - \operatorname{diam} \Upsilon - 2\ell)} \\ &\leq \mathrm{e}^{-m_3^*(\|y-x\| - 7\ell)} \\ &\leq \mathrm{e}^{-m_3^* \|y-x\|(1 - 7\ell^{1-\gamma_1 \tau})} \\ &\leq \mathrm{e}^{M \|y-x\|}, \end{aligned}$$

where we used (4.8) and took

$$\begin{aligned} M^* &= m_3^*(1 - 7\ell^{1-\gamma_1\tau}) \\ &\geq \left(m^*\left(1 - 4\ell^{\frac{\tau-1}{2}}\right) - C_{d,\varepsilon_0} \gamma_1 q \frac{\log \ell}{\ell_{\tau}}\right)(1 - 7\ell^{1-\gamma_1\tau}) \\ &\geq m^*(1 - 4\ell^{\frac{\tau-1}{2}} - C_{d,\varepsilon_0}\gamma_1 q \ell^{\kappa-\tau})(1 - 7\ell^{1-\gamma_1\tau}) \\ &\geq m^*(1 - C_{d,\varepsilon_0}\gamma_1 q \ell^{-\min\{\frac{1-\tau}{2}, \gamma_1\tau - 1, \tau-\kappa\}}) \\ &\geq \tfrac{1}{2}\ell^{-\kappa} \\ &\geq \ell^{-\gamma_1\kappa} \\ &= L^{-\kappa} \end{aligned}$$

for $\ell$ sufficiently large, where we used (2.13) and $m^* \geq \ell^{-\kappa}$.

We conclude that $\{(\psi_x, \lambda_x)\}_{x \in \Lambda_L}$ is an $M^*$-localized eigensystem for $\Lambda_L$, where $M^*$ is given in (4.32), so the box $\Lambda_L$ is $M^*$-mix localizing for $H_{\varepsilon,\omega}$. □

*Proof of Proposition* 4.4. We assume (4.29) and set $L_{k+1} = L_k^{\gamma_1}$ for $k = 0, 1, \ldots$. If $L_0$ is sufficiently large it follows from Lemma 4.5 by an induction argument that

$$\inf_{x\in\mathbb{R}^d} \mathbb{P}\{\Lambda_{L_k}(x) \text{ is } m_k^*\text{-localizing for } H_{\varepsilon,\omega}\} \geq 1 - L_k^{-p} \quad \text{for } k = 0, 1, \ldots,$$

where for $k = 1, 2, \ldots$ we have

$$m_k^* \geq m_{k-1}^*(1 - C_{d,\varepsilon_0}\gamma_1 q L_{k-1}^{-\varrho}), \text{ with } \varrho = \min\left\{\tfrac{1-\tau}{2}, \gamma_1\tau - 1, \tau - \kappa\right\}.$$

Thus for all $k = 1, 2, \ldots$, taking $L_0$ sufficiently large we get

$$m_k^* \geq m_0^* \prod_{j=0}^{k-1}(1 - C_{d,\varepsilon_0}\gamma_1 q L_0^{-\varrho\gamma^j}) \geq m_0^* \prod_{j=0}^{\infty}(1 - C_{d,\varepsilon_0}\gamma_1 q L_0^{-\varrho\gamma_1^j}) \geq \frac{m_0^*}{2},$$

finishing the proof of Proposition 4.4. □

### 4.4. The third multiscale analysis

**Proposition 4.6.** *Fix* $\varepsilon_0 > 0$, $Y \geq 400^{\frac{1}{1-s}}$, *and* $\widetilde{P}_0 < (2(2Y)^{(\lfloor Y^s\rfloor+1)d})^{-\frac{1}{\lfloor Y^s\rfloor}}$. *There exists a finite scale* $\mathcal{L}(\varepsilon_0, Y)$ *with the following property: Suppose for some scale* $L_0 \geq \mathcal{L}(\varepsilon_0, Y)$ *and* $0 < \varepsilon \leq \varepsilon_0$ *we have*

$$\inf_{x\in\mathbb{R}^d} \mathbb{P}\{\Lambda_{L_0}(x) \text{ is } s\text{-SEL for } H_{\varepsilon,\omega}\} \geq 1 - \widetilde{P}_0. \tag{4.39}$$

*Then, setting* $L_{k+1} = Y L_k$ *for* $k = 0, 1, \ldots$, *there exists* $K_0 = K_0(Y, L_0, \widetilde{P}_0) \in \mathbb{N}$ *such that*

$$\inf_{x\in\mathbb{R}^d} \mathbb{P}\{\Lambda_{L_k}(x) \text{ is } s\text{-SEL for } H_{\varepsilon,\omega}\} \geq 1 - \mathrm{e}^{-L_k^{\zeta}} \quad \text{for } k \geq K_0. \tag{4.40}$$

Proposition 4.6 follows from the following induction step for the multiscale analysis.

**Lemma 4.7.** *Fix* $\varepsilon_0 > 0$, $Y \geq 400^{\frac{1}{1-s}}$, *and* $0 \leq P \leq 1$. *Suppose for some scale* $\ell$ *and* $0 < \varepsilon \leq \varepsilon_0$ *we have*

$$\inf_{x\in\mathbb{R}^d} \mathbb{P}\{\Lambda_\ell(x) \text{ is } s\text{-SEL for } H_{\varepsilon,\omega}\} \geq 1 - P. \tag{4.41}$$

*If* $\ell$ *is sufficiently large, for* $L = Y\ell$, *then*

$$\inf_{x\in\mathbb{R}^d} \mathbb{P}\{\Lambda_L(x) \text{ is } s\text{-SEL for } H_{\varepsilon,\omega}\} \geq 1 - ((2Y)^{(\lfloor Y^s\rfloor+1)d} P^{\lfloor Y^s\rfloor+1} + \tfrac{1}{2}\mathrm{e}^{-L^{\zeta}}). \tag{4.42}$$

*Proof.* We follow the proof of Lemma 4.2. For $N \in \mathbb{N}$, let $\mathcal{B}_N$ denote the event that there exist at most $N$ disjoint boxes in $\mathcal{C}_{L,\ell}$ that are not $s$-SEL for $H_{\varepsilon,\omega}$. Using (4.41), (2.2) and the fact that events on disjoint boxes are independent, if $N = \lfloor Y^s \rfloor$, then

$$\mathbb{P}\{\mathcal{B}^c\} \leq \big(\tfrac{2L}{\ell}\big)^{(N+1)d} P^{N+1} = (2Y)^{(\lfloor Y^s \rfloor+1)d} P^{\lfloor Y^s \rfloor+1}. \tag{4.43}$$

We now fix $\omega \in \mathcal{B}_N$, and proceed as in the proof of Lemma 4.2 with $\sharp$ being $s$-SEL. Then we have $\Upsilon_r$, $r = 1, 2, \ldots, R$ such that each $\Upsilon_r$ satisfies all the requirements to be an $s$-SEL-buffered subset of $\Lambda_L$ with $\mathcal{G}_{\Upsilon_r} = \partial_{\mathrm{ex}}^{\mathbb{G}_1} \widetilde{\Phi}_r$, except we do not know if $\Upsilon_r$ is $L$-level spacing for $H_{\varepsilon,\omega}$.

It follows from Lemma 3.1 that, for any $\Theta \subset \Lambda_L$,

$$\mathbb{P}\{\Theta \text{ is } L\text{-level spacing for } H_{\varepsilon,\omega}\} \geq 1 - Y_{\varepsilon_0} \mathrm{e}^{-(2\alpha-1)L^\beta} (L+1)^{2d}. \tag{4.44}$$

Given a $\mathbb{G}_2$-connected subset $\Phi$ of $\Xi_{L,\ell}$, let $\Upsilon(\Phi) \subset \Lambda_L$ be constructed from $\Phi$ as in (4.7) with $\sharp$ being $s$-SEL. Let $\mathcal{S}_N$ denote the event that the box $\Lambda_L$ and the subsets the subsets $\{\Upsilon(\Phi)\}_{\Phi\in\mathcal{F}_N}$ are all $L$-level spacing for $H_{\varepsilon,\omega}$. Using (4.44) and (4.10), if $N = \lfloor Y^s \rfloor$ we have

$$\mathbb{P}\{\mathcal{S}_N^c\} \leq Y_{\varepsilon_0}(1 + (L+1)^d N!(d4^d)^{N-1})(L+1)^{2d} \mathrm{e}^{-(2\alpha-1)L^\beta} < \tfrac{1}{2}\mathrm{e}^{-L^\zeta} \tag{4.45}$$

for sufficiently large $L$, since $\zeta < \beta$.

Let $\mathcal{E}_\mathcal{N} = \mathcal{B}_\mathcal{N} \cap \mathcal{S}_\mathcal{N}$. Combining (4.43) and (4.45), we conclude that

$$\mathbb{P}\{\mathcal{E}_N\} > 1 - \big((2Y)^{(\lfloor Y^s \rfloor+1)d} P^{\lfloor Y^s \rfloor+1} + \tfrac{1}{2}\mathrm{e}^{-L^\zeta}\big).$$

To finish the proof we need to show that for all $\omega \in \mathcal{E}_N$ the box $\Lambda_L$ is $s$-SEL for $H_{\varepsilon,\omega}$.

We fix $\omega \in \mathcal{E}_N$. Then we have (4.6), $\Lambda_L$ is level spacing for $H_{\varepsilon,\omega}$, and the subsets $\{\Upsilon_r\}_{r=1}^R$ constructed in (4.7) are $s$-SEL-buffered subsets of $\Lambda_L$ for $H_{\varepsilon,\omega}$. We proceed as in the proof of Lemma 4.2. To claim (4.14), we assume $\lambda \in \sigma_\mathcal{G} \setminus (\sigma_\mathcal{G}(H_{\Lambda_L}) \cup \sigma_\mathcal{B}(H_{\Lambda_L}))$. Since $\Lambda_L$ is level spacing for $H$, Lemma 2.4(ii)(c) gives

$$|\psi_\lambda(y)| \leq \mathrm{e}^{-c_2\ell^s} \quad \text{for all } y \in \bigcup_{a\in\mathcal{G}} \Lambda_\ell^{\Lambda_L,2\ell'}(a),$$

and Lemma 2.7(ii) gives

$$|\psi_\lambda(y)| \leq \mathrm{e}^{-c_4\ell^s} \quad \text{for all } y \in \bigcup_{r=1}^R \Upsilon_r^{\Lambda_L,2\ell'}.$$

Using (4.12), we conclude that (note $c_4 \le c_2$)

$$1 = \|\psi_\lambda(y)\| \le \mathrm{e}^{-c_4 \ell^s}(L+1)^{\frac{d}{2}} < 1,$$

a contradiction. This establishes the claim.

To index the eigenvalues and eigenvectors of $H_{\Lambda_L}$ by sites in $\Lambda_L$, we define $\mathcal{N}(x)$ as in (4.15) proceed as in the proof of Lemma 4.2. We have:

- If $x \in \Lambda_L$ and $\lambda \in \sigma_{\mathcal{G}}(H_{\Lambda_L}) \setminus \mathcal{N}_0(x)$, we have $\lambda = \tilde{\lambda}_{x_\lambda}^{(a_\lambda)}$ with $\|x_\lambda - x\| \ge \ell'$, so, using (1.2) and (2.28),

$$|\psi_\lambda(x)| \le |\varphi_{x_\lambda}^{(a_\lambda)}(x)| + \|\varphi_{x_\lambda}^{(a_\lambda)} - \psi_\lambda\| \le \mathrm{e}^{-\ell^s} + 2\mathrm{e}^{-c_1\ell^s}\mathrm{e}^{L^\beta} \le 3\mathrm{e}^{-c_1\ell^s}\mathrm{e}^{L^\beta}.$$

- If $x \in \Lambda_L \setminus \widehat{\Upsilon}_r'$ and $\lambda \in \sigma_{\Upsilon_r}(H_{\Lambda_L})$, then $\lambda = \tilde{\nu}^{(r)}$ for some $\nu^{(r)} \in \sigma_{\mathcal{B}}(H_{\Upsilon_r})$, and, using (2.43) and (2.47), (Note $\phi_{\nu^{(r)}}(x) = 0$ if $x \notin \Upsilon_r$.)

$$|\psi_\lambda(x)| \le |\phi_\nu(x)| + \|\phi_\nu(x) - \psi_\lambda\| \le \mathrm{e}^{-c_2\ell^s} + 2\mathrm{e}^{-c_3\ell^s}\mathrm{e}^{L^\beta} \le 3\mathrm{e}^{-c_3\ell^s}\mathrm{e}^{L^\beta}.$$

Therefore for all $x \in \Lambda_L$ and $\lambda \in \sigma(H_{\Lambda_L}) \setminus \mathcal{N}(x)$ we have

$$|\psi_\lambda(x)| \le 3\mathrm{e}^{-c_3\ell^s}\mathrm{e}^{L^\beta} \le \mathrm{e}^{-\frac{1}{2}c_3\ell^s}. \tag{4.46}$$

If $\lambda \notin \mathcal{N}(\Theta)$, for all $x \in \Theta$ we have (4.46); thus

$$\|(1 - Q_\Theta)\chi_\Theta\| \le |\Lambda_L|^{\frac{1}{2}}|\Theta|^{\frac{1}{2}}\mathrm{e}^{-\frac{1}{2}c_3\ell^s} \le (L+1)^d\mathrm{e}^{-\frac{1}{2}c_3\ell^s} < 1.$$

Following the proof of Lemma 4.2, we can apply Hall's Marriage Theorem to obtain an eigensystem $\{(\psi_x, \lambda_x)\}_{x\in\Lambda_L}$ for $H_{\Lambda_L}$.

To finish the proof we need to show that $\{(\psi_x, \lambda_x)\}_{x\in\Lambda_L}$ is an $s$-subexponentially localized eigensystem for $\Lambda_L$. We fix $N = \lfloor Y^s \rfloor$, $x \in \Lambda_L$, take $y \in \Lambda_L$, and consider several cases.

(i) Suppose $\lambda_x \in \sigma_{\mathcal{G}}(\Lambda_L)$. Then $x \in \Lambda_\ell(a_{\lambda_x})$ with $a_{\lambda_x} \in \mathcal{G}$, and $\lambda_x \in \sigma_{\{a_{\lambda_x}\}}(H_{\Lambda_L})$. In view of (4.12) we consider two cases.

(a) If $y \in \Lambda_\ell^{\Lambda_L, \frac{\ell}{10}}(a)$ for some $a \in \mathcal{G}$ and $\|y - x\| \ge 2\ell$, we must have $\Lambda_\ell(a_{\lambda_x}) \cap \Lambda_\ell(a) = \emptyset$, so it follows from (2.36) that $\lambda_x \notin \sigma_{\{a\}}(H_{\Lambda_L})$, and (2.32) gives

$$|\psi_x| \le \mathrm{e}^{-c_2\ell^s}|\psi_x(y_1)| \quad \text{for some } y_1 \in \partial^{\Theta, 2\ell'}\Lambda_\ell(a). \tag{4.47}$$

(b) If $y \in \Upsilon_r^{\Lambda_L, \frac{\ell}{10}}$ for some $r \in \{1, 2, \ldots, R\}$, and $\|y - x\| \ge \ell + \operatorname{diam} \Upsilon_r$, we must have $\Lambda_\ell(a_{\lambda_x}) \cap \Upsilon_r = \emptyset$, so it follows from (2.36) that $\lambda_x \notin \sigma_{\mathcal{G}_{\Upsilon_r}}(H_{\Lambda_L})$, and clearly $\lambda_x \notin \sigma_{\Upsilon_r}(H_{\Lambda_L})$ in view of (4.13). Thus Lemma 2.7(ii) gives

$$|\psi_x(y)| \le \mathrm{e}^{-c_4\ell^s}|\psi_x(v)| \quad \text{for some } v \in \partial^{\Lambda_L, 2\ell'}\Upsilon_r. \tag{4.48}$$

(ii) Suppose $\lambda_x \notin \sigma_{\mathcal{G}}(\Lambda_L)$. Then it follows from (4.14) that we must have $\lambda_x \in \sigma_{\Upsilon_{\tilde{r}}}(H_{\Lambda_L})$ for some $\tilde{r} \in \{1, 2, \dots, R\}$. In view of (4.12) we consider two cases.

(a) If $y \in \Lambda_\ell^{\Lambda_L, \frac{\ell}{10}}(a)$ for some $a \in \mathcal{G}$, and $\|y - x\| \geq \ell + \operatorname{diam} \Upsilon_{\tilde{r}}$, we must have $\Lambda_\ell(a) \cap \Upsilon_{\tilde{r}} = \emptyset$, and (2.32) gives (4.47).

(b) If $y \in \Upsilon_r^{\Lambda_L, \frac{\ell}{10}}$ for some $r \in \{1, 2, \dots, R\}$, and $\|y - x\| \geq \operatorname{diam} \Upsilon_{\tilde{r}} + \operatorname{diam} \Upsilon_r$, we must have $r \neq \tilde{r}$. Thus Lemma 2.7(ii) gives (4.48).

Now we fix $x \in \Lambda_L$, and take $y \in \Lambda_L$ such that $\|y - x\| \geq L'$. Suppose $|\psi_x(y)| > 0$ without loss of generality. We estimate $|\psi_x(y)|$ using either (4.47) or (4.48) repeatedly, as appropriate, stopping when we get too close to $x$ so we are not in any case described above. (Note that this must happen since $|\psi_x(y)| > 0$.) We accumulate decay only when we use (4.47), and just use $\mathrm{e}^{-c_4 \ell^s} < 1$ when using (4.48), recalling $L = Y\ell$, then we get

$$|\psi_x(y)| \leq (\mathrm{e}^{-c_2 \ell^s})^{n(Y)}, \tag{4.49}$$

where $n(Y)$ is the number of times we used (4.47). We have

$$n(Y)(\ell + 1) + \sum_{r=1}^{R} \operatorname{diam} \Upsilon_r + 2\ell \geq L'.$$

Thus, using (4.8), we have

$$n(Y) \geq \tfrac{1}{\ell+1}(L' - 5\ell \lfloor Y^s \rfloor - 2\ell) \geq \tfrac{\ell}{\ell+1}\left(\tfrac{Y}{40} - 5Y^s - 2\right) \geq 2Y^s.$$

for sufficiently large $\ell$ since $Y \geq 400^{\frac{1}{1-s}}$. It follows from (4.49),

$$|\psi_x(y)| \leq (\mathrm{e}^{-c_2 \ell^s})^{2Y^s} \leq \mathrm{e}^{-L^s},$$

for sufficiently large $\ell$.

We conclude that $\{(\psi_x, \lambda_x)\}_{x \in \Lambda_L}$ is an $s$-subexponentially localized eigensystem for $\Lambda_L$, so the box $\Lambda_L$ is $s$-SEL for $H_{\varepsilon,\omega}$. □

*Proof of Proposition* 4.6. We assume (4.39) and set $L_{k+1} = Y L_k$ for $k = 0, 1, \dots$. We set

$$\widetilde{P}_k = \sup_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_{L_k}(x) \text{ is not } s\text{-SEL for } H_{\varepsilon,\omega}\} \quad \text{for } k = 1, 2, \dots.$$

Then by Lemma 4.7,

$$\widetilde{P}_{k+1} \leq (2Y)^{(\lfloor Y^s \rfloor + 1)d}\, \widetilde{P}_k^{\lfloor Y^s \rfloor + 1} + \tfrac{1}{2}\mathrm{e}^{-L_{k+1}^{\xi}} \quad \text{for } k = 0, 1, \dots. \tag{4.50}$$

If $\widetilde{P}_k \le \mathrm{e}^{-L_k^{\zeta}}$ for some $k \ge 0$, then

$$\begin{aligned}\widetilde{P}_{k+1} &\le (2Y)^{(\lfloor Y^s\rfloor+1)d}(\mathrm{e}^{-L_k^{\zeta}})^{\lfloor Y^s\rfloor+1} + \tfrac{1}{2}\mathrm{e}^{-L_{k+1}^{\zeta}}\\ &\le (2Y)^{(\lfloor Y^s\rfloor+1)d}\mathrm{e}^{-\frac{\lfloor Y^s\rfloor+1}{Y^{\zeta}}L_{k+1}^{\zeta}} + \tfrac{1}{2}\mathrm{e}^{-L_{k+1}^{\zeta}}\\ &\le \mathrm{e}^{-L_{k+1}^{\zeta}}\end{aligned}$$

for $L_0$ sufficiently large, since $\zeta < s$. Therefore to finish the proof, we need to show that

$$K_0 = \inf\{k \in \mathbb{N};\ \widetilde{P}_k \le \mathrm{e}^{-L_k^{\zeta}}\} < \infty.$$

It follows from (4.50) that for any $1 \le k < K_0$,

$$\widetilde{P}_k \le (2Y)^{(\lfloor Y^s\rfloor+1)d}\,\widetilde{P}_{k-1}^{\lfloor Y^s\rfloor+1} + \tfrac{1}{2}\mathrm{e}^{-L_k+^{\zeta}} < (2Y)^{(\lfloor Y^s\rfloor+1)d}\,\widetilde{P}_{k-1}^{\lfloor Y^s\rfloor+1} + \tfrac{1}{2}\widetilde{P}_k,$$

so

$$(2(2Y)^{(\lfloor Y^s\rfloor+1)d})^{\frac{1}{\lfloor Y^s\rfloor}}\,\widetilde{P}_k < ((2(2Y)^{(N+1)d})^{\frac{1}{\lfloor Y^s\rfloor}}\,\widetilde{P}_{k-1})^{\lfloor Y^s\rfloor+1}.$$

For $1 \le k < K_0$, since $\big(2(2Y)^{(\lfloor Y^s\rfloor+1)d}\big)^{\frac{1}{\lfloor Y^s\rfloor}}\,\widetilde{P}_0 < 1$,

$$\begin{aligned}(2(2Y)^{(\lfloor Y^s\rfloor+1)d})^{\frac{1}{\lfloor Y^s\rfloor}}\mathrm{e}^{-Y^{k\zeta}L_0^{\zeta}} &= (2(2Y)^{(\lfloor Y^s\rfloor+1)d})^{\frac{1}{\lfloor Y^s\rfloor}}\mathrm{e}^{-L_k^{\zeta}}\\ &< (2(2Y)^{(\lfloor Y^s\rfloor+1)d})^{\frac{1}{\lfloor Y^s\rfloor}}\,\widetilde{P}_k\\ &< ((2(2Y)^{(\lfloor Y^s\rfloor+1)d})^{\frac{1}{\lfloor Y^s\rfloor}}\,\widetilde{P}_0)^{(\lfloor Y^s\rfloor+1)^k}\\ &\le ((2(2Y)^{(\lfloor Y^s\rfloor+1)d})^{\frac{1}{\lfloor Y^s\rfloor}}\,\widetilde{P}_0)^{Y^{ks}}.\end{aligned} \tag{4.51}$$

Since $\zeta < s$, $\big(2(2Y)^{(\lfloor Y^s\rfloor+1)d}\big)^{\frac{1}{\lfloor Y^s\rfloor}}\,\widetilde{P}_0 < 1$, (4.51) cannot be satisfied for large $k$. We conclude that $K_0 < \infty$. □

### 4.5. The second intermediate step

**Proposition 4.8.** *Fix $\varepsilon_0 > 0$. Suppose that, for some scale $\ell$ and $0 < \varepsilon \le \varepsilon_0$,*

$$\inf_{x\in\mathbb{R}^d} \mathbb{P}\{\Lambda_\ell(x) \text{ is } s\text{-SEL for } H_{\varepsilon,\omega}\} \ge 1 - \mathrm{e}^{-\ell^{\zeta}}. \tag{4.52}$$

*If $\ell$ is sufficiently large, then for $L = \ell^{\gamma}$*

$$\inf_{x\in\mathbb{R}^d} \mathbb{P}\{\Lambda_L(x) \text{ is } m_0\text{-localizing for } H_{\varepsilon,\omega}\} \ge 1 - \mathrm{e}^{-L^{\zeta}}, \tag{4.53}$$

*where*

$$m_0 \ge \tfrac{1}{8}L^{-(1-\tau+\frac{1-s}{\gamma})}. \tag{4.54}$$

*Proof.* We let $\mathcal{B}_N$, $\mathcal{S}_N$ and $\mathcal{E}_N$ as in the proof of Lemma 4.7. We proceed as in the proof of Lemma 4.7. Using (4.52), (2.2) and the fact that events on disjoint boxes are independent, we have

$$\begin{aligned}\mathbb{P}\{\mathcal{B}^c\} &\le \left(\tfrac{2L}{\ell}\right)^{(N+1)d} \mathrm{e}^{-(N+1)\ell^{\zeta}} \\ &= 2^{(N+1)d}\,\ell^{(\gamma-1)(N+1)d}\,\mathrm{e}^{-(N+1)\ell^{\zeta}} \\ &< \tfrac{1}{2}\mathrm{e}^{-\ell^{\gamma\zeta}} \\ &= \tfrac{1}{2}\mathrm{e}^{-L^{\zeta}},\end{aligned} \tag{4.55}$$

if $N+1 > \ell^{(\gamma-1)\zeta}$ and $\ell$ is sufficiently large. For this reason we take

$$N = N_\ell = \lfloor \ell^{(\gamma-1)\tilde{\xi}} \rfloor \implies \mathbb{P}\{\mathcal{B}^c_{N_\ell}\} \le \tfrac{1}{2}\mathrm{e}^{-L^{\zeta}} \quad \text{for all } \ell \text{ sufficiently large.}$$

Also, using (4.44) and (4.10),

$$\mathbb{P}\{\mathcal{S}^c_N\} \le Y_{\varepsilon_0}(1+(L+1)^d N_\ell!(d4^d)^{N|\ell-1})(L+1)^{2d}\mathrm{e}^{-(2\alpha-1)L^{\beta}} < \tfrac{1}{2}\mathrm{e}^{-L^{\zeta}} \tag{4.56}$$

for sufficiently large $L$, since $(\gamma-1)\tilde{\xi} < (\gamma-1)\beta < \gamma\beta$ and $\zeta < \beta$. Combining (4.55) and (4.56), we conclude that

$$\mathbb{P}\{\mathcal{E}_N\} > 1 - \mathrm{e}^{-L^{\zeta}}.$$

To finish the proof we need to show that for all $\omega \in \mathcal{E}_N$ the box $\Lambda_L$ is $m_0$-localizing for $H_{\varepsilon,\omega}$, where $m_0$ is given in (4.54). Following the proof of Lemma 4.7, we get $\sigma(H_{\Lambda_L}) = \sigma_{\mathcal{G}}(H_{\Lambda_L}) \cup \sigma_{\mathcal{B}}(H_{\Lambda_L})$ and obtain an eigensystem $\{(\psi_x,\lambda_x)\}_{x\in\Lambda_L}$ for $H_{\Lambda_L}$. To finish the proof we need to show that $\{(\psi_x,\lambda_x)\}_{x\in\Lambda_L}$ is an $m_0$-localized eigensystem for $\Lambda_L$. We proceed as in the proof of Lemma 4.7. We fix $N=1$, $x\in\Lambda_L$, and take $y\in\Lambda_L$ such that $\|y-x\| \ge L_\tau$. We have

$$n(\ell)(\ell+1) + \sum_{r=1}^{R} \operatorname{diam} \Upsilon_r + 2\ell \ge L_\tau.$$

where $n(\ell)$ is the number of times we used (4.47). Thus, recalling $N = \lfloor \ell^{(\gamma-1)\tilde{\xi}} \rfloor$ and using (4.8),

$$n(\ell) \ge \tfrac{1}{\ell+1}(L_\tau - 5\ell\lfloor \ell^{(\gamma-1)\tilde{\xi}} \rfloor - 2\ell) \ge \tfrac{\ell}{\ell+1}\left(\tfrac{1}{2}\ell^{\gamma\tau-1} - 5\ell^{(\gamma-1)\tilde{\xi}} - 2\right) \ge \tfrac{1}{4}\ell^{\gamma\tau-1}.$$

for sufficiently large $\ell$ since $(\gamma-1)\tilde{\xi}+1 < \gamma\tau$. It follows from (4.49),

$$|\psi_x(y)| \le (\mathrm{e}^{-c_2\ell^{s}})^{\frac{1}{4}\ell^{\gamma\tau-1}} \le \mathrm{e}^{-\frac{1}{8}L^{-(1-\tau+\frac{1-s}{\gamma})}\|y-x\|}$$

for sufficiently large $\ell$.

We conclude that $\{(\psi_x,\lambda_x)\}_{x\in\Lambda_L}$ is an $m_0$-localized eigensystem for $\Lambda_L$, where $m_0$ is given in (4.54), so the box $\Lambda_L$ is $m_0$-localizing for $H_{\varepsilon,\omega}$. □

### 4.6. The fourth multiscale analysis

**Proposition 4.9.** *Fix $\varepsilon_0 > 0$. There exists a finite scale $\mathcal{L}(\varepsilon_0)$ with the following property: Suppose for some scale $L_0 \geq \mathcal{L}(\varepsilon_0)$, $0 < \varepsilon \leq \varepsilon_0$, and $m_0 \geq L_0^{-\kappa}$, where $0 < \kappa < \tau - \gamma\beta$, we have*

$$\inf_{x\in\mathbb{R}^d} \mathbb{P}\{\Lambda_{L_0}(x) \text{ is } m_0\text{-localizing for } H_{\varepsilon,\omega}\} \geq 1 - \mathrm{e}^{-L_0^{\zeta}}. \tag{4.57}$$

*Then, setting $L_{k+1} = L_k^{\gamma}$ for $k = 0, 1, \ldots,$*

$$\inf_{x\in\mathbb{R}^d} \mathbb{P}\{\Lambda_{L_k}(x) \text{ is } \tfrac{m_0}{2}\text{-localizing for } H_{\varepsilon,\omega}\} \geq 1 - \mathrm{e}^{-L_k^{\zeta}} \quad \text{for } k = 0, 1, \ldots.$$

*Moreover,*

$$\inf_{x\in\mathbb{R}^d} \mathbb{P}\{\Lambda_{L_k}(x) \text{ is } \tfrac{m_0}{4}\text{-localizing for } H_{\varepsilon,\omega}\} \geq 1 - \mathrm{e}^{-L_k^{\xi}} \quad \text{for all } L \geq L_0^{\gamma}. \tag{4.58}$$

**Lemma 4.10.** *Fix $\varepsilon_0 > 0$. Suppose for some scale $\ell$, $0 < \varepsilon \leq \varepsilon_0$, and $m \geq \ell^{-\kappa}$, where $0 < \kappa < \tau - \gamma\beta$, we have*

$$\inf_{x\in\mathbb{R}^d} \mathbb{P}\{\Lambda_{\ell}(x) \text{ is } m\text{-localizing for } H_{\varepsilon,\omega}\} \geq 1 - \mathrm{e}^{-\ell^{\zeta}}.$$

*Then, if $\ell$ is sufficiently large, for $L = \ell^{\gamma}$*

$$\inf_{x\in\mathbb{R}^d} \mathbb{P}\{\Lambda_{L}(x) \text{ is } M\text{-localizing for } H_{\varepsilon,\omega}\} \geq 1 - \mathrm{e}^{-L^{\zeta}},$$

*where*

$$M \geq m(1 - C_{d,\varepsilon_0}\ell^{-\min\{\frac{1-\tau}{2}, \gamma\tau-(\gamma-1)\tilde{\xi}-1, \tau-\gamma\beta-\kappa\}}) \geq \tfrac{1}{L^{\kappa}}.$$

Lemma (4.10) and Proposition (4.9) follow from [10, Lemma 4.5], [10, Proposition 4.3], and [10, Section 4.3]. (Note that in [10], they assume $m \geq m_-$ for a fixed $m_-$. However, all the results still hold when $m \geq \ell^{-\kappa}, 0 < \kappa < \tau - \gamma\beta$. (See the Lemmas for $\sharp$ being LOC in Sections 2.2 and 2.3.))

**4.7. The proof of the bootstrap multiscale analysis.** To prove Theorem 1.6, first we assume (1.6), which is the same as (4.1) with letting $Y = 400$, for some length scales. We apply Proposition 4.1, obtaining a sequence of length scales satisfying (4.2). Therefore (4.22) is satisfied for some length scales. Applying Proposition 4.3, we get a length scale satisfying (4.23). It follows that (4.29) is satisfied since $0 < 1 - \tau + \frac{1}{\gamma_1} < \tau$. We apply Proposition 4.4, obtaining a sequence of length scales satisfying (4.30). Therefore, In view of Remark 1.5, (4.39) is satisfied with letting $Y = 400^{\frac{1}{1-s}}$. We apply Proposition 4.6, obtaining a sequence of length scales satisfying (4.40). Therefore (4.52) is satisfied for some length scales. Applying Proposition 4.8, we get a length scale satisfying (4.53). It follows that (4.57) is satisfied since $0 < 1 - \tau + \frac{1-s}{\gamma} < \tau - \gamma\beta$. We apply Proposition 4.9, getting (4.58), so (1.6) holds.

## 5. The initial step for the bootstrap multiscale analysis

Theorem 1.7 is an immediate consequence of Theorem 1.6 and Proposition 5.1.

**Proposition 5.1.** *Given $q > \frac{2d}{\alpha}$ and $\varepsilon > 0$, set*

$$\theta_{\varepsilon,L} = \frac{\left\lfloor \frac{L}{20} \right\rfloor}{\log L} \log\left(1 + \tfrac{L^{-q}}{2d\varepsilon}\right). \tag{5.1}$$

*Then*

$$\begin{aligned} &\inf_{x\in\mathbb{R}^d} \mathbb{P}\{\Lambda_L(x) \text{ is } \theta_{\varepsilon,L}\text{-polynomially localizing for } H_{\varepsilon,\omega}\} \\ &\geq 1 - \tfrac{1}{2} K (L+1)^{2d} \left(8d\varepsilon + 2L^{-q}\right)^{\alpha}. \end{aligned} \tag{5.2}$$

*In particular, given $\theta > 0$ and $P_0 > 0$, there exists a finite scale $\mathcal{L}(q,\theta,P_0)$ such that for all $L \geq \mathcal{L}(q,\theta,P_0)$ and $0 < \varepsilon \leq \frac{1}{4d} L^{-q}$,*

$$\inf_{x\in\mathbb{R}^d} \mathbb{P}\{\Lambda_L(x) \text{ is } \theta\text{-polynomially localizing for } H_{\varepsilon,\omega}\} \geq 1 - P_0.$$

Proposition 5.1 shows that the starting hypothesis for the bootstrap multiscale analysis of Theorem 1.6 can be fulfilled .

To prove Proposition 5.1, we will use the following lemma given in [10, Lemma 4.4].

**Lemma 5.2** ([10, Lemma 4.4]). *Let $H_\varepsilon = -\varepsilon\Delta + V$ on $\ell^2(\mathbb{Z}^d)$, where $V$ is a bounded potential and $\varepsilon > 0$. Let $\Theta \subset \mathbb{Z}^d$, and suppose there is $\eta > 0$ such that*

$$|V(x) - V(y)| \geq \eta \quad \text{for all } x, y \in \Theta, x \neq y.$$

*Then for $\varepsilon < \frac{\eta}{4d}$ the operator $H_{\varepsilon,\Theta}$ has an eigensystem $\{(\psi_x, \lambda_x)\}_{x\in\Theta}$ such that*

$$|\lambda_x - \lambda_y| \geq \eta - 4d\varepsilon > 0 \quad \text{for all } x, y \in \Theta, x \neq y, \tag{5.3}$$

*and for all $y \in \Theta$ we have*

$$|\psi_y(x)| \leq \left(\tfrac{2d\varepsilon}{\eta - 2d\varepsilon}\right)^{|x-y|_1} \quad \text{for all } x \in \Theta. \tag{5.4}$$

*Proof of Proposition* 5.1. Let $\varepsilon > 0$ and $\Lambda_L = \Lambda_L(x_0)$ for some $x_0 \in \mathbb{R}^d$. Let $\eta = 4d\varepsilon + L^{-q}$ and suppose

$$|V(x) - V(y)| \geq \eta \quad \text{for all } x, y \in \Theta, x \neq y. \tag{5.5}$$

It follows from Lemma 5.2 that $H_{\varepsilon,\Lambda_L}$ has an eigensystem $\{(\psi_x, \lambda_x)\}_{x \in \Lambda_L}$ satisfying (5.3) and (5.4). We conclude from (5.3) that $\Lambda_L$ is polynomially level spacing for $H_\varepsilon$. Moreover, using (5.4) and $\|x\| \leq |x|_1$, for all $y, x \in \Lambda_L$ with $\|x - y\| \geq L'$ we have

$$\begin{aligned} |\psi_y(x)| &\leq \left(\tfrac{2d\varepsilon}{\eta - 2d\varepsilon}\right)^{\|x-y\|} \\ &= L^{-\frac{\|x-y\|}{\log L}\log\left(\frac{\eta-2d\varepsilon}{2d\varepsilon}\right)} \\ &= L^{-\frac{\|x-y\|}{\log L}\log\left(1+\frac{L^{-q}}{2d\varepsilon}\right)} \\ &\leq L^{-\theta_{\varepsilon,L}} \end{aligned}$$

with $\theta_{\varepsilon,L}$ as in (5.1). Therefore $\Lambda_L(x)$ is $\theta$-polynomially localizing.

We have

$$\begin{aligned} &\mathbb{P}\{\Lambda_L \text{ is not } \theta_{\varepsilon,L}\text{-polynomially localizing}\} \leq \mathbb{P}\{(5.5) \text{ does not hold}\} \\ &\quad \leq \tfrac{(L+1)^{2d}}{2} S_\mu(2(4d\varepsilon + L^{-q})) \\ &\quad \leq \tfrac{1}{2} K(L+1)^{2d}(8d\varepsilon + 2L^{-q})^\alpha, \end{aligned}$$

which yields (5.2). (We assumed $8d\varepsilon + 2Ł^{-q} \leq 1$; if not (5.2) holds trivially.)

If $0 < \varepsilon \leq \frac{1}{4d} L^{-q}$, for sufficiently large $L$ we have $\theta_{\varepsilon,L} \geq \theta$, and

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_L(x) \text{ is } \theta\text{-polynomially localizing for } H_{\varepsilon,\omega}\} \geq 1 - P_0,$$

since $\alpha q - 2d > 0$. □

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