Uniform distribution of eigenstates
on a torus with two point scatterers

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Abstract. We study the Laplacian perturbed by two delta potentials on a two-dimensional flat torus. There are two types of eigenfunctions for this operator: old, or unperturbed eigenfunctions which are eigenfunctions of the standard Laplacian, and new, perturbed eigenfunctions which are affected by the scatterers. We prove that along a density one sequence, the new eigenfunctions are uniformly distributed in configuration space, provided that the difference of the scattering points is Diophantine.

1. Introduction

1.1. Toral point scatterers. In the field of Quantum Chaos, one of the fundamental questions concerns the quantum ergodicity of a quantum system, i.e., equidistribution of almost all eigenstates of the system in the high energy limit. A key result is Shnirelman's quantum ergodicity theorem [12, 3, 19], which asserts that a quantum system whose classical counterpart has ergodic dynamics is quantum ergodic. On the other hand, there are quantum systems whose classical counterpart has integrable dynamics, for which the eigenstates tend to localize ("scar") in phase space.

A point scatterer on a flat torus is a popular model to study the transition between integrable and chaotic systems. Formally, it is defined as a rank one perturbation of the Laplacian, namely

$$-\Delta + \alpha \langle \delta_{x_0}, \cdot \rangle \delta_{x_0} \tag{1.1}$$

where $\alpha \in \mathbb{R}$ is a coupling parameter, and x_0 is the scattering point. It is an intermediate model, in the sense that the delta potential at x_0 does not change

the (integrable) classical dynamics of the system except for a measure zero set of trajectories, whereas it has a chaotic influence on the behaviour of the quantum system.

A standard way to rigorously define a point scatterer is via the theory of self-adjoint extensions, as described in depth in [2]. One defines the operator (1.1) as a self-adjoint extension of the Laplacian vanishing near the point x_0 (there are non-trivial self-adjoint extensions only in dimensions $d \leq 3$); such extensions are parametrized by a phase $\phi \in (-\pi, \pi]$, where $\phi = \pi$ corresponds to $\alpha = 0$ in (1.1), i.e., to the standard Laplacian. Consider the other, non-trivial extensions. Their eigenfunctions can be split into eigenfunctions of the standard Laplacian, referred to as the old, or unperturbed eigenfunctions, as well as new, or perturbed eigenfunctions which are affected by the presence of the scatterer, and therefore are the main object of study.

The semiclassical limits for the perturbed eigenfunctions of a point scatterer on flat tori have been studied extensively in recent years (see [13] for a survey on some of the results). Rudnick and Ueberschär proved uniform distribution in configuration space of the perturbed eigenfunctions for a point scatterer on two-dimensional flat tori [9]. This was also proved for three-dimensional flat tori [17], both on the standard square torus and on irrational tori with a Diophantine condition on the side lengths, where in the former case of the standard torus *all* of the perturbed eigenfunctions equidistribute in configuration space. As for quantum ergodicity in full phase space, it was proved both on the standard two-dimensional flat torus [6] and on the standard three-dimensional torus [18].

Scarring behavior has also been studied in several settings. Kurlberg and Ueberschär showed [7] that for an irrational two-dimensional torus (also known as the "Šeba billiard" as introduced in [10]) with a Diophantine condition on the side lengths, quantum ergodicity does not hold in full phase space; in fact, almost all new eigenfunctions strongly localize in momentum space. More recently, Kurlberg and Rosenzweig studied scarring behaviour on standard tori both in two and three dimensions [5].

1.2. Two point scatterers.
Recently, Ueberschär raised the natural question of the behavior of a system with several scatterers [14, 15, 16]. For a standard torus with n i.i.d uniform random scatterers, he showed [15] that uniform distribution in configuration space of almost all of the perturbed eigenfunctions holds with probability $\gg 1/n$. Our goal in this paper is to prove a deterministic result for two point scatterers on the torus.

Interestingly, our techniques do not generalize to the case of three or more scatterers, for which the symmetries that we exploit fail to hold. Indeed, it seems that even one additional (third) scatterer significantly complicates the nature of the system, so deterministic results for three or more scatterers require additional arguments. As an example, note that even with the presence of only a few scatterers, some unique phenomena occur, such as Laplace eigenspaces of dimensions smaller than the number of scatterers (this can occur for all eigenspaces, e.g. for irrational tori). Additional arguments are also required in order to extend our equidistribution results to phase space (as in [6]) or to study scarring behaviour for systems with multiple scatterers.

For the clarity of the paper, we will not work in the most general setting. Here we consider the two-dimensional standard flat torus $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$ with two scatterers at the points $x_1, x_2 \in \mathbb{T}^2$, whose normalized difference $(x_2 - x_1)/\pi$ is Diophantine. Our results can be easily generalized to non-square tori with a Diophantine condition on the difference of the scatterers – see Theorem 1.5 below. In addition, using the methods of [17], Theorem 1.3 can be extended to the standard three-dimensional torus, and also to irrational tori with the same Diophantine condition on the side lengths as in [17].

To give a more detailed account of our results, recall the definition of a Diophantine vector.

Definition 1.1. A vector $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ is said to be Diophantine of type κ, if there exists a constant $C > 0$ such that

$$\max_{j=1,2} \left| \alpha_j - \frac{p_j}{q} \right| > \frac{C}{q^\kappa}$$

for all $p_1, p_2, q \in \mathbb{Z}, q > 0$. By Dirichlet's theorem, the smallest possible value for κ is $3/2$.

Let $x_0 = x_2 - x_1$, and assume that the vector x_0/π is Diophantine. Note that by Khinchin's theorem on Diophantine approximations, our assumption holds for almost all pairs x_1, x_2.

Consider the Laplacian perturbed by two delta potentials at x_1, x_2, which is formally defined as a symmetric rank two perturbation of the Laplacian, namely

$$-\Delta + \sum_{i,j=1}^{2} h_{ji} \langle \delta_{x_i}, \cdot \rangle \delta_{x_j} \tag{1.2}$$

where $H = \{h_{ij}\}_{i,j=1}^{2}$ is a Hermitian matrix, and $x_1, x_2 \in \mathbb{T}^2$ are the scattering points (in particular, the case where we assume no non-local interaction between

the scatterers corresponds to perturbations with a diagonal H, i.e.,

$$-\Delta + \alpha \langle \delta_{x_1}, \cdot \rangle \delta_{x_1} + \beta \langle \delta_{x_2}, \cdot \rangle \delta_{x_2}$$

where $\alpha, \beta \in \mathbb{R}$ are coupling constants). As in the case of a single scatterer, the formal operators (1.2) can be defined rigorously using the theory of self-adjoint extensions of the standard Laplacian vanishing at x_1, x_2. The self-adjoint extensions are parametrized by the unitary group $U(2)$. As we will see, the standard Laplacian is retrieved by the extension corresponding to the matrix $U = -I$, and the eigenvalues of the other extensions can be again divided into old or unperturbed eigenvalues, i.e., eigenvalues of the standard Laplacian, as well as a set of new, perturbed eigenvalues, which we denote by $\Lambda = \Lambda_U$.

To establish a link between the old and the new eigenvalues of a self-adjoint extension $-\Delta_U$, we define the "weak interlacing" property.

Definition 1.2. We say that a set $A \subseteq \mathbb{R}$ weakly interlaces with a set $B \subseteq \mathbb{R}$, if $A \cap B = \emptyset$, and there exists a constant $C > 0$ such that between any two elements of A there are at most C elements of B, and vice versa.

It is a general fact [1] that for n point scatterers (which are similarly defined via self-adjoint extensions), the difference between the spectral counting function of $-\Delta_U$ (with multiplicities) and the spectral counting function of the standard Laplacian is uniformly bounded by n. In Appendix A, we will see that for each $0 \neq \lambda \in \sigma(-\Delta)$, the dimension of the corresponding eigenspace of $-\Delta_U$ is equal to the dimension of the Laplace eigenspace minus $\operatorname{rank}(I + U)$. It follows that the set Λ of new eigenvalues weakly interlaces with the Laplace eigenvalues.

1.3. Statement of the main result. We now state our main result. Let Λ_0 be any set of real numbers which weakly interlaces with the Laplace eigenvalues. For $\lambda \in \Lambda_0$ and $(d_1, d_2) \neq (0, 0)$, let

$$G_\lambda(x) = G_\lambda(x; d_1, d_2) = d_1 G_\lambda(x, x_1) + d_2 G_\lambda(x, x_2)$$

be any non-zero superposition of the Green's functions

$$G_\lambda(x, x_j) = (\Delta + \lambda)^{-1} \delta_{x_j}(x) \quad j = 1, 2,$$

and let $g_\lambda = G_\lambda / \|G_\lambda\|_2$.

Theorem 1.3. *Let $x_0 = x_2 - x_1$, and assume that x_0/π is Diophantine. For any $\epsilon > 0$ and for any set $\Lambda_0 \subseteq \mathbb{R}$ which weakly interlaces with the Laplace eigenvalues, there exists a subset $\Lambda_\infty \subseteq \Lambda_0$ of density one so that for all observables $a \in C^\infty(\mathbb{T}^2)$,*

$$\int_{\mathbb{T}^2} a(x)|g_\lambda(x; d_1, d_2)|^2 \, dx = \frac{1}{4\pi^2} \int_{\mathbb{T}^2} a(x) \, dx + O_{a,\epsilon}(\lambda^{-1/8+\epsilon})$$

as $\lambda \to \infty$ along Λ_∞.

Let $-I \neq U \in U(2)$, and let $-\Delta_U$ the corresponding self-adjoint extension. For a new eigenvalue $\lambda \in \Lambda$, the corresponding eigenfunction is a superposition of the Green's functions $G_\lambda(x, x_j)$. Thus, given an orthonormal basis $\{\varphi_k\}$ for the subspace of the perturbed eigenfunctions, it follows from Theorem 1.3 and from the weak interlacing of Λ with the eigenvalues of $-\Delta$ that $\{\varphi_k\}$ is uniformly distributed in configuration space along a density one subsequence.

Corollary 1.4. *Let $x_0 = x_2 - x_1$, and assume that x_0/π is Diophantine. For any $-I \neq U \in U(2)$, let $\{\varphi_k\}$ be an orthonormal basis for the subspace of the perturbed eigenfunctions of $-\Delta_U$ with eigenvalues $\{\lambda_k\}$. For any $\epsilon > 0$, there exists a density one sequence $\{\lambda_{k_j}\}$ so that for all observables $a \in C^\infty(\mathbb{T}^2)$,*

$$\int_{\mathbb{T}^2} a(x)|\varphi_{k_j}(x)|^2 dx = \frac{1}{4\pi^2} \int_{\mathbb{T}^2} a(x) dx + O_{a,\epsilon}(\lambda_{k_j}^{-1/8+\epsilon})$$

as $j \to \infty$.

In particular, we improve on the result of Ueberschär [15, Theorem 1.1] for two scatterers, as in that case his result only gives the result for random x_1, x_2 in a set of positive, but not necessarily full measure. Our result is deterministic and applies for almost all x_1, x_2.

Note that the formulation of Theorem 1.3 is fairly general, and is independent of the self-adjoint extension U, which is advantageous since in the physics literature one often considers self-adjoint extensions which are not fixed but vary with λ. For a single scatterer, for example, there is a popular quantization condition known as the "strong coupling limit" where $\tan\frac{\phi}{2} \sim -C \log \lambda$ (see [11, 13]), in which phenomena such as level repulsion between the new eigenvalues are observed, as opposed to the "weak coupling limit" where the self-adjoint extension is fixed. In particular, it follows from Theorem 1.3 that uniform distribution in configuration space holds even if the self-adjoint extensions change with λ.

As stated above, the proof of Theorem 1.3 is easily generalized to non-square tori. For $a > 0$, define a lattice $\mathcal{L}_0 = \mathbb{Z}(1/a, 0) \oplus \mathbb{Z}(0, a)$ in $\mathbb{R}^2$, and let

$$\mathcal{L} = \{x \in \mathbb{R}^2 : \langle x, l \rangle \in \mathbb{Z}, \text{ for all } l \in \mathcal{L}_0\} = \mathbb{Z}(a, 0) \oplus \mathbb{Z}(0, 1/a)$$

be the dual lattice. Consider the torus $\mathbb{T}^2_{\mathcal{L}_0} = \mathbb{R}^2/2\pi\mathcal{L}_0$ with scattering points $x_1, x_2 \in \mathbb{T}^2_{\mathcal{L}_0}$, whose difference we denote by $x_2 - x_1 = (\alpha_1, \alpha_2)$, and assume that $(\alpha_1 a/\pi, \alpha_2/(\pi a))$ is Diophantine. Let Λ_0 be any set of real numbers which weakly interlaces with the Laplace eigenvalues, which are the norms $|\xi|^2$ of the elements $\xi \in \mathcal{L}$. For $\lambda \in \Lambda_0$ and $(d_1, d_2) \neq (0, 0)$, let

$$G_\lambda(x) = G_\lambda(x; d_1, d_2) = d_1 G_\lambda(x, x_1) + d_2 G_\lambda(x, x_2)$$

be any non-zero superposition of the Green's functions $(\Delta + \lambda)^{-1}\delta_{x_j}$ and let $g_\lambda = G_\lambda/\|G_\lambda\|_2$.

Theorem 1.5. *Let $x_2 - x_1 = (\alpha_1, \alpha_2)$, and assume that $(\alpha_1 a/\pi, \alpha_2/(\pi a))$ is Diophantine. There exists a constant $\gamma > 0$, such that for any $\epsilon > 0$ and for any set $\Lambda_0 \subseteq \mathbb{R}$ which weakly interlaces with the Laplace eigenvalues, there exists a subset $\Lambda_\infty \subseteq \Lambda_0$ of density one so that for all observables $a \in C^\infty(\mathbb{T}^2_{\mathcal{L}_0})$,*

$$\int_{\mathbb{T}^2_{\mathcal{L}_0}} a(x)|g_\lambda(x; d_1, d_2)|^2 \, dx = \frac{1}{4\pi^2} \int_{\mathbb{T}^2_{\mathcal{L}_0}} a(x) \, dx + O_{a,\epsilon}(\lambda_{k_j}^{-\gamma+\epsilon})$$

as $\lambda \to \infty$ along Λ_∞.

Remark 1.6. One can take $\gamma = (1 - 3\theta)/2$, where θ is the exponent in the remainder term in Weyl's law on the torus (see [15]). The best known exponent $\theta = 131/416$ is due to Huxley [4], so one can take $\gamma = 23/832$.

2. Two points scatterers on the torus

2.1. Self-adjoint extensions. Let $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$ be the standard two-dimensional flat torus. Let $x_1, x_2 \in \mathbb{T}^2$ two points on the torus, and denote the difference of x_1 and x_2 by $x_0 = x_2 - x_1 = (\alpha_1, \alpha_2)$. Recall that we assume that x_0/π is Diophantine. We rigorously define the Laplacian perturbed by potentials at x_1, x_2

using the theory of self-adjoint extensions of unbounded symmetric operators. We give here a brief summary of the procedure – a more general calculation for n scatterers can be found in [15], however in the case of two scatterers we are able to give a more explicit computation.

Let $D_0 = C_c^\infty(\mathbb{T}^2 \setminus \{x_1, x_2\})$ be the space of smooth functions supported away from the points x_1, x_2, and let $-\Delta_0 = -\Delta_{\upharpoonright D_0}$ be the Laplacian restricted to this domain. We realize the perturbed operator as a self-adjoint extension of the operator $-\Delta_0$. In fact, it can be shown that the deficiency indices of $-\Delta_0$ are $(2, 2)$, hence the self-adjoint extensions are parametrized by the unitary group $U(2)$.

For $\lambda \notin \sigma(-\Delta)$, let

$$G_\lambda(x, y) = (\Delta + \lambda)^{-1} \delta_y(x)$$

be the Green's function of the Laplacian on $\mathbb{T}^2$. In particular, it has the L^2-expansion

$$G_\lambda(x, y) = -\frac{1}{4\pi^2} \sum_{\xi \in \mathbb{Z}^2} \frac{e^{i\langle \xi, x-y \rangle}}{|\xi|^2 - \lambda}.$$

The deficiency subspaces of $-\Delta_0$, namely $\ker(\Delta_0^* \pm i)$ are spanned by

$$\{G_i(x, x_1), G_i(x, x_2)\}, \{G_{-i}(x, x_1), G_{-i}(x, x_2)\}.$$

Note that $G_{-i}(x, x_j) = \overline{G_i(x, x_j)}$, and that for $\lambda \in \mathbb{R}$, $G_\lambda(x, x_j)$ is real.

Let

$$c_1 = \|G_{\pm i}(x, x_j)\|_2^2 = \frac{1}{16\pi^4} \sum_{\xi \in \mathbb{Z}^2} \frac{1}{|\xi|^4 + 1},$$

$$c_2 = \int G_i(x, x_1) \overline{G_i(x, x_2)} \, dx$$

$$= \int G_{-i}(x, x_1) \overline{G_{-i}(x, x_2)} \, dx$$

$$= \frac{1}{16\pi^4} \sum_{\xi \in \mathbb{Z}^2} \frac{\cos(\langle \xi, x_0 \rangle)}{|\xi|^4 + 1}.$$

Thus, defining

$$\mathbb{G}_\lambda(x) = (G_\lambda(x, x_1), G_\lambda(x, x_2))$$

(for notational convenience we treat $\mathbb{G}_\lambda(x)$ as a vector with two coordinates) and

$$T = \begin{pmatrix} \frac{1}{\sqrt{c_1}} & 0 \\ -\frac{c_2}{\sqrt{c_1(c_1^2 - c_2^2)}} & \sqrt{\frac{c_1}{c_1^2 - c_2^2}} \end{pmatrix},$$

we get that $T\mathbb{G}_i(x)$ and $T\mathbb{G}_{-i}(x)$ form orthonormal bases for the deficiency spaces $\ker(\Delta_0^* \pm i)$.

Denote the self-adjoint extension of $-\Delta_0$ corresponding to $U \in U(2)$ by $-\Delta_U$. The domain of $-\Delta_U$ consists of the functions in the Sobolev space

$$H^2(\mathbb{T}^2 \setminus \{x_1, x_2\})$$

of the form

$$g(x) = f(x) + \langle v, T\mathbb{G}_i(x) \rangle + \langle v, UT\mathbb{G}_{-i}(x) \rangle \tag{2.1}$$

where $f \in H^2(\mathbb{T}^2)$ such that $f(x_1) = f(x_2) = 0$ and $v \in \mathbb{C}^2$.

We can also rewrite (2.1) as

$$g(x) = f(x) + \langle T^*(I + U^*)v, \operatorname{Re}\mathbb{G}_i(x) \rangle + i \langle T^*(I - U^*)v, \operatorname{Im}\mathbb{G}_i(x) \rangle.$$

Since $\operatorname{Im} G_i(x, x_j) \in H^2(\mathbb{T}^2)$, the extension $-\Delta_{-I}$ retrieves the standard Laplacian $-\Delta$ on $H^2(\mathbb{T}^2)$, and we have $g \in H^2(\mathbb{T}^2)$ if and only if $(I + U^*)v = 0$.

Another class of special extensions are $-\Delta_U$ where $\operatorname{rank}(I + U) = 1$. For these extensions, there is a non-zero v_0 (unique up to multiplication by a scalar) such that $(I + U^*)v_0 = 0$, and therefore for the choice $v = cv_0$, (2.1) reads

$$g(x) = f(x) + 2ic\langle T^*v_0, \operatorname{Im}\mathbb{G}_i(x) \rangle$$

so $g \in H^2(\mathbb{T}^2)$. Since

$$\operatorname{Im} G_i(x_1, x_1) = \operatorname{Im} G_i(x_2, x_2) = -4\pi^2 c_1, \tag{2.2}$$

$$\operatorname{Im} G_i(x_1, x_2) = \operatorname{Im} G_i(x_2, x_1) = -4\pi^2 c_2 \tag{2.3}$$

we see that $\langle T^*v_0, \operatorname{Im}\mathbb{G}_i(x) \rangle$ and therefore g do not vanish simultaneously at x_1, x_2. Thus, if $\operatorname{rank}(I + U) = 1$, then there exists $g \in \operatorname{Dom}(-\Delta_U)$ such that $g \in H^2(\mathbb{T}^2)$ with either $g(x_1) \neq 0$ or $g(x_2) \neq 0$, a phenomenon which does not occur in the case of a single scatterer.

2.2. Spectrum and eigenfunctions. The eigenvalues of $-\Delta_U$ for $U \neq -I$, and the corresponding eigenfunctions, fall into two kinds. First, there are the "old", or "unperturbed" eigenvalues, which are the eigenvalues of the standard Laplacian $-\Delta$ on $\mathbb{T}^2$, i.e., belong to the set $\mathcal{N}$ of integers which are representable as a sum of two squares. For each $0 \neq \lambda \in \sigma(-\Delta)$, we will see in Appendix A that every eigenfunction of $-\Delta_U$ with an eigenvalue λ is also an eigenfunction of $-\Delta$. From this we will deduce that the dimension of the corresponding eigenspace of $-\Delta_U$ is equal to the dimension of the Laplace eigenspace minus $\operatorname{rank}(I + U)$.

The second group of eigenvalues of $-\Delta_U$ will be referred to as the group of new, or perturbed eigenvalues. These are the eigenvalues that are affected by the scatterers, and therefore are the main object of our study. Denote the set of the perturbed eigenvalues of $-\Delta_U$ by $\Lambda = \Lambda_U$.

For $\lambda \in \Lambda$, the corresponding eigenfunction G_λ is of the form

$$G_\lambda(x) = f(x) + \langle T^*v, \mathbb{G}_i(x) \rangle + \langle (UT)^*v, \mathbb{G}_{-i}(x) \rangle$$

where $f(x_1) = f(x_2) = 0$ and $v \notin \mathrm{Ker}(I + U^*)$.

Since G_λ is an eigenvalue of $-\Delta_U$, it is also an eigenvalue of the adjoint operator $-\Delta_0^*$. In addition, we have $\Delta_0^* \mathbb{G}_{\pm i}(x, x_j) = \mp i \mathbb{G}_{\pm i}(x, x_j)$, so

$$0 = (\Delta_0^* + \lambda)G_\lambda = (\Delta + \lambda)f + (-i + \lambda)\langle T^*v, \mathbb{G}_i \rangle + (i + \lambda)\langle (UT)^*v, \mathbb{G}_{-i} \rangle \quad (2.4)$$

and after simplifying using the resolvent identity

$$\frac{\mp i + \lambda}{(\Delta + \lambda)(\Delta \pm i)} = \frac{-1}{\Delta + \lambda} + \frac{1}{\Delta \pm i}$$

we get

$$0 = f + \langle T^*v, \mathbb{G}_i - \mathbb{G}_\lambda \rangle + \langle (UT)^*v, \mathbb{G}_{-i} - \mathbb{G}_\lambda \rangle = f + \langle v, \mathbb{A}_\lambda \rangle$$

where

$$\mathbb{A}_\lambda(x) = T(\mathbb{G}_i - \mathbb{G}_\lambda)(x) + UT(\mathbb{G}_{-i} - \mathbb{G}_\lambda)(x).$$

Evaluating at $x = x_1, x_2$, we see that a necessary condition on λ being a new eigenvalue is that

$$\det(\mathbb{A}_\lambda(x_1), \mathbb{A}_\lambda(x_2)) = 0.$$

We remark that the condition is also sufficient, since if the determinant is zero, we can easily construct G_λ. Also note that

$$G_\lambda(x) = \langle T^*(I + U^*)v, \mathbb{G}_\lambda(x) \rangle,$$

so the perturbed eigenfunctions are linear combinations of the Green's functions $G_\lambda(x, x_j)$.

3. Uniform distribution in configuration space

3.1. Density one subsequence. Let Λ_0 be a set of real numbers which weakly interlaces with the Laplace eigenvalues. We first build a density one subsequence in Λ_0 along which we will be able to obtain a lower bound for the L^2-norm of G_λ.

Recall that $\mathbb{N}$ is the set of integers representable as a sum of two squares, i.e., the eigenvalues of $-\Delta$, and let $r_2(n)$ be the number of such representations. For any $\lambda \in \Lambda_0$, define

$$n_\lambda = \max\{n \in \mathbb{N} : n < \lambda\}.$$

Fix a small $\epsilon > 0$.

Lemma 3.1. *There exists a density one subsequence Λ_1 in Λ_0, such that for every $\lambda \in \Lambda_1$, we have $\lambda - n_\lambda \ll \lambda^\epsilon$.*

Proof. Denote $\mathbb{N} = \{n_1, n_2, \dots\}$. By Lemma 2.1 in [9], for a density one sequence $\mathbb{N}_1$ in $\mathbb{N}$ we have $n_{k+1} - n_k \ll n_k^\epsilon$. Let $\Lambda_1 = \{\lambda \in \Lambda_0 : n_\lambda \in \mathbb{N}_1\}$. The statement of the lemma then follows from the weak interlacing of Λ_0 with $\mathbb{N}$. $\qquad\square$

Lemma 3.2. *There exists a density one subsequence Λ_2 in Λ_0, such that for all $\lambda \in \Lambda_2$ and $\xi \in \mathbb{Z}^2$ such that $|\xi|^2 = n_\lambda$, we have $\max_{j=1,2} |\sin(\xi_j \alpha_j)| \gg \lambda^{-\epsilon}$.*

Proof. Denote the distance to the nearest integer by

$$\|t\| = \min_{n \in \mathbb{Z}} |t - n|.$$

Since $|\sin(\xi_j \alpha_j)| \asymp \|\xi_j \alpha_j / \pi\|$, it is enough to find a density one subsequence along which $\max_{j=1,2} \|\xi_j \alpha_j / \pi\| \gg \lambda^{-\epsilon}$.

Let κ be the type of x_0/π. Let

$$A = \{\xi \in \mathbb{Z}^2 : |\xi|^2 \leq X, \ \max_{j=1,2} \|\xi_j \alpha_j / \pi\| \leq X^{-\epsilon}\}.$$

Then by writing $\xi_1 = n, \xi_2 = n + h$, we have

$$\#A \leq \sum_{|h| \leq 2X^{1/2}} \#A_h$$

where

$$A_h = \{n \in \mathbb{Z} : |n| \leq X^{1/2}, \ \|n\alpha_1/\pi\| \leq X^{-\epsilon}, \ \|(n+h)\alpha_2/\pi\| \leq X^{-\epsilon}\}.$$

Fix $|h| \leq 2X^{1/2}$, and divide the interval $[-X^{1/2}, X^{1/2}]$ into subintervals of length $X^{\epsilon/(\kappa-1)}$, so the number of such intervals is $\asymp X^{1/2 - \epsilon/(\kappa-1)}$. For any $n \neq m$ which lie in one of these intervals, the distance between the points $(\|n\alpha_1/\pi\|, \|(n+h)\alpha_2/\pi\|)$ and $(\|m\alpha_1/\pi\|, \|(m+h)\alpha_2/\pi\|)$ is bounded from below by

$$\max_{j=1,2} \|(n-m)\alpha_j/\pi\| \gg \frac{1}{(n-m)^{\kappa-1}} \gg X^{-\epsilon},$$

so the number of points in each of the intervals belonging to A_h is bounded. Therefore $\#A_h \ll X^{1/2-\epsilon/(k-1)}$, and $\#A \ll X^{1-\epsilon/(k-1)}$. Moreover, it follows that

$$\#\{\xi \in \mathbb{Z}^2 : |\xi|^2 \leq X, \, \max_{j=1,2} \|\xi_j \alpha_j / \pi\| \leq (1 + |\xi|^2)^{-\epsilon}\} \ll X^{1-\epsilon/2(k-1)},$$

and thus the set

$$B = \{n \in \mathbb{N} : n \leq X \text{ such that there exists } \xi \in \mathbb{Z}^2 \text{ such that } |\xi|^2 = n \text{ and}$$
$$\max_{j=1,2} \|\xi_j \alpha_j / \pi\| \leq (1 + n)^{-\epsilon}\}$$

satisfies $\#B \ll X^{1-\epsilon/2(k-1)}$. On the other hand, since $r_2(n) \ll n^\eta$ for all $\eta > 0$, we have

$$\#\{n \in \mathbb{N} : n \leq X\} \gg X^{1-\eta}$$

(in fact, by Landau's theorem [8] we have $\#\{n \in \mathbb{N} : n \leq X\} \sim C \frac{x}{\sqrt{\log x}}$). Thus $\mathbb{N} \setminus B$ is a density one set in $\mathbb{N}$. Let $\Lambda_2 = \{\lambda \in \Lambda_0 : n_\lambda \in \mathbb{N} \setminus B\}$. The statement of the lemma again follows from the weak interlacing of Λ_0 with $\mathbb{N}$. $\qquad\square$

Finally, we define $\Lambda' = \Lambda_1 \cap \Lambda_2$ which is a density one set in Λ_0 by Lemmas 3.1 and 3.2.

3.2. Lower bound for the L^2-norm of G_λ. For $\lambda \in \Lambda'$, let

$$G_\lambda(x) = d_1 G_\lambda(x, x_1) + d_2 G_\lambda(x, x_2),$$

normalized such that $|d_1|^2 + |d_2|^2 = 1$. Assume without loss of generality that $|d_2|^2 \geq 1/2$.

We now give a lower bound for the L^2-norm of G_λ along $\lambda \in \Lambda'$.

Lemma 3.3. *For all $\lambda \in \Lambda'$, we have $\|G_\lambda\|_2^2 \gg \lambda^{-4\epsilon}$.*

Proof. Let $\lambda \in \Lambda'$. We have

$$\|G_\lambda\|_2^2 = \frac{1}{16\pi^4} \sum_{\xi \in \mathbb{Z}^2} \frac{|d_1 e^{i\langle\xi, x_1\rangle} + d_2 e^{i\langle\xi, x_2\rangle}|^2}{(|\xi|^2 - \lambda)^2}$$

$$= \frac{1}{16\pi^4} |d_2|^2 \sum_{\xi \in \mathbb{Z}^2} \frac{|d_1/d_2 + e^{i\langle\xi, x_0\rangle}|^2}{(|\xi|^2 - \lambda)^2}.$$

Choose $\xi = (\xi_1, \xi_2)$ such that $|\xi|^2 = n_\lambda$. From Lemmas 3.1 and 3.2, we have $\max\limits_{j=1,2} |\sin(\xi_j \alpha_j)| \gg \lambda^{-\epsilon}$ and $n_\lambda - \lambda \ll \lambda^\epsilon$. We can assume that $|\sin(\xi_1 \alpha_1)| \gg \lambda^{-\epsilon}$ (so in particular $\xi_1 \neq 0$). Thus

$$
\begin{aligned}
\|G_\lambda\|_2^2 &\gg \sum_{\xi \in \mathbb{Z}^2} \frac{|d_1/d_2 + e^{i\langle \xi, x_0 \rangle}|^2}{(|\xi|^2 - \lambda)^2} \\
&\gg \lambda^{-2\epsilon}(|d_1/d_2 + e^{i\langle (\xi_1, \xi_2), x_0 \rangle}|^2 + |d_1/d_2 + e^{i\langle (-\xi_1, \xi_2), x_0 \rangle}|^2) \\
&\gg \lambda^{-2\epsilon}|e^{i\langle (\xi_1, \xi_2), x_0 \rangle} - e^{i\langle (-\xi_1, \xi_2), x_0 \rangle}|^2 \asymp \lambda^{-2\epsilon} \sin^2(\xi_1 \alpha_1) \\
&\gg \lambda^{-4\epsilon}. \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square
\end{aligned}
$$

3.3. Truncation. Let $0 < \delta < 1/4$ and let $L = \lambda^\delta$. We define $G_{\lambda, L} = d_1 G_{\lambda, L}(x, x_1) + d_2 G_{\lambda, L}(x, x_2)$ where

$$
G_{\lambda, L}(x, x_j) = -\frac{1}{4\pi^2} \sum_{||\xi|^2 - \lambda| \leq L} \frac{e^{i\langle \xi, x - x_j \rangle}}{|\xi|^2 - \lambda}
$$

is the truncated Green's function. Denote

$$
g_\lambda = \frac{G_\lambda}{\|G_\lambda\|_2}, \quad g_{\lambda, L} = \frac{G_{\lambda, L}}{\|G_{\lambda, L}\|_2}.
$$

Lemma 3.4. *For all $\lambda \in \Lambda'$ we have $\|g_\lambda - g_{\lambda, L}\|_2^2 \ll \lambda^{5\epsilon}/L$.*

Proof. We have

$$
\|g_\lambda - g_{\lambda, L}\|_2^2 \leq 4 \frac{\|G_\lambda - G_{\lambda, L}\|_2^2}{\|G_\lambda\|_2^2} \ll \lambda^{4\epsilon} \|G_\lambda - G_{\lambda, L}\|_2^2.
$$

But

$$
\begin{aligned}
\|G_\lambda - G_{\lambda, L}\|_2^2 &= \frac{1}{16\pi^4} \sum_{||\xi|^2 - \lambda| > L} \frac{|d_1 e^{i\langle \xi, x_1 \rangle} + d_2 e^{i\langle \xi, x_2 \rangle}|^2}{(|\xi|^2 - \lambda)^2} \\
&\ll \sum_{||\xi|^2 - \lambda| > L} \frac{1}{(|\xi|^2 - \lambda)^2} \\
&\ll \sum_{|n - \lambda| > L} \frac{n^\epsilon}{(n - \lambda)^2} \\
&\ll \lambda^\epsilon/L. \qquad\qquad\qquad\qquad\qquad\qquad\square
\end{aligned}
$$

For all $a \in C^\infty(\mathbb{T}^2)$, we have (see [9])

$$|\langle ag_\lambda, g_\lambda \rangle - \langle ag_{\lambda,L}, g_{\lambda,L} \rangle| \le 2\|a\|_\infty \|g_\lambda - g_{\lambda,L}\|_2.$$

Thus,

$$\langle ag_\lambda, g_\lambda \rangle = \langle ag_{\lambda,L}, g_{\lambda,L} \rangle + O(\lambda^{(5\epsilon-\delta)/2}).$$

Taking $\delta = 1/4 - \epsilon$, we see that in order to prove Theorem 1.3, it is enough to find a density one sequence Λ_∞ in Λ_0 so that for all $a \in C^\infty(\mathbb{T}^2)$ and all $M > 0$,

$$\langle ag_{\lambda,L}, g_{\lambda,L} \rangle = \frac{1}{4\pi^2} \int_{\mathbb{T}^2} a(x)\,\mathrm{d}x + O(\lambda^{-M}) \qquad (3.1)$$

as $\lambda \to \infty$ along Λ_∞.

3.4. Proof of Theorem 1.3.

We first show that for every fixed $\zeta \in \mathbb{Z}^2 \setminus \{(0,0)\}$

$$\langle e^{i\langle \zeta, x \rangle} g_{\lambda,L}, g_{\lambda,L} \rangle = 0$$

along a density one subsequence.

Let

$$S_\zeta = \{\xi \in \mathbb{Z}^2 \colon |\langle \xi, \zeta \rangle| \le 2|\xi|^{2\delta}\},$$

and let

$$\Lambda_\zeta = \{\lambda \in \Lambda' \colon \text{for all } \xi \in S_\zeta. \, ||\xi|^2 - \lambda| > L\}.$$

Lemma 3.5. Λ_ζ *is a density one set in* Λ_0.

Proof. We follow the proof of Proposition 6.1 in [9]. Write $\zeta = (p, q), \zeta^\perp = (-q, p)$. Then every $\xi \in S_\zeta$ can be written as $\xi = u\frac{\zeta}{|\zeta|} + v\frac{\zeta^\perp}{|\zeta^\perp|}$, and therefore the set of lattice points $\{\xi \in S_\zeta \colon |\xi|^2 \le X\}$ is contained in the rectangle

$$\left\{ u\frac{\zeta}{|\zeta|} + v\frac{\zeta^\perp}{|\zeta^\perp|} \colon u \le \frac{2X^\delta}{|\zeta|}, \, v \le \sqrt{X} \right\}.$$

Since the number of lattice points inside a rectangle is bounded (up to a constant) by the area of the rectangle, we see that

$$\#\{\xi \in S_\zeta \colon |\xi|^2 \le X\} \ll \frac{X^{1/2+\delta}}{|\zeta|}.$$

Let $\mathcal{N} \subseteq \mathbb{N}$ be the set of norms $|\xi|^2$ in S_ζ. Define a map $\phi \colon \Lambda' \setminus \Lambda_\zeta \to \mathcal{N}_\zeta$ which takes $\lambda \in \Lambda' \setminus \Lambda_\zeta$ to the closest element $n \in \mathcal{N}$ to λ (if there are two elements with the same distance take the smallest of them). For every $\mathcal{N}_\zeta$ we have

$$\#\phi^{-1}(n) \le \#\{\lambda \in \Lambda' \setminus \Lambda_\zeta \colon \text{there exists } \xi \in S_\zeta$$
$$\text{such that } |\xi|^2 = n, |n - \lambda| \le L\} \ll n^\delta.$$

Thus,

$$\#\{\lambda \in \Lambda' \setminus \Lambda_\zeta : \lambda \leq X\} \leq \sum_{\substack{n \in \mathcal{N}_\zeta \\ n \leq 2X}} \#\phi^{-1}(n) \ll X^\delta \#\{n \in \mathcal{N}_\zeta : n \leq 2X\}$$

$$\leq X^\delta \#\{\xi \in S_\zeta : |\xi|^2 \leq 2X\}$$

$$\ll \frac{X^{1/2+2\delta}}{|\zeta|},$$

so Λ_ζ is a density one set in Λ_0 (since $\delta < 1/4$). $\qquad\square$

Lemma 3.6. *For all $\lambda \in \Lambda_\zeta$ such that $\lambda^\delta \gg |\zeta|^2$,*

$$||\xi|^2 - \lambda| \leq L \implies ||\xi + \zeta|^2 - \lambda| > L.$$

Proof. For all $\lambda \in \Lambda_\zeta$ such that $\lambda^\delta \gg |\zeta|^2$, if $||\xi|^2 - \lambda| \leq L$, then $\xi \notin S_\zeta$, i.e., $|\langle \xi, \zeta \rangle| > 2|\xi|^{2\delta}$, and therefore

$$||\xi + \zeta|^2 - \lambda| \geq 2|\langle \xi, \zeta \rangle| - ||\xi|^2 - \lambda| - |\zeta|^2 > L. \qquad\square$$

Proof of Theorem 1.3. We have

$$\langle e^{i\langle \zeta, x \rangle} g_{\lambda,L}, g_{\lambda,L} \rangle = \frac{\langle e^{i\langle \zeta, x \rangle} G_{\lambda,L}, G_{\lambda,L} \rangle}{\|G_{\lambda,L}\|_2^2}.$$

Denoting $c(\xi) = d_1 e^{-i\langle \xi, x_1 \rangle} + d_2 e^{-i\langle \xi, x_2 \rangle}$, note that

$$\langle e^{i\langle \zeta, x \rangle} G_{\lambda,L}, G_{\lambda,L} \rangle = \frac{16}{\pi^4} \sum_{\substack{||\xi|^2 - \lambda| \leq L \\ ||\xi + \zeta|^2 - \lambda| \leq L}} \frac{c(\xi)\overline{c(\xi + \zeta)}}{(|\xi|^2 - \lambda)(|\xi + \zeta|^2 - \lambda)}.$$

However by Lemma 3.6, the last sum is empty along $\lambda \in \Lambda_\zeta$ such that $\lambda^\delta \gg |\zeta|^2$, so along this sequence $\langle e^{i\langle \zeta, x \rangle} g_{\lambda,L}, g_{\lambda,L} \rangle = 0$.

We conclude (3.1) by an argument which can be found in [15]. We expand a into a Fourier series:

$$a(x) = \sum_{\zeta \in \mathbb{Z}^2} \hat{a}(\zeta) e^{i\langle \zeta, x \rangle}.$$

By the rapid decay of the Fourier coefficients $\hat{a}(\zeta)$, we see that for any $M > 0$

$$a(x) = \sum_{|\zeta| \leq \lambda^\epsilon} \hat{a}(\zeta) e^{i\langle \zeta, x \rangle} + O(\lambda^{-M}).$$

Define

$$\Lambda_\infty = \{\lambda \in \Lambda' : \text{for all } |\zeta| \le \lambda^\epsilon. \lambda \in \Lambda_\zeta\}.$$

We have

$$\#\{\lambda \in \Lambda' \setminus \Lambda_\infty : \lambda \le X\} \le \sum_{|\zeta| \le X^\epsilon} \#\{\lambda \in \Lambda' \setminus \Lambda_\zeta : \lambda \le X\}$$

$$\ll X^{1/2+2\delta} \sum_{0 < |\zeta| \le X^\epsilon} \frac{1}{|\zeta|} \ll X^{1/2+2\delta+\epsilon},$$

so Λ_∞ is a density one set in Λ_0. Finally, for any $\lambda \in \Lambda_\infty$

$$\langle a g_{\lambda,L}, g_{\lambda,L} \rangle = \hat{a}(0) + \sum_{0 < |\zeta| \le \lambda^\epsilon} \hat{a}(\zeta) \langle e^{i\langle \zeta, x \rangle} g_{\lambda,L}, g_{\lambda,L} \rangle + O(\lambda^{-M})$$

$$= \hat{a}(0) + O(\lambda^{-M}),$$

since for each $|\zeta| \le \lambda^\epsilon$, $\langle e^{i\langle \zeta, x \rangle} g_{\lambda,L}, g_{\lambda,L} \rangle = 0$. Thus, the limit (3.1) holds along Λ_∞, and Theorem 1.3 follows. □

Appendix A.

We study the eigenspaces of a self-adjoint extension $-\Delta_U$ corresponding to old eigenvalues of $-\Delta$. Our goal is to show that their dimensions are equal to the dimensions of the eigenspaces of $-\Delta$ minus $\text{rank}(I + U)$. We first prove three auxiliary lemmas.

Lemma A.1. *Let* $0 \ne \lambda \in \sigma(-\Delta)$, *and let d be the dimension of the corresponding eigenspace* E_λ. *Then the dimension of the subspace*

$$\{f \in E_\lambda : f(x_1) = f(x_2) = 0\}$$

is equal to $d - 2$.

Proof. Since x_0/π is Diophantine, we can assume that $\alpha_1/\pi \notin \mathbb{Q}$. Fix $\xi = (\xi_1, \xi_2)$ such that $\xi_1 \ne 0$, $|\xi|^2 = \lambda$. The functions

$$\{e^{i\langle x-x_1, \eta \rangle} - e^{i\langle x-x_1, \xi \rangle}\}_{|\eta|^2 = \lambda, \eta \ne \xi}$$

form a basis for the subspace $\{f \in E_\lambda : f(x_1) = 0\}$. Choose $\eta = (-\xi_1, \xi_2)$, and let $g(x) = e^{i\langle x-x_1, \eta \rangle} - e^{i\langle x-x_1, \xi \rangle}$. Then

$$|g(x_2)| = 2|\sin(\xi_1 \alpha_1)| \ne 0$$

since $\alpha_1/\pi \notin \mathbb{Q}$, and therefore the functions

$$\{e^{i\langle x-x_1,\zeta\rangle} - e^{i\langle x-x_1,\xi\rangle} - \frac{g(x)}{g(x_2)}(e^{i\langle x_0,\zeta\rangle} - e^{i\langle x_0,\xi\rangle})\}_{|\zeta|^2=\lambda,\,\zeta\neq\xi,\eta}$$

form a basis for the subspace $\{f \in E_\lambda : f(x_1) = f(x_2) = 0\}$.   □

Recall that if $\mathrm{rank}(I + U) = 1$, v_0 is defined to be the (unique up to scalar) non-zero vector such that $(I + U^*)v_0 = 0$. We have the following property for $\langle T^*v_0, \mathrm{Im}\,\mathbb{G}_i(x)\rangle$.

Lemma A.2. *Assume that* $\mathrm{rank}(I + U) = 1$. *Then* $\langle T^*v_0, \mathrm{Im}\,\mathbb{G}_i(x)\rangle$ *is not an eigenfunction of* $-\Delta$.

Proof. Denote $T^*v_0 = (v_1, v_2)$, so

$$\langle T^*v_0, \mathrm{Im}\,\mathbb{G}_i(x)\rangle = -\frac{1}{4\pi^2}\sum_{\xi\in\mathbb{Z}^2}(v_1 e^{-i\langle\xi,x_1\rangle} + v_2 e^{-i\langle\xi,x_2\rangle})\frac{e^{i\langle\xi,x\rangle}}{|\xi|^4 + 1}.$$

Assume that $\langle T^*v_0, \mathrm{Im}\,\mathbb{G}_i(x)\rangle$ is an eigenfunction of $-\Delta$ with an eigenvalue λ. Then for all ξ such that $|\xi|^2 = m \neq \lambda$ we have

$$v_1 e^{-i\langle\xi,x_1\rangle} + v_2 e^{-i\langle\xi,x_2\rangle} = 0.$$

We can again assume that $\alpha_1/\pi \notin \mathbb{Q}$. Choosing any $\xi = (\xi_1, \xi_2)$ with $\xi_1 \neq 0$ and $|\xi|^2 = m \neq \lambda$, we get in particular that

$$\det\begin{pmatrix} 1 & e^{-i\langle(\xi_1,\xi_2),x_0\rangle} \\ 1 & e^{-i\langle(-\xi_1,\xi_2),x_0\rangle} \end{pmatrix} = 0,$$

however since $\alpha_1/\pi \notin \mathbb{Q}$, we have

$$\left|\begin{pmatrix} 1 & e^{-i\langle(\xi_1,\xi_2),x_0\rangle} \\ 1 & e^{-i\langle(-\xi_1,\xi_2),x_0\rangle} \end{pmatrix}\right| = 2|\sin(\xi_1\alpha_1)| \neq 0$$

a contradiction.   □

Lemma A.3. *Assume that* $\mathrm{rank}(I + U) = 1$. *Let* $0 \neq \lambda \in \sigma(-\Delta)$, *and let* d *be the dimension of the corresponding eigenspace* E_λ. *Then the dimension of the subspace*

$$\{g \in E_\lambda : g(x) = f(x) + c\langle T^*v_0, \mathrm{Im}\,\mathbb{G}_i(x)\rangle, f(x_1) = f(x_2) = 0, c \in \mathbb{C}\}$$

is equal to $d - 1$.

Proof. From Lemma A.2, $\langle T^* v_0, \operatorname{Im} \mathbb{G}_i(x) \rangle \notin E_\lambda$, and therefore

$$\dim(E_\lambda + \langle T^* v_0, \operatorname{Im} \mathbb{G}_i(x) \rangle) = d + 1.$$

The proof of the statement of the lemma now follows similarly to the proof of Lemma A.1. □

Proposition A.4. *Let* $0 \neq \lambda \in \sigma(-\Delta)$, *and assume that g is an eigenfunction of* $-\Delta_U$ *with an eigenvalue* λ. *Then g is an eigenvalue of* $-\Delta$.

Proof. Assume otherwise, so there exist $v \notin \operatorname{Ker}(I + U^*)$ and $f \in H^2(\mathbb{T}^2)$ with $f(x_1) = f(x_2) = 0$ such that

$$g(x) = f(x) + \langle T^* v, \mathbb{G}_i(x) \rangle + \langle (UT)^* v, \mathbb{G}_{-i}(x) \rangle$$

where g is an eigenvalue of $-\Delta_U$, and hence of the adjoint operator $-\Delta_0^*$. Thus, as in (2.4)

$$0 = (\Delta_0^* + \lambda) g = (\Delta + \lambda) f + (-i + \lambda) \langle T^* v, \mathbb{G}_i \rangle + (i + \lambda) \langle (UT)^* v, \mathbb{G}_{-i} \rangle.$$

Assume that $\alpha_1/\pi \notin \mathbb{Q}$, and let $\xi = (\xi_1, \xi_2)$ such that $\xi_1 \neq 0$ and $|\xi|^2 = \lambda$. Evaluating the Fourier coefficient at $(\pm \xi_1, \xi_2)$, we see that

$$\langle T^*(I + U^*) v, (1, e^{-i \langle (\xi_1, \xi_2), x_0 \rangle}) \rangle = 0$$

and

$$\langle T^*(I + U^*) v, (1, e^{-i \langle (-\xi_1, \xi_2), x_0 \rangle}) \rangle = 0.$$

Since $v \notin \operatorname{Ker}(I + U^*)$ it implies that

$$\left| \det \begin{pmatrix} 1 & e^{-i \langle (\xi_1, \xi_2), x_0 \rangle} \\ 1 & e^{-i \langle (-\xi_1, \xi_2), x_0 \rangle} \end{pmatrix} \right| = 0,$$

a contradiction. □

Corollary A.5. *Let* $0 \neq \lambda \in \sigma(-\Delta)$. *Let d be the dimension of the eigenspace of* $-\Delta$ *corresponding to* λ. *Then the dimension of the eigenspace of* $-\Delta_U$ *corresponding to* λ *is equal to* $d - \operatorname{rank}(I + U)$.

Proof. If rank$(I + U) = 2$ and g is an eigenvalue of $-\Delta_U$ with $0 \neq \lambda \in \sigma(-\Delta)$, then from the proof of Lemma A.4, we have

$$g(x) = f(x) + \langle T^*v, \mathbb{G}_i(x)\rangle + \langle (UT)^*v, \mathbb{G}_{-i}(x)\rangle$$

with $f(x_1) = f(x_2) = 0$ and $(I + U^*)v = 0$, and therefore $v = 0$, so $g \in H^2(\mathbb{T}^2)$ with $g(x_1) = g(x_2) = 0$, and the statement follows from Lemma A.1 .

If rank$(I + U) = 1$, then there exist $0 \neq v_0 \in \mathbb{C}^2$, so that

$$g(x) = f(x) + c\langle T^*v_0, \operatorname{Im}\mathbb{G}_i(x)\rangle$$

with $f(x_1) = f(x_2) = 0$, $c \in \mathbb{C}$. Thus, in this case the statement follows from Lemma A.3. $\qquad\square$

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