At the request of the authors, this article has been partially retracted.

Attached is a note from the authors explaining their reasons for partially retracting the article and the original article.

The Editors of the *Journal of Spectral Theory*

J. Spectr. Theory (online first), 1–2
DOI 10.4171/JST/353

Partial retraction of "Two-term, asymptotically sharp estimates for eigenvalue means of the Laplacian"

Evans M. Harrell II and Joachim Stubbe

We regret that we have to retract portions of the article "Two-term, asymptotically sharp estimates for eigenvalue means of the Laplacian" [J. Spectral Theory 8 (2018), 1529–1550] due to an essential error in the proof of Theorem 1.2, which is used in other places in the paper.

An error in the proof of Theorem 1.2 was pointed out to us by S. Larson. The proof relies on an average over certain translations, but the parameter L there cannot be chosen independently of the spectral parameter z in order to eliminate the remainder term called $G(z)$ in the proof. Since we have been unable to remedy the error and Theorem 1.2 is used throughout, we retract Theorem 1.2 and all claims depending on it.

Several salient claims of the paper do not depend on the erroneous averaging and remain unaffected. Before listing them we recall some definitions for the reader's convenience:

The eigenvalues of the Neumann Laplacian on a bounded domain Ω are denoted

$$0 = \mu_1 < \mu_2 \le \mu_3 \le \cdots, \tag{1.2}$$

and some related quantities that will appear are

$$m_k := C_d \Big(\frac{k}{|\Omega|}\Big)^{\frac{2}{d}}, \quad S_k := \frac{\frac{d+2}{d}\frac{1}{k}\sum_{j=1}^{k}\mu_j}{m_k}.$$

The "classical constant" is written $C_d = (2\pi)^2 B_d^{-\frac{2}{d}}$, where $B_d = \frac{\pi^{\frac{d}{2}}}{\Gamma(1+\frac{d}{2})}$ is the volume of the d-dimensional unit ball. Pólya's conjecture for Neumann domains reads

$$\mu_j \le C_d |\Omega|^{-\frac{2}{d}} (j-1)^{-\frac{2}{d}}. \tag{1.5}$$

Claims that remain valid include the following:

Theorem 1.1 (a sharpening of Kröger's inequality). *Let $d \geq 2$. Then for all $k \geq 0$ the Neumann eigenvalue μ_{k+1} satisfies*

$$m_k^2(1 - S_k) \geq (\mu_{k+1} - m_k)^2, \tag{1.13}$$

i.e.,

$$m_k(1 - \sqrt{1 - S_k}) \leq \mu_{k+1} \leq m_k(1 + \sqrt{1 - S_k}). \tag{1.14}$$

Corollary 1.3. *Let $d \geq 2$ and $\Omega = \Omega' \times [0, \delta]$ be a bounded domain. Then for all $z \geq 0$,*

$$\begin{aligned} &\sum_{j=1} (z - \mu_j)_+ \\ &\geq L_{1,d}^{cl} |\Omega| z^{\frac{d}{2}+1} + \frac{1}{2} L_{1,d-1}^{cl} \frac{|\Omega|}{\delta} z^{\frac{d}{2}+\frac{1}{2}} - \frac{1}{24} (2\pi)^{2-d} B_d \frac{|\Omega|}{\delta^2} z^{\frac{d}{2}}. \end{aligned} \tag{1.20}$$

The statement of Corollary 1.4 needs to drop a lower-order nonnegative contribution derived from Theorem 1.2. After correction, it reads:

Corollary 1.4 (Pólya's conjecture for Cartesian products). *Suppose that $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{R}^d$ where $\Omega_1 \subset \mathbb{R}^{d_1}$ and $\Omega_2 \subset \mathbb{R}^{d_2}$ are two bounded domains with spectra consisting of increasing eigenvalues satisfying eq.* (1.2)*, and where Pólya's conjecture* (1.5) *holds for Ω_1. Then*

$$\mathcal{N}(z) \geq 1 + |\Omega| L_{0,d}^{cl} z^{\frac{d}{2}}. \tag{1.22}$$

This implies Pólya's conjecture for Ω of the form $\Omega_1 \times \Omega_2$.

Section 3, containing detailed calculations for rectangles, and the Appendix, discussing refinements of Young's and Hölder's inequalities, are entirely independent of Theorem 1.2 and hence unaffected by the error.

Other parts of Section 1 aside from those listed above and Lemma 1.5 (which is from an earlier work), as well as Section 2 and Section 4, can be disregarded.

Received May 6, 2021

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J. Spectr. Theory 8 (2018), 1529–1550
DOI 10.4171/JST/234

Journal of Spectral Theory

Two-term, asymptotically sharp estimates for eigenvalue means of the Laplacian

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Abstract. We present asymptotically sharp inequalities for the eigenvalues μ_k of the Laplacian on a domain with Neumann boundary conditions, using the averaged variational principle introduced in [14]. For the Riesz mean $R_1(z)$ of the eigenvalues we improve the known sharp semiclassical bound in terms of the volume of the domain with a second term with the best possible expected power of z.

In addition, we obtain two-sided bounds for individual μ_k, which are semiclassically sharp, and we obtain a Neumann version of Laptev's result that the Pólya conjecture is valid for domains that are Cartesian products of a generic domain with one for which Pólya's conjecture holds. In a final section, we remark upon the Dirichlet case with the same methods.

Mathematics Subject Classification (2010). 58J50, 47F05, 47A75.

Keywords. Neumann Laplacian, Dirichlet Laplcian, semiclassical bounds for eigenvalues.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with boundary $\partial\Omega$. We mainly consider here the eigenvalue problem for the Laplacian with Neumann boundary conditions,

$$\begin{aligned} -\Delta u &= \mu u \quad \text{on } \Omega, \\ \frac{\partial u}{\partial n} &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

We suppose that the spectrum (1.1) consists of an ordered sequence of eigenvalues μ_j tending to infinity,

$$0 = \mu_1 < \mu_2 \leq \mu_3 \leq \dots. \tag{1.2}$$

The corresponding normalized eigenfunctions are denoted u_j. This assumption holds when Ω satisfies some regularity assumptions, see e.g. [25], and is different from the situation for the Dirichlet Laplacian which admits a spectrum consisting of an strictly positive eigenvalues $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$ for any bounded domain whatsoever. Neumann eigenvalues satisfy the same Weyl asymptotic relation as the better-studied Dirichlet eigenvalues, *viz.*,

$$\lim_{j\to\infty} \mu_j j^{-\frac{2}{d}} = \lim_{j\to\infty} \lambda_j j^{-\frac{2}{d}} = C_d |\Omega|^{-\frac{2}{d}}, \tag{1.3}$$

where $|\Omega|$ denotes the volume of Ω and the "classical constant" C_d is given by

$$C_d = (2\pi)^2 B_d^{-\frac{2}{d}}, \tag{1.4}$$

where $B_d = \dfrac{\pi^{\frac{d}{2}}}{\Gamma(1+\frac{d}{2})}$ is the volume of the d-dimensional unit ball. An important question in the spectral theory of Laplacian operators concerns the relation between the eigenvalues and the geometry of the domain Ω, for example through estimates of eigenvalues in terms of the Weyl limit (1.3) or, more generally, in terms of asymptotic expansions beyond the Weyl limit, as we shall discuss below. In 1961, Pólya showed that

$$\mu_j \leq C_d |\Omega|^{-\frac{2}{d}} (j-1)^{-\frac{2}{d}} \tag{1.5}$$

for all positive integers j when Ω is any tiling domain of $\mathbb{R}^d$, and the opposite inequality for the Dirichlet eigenvalues,

$$\lambda_j \geq C_d |\Omega|^{-\frac{2}{d}} j^{-\frac{2}{d}}. \tag{1.6}$$

His still unproven conjecture is that these inequalities hold for all bounded domains $\Omega \subset \mathbb{R}^d$. In other words the Weyl limit (1.3) is approached from below in the Neumann case and above for Dirichlet.

Whereas there are universal domain-independant and hence scale-invariant constraints for eigenvalues of the Dirichlet problem, of the form

$$F_d(k, \lambda_1/\lambda_{k+1}, \dots, \lambda_k/\lambda_{k+1}) \leq 0,$$

for the Neumann problem Colin-de-Verdière showed in 1987 [9] that for any finite nondecreasing $0 = \mu_1 < \mu_2 \leq \mu_3 \leq \cdots \leq \mu_k$, there exists a bounded domain having these values as the first k eigenvalues. Therefore inequalities among Neumann eigenvalues must incorporate geometric properties of Ω to be of interest. (See, e.g. [2, 4, 3], for discussions of universal eigenvalue bounds and related references.)

Other convenient ways to study the spectrum rely on the counting function,

$$\mathcal{N}(\mu) := \sharp\{\mu_j : \mu_j < \mu\}, \tag{1.7}$$

and, in a tradition going back to Berezin [6], Riesz means, $R_\sigma(z) := \sum_j (z-\mu_j)_+^\sigma$, or, resp., $\sum_j (z-\lambda_j)_+^\sigma$. Here x_+ denotes the positive part of x. $\mathcal{N}(z)$ can be interpreted as the limit of $R_\sigma(z)$ when $\sigma \to 0$. For instance, Berezin proved the equivalent of the summed version of (1.6) in the Riesz mean form,

$$\sum_j (z-\lambda_j)_+ \leq L_{1,d}^{cl}\, |\Omega| z^{1+\frac{d}{2}}, \tag{1.8}$$

where

$$L_{\gamma,d}^{cl} := \frac{\Gamma(\gamma+1)}{(4\pi)^{\frac{d}{2}}\Gamma\left(\gamma+1+\frac{d}{2}\right)}. \tag{1.9}$$

In recent years, beginning with a paper by Melas [21], there has arisen an industry to improve (1.8) by including further terms in lower powers of z. An improvement incorporating the best expected succeeding power in (1.8), $z^{d+\frac{1}{2}}$ was obtained in the Dirichlet case by Weidl [26] and later improved by Geisinger, Laptev, and Weidl [13], and we refer to those papers for further background.

Our main goal here is to achieve analogous improvements in Riesz means for Neumann eigenvalues in terms of z to the expected powers. In addition, we obtain two-sided bounds for individual eigenvalues μ_k, which are semiclassically sharp. For this we rely on the averaged variational introduced in [14] and a series of analytic inequalities. In a final section, we also treat the Dirichlet case with the same methods. An appendix contains a discussion of refinements of Young's and Hölder's inequalities, including some results going beyond those we use in the main part of this article.

An important step towards Pólya's conjecture in the Neumann case was taken in 1991 by Kröger, who by applying a variational estimate for the sum of the first k eigenvalues, obtained the asymptotically sharp inequality

$$\frac{d+2}{d}\sum_{j=1}^{k} \mu_j \leq C_d |\Omega|^{-\frac{2}{d}} k^{1-\frac{2}{d}}. \tag{1.10}$$

Later, using the Fourier transforms of the eigenfunctions u_j, Laptev [18] proved the Riesz mean inequality equivalent to Kröger's estimate (1.10),

$$\sum_j (z-\mu_j)_+ \geq L_{1,d}^{cl}\, |\Omega| z^{1+\frac{d}{2}}, \tag{1.11}$$

for all $z \geq 0$. (See also [19].)

Our first result is an improvement of (1.10) using a refinement of Young's inequality for real numbers, which not only improves the estimates of Riesz means and sums, but also provides a bound on individual eigenvalues. It will be useful to introduce the following notation.

$$m_k := C_d \left(\frac{k}{|\Omega|}\right)^{\frac{2}{d}}, \quad S_k := \frac{\frac{d+2}{d}\frac{1}{k}\sum_{j=1}^{k}\mu_j}{m_k}. \tag{1.12}$$

In these terms m_k is the Weyl expression, and Kröger's inequality (1.10) is expressed as $S_k \leq 1$. We shall prove the following refinement of Kröger's inequality.

Theorem 1.1. *Let $d \geq 2$. Then for all $k \geq 0$ the Neumann eigenvalue μ_{k+1} satisfies*

$$m_k^2(1 - S_k) \geq (\mu_{k+1} - m_k)^2. \tag{1.13}$$

I.e.,

$$m_k(1 - \sqrt{1 - S_k}) \leq \mu_{k+1} \leq m_k(1 + \sqrt{1 - S_k}). \tag{1.14}$$

Kröger's bound corresponds to replacing the right side of (1.13) by 0. One may further ask whether there is an additional remainder term improving the right side of the universal inequality (1.13), which contains more explicit information on the geometry of Ω. The asymptotic expansion of the counting function suggests that under sufficient regularity conditions the $(d-1)$-dimensional volume of the boundary $\partial\Omega$ (see [15, 22]) may appear:

$$\mathcal{N}(\mu) \approx C_d^{\frac{d}{2}}|\Omega|\mu^{\frac{d}{2}} + \frac{1}{4}C_{d-1}^{\frac{d-1}{2}}|\partial\Omega|\mu^{\frac{d-1}{2}}, \tag{1.15}$$

and therefore, for the Riesz mean,

$$R_1(z) := \sum_{j=1}(z - \mu_j)_+ \approx L_{1,d}^{cl}|\Omega|z^{1+\frac{d}{2}} + \frac{1}{4}L_{1,d-1}^{cl}|\partial\Omega|z^{\frac{d+1}{2}}. \tag{1.16}$$

In the present paper we present a two-term bound for $R_1(\mu)$, using additional geometrical information on Ω. To this end, for any unit vector $\mathbf{v} \in \mathbb{R}^d$ we let $\delta_{\mathbf{v}}$ be the width of Ω in the $\mathbf{v}$-direction, that is,

$$\delta_{\mathbf{v}}(\Omega) := \sup\{\mathbf{v}\cdot(\mathbf{x}-\mathbf{y}) : \mathbf{x},\mathbf{y} \in \Omega\} = \max\{v\cdot(\mathbf{x}-\mathbf{y}) : \mathbf{x},\mathbf{y} \in \partial\Omega\}. \tag{1.17}$$

We note that $\delta_{\mathbf{v}}(\Omega)$ always lies between twice the inradius and the diameter of Ω. We prove the following.

Theorem 1.2. *Let $d \geq 2$. Then for each unit vector $\mathbf{v} \in \mathbb{R}^d$ and for all $z \geq 0$,*

$$\begin{aligned}\sum (z-\mu_j)_+ \geq & L_{1,d}^{cl} |\Omega| z^{\frac{d}{2}+1} + \frac{1}{4} L_{1,d-1}^{cl} \frac{|\Omega|}{\delta_{\mathbf{v}}(\Omega)} z^{\frac{d}{2}+\frac{1}{2}} \\ & - \frac{1}{96} (2\pi)^{2-d} B_d \frac{|\Omega|}{\delta_{\mathbf{v}}(\Omega)^2} z^{\frac{d}{2}}.\end{aligned} \tag{1.18}$$

Together with the semiclassical bound (1.11) this implies the improved estimate

$$\begin{aligned}\sum (z-\mu_j)_+ \geq & L_{1,d}^{cl} |\Omega| z^{\frac{d}{2}+1} \\ & + \left(\frac{1}{4} L_{1,d-1}^{cl} \frac{|\Omega|}{\delta_{\mathbf{v}}(\Omega)} z^{\frac{d}{2}+\frac{1}{2}} - \frac{1}{96} (2\pi)^{2-d} B_d \frac{|\Omega|}{\delta_{\mathbf{v}}(\Omega)^2} z^{\frac{d}{2}} \right)_+ .\end{aligned} \tag{1.19}$$

Both inequalities (1.18) and (1.19) will follow from our proof. Although the bound (1.19) improves (1.18), we work in most cases with (1.18) since we are mainly interested in large z. An exception is Corollary 1.4 below, where we use the estimate (1.19). We also remark that while the first term is sharp, the second term in eq. (1.18) appears too small by a factor 1/2. Indeed, for the box $\Omega = [0,1]^{d-1} \times [0,\delta]$ the bound (1.18) differs from the the asymptotic formula (1.15) by a factor 1/2, since with $\delta_{\mathbf{v}}(\Omega) = \delta$ in the comparison of the second term of (1.18) and the asymptotic expansion (1.16) we have

$$\frac{|\Omega|}{\delta_{\mathbf{v}}(\Omega)} = 1, \quad |\partial\Omega| = 2 + 2(d-1)\delta,$$

in which δ can be chosen arbitrarily small. More precisely, according to the asymptotic formula (1.16), we find

$$\lim_{z\to\infty} \frac{\sum (z-\mu_j)_+ - L_{1,d}^{cl} |\Omega| z^{\frac{d}{2}+1}}{z^{\frac{d}{2}+\frac{1}{2}}} = \frac{1}{4} L_{1,d-1}^{cl} (2 + 2(d-1)\,\delta),$$

while Theorem 1.2 yields the lower bound $\frac{1}{4} L_{1,d-1}^{cl}$ for this limit. Furthermore, this argument applies to any domain of the form $\Omega = \Omega' \times [0,\delta]$ such that Ω' is bounded in $\mathbb{R}^{d-1}$ with finite boundary, since

$$\frac{|\Omega|}{\delta_{\mathbf{v}}(\Omega)} = |\Omega'|, \quad |\partial\Omega| = 2|\Omega'| + |\partial\Omega'|\,\delta.$$

From our method of proof it will be seen that for these kinds of domains the lower bound (1.18) can be improved to the optimal lower bound consistent with the asymptotic formula (1.16). It is less clear whether the improvement can be obtained in the absence of a product structure.

Corollary 1.3. *Let $d \geq 2$ and $\Omega = \Omega' \times [0, \delta]$ be a bounded domain. Then for all $z \geq 0$,*

$$\sum_{j=1} (z - \mu_j)_+ \geq L_{1,d}^{cl} |\Omega| z^{\frac{d}{2}+1} + \frac{1}{2} L_{1,d-1}^{cl} \frac{|\Omega|}{\delta(\Omega)} z^{\frac{d}{2}+\frac{1}{2}} - \frac{1}{24} (2\pi)^{2-d} B_d \frac{|\Omega|}{\delta(\Omega)^2} z^{\frac{d}{2}}. \tag{1.20}$$

Note that by means of the integral

$$\int_0^\infty (z - \lambda - t)_+ t^{\gamma-2}\, dt = \frac{(z-\lambda)_+^\gamma}{\gamma(\gamma-1)}, \quad \gamma > 1,$$

Eq. (1.18) implies further bounds for higher Riesz means, *viz.*,

$$\sum_{j=1} (z - \mu_j)_+^\gamma \geq L_{\gamma,d}^{cl} |\Omega| z^{\frac{d}{2}+\gamma} + L_{\gamma,d-1}^{cl} \frac{|\Omega|}{4\delta_{\mathbf{v}}(\Omega)} z^{\frac{d}{2}+\gamma-\frac{1}{2}} - \frac{\pi}{96} L_{\gamma,d-2}^{cl} \frac{|\Omega|}{\delta_{\mathbf{v}}(\Omega)^2} z^{\frac{d}{2}+\gamma-1}, \tag{1.21}$$

for any $\gamma \geq 1$, as well as a strengthened version by means of eq.(1.19). This moreover implies that Pólya's conjecture (1.5) can be proved with an improvement for domains in product form.

Corollary 1.4. *Suppose that $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{R}^d$ where $\Omega_1 \subset \mathbb{R}^{d_1}$ and $\Omega_2 \subset \mathbb{R}^{d_2}$ are two bounded domains with spectra consisting of increasing eigenvalues satisfying eq. (1.2), and where Pólya's conjecture (1.5) holds for Ω_1. Then*

$$\mathcal{N}(z) \geq 1 + |\Omega| L_{0,d}^{cl} z^{\frac{d}{2}} + |\Omega| \left(\frac{L_{0,d+1}^{cl}}{\sqrt{4\pi} \cdot 4\delta_{\mathbf{v}}(\Omega_2)} z^{\frac{d}{2}-\frac{1}{2}} - \frac{L_{0,d+2}^{cl}}{384\delta_{\mathbf{v}}(\Omega_2)^2} z^{\frac{d}{2}-1} \right)_+. \tag{1.22}$$

This implies Pólya's conjecture for Ω, when only the first two terms in this expression are kept.

The proof of the main Theorem 1.2 is based on an averaged variational principle introduced by the authors [14], which was later used in [11] to extend and simplify Kröger's results for certain operators on manifolds. The averaged variational principle uses only basic properties of quadratic forms and an averaging over an orthormal basis or, more generally, a frame. Quoting from the formulation in [11]:

Lemma 1.5. *Consider a self-adjoint operator H on a Hilbert space $\mathcal{H}$, the spectrum of which is discrete at least in its lower portion, so that $-\infty < \mu_0 \leq \mu_1 \leq \cdots$. The corresponding orthonormalized eigenvectors are denoted $\{\psi^{(\ell)}\}$. The closed quadratic form corresponding to H is denoted $Q(\varphi, \varphi)$ for vectors φ in the quadratic-form domain $\mathcal{Q}(H) \subset \mathcal{H}$. Let $f_\zeta \in \mathcal{Q}(H)$ be a family of vectors indexed by a variable ζ ranging over a measure space $(\mathfrak{M}, \Sigma, \sigma)$. Suppose that $\mathfrak{M}_0$ is a subset of $\mathfrak{M}$. Then for any $z \in \mathbb{R}$,*

$$\sum_j (z - \mu_j)_+ \int_{\mathfrak{M}} |\langle \psi^{(j)}, f_\zeta \rangle|^2 \, d\sigma \geq \int_{\mathfrak{M}_0} (z \| f_\zeta \|^2 - Q(f_\zeta, f_\zeta)) d\sigma, \qquad (1.23)$$

provided that the integrals converge.

2. Proofs of the main results

2.1. Refinement of Kröger's inequality: Theorem 1.1. The quadratic-form domain of the Neumann Laplacian $-\Delta^N$ on a Euclidean domain Ω is the restriction to Ω of functions in the Sobolev space $H_0^1(\mathbb{R}^d)$ [10] (which is normally but not always the same as $H^1(\Omega)$), and the quadratic form corresponding to $-\Delta^N$ is

$$Q(f, f) = \int_\Omega |\nabla f|^2 \, dx. \qquad (2.1)$$

The trial functions $f(\mathbf{x}) = e^{i\mathbf{p}\cdot\mathbf{x}}$ are admissible, so choosing them as in [17] leads after a calculation to the following bound for the eigenvalues of the Neumann Laplacian (the set $\mathfrak{M}$ is chosen as $\{\mathbf{p} \in \mathbb{R}^d\}$ with Lebesgue measure, and $\mathfrak{M}_\text{o}$ is the ball of radius R; see [17, 11] for details of the calculation),

$$\mu_{k+1} R^d - \frac{d}{d+2} R^{d+2} \leq m_k^{d/2} \Big(\mu_{k+1} - \frac{1}{k} \sum_{i=1}^{k} \mu_i \Big) \qquad (2.2)$$

for all $R > 0$, cf. (1.12). Putting $R^d = m_k^{d/2} x^{d/2}$, we get the bound

$$\frac{d+2}{d} \frac{1}{k} \sum_{i=1}^{k} \mu_i \leq m_k \Big(\frac{d+2}{d} \frac{\mu_{k+1}}{m_k} - \frac{d+2}{d} \frac{\mu_{k+1}}{m_k} x^{\frac{d}{2}} + x^{\frac{d+2}{2}} \Big).$$

We choose $x = x_k = \dfrac{\mu_{k+1}}{m_k}$. This yields

$$\frac{d+2}{d} \frac{1}{k} \sum_{i=1}^{k} \mu_i - m_k \leq m_k \frac{2}{d} \Big(\frac{d+2}{2} x_k - \frac{d}{2} - x_k^{\frac{d+2}{2}} \Big). \qquad (2.3)$$

We may assume that $d \geq 2$, since when $d = 1$ all eigenvalues are explicitly known. Then $p = \dfrac{d}{2} \geq 1$, and, therefore, the function $g_p(x)$ defined in (A.6) is ≤ 0. Hence we obtain

$$\frac{d+2}{d}\frac{1}{k}\sum_{i=1}^{k}\mu_i - m_k \leq -m_k\,(x_k-1)^2, \tag{2.4}$$

which strengthens Kröger's estimate

$$\frac{d+2}{d}\frac{1}{k}\sum_{i=1}^{k}\mu_i \leq m_k = C_d\frac{k^{2/d}}{|\Omega|^{2/d}}$$

and yields the bound on μ_{k+1} claimed in (1.14).

2.2. Two-term spectral bounds: Proof of Theorem 1.2.

Proof. Let $\mathbf{v} \in \mathbb{R}^d$ be a unit vector. After a translation we may suppose that $\Omega \subset \mathbb{R}^d$ is a bounded domain such that $\Omega \subset \{\mathbf{x} \in \mathbb{R}^d : 0 \leq \mathbf{v}\cdot\mathbf{x} \leq L\}$, that is, in the $\mathbf{v}$ direction all $\mathbf{x} \in \Omega$ are contained in an interval of length L. We shall choose L later as $L = 2\delta_{\mathbf{v}}(\Omega)$. Fixing $\mathbf{v}$, we may choose a coordinate system such that $\mathbf{v}$ is a standard unit vector of the canonical basis of $\mathbb{R}^d$. We apply the averaged variational principle 1.5 with test functions of the form

$$f(\mathbf{x}) = (2\pi)^{-\frac{d-1}{2}}e^{i\mathbf{p}_\perp\cdot\mathbf{x}}\phi_n(\mathbf{v}\cdot\mathbf{x}), \tag{2.5}$$

where $\mathbf{p}_\perp = \mathbf{p} - (\mathbf{p}\cdot\mathbf{v})\mathbf{v}$ and ϕ_n is an eigenfunction of the Neumann Laplacian on an interval of length L, that is,

$$-\phi_n''(y) = \kappa_n\phi_n(y) \quad \text{on}\,]0,L[\,\text{and}\,\phi_n'(0) = \phi_n'(L) = 0. \tag{2.6}$$

Recall that the eigenvalues κ_n are given by $\kappa_n = \frac{(\pi n)^2}{L^2}$, $n \in \mathbb{N}$ and the (normalized) eigenfunctions are given by $\phi_0(y) = L^{-1/2}$ and $\phi_n(y) = \sqrt{\frac{2}{L}}\cos\left(\frac{\pi n y}{L}\right)$, where n ranges over the positive integers. With these test functions, the variational principle implies that

$$\begin{aligned}&\sum_{j=1}^{k}(z-\mu_j)|\langle f,u_j\rangle|^2\\ &\quad\geq (2\pi)^{1-d}(z-|\mathbf{p}_\perp|^2)\int_\Omega \phi_n(\mathbf{v}\cdot\mathbf{x})^2 - (2\pi)^{1-d}\int_\Omega \phi_n'(\mathbf{v}\cdot\mathbf{x})^2\end{aligned} \tag{2.7}$$

for any $z \in [\mu_k, \mu_{k+1}]$, where u_j are again the orthonormalized eigenfunctions of the Neumann Laplacian. When $n > 0$ we apply the trigonometric identities

$\cos^2 t = \frac{1+\cos 2t}{2}$ and $\sin^2 t = \frac{1-\cos 2t}{2}$ to $\phi_n(\mathbf{v}\cdot\mathbf{x})^2$ and $\phi_n'(\mathbf{v}\cdot\mathbf{x})^2$, respectively. Then for all $n \geq 0$, (2.7) becomes

$$\begin{aligned}
&\sum_{j=1}^{k}(z-\mu_j)|\langle f, u_j\rangle|^2 \\
&\quad\geq (2\pi)^{1-d}L^{-1}|\Omega|\Big(z-|\mathbf{p}_\perp|^2-\frac{(\pi n)^2}{L^2}\Big) \\
&\qquad+(2\pi)^{1-d}L^{-1}\Big(z-|\mathbf{p}_\perp|^2+\frac{(\pi n)^2}{L^2}\Big)(1-\delta_{0,n})\int_\Omega \cos\Big(\frac{2\pi n\mathbf{v}\cdot\mathbf{x}}{L}\Big),
\end{aligned}\tag{2.8}$$

where $\delta_{0,n}$ denotes the Kronecker delta. On the right side we integrate over the set $\Phi_k = \big\{(\mathbf{p}_\perp, n) \in \mathbb{R}^{d-1}\times\mathbb{N}\colon |\mathbf{p}_\perp|^2 + \frac{\pi^2 n^2}{L^2} \leq z\big\}$ while on the left side over the larger set $\mathbb{R}^{d-1}\times\mathbb{N}$, using Parseval's identity. We shall prove in Lemma 3.2 below that for all $R > 0$,

$$\sum_{k\geq 0}(R^2-k^2)_+ \geq \max\Big(\frac{2R^3}{3}+\frac{R^2}{2}-\frac{R}{6}, R^2\Big). \tag{2.9}$$

By applying the lower bound (2.9) to the sum over n and then integrating over $\mathbf{p}_\perp$ we obtain an explicit lower bound for $\int\sum_{\Phi_k}\big(z-|\mathbf{p}_\perp|^2-\frac{(\pi n)^2}{L^2}\big)$. Since $\int\max(f,g) \geq \max(\int f, \int g)$, this yields

$$\begin{aligned}
\sum_{j=1}^{k}(z-\mu_j) \geq{}& \frac{2}{d+2}(2\pi)^{-d}B_d|\Omega|\,z^{\frac{d}{2}+1} \\
&+\frac{1}{d+1}(2\pi)^{1-d}B_{d-1}|\Omega|L^{-1}z^{\frac{d+1}{2}} \\
&-\frac{1}{24}(2\pi)^{2-d}B_d|\Omega|L^{-2}z^{\frac{d}{2}}+G(z),
\end{aligned}\tag{2.10}$$

where

$$G(z) := \int\sum_{\Phi_k}(2\pi)^{1-d}\Big(z-|\mathbf{p}_\perp|^2+\frac{(\pi n)^2}{L^2}\Big)(1-\delta_{0,n})\int_\Omega\cos\Big(\frac{2\pi n\mathbf{v}\cdot\mathbf{x}}{L}\Big).$$

It remains to control $G(z)$, which could in principle be positive or negative. In fact, by averaging (2.10) in a certain way we shall show that G can be dropped altogether. To this end we choose L large enough that Ω is also contained in $\{\mathbf{x}\in\mathbb{R}^d\colon 0\leq\mathbf{v}\cdot\mathbf{x}\leq L\}$ when translated by $L/2$. This means nothing else than assuming that $\Omega\subset\{\mathbf{x}\in\mathbb{R}^d\colon 0\leq\mathbf{v}\cdot\mathbf{x}\leq L/2\}$. In the corresponding Neumann eigenfunctions we have to replace $\mathbf{v}\cdot\mathbf{x}$ by $\mathbf{v}\cdot\mathbf{x}+L/2$. We may apply the averaged variational principle on both sets (the eigenvalues $\frac{(\pi n)^2}{L^2}$, $n \in \mathbb{N}$ remain unchanged). Since

$$\frac{1}{2}\Big(\cos\Big(\frac{2\pi n\mathbf{v}\cdot\mathbf{x}}{L}\Big)+\cos\Big(\frac{2\pi n(\mathbf{v}\cdot\mathbf{x}+L/2)}{L}\Big)\Big)
=\begin{cases}\cos\Big(\frac{2\pi n\mathbf{v}\cdot\mathbf{x}}{L}\Big) & \text{if } n \text{ is even,}\\ 0 & \text{if } n \text{ is odd,}\end{cases}$$

all odd n may be dropped from $G(z)$, leaving only cosine functions of the form $\cos\big(\frac{4\pi n\mathbf{v}\cdot\mathbf{x}}{L}\big)$ with n a positive integer. We apply the same averaging procedure with a translation by $L/4$. Since

$$\frac{1}{2}\Big(\cos\Big(\frac{4\pi n\mathbf{v}\cdot\mathbf{x}}{L}\Big)+\cos\Big(\frac{4\pi n(\mathbf{v}\cdot\mathbf{x}+L/4)}{L}\Big)\Big)
=\begin{cases}\cos\Big(\frac{4\pi n\mathbf{v}\cdot\mathbf{x}}{L}\Big) & \text{if } n \text{ is even,}\\ 0 & \text{if } n \text{ is odd,}\end{cases}$$

again the terms containing odd integers may be dropped. Since $G(z)$ contains only a finite number of contributions, after a finite sequence of averages with shifts $L/2^n$, every contribution will be eliminated. Hence

$$\begin{aligned}\sum_{j=1}^{k}(z-\mu_j) \geq{}& \frac{2}{d+2}(2\pi)^{-d}B_d|\Omega|\,z^{\frac{d}{2}+1}\\ &+\frac{1}{d+1}(2\pi)^{1-d}B_{d-1}|\Omega|L^{-1}\,z^{\frac{d+1}{2}}\\ &-\frac{1}{24}(2\pi)^{2-d}B_d|\Omega|L^{-2}z^{\frac{d}{2}}.\end{aligned} \tag{2.11}$$

We may now choose $L = 2\delta_{\mathbf{v}}(\Omega)$, which yields the statement of the theorem. □

To prove Corollary 1.3 we note that when $\Omega = \Omega' \times [0, \delta]$ we may choose $\mathbf{v} = \mathbf{e}_d$. As a consequence

$$\int_\Omega \phi_n(\mathbf{v}\cdot\mathbf{x})^2 = |\Omega'|, \quad \int_\Omega \phi_n'(\mathbf{v}\cdot\mathbf{x})^2 = \frac{(\pi n)^2}{L^2}|\Omega'|,$$

and no translations are needed. Therefore we may choose $L = \delta$ which yields the bound (1.20).

From the bound (2.9) it is straightforward to derive the simpler expression

$$\sum_{k\geq 0}(R^2-k^2)_+ \geq \frac{2R^3}{3}+\frac{R^2}{3}, \tag{2.12}$$

containing only two terms. This yields the following spectral bound.

Corollary 2.1. *Let $d \geq 2$. Then for any unit vector $v \in \mathbb{R}^d$ and for all $\mu \geq 0$*

$$\sum_{j=1}^{k}(z-\mu_j)_+ \geq L^{cl}_{1,d}|\Omega| z^{\frac{d}{2}+1} + L^{cl}_{1,d-1}\frac{|\Omega|}{6\delta_{\mathbf{v}}(\Omega)} z^{\frac{d}{2}+\frac{1}{2}}. \tag{2.13}$$

The term containing the width $\delta_{\mathbf{v}}$ can be estimated by geometric properties of the convex hull of Ω, since $\delta_{\mathbf{v}}(\Omega)$ coincides with $\delta_{\mathbf{v}}(\mathrm{hull}(\Omega))$. For example, in 2 dimensions,

$$\int_{S^1}\delta_{\mathbf{v}} = 2|\partial\, \mathrm{hull}(\Omega)|. \tag{2.14}$$

With Corollary 2.1, by choosing $\mathbf{v}$ so that $\delta_{\mathbf{v}}$ equals the mean width w of hull(Ω) (= the average of $\delta_{\mathbf{v}}$ uniformly over directions $\mathbf{v}$), we obtain a correction involving the isoperimetric ratio of Ω,

$$\sum_{j=1}^{k}(\mu-\mu_j)_+ \geq L^{cl}_{1,2}|\Omega|\mu^2 + L^{cl}_{1,1}\frac{\pi|\Omega|}{6|\partial\, \mathrm{hull}(\Omega)|}\mu^{3/2}. \tag{2.15}$$

In arbitrary dimensions, if $\delta_{\mathbf{v}}$ is chosen equal to w, then, following Bourgain [7], the final term in (2.13) can be bounded from below in terms of the *isotropic constant*,

$$L^2_{\mathrm{hull}(\Omega)|} := \frac{\det(M_{\mathrm{hull}(\Omega)|})^{\frac{1}{d}}}{\mathrm{Vol}(\mathrm{hull}(\Omega)|)^{1+\frac{2}{d}}},$$

where the inertia matrix $M_{ij} = \int_{\mathrm{hull}(\Omega)|} x_i x_j\, dx$ has been minimized with respect to the choice of the origin. Finding the optimal upper bound for the ratio $\frac{w}{L_\Omega}$ for convex Ω is an open problem in analysis. In [23], Milman has, for example, proved an upper bound for w in the form of a universal constant times $\sqrt{d}\,\log(d)^2$.

It has been known since the work of Ball [5] that under various further assumptions convex bodies satisfy reverse isoperimetric inequalities, with which Inequality (2.15) can be connected to additional geometric properties of hull(Ω). See, e.g., [24]. We also recall that for convex domains a remainder term with the surface area for Riesz means with power $\gamma \geq 3/2$ has been obtained by Larson [20].

Finally, we prove Corollary 1.4. For the Dirichlet case it was shown by Laptev [18] that if Pólya's conjecture holds for a domain Ω_1, then it holds on arbitrary Cartesian products of the form $\Omega_1 \times \Omega_2$. In fact, the same argument allows improved bounds on the counting function, benefitting from the improved bounds for sums coming from Ω_2, as follows.

Proof. Suppose that $\Omega_1 \subset \mathbb{R}^{d_1}$ $d_1 \geq 2$ is a domain for which Pólya's conjecture

$$\mathcal{N}(z) = \sum_j (z - \mu_j)_+^0 \geq 1 + L_{0,d_1}^{cl} |\Omega_1| z^{d_1/2}$$

is valid. Let $d_1 + d_2 = d$, $\Omega = \Omega_1 \times \Omega_2$ with $\Omega_2 \subset \mathbb{R}^{d_2}$. The Neumann eigenvalues μ_j of Ω are of the form $\mu_j = \mu_{j_1} + \mu_{j_2}$ where μ_{j_1}, μ_{j_2} are the Neumann eigenvalues of Ω_1, Ω_2, respectively. Therefore,

$$\sum_j (z - \mu_j)_+^0 = \sum_{j_2} \sum_{j_1} (z - \mu_{j_2} - \mu_{j_1})_+^0 \geq 1 + L_{0,d_1}^{cl} |\Omega_1| \sum_{j_2} (z - \mu_{j_2})_+^{d_1/2}.$$

Since $d_1/2 \geq 1$, using (1.21) and (1.9) we obtain

$$\begin{aligned} \mathcal{N}(z) \geq 1 + L_{0,d_1}^{cl} |\Omega_1| |\Omega_2| \Bigg(& L_{\frac{d_1}{2},d_2}^{cl} z^{\frac{d}{2}} + \frac{L_{\frac{d_1}{2},d_2-2}^{cl}}{4\delta_{\mathbf{v}}(\Omega_2)} z^{\frac{d}{2}-\frac{1}{2}} \\ & - \frac{\pi}{96} \frac{L_{\gamma,d-2}^{cl}}{\delta_{\mathbf{v}}(\Omega_2)^2} z^{\frac{d}{2}+\gamma-1} \Bigg) \\ \geq 1 + |\Omega| \Bigg(L_{0,d}^{cl} z^{\frac{d}{2}} + & \frac{L_{0,d+1}^{cl}}{\sqrt{4\pi} \cdot 4\delta_{\mathbf{v}}(\Omega_2)} z^{\frac{d}{2}-\frac{1}{2}} - \frac{L_{0,d+2}^{cl}}{384\delta_{\mathbf{v}}(\Omega_2)^2} z^{\frac{d}{2}-1} \Bigg), \end{aligned}$$

as well as

$$\mathcal{N}(z) \geq 1 + |\Omega| L_{0,d}^{cl} z^{\frac{d}{2}}.$$

Combining both estimates we prove the claim.

□

3. Riesz means of Laplacians on rectangles

In this section we derive upper and lower bounds for Riesz means of Neumann and Dirichlet Laplacians, respectively, on the rectangle $R := [0, l_1] \times [0, l_2]$.

Theorem 3.1. *Let μ_i^R, λ_i^R denote the eigenvalues of the Neumann Laplacian and the Dirichlet Laplacian on $R = [0, l_1] \times [0, l_2]$. Suppose that $l_1 \le l_2$. Then the following estimates hold*

$$\begin{aligned} &\frac{3\pi}{128}\Big(\frac{l_2}{l_1} + \frac{l_1}{l_2} + \frac{32}{3\pi}\Big)\mu + \frac{3\pi}{64}\Big(\frac{1}{l_1} + \frac{1}{l_2}\Big)\mu^{1/2} + \frac{3\pi^{3/2}2^{1/2}}{64 l_2 l_1^{1/2}}\,\mu^{1/4} \\ &\quad \ge \sum_{j=1}^{k}(\mu - \mu_j^R)_+ - \frac{|R|}{8\pi}\,\mu^2 - \frac{|\partial R|}{6\pi}\,\mu^{3/2} \\ &\quad \ge -\frac{\pi}{24}\Big(\frac{l_2}{l_1} + \frac{l_1}{l_2} - \frac{6}{\pi}\Big)\mu - \frac{\pi}{12}\Big(\frac{1}{l_1} + \frac{1}{l_2}\Big)\mu^{1/2} - \frac{\pi^{3/2}2^{1/2}}{12 l_2 l_1^{1/2}}\,\mu^{1/4}, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} &\frac{3\pi}{128}\Big(\frac{l_2}{l_1} + \frac{l_1}{l_2} + \frac{32}{3\pi}\Big)\lambda + \frac{\pi}{12}\Big(\frac{1}{l_2} - \frac{9}{16 l_1}\Big)\lambda^{1/2} + \frac{3\pi^{3/2}2^{1/2}}{64 l_2 l_1^{1/2}}\,\lambda^{1/4} \\ &\quad \ge \sum_{j=1}^{k}(\lambda - \lambda_j^R)_+ - \frac{|R|}{8\pi}\,\lambda^2 + \frac{|\partial R|}{6\pi}\,\lambda^{3/2} \\ &\quad \ge -\frac{\pi}{24}\Big(\frac{l_2}{l_1} + \frac{l_1}{l_2} - \frac{6}{\pi}\Big)\lambda + \frac{\pi}{12}\Big(\frac{1}{l_1} - \frac{9}{16 l_2}\Big)\lambda^{1/2} - \frac{\pi^{3/2}2^{1/2}}{12 l_2 l_1^{1/2}}\,\lambda^{1/4}. \end{aligned} \tag{3.2}$$

Proof. The Riesz mean for the Neumann Laplacian on R is given by

$$R_1^N(z) = \sum\sum_{n_1, n_2 \ge 0}\Big(z - \frac{(\pi n_1)^2}{l_1^2} - \frac{(\pi n_2)^2}{l_2^2}\Big)_+.$$

We need the following polynomial upper and lower bounds for one-dimensional Riesz means $\sum(R^2 - k^2)_+^p$, in particular (2.9).

Lemma 3.2. *For all $R > 0$,*

$$\max\Big(\frac{2R^3}{3} + \frac{R^2}{2} - \frac{R}{6}, R^2\Big) \le \sum_{k\ge 0}(R^2 - k^2)_+ \le \frac{2R^3}{3} + \frac{R^2}{2} + \frac{3R}{32}, \tag{3.3}$$

and for all $R > 0$, $\beta > 0$,

$$\begin{aligned} &\max\Big(\frac{\sqrt{\pi}\,\Gamma(\beta+2)}{2\,\Gamma(\beta+5/2)}\,R^{2\beta+3} + \frac{1}{2}\,R^{2\beta+2} - \frac{\sqrt{\pi}\,\Gamma(\beta+2)}{12\,\Gamma(\beta+3/2)}\,R^{2\beta+1}, R^{2\beta+2}\Big) \\ &\quad \le \sum_{k\ge 0}(R^2 - k^2)_+^{\beta+1} \\ &\quad \le \frac{\sqrt{\pi}\,\Gamma(\beta+2)}{2\,\Gamma(\beta+5/2)}\,R^{2\beta+3} + \frac{1}{2}\,R^{2\beta+2} + \frac{3\sqrt{\pi}\,\Gamma(\beta+2)}{64\,\Gamma(\beta+3/2)}\,R^{2\beta+1}. \end{aligned} \tag{3.4}$$

Finally, for all $R > 0$,

$$\sum_{k\geq 0}\sqrt{(R^2-k^2)_+} \leq \frac{\pi R^2}{4}+\frac{R}{2}+\frac{\sqrt{2R}}{2}. \tag{3.5}$$

The lemma will be proved below. Assuming it for now, we continue the proof of the theorem for the Neumann Laplacian on the rectangle $[0,l_1]\times[0,l_2]$. Since

$$R_1^N(z) = \frac{\pi^2}{l_2^2}\sum\sum_{n_1,n_2\geq 0}\Big(\frac{l_2^2 z}{\pi^2}-\frac{l_2^2 n_1^2}{l_1^2}-n_2^2\Big)_+,$$

by applying the lower bound (3.3) we get

$$\begin{aligned} R_1^N(z) \geq{}& \frac{2\pi^2 l_2}{3l_1^3}\sum_{n_1\geq 0}\Big(\frac{l_1^2 z}{\pi^2}-n_1^2\Big)_+^{3/2}+\frac{\pi^2}{2l_1^2}\sum_{n_1\geq 0}\Big(\frac{l_1^2 z}{\pi^2}-n_1^2\Big)_+ \\ &-\frac{\pi^2}{6l_1 l_2}\sum_{n_1\geq 0}\Big(\frac{l_1^2 z}{\pi^2}-n_1^2\Big)_+^{1/2}. \end{aligned}$$

Applying the lower bounds (3.3), (3.4) and the upper bound (3.5) we get

$$\frac{2\pi^2 l_2}{3l_1^3}\sum_{n_1\geq 0}\Big(\frac{l_1^2 z}{\pi^2}-n_1^2\Big)_+^{3/2} \geq \frac{l_1 l_2}{8\pi}z^2+\frac{l_2}{3\pi}z^{3/2}-\frac{\pi}{24}\frac{l_2}{l_1}z,$$

$$\frac{\pi^2}{2l_1^2}\sum_{n_1\geq 0}\Big(\frac{l_1^2 z}{\pi^2}-n_1^2\Big)_+ \geq \frac{l_1}{3\pi}z^{3/2}+\frac{z}{4}-\frac{\pi}{12l_1}z^{1/2},$$

and

$$-\frac{\pi^2}{6l_1 l_2}\sum_{n_1\geq 0}\Big(\frac{l_1^2 z}{\pi^2}-n_1^2\Big)_+^{1/2} \geq -\frac{\pi}{24}\frac{l_1}{l_2}z-\frac{\pi}{12l_2}z^{1/2}-\frac{\pi^{3/2}2^{1/2}}{12l_2 l_1^{1/2}}z^{1/4}.$$

Summarizing all estimates, we get the lower bound of (3.1). Similarly, we get the upper bound of (3.1) interchanging l_1 and l_2. The Riesz mean for the Dirichlet Laplacian on R is given by

$$R_1^D(z) = \frac{\pi^2}{l_2^2}\sum\sum_{n_1,n_2\geq 1}\Big(\frac{l_2^2 z}{\pi^2}-\frac{l_2^2 n_1^2}{l_1^2}-n_2^2\Big)_+.$$

The corresponding one-dimensional bounds are those of Lemma 3.2 subtracting R^2, $R^{2\beta+2}$, and respectively R in (3.3), (3.4) and the upper bound (3.5), leading to a change of the sign of the second term, from which we get the bounds (3.2) of the theorem. □

We next prove Lemma 3.2.

Proof. Start from the identity

$$\sum_{k\geq 0}(R^2-k^2)_+ = R^2 + R^2[R] - \frac{[R]^3}{3} - \frac{[R]^2}{2} - \frac{[R]}{6}, \tag{3.6}$$

where $[R]$ denotes the integer part of R. We substitute the periodic sawtooth function $\psi(t) = \big(t - [t] - \frac{1}{2}\big)$, in terms of which

$$\begin{aligned}\sum_{k\geq 0}(R^2-k^2)_+ &= \frac{2R^3}{3} + \frac{R^2}{2} - \frac{R}{6} + \Big(\frac{1}{4} - \psi(R)^2\Big)\Big(R - \frac{\psi(R)}{3}\Big)\\ &\geq \frac{2R^3}{3} + \frac{R^2}{2} - \frac{R}{6},\end{aligned} \tag{3.7}$$

since both factors of the product are nonnegative. This lower bound is exact when R is an integer. Since $\sum_{k\geq 0}(R^2-k^2)_+ = R^2$ trivially for all $0 < R < 1$, the lower bound follows. For the upper bound, we wish to replace $\big(\frac{1}{4} - \psi(R)^2\big)\big(R - \frac{\psi(R)}{3}\big)$ by a linear expression in R for $R \geq 0$, or, equivalently, find an upper bound for

$$F(R) := \Big(\frac{1}{4} - \psi(R)^2\Big)\Big(1 - \frac{\psi(R)}{3R}\Big).$$

Because on each interval $(n, n+1)$ the function $\psi(R)$ is antisymmetric about $n+\frac{1}{2}$ and negative on $(n, n+\frac{1}{2})$, the maximum is to be sought in an interval of the form $(n, n+\frac{1}{2})$. On these subintervals, the second factor decreases when R is replaced by $R+1$, while the first factor is positive and unchanged. Hence, the maximum of $F(R)$ occurs where $0 < R < \frac{1}{2}$. In this interval, however, an elementary calculus exercise shows that the maximizing value is $R = \frac{3}{8}$, and thus $F(R) \leq F(\frac{3}{8}) = \frac{25}{96}$. Substituting this into the first line of (3.7) yields the claim. We observe that the upper and lower bounds in (3.3) coincide uniquely when $R = \frac{3}{8}$.

To prove (3.4) we note that for all $\beta > 0$,

$$\begin{aligned}\int_0^\infty \sum_{k\geq 0}(R^2-t-k^2)_+ t^{\beta-1}\,dt &= \frac{1}{\beta(\beta+1)}\sum_{k\geq 0}(R^2-k^2)_+^{\beta+1}\\ &= 2\int_0^\infty \sum_{k\geq 0}(s^2-k^2)_+\, s\,(R^2-s^2)_+^{\beta-1}\,ds,\end{aligned}$$

and then apply the bounds (3.3).

It remains to show (3.5). We start from the identity

$$\sum_{k\geq 0}\sqrt{(R^2-k^2)_+} = \frac{\pi R^2}{4} + \frac{R}{2} - \int_0^R t(R^2-t^2)^{-1/2}\Big(t-[t]-\frac{1}{2}\Big)dt.$$

For any continuous increasing function $f\colon[0,R]\to\mathbb{R}$ and any positive integer $k\leq R$,

$$\int_{k-1}^{k}\psi(t)f(t)\,dt = \int_0^{\frac{1}{2}}\Big(\frac{1}{2}-s\Big)(f(k-s)-f(k-1+s))\,ds \geq 0. \tag{3.8}$$

Consequently,

$$\sum_{k\geq 0}\sqrt{(R^2-k^2)_+} \leq \frac{\pi R^2}{4} + \frac{R}{2} - \int_{[R]}^R t(R^2-t^2)^{-1/2}\Big(t-[t]-\frac{1}{2}\Big)\,dt. \tag{3.9}$$

The integral between $[R]$ and R can also be computed explicitly. Define $\rho = \frac{[R]}{R}$ and $\kappa = \sqrt{1-\rho^2}$. Then for all $R>0$ we have $1-\min\big(1,\frac{1}{R}\big)\leq\rho\leq 1$. Hence $0<\kappa<1$ if $R<1$ and $0<\kappa<R^{-1}\sqrt{2R-1}$ otherwise. Then

$$\begin{aligned}\int_{[R]}^R\frac{t\,\psi(t)}{\sqrt{R^2-t^2}}\,dt &= R^2\int_\rho^1\frac{s^2-\rho s-\frac{s}{2R}}{\sqrt{1-s^2}}\,ds\\ &= \frac{R^2}{2}(\arcsin\kappa-\kappa\sqrt{1-\kappa^2}) - \frac{R}{2}\kappa.\end{aligned}$$

We also note that $\kappa\mapsto\arcsin\kappa-\kappa\sqrt{1-\kappa^2}-\dfrac{2\kappa^3}{3}$ is increasing. It follows that

$$\int_{[R]}^R\frac{t\,\psi(t)}{\sqrt{R^2-t^2}}\,dt \leq -\frac{R^2\kappa^3}{3}+\frac{R\kappa}{2},$$

proving the claim. □

4. Two-term estimates for Dirichlet Laplacians by averaging

For Dirichlet Laplacians on a bounded domain Ω our strategy will be to enclose Ω in a box B and then to use the averaged variational principle to estimate the Riesz means of the Dirichlet Laplacian on B in terms of expectations with the eigenfunctions of $-\Delta_\Omega$. Thus suppose that $\Omega\subset B$ where $B=\prod_{\alpha=1}^d]0,L_\alpha[$ is a box of volume $|B|=\prod_{\alpha=1}^d L_\alpha$. We let v_k^Ω denote the Dirichlet eigenfunctions on Ω, and, similarly, for B we define

$$v_k^B(x) = \prod_{\alpha=1}^d\psi_{n_\alpha}(x_\alpha),$$

where

$$\psi_{n_\alpha}(x_\alpha) := \sqrt{\frac{2}{l_\alpha}} \sin\left(\frac{n\pi x_\alpha}{l_\alpha}\right),$$

corresponding to eigenvalues

$$\lambda_k^B = \sum_{\alpha=1}^{d} \frac{\pi^2 n_\alpha^2}{l_\alpha^2}$$

with $n_\alpha \in \mathbb{Z}_+$. By the variational principle,

$$\sum (z - \lambda_j^B)_+ \left|\langle v_k^\Omega, v_j^B \rangle_B\right|^2 \geq z \int_B |v_k^\Omega|^2 \, dx - \int_B |\nabla v_k^\Omega|^2 \, dx. \qquad (4.1)$$

Since $v_k^\Omega \in H_0^1(\Omega)$, all integrals reduce to integrals on Ω. On the right side we take a finite sum in k while on the left we sum over all k and apply the completeness relation, obtaining

$$\sum (z - \lambda_j^B)_+ \int_\Omega |v_j^B(x)|^2 \, dx \geq \sum (z - \lambda_j^\Omega)_+. \qquad (4.2)$$

To apply the translation argument as above we suppose that l_α is at least twice the width of Ω in the α direction. Using again the trigonometric identity $\sin^2 t = \frac{1-\cos 2t}{2}$, we may apply the same iteration of averages as earlier in the proof of Theorem 1.2. Repeating this for all α we get

$$\frac{|\Omega|}{|B|} \sum (z - \lambda_j^B)_+ \geq \sum (z - \lambda_j^\Omega)_+, \qquad (4.3)$$

which improves Berezin–Li–Yau. Consider, for example, the case $d = 2$ where applying the upper bound in (3.2) of Theorem 3.1 for the Dirichlet Laplacian on a rectangle B with side lengths l_1, l_2 we obtain the explicit upper bound

$$\sum (\lambda - \lambda_j^\Omega)_+ \leq L_{1,2}^{cl} |\Omega| \, \lambda^2 - \frac{1}{4} L_{1,1}^{cl} \frac{|\partial B||\Omega|}{|B|} \lambda^{3/2} + F(l_1, l_2, \lambda)|\Omega|, \qquad (4.4)$$

where $F(l_1, l_2, \lambda)$ is shorthand notation for the lower-order terms of the left side in (3.2)

Appendix A. Refinements of Young's and Hölder's inequality

In §2.1, we rely on an improvement of Young's inequality in order to strengthen Kröger's inequality with (2.4). Improvements of Young's inequality that are adequate for this purpose already exist in the literature [1, 16, 12], but we take the opportunity in this appendix to present an efficient approach to deriving improvements to Young's and Hölder's inequalities.

To begin, let $p > -1$. For $x \geq 0$ we define the function $y_p(x)$ by

$$y_p(x) := (p+1)x - p - x^{p+1}. \tag{A.1}$$

The unique critical point of $y_p(x)$ occurs at $x = 1$. Since $y_p(1) = 0$, Young's inequality follows in the following formulation:

(1) $y_p(x) \leq 0$ for all $x \geq 0$ if $p \geq 0$;

(2) $y_p(x) \geq 0$ for all $x \geq 0$ if $-1 < p \leq 0$.

Before deriving an improvement, we first note that the case $-1 < p \leq 0$ is equivalent to the case $p \geq 0$ by means of the duality

$$y_p(x) = -(p+1)y_q(z), \quad (p+1)(q+1) = 1, \quad z = x^{q+1},$$

the fixed point of which is the trivial case $p = q = 0$. In the following we therefore only consider the case $p > 0$. Putting $x = a/b^{1/p}$, defining $s = p+1$, $r = \frac{p+1}{p}$, such that $\frac{1}{r} + \frac{1}{s} = 1$, and dividing by $p+1$, we obtain the classical version of Young's inequality:

$$ab - \frac{b^r}{r} - \frac{a^s}{s} \leq 0, \quad a, b \geq 0. \tag{A.2}$$

There are basically two refinements discussed in [1, 16, 12], which as we shall show follow directly from identities for the functions $y_p(x)$. First, we consider the family of functions f_p defined by

$$f_p(x) := y_p(x) + (x^{(p+1)/2} - 1)^2 = 2y_{(p-1)/2}(x). \tag{A.3}$$

Clearly

(1) $f_p(x) \leq 0$ for all $x > 0$ if $p \geq 1$,

(2) $f_1(x) = 0$ for all $x > 0$,

(3) $f_p(x) \geq 0$ for all $x > 0$ if $0 < p \leq 1$.

When $p \geq 1$ we have $s = p + 1 \geq 2$, and with $x = a/b^{1/p}$ the refinement of Young's inequality becomes:

$$ab - \frac{b^r}{r} - \frac{a^s}{s} \leq -\frac{1}{s}(a^{s/2} - b^{r/2})^2, \quad a, b \geq 0,\ s \geq 2 \geq r > 1. \tag{A.4}$$

When $0 \leq p \leq 1$ the inequality is reversed. Exchanging a and b as well as r and s, we get

$$ab - \frac{b^r}{r} - \frac{a^s}{s} \geq -\frac{1}{r}(a^{s/2} - b^{r/2})^2, \quad a, b \geq 0,\ s \geq 2 \geq r > 1. \tag{A.5}$$

Another refinement follows from considering the family of functions g_p defined by

$$g_p(x) = y_p(x) + p(x-1)^2 = px^2 - (p-1)x - x^{p+1} = x\,y_{p-1}(x). \quad \text{(A.6)}$$

We observe that

(1) $g_p(x) \le 0$ for all $x > 0$ if $p \ge 1$,
(2) $g_1(x) = 0$ for all $x > 0$,
(3) $g_p(x) \ge 0$ for all $x > 0$ if $0 < p \le 1$.

When $p \ge 1$ we have $s = p + 1 \ge 2$, and with $x = a/b^{1/p}$ the refinement of Young's inequality becomes

$$ab - \frac{b^r}{r} - \frac{a^s}{s} \le -\frac{1}{r}(a - b^{r-1})^2 b^{2-r}, \quad a, b \ge 0,\ s \ge 2 \ge r > 1. \quad \text{(A.7)}$$

When $0 \le p \le 1$ we find a reversed inequality. Exchanging a and b as well as r and s, we obtain

$$ab - \frac{b^r}{r} - \frac{a^s}{s} \ge -\frac{1}{s}(b - a^{s-1})^2 a^{2-s}, \quad a, b \ge 0,\ s \ge 2 \ge r > 1. \quad \text{(A.8)}$$

Although we do not use it in this paper, we further note that refinements of Hölder's inequality, cf. [8], are easily obtained from the inequalities above as follows.

Let M be a measure space and $a \in L^s(M)$, $b \in L^r(M)$ such that $\|a\|_s = \|b\|_r = 1$ where $r^{-1} + s^{-1} = 1$, $s \ge 2 \ge r > 1$ and $\|\cdot\|_p$ denotes the usual norm in $L^p(M)$. Then by integrating the pointwise inequalities (A.4) and (A.5),

$$1 - \frac{1}{r}\int (|a|^{s/2} - |b|^{r/2})^2 \le \int |ab| \le 1 - \frac{1}{s}\int (|a|^{s/2} - |b|^{r/2})^2, \quad \text{(A.9)}$$

with equality if and only if $|a|^s = |b|^r$ pointwise almost everywhere. We also may directly make the replacements $a \to t^{-1}a$, $b \to tb$ in (A.4) and (A.5) and after integration optimize with respect to t. This yields the slightly improved inequalities:

$$\left(1 - \frac{1}{2}\int (|a|^{s/2} - |b|^{r/2})^2\right)^{2/r} \le \int |ab| \le \left(1 - \frac{1}{2}\int (|a|^{s/2} - |b|^{r/2})^2\right)^{2/s}. \quad \text{(A.10)}$$

When integrating the pointwise inequalities (A.7) and (A.8):

$$1 - \frac{1}{s}\int (|b| - |a|^{s-1})^2 |a|^{2-s} \le \int |ab| \le 1 - \frac{1}{r}\int (|a| - |b|^{r-1})^2 |b|^{2-r}, \quad \text{(A.11)}$$

with equality if and only if $|a|^s = |b|^r$ pointwise almost everywhere.

References

[1] J. M. Aldaz, A stability version of Hölder's inequality. *J. Math. Anal. Appl.* **343** (2008), no. 2, 842–852. MR 2401540 Zbl 1138.26308

[2] M. S Ashbaugh, Universal eigenvalue bounds of Payne–Polya–Weinberger, Hile–Protter and H. C. Yang. *Proc. Indian Acad. Sci. Math. Sci.* **112** (2002), no. 1, 3–30. MR 1894540 Zbl 1199.35261

[3] M. S Ashbaugh, Universal inequalities for the eigenvalues of the Dirichlet Laplacian. In A. Henrot (eds.), *Shape optimization and spectral theory.* De Gruyter Open, Warsaw, 2017, 282–324. MR 3681152 Zbl 1376.49053

[4] M. S Ashbaugh and L. Hermi, A unified approach to universal inequalities for eigenvalues of elliptic operators. *Pacific J. Math.* **217** (2004), no. 2, 201–219. MR 2109931 Zbl 1078.35080

[5] K. Ball, Volume ratios and a reverse isoperimetric inequality. *J. London Math. Soc.* (2) **44** (1991), no. 2, 351–359. MR 1136445 Zbl 0694.46010

[6] F. Berezin, Convariant and contravariant symbols of operators. *Izv. Akad. Nauk SSSR* **37** (1973), 1134–1167. In Russian, English translation, *Math. USSR-Izv.* **6** (1973), 1117–1151. Zbl 0259.47004

[7] J. Bourgain, On the distribution of polynomials on high-dimensional convex sets. In J. Lindenstrauss and V. D. Milman (eds.), *Geometric aspects of functional analysis.* Proceedings of the Israel Seminar held in 1989–90. Lecture Notes in Mathematics, 1469. Springer-Verlag, Berlin, 1991, 127–137. MR 1122617 Zbl 0773.46013

[8] E. A. Carlen, R. L. Frank, and E. H. Lieb, Stability estimates for the lowest eigenvalue of a Schrödinger operator, *Geom. Funct. Anal.* **24** (2014), no. 1, 63–84. MR 3177378 Zbl 1291.35145

[9] Y. Colin de Verdière, Construction de laplaciens dont une partie finie du spectre est donnée. *Ann. Sci. École Norm. Sup.* (4) **20** (1987), no. 4, 599–615. MR 0932800 Zbl 0636.58036

[10] D. E. Edmunds and W. D. Evans, *Spectral theory and differential operators.* Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1987. MR 0929030 Zbl 0628.47017

[11] A. El Soufi, E. M. Harrell II, S. Ilias, and J. Stubbe, On sums of eigenvalues of elliptic operators on manifolds. *J. Spectr. Theory* **7** (2017), no. 4, 985–1022. MR 3737886 Zbl 1382.58023

[12] S. Furuichi, On refined Young inequalities and reverse inequalities. *J. Math. Inequal.* **5** (2011), no. 1, 21–31. MR 2799055 Zbl 1213.15017

[13] L. Geisinger, A. Laptev, and T. Weidl, Geometrical versions of improved Berezin–Li–Yau inequalities. *J. Spectr. Theory* **1** (2011), no. 1, 87–109. MR 2820887 Zbl 1257.35138

[14] E. M. Harrell II and J. Stubbe, On sums of graph eigenvalues. *Linear Algebra Appl.* **455** (2014), 168–186. Corrigendum, ibid. **458** (2014), 699–700. MR 3217405 MR 3231844 (corrigendum) Zbl 1305.05127 Zbl 1305.05128 (corrigendum)

[15] V. Y. Ivrii, Second term of the spectral asymptotic expansion of the Laplace–Beltrami operator on manifolds with boundary. *Funktsional. Anal. i Prilozhen.* **14** (1980), no. 2, 25–34. In Russian. English translation, *Functional Anal. Appl.* **14** (1980), no. 2, 98–106. MR 0575202 Zbl 0453.35068

[16] F. Kittaneh and Y. Manasrah, Improved Young and Heinz inequalities for matrices. *J. Math. Anal. Appl.* **361** (2010), no. 1, 262–269. MR 2567300 Zbl 1180.15021

[17] P. Kröger, Upper bounds for the Neumann eigenvalues on a bounded domain in Euclidean space. *J. Funct. Anal.* **106** (1992), no. 2, 353–357. MR 1165859 Zbl 0777.35044

[18] A. Laptev, Dirichlet and Neumann eigenvalue problems on domains in Eucilidean spaces. *J. Funct. Anal.* **151** (1997), no. 2, 531–545. MR 1491551 Zbl 0892.35115

[19] A. Laptev, *Spectral inequalities for partial differential equations and their applications.* In L. Ji, Y. Sun Poon, L. Yang and S.-T. Yau, *Fifth International Congress of Chinese Mathematicians.* Part 2. Proceedings of the congress (ICCM 2010) held at Tsinghua University, Beijing, December 17–22, 2010. AMS/IP Studies in Advanced Mathematics, 51, Part 2. American Mathematical Society, Providence, R.I., and International Press, Boston, M.A., 2012, Part 2, 629–643. MR 2908096 Zbl 2919976

[20] S. Larson, On the remainder term of the Berezin inequality on a convex domain. *Proc. Amer. Math. Soc.* **145** (2017), no. 5, 2167–2181. MR 3611329 Zbl 1377.35206

[21] A. D. Melas, A lower bound for sums of eigenvalues of the Laplacian. *Proc. Amer. Math. Soc.* **131** (2003), no. 2, 631–636. MR 1933356 Zbl 1015.58011

[22] R. Melrose, Weyl's conjecture for manifolds with concave boundary. In R. Osserman and A. Weinstein (eds.), *Geometry of the Laplace operator.* Proceedings of the Symposium in Pure Mathematics of the American Mathematical Society held at the University of Hawaii, Honolulu, Hawaii, March 27–30, 1979. Proceedings of Symposia in Pure Mathematics, XXXVI. American Mathematical Society, Providence, R.I., 1980, 257–274. MR 0573438 Zbl 0436.58024

[23] E. Milman, On the mean-width of isotropic convex bodies and their associated L_p-centroid bodies, *Int. Math. Res. Not. IMRN* **2015**, no. 11, 3408–3423. MR 3373054 Zbl 1319.52007

[24] S.-L. Pan and H. Zhang, A reverse isoperimetric inequality for convex plane curves. *Beiträge Algebra Geom.* **48** (2007), no. 1, 303–308. MR 2327085 Zbl 1121.52023

[25] M. Reed and B. Simon, *Methods of modern mathematical physics*. IV. Analysis of operators. Academic Press, New York and London, 1978. MR 0493421 Zbl 0401.47001

[26] T. Weidl, Improved Berezin–Li–Yau inequalities with a remainder term. In T. Suslina and D. Yafaev (eds.), *Spectral theory of differential operators.* M. Sh. Birman 80th anniversary collection. American Mathematical Society Translations, Series 2, 225. Advances in the Mathematical Sciences, 62. American Mathematical Society, Providence, R.I., 2008, 253–263. MR 2509788 Zbl 1168.35404

Received July 8, 2016

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