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Subellipticity at Higher Degree of a Boundary Condition Associated with Construction of the Versal Family of Strongly Pseudo-Convex Domains

By

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Introduction

Let N be a complex manifold of $\dim_C N=n\geq 4$, Ω a relatively compact domain of N with a strongly pseudoconvex boundary $\partial\Omega=M$ and ${}^{\circ}T''$ a CR structure on M induced from the complex structure on N.

In the construction of the versal family of complex structures on $\bar{\Omega}$, it was useful to restrict ourselves to the argument on $T'N$-valued forms which are ${}^{\circ}T'$-valued on M (cf. [2]). In order to accomplish this argument, a new boundary condition for $T'N$-valued $\bar{\partial}$-complex on $\bar{\Omega}$ was introduced (cf. [2]).

A priori estimate for this new boundary condition has not been established at degree q except for $q=2$, though its cohomology groups are isomorphic to usual ones at $2\leq q\leq n-1$ (cf. [2], [3]). The pourpose of this paper is to show that a priori estimate also holds at higher degree:

Main Theorem. *If $2\leq q\leq n-2$, then there exist positive constants c and c' such that*

$$c'\|\phi\|_{1/2}^2\leq\|\phi\|'^2\leq c(\|\bar{\partial}\phi\|^2+\|\vartheta\phi\|^2+\|\phi\|^2)$$

for any $\phi\in\Gamma(\bar{\Omega}, T'N\otimes\Lambda^q(T''N)^*)$ satisfying

$$\tau\phi\in\Gamma(M, E_q) \quad \text{and} \quad \langle\sigma(\vartheta, dr)\phi, y\rangle=0 \quad \text{on } M \text{ for all} \quad y\in E_{q-1}$$

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By Akahori's criterion, to prove Main Theorem is reduced to establish the following a priori estimates for a subbundle E_q of $T'N_{|M}\otimes\wedge^q({}^\circ T'')^*$ and its orthogonal complement $E_q^\perp$ with respect to the Levi metric (cf. [3]).

Theorem 1. *If $2\leq q\leq n-2$, then there exists a positive constant c such that*

$$\|\phi\|'^2\leq c(\|\bar\partial_b\phi\|^2+\|(\vartheta_b\phi)_{E_{q-1}}\|^2+\|\phi\|^2)$$

for any $\phi\in\Gamma(M, E_q)$, where $(\vartheta_b\phi)_{E_{q-1}}$ denotes the orthogonal projection onto E_{q-1} with respect to the Levi metric.

Theorem 2. *If $1\leq q\leq n-3$, then there exists a positive constant c such that*

$$\|\phi\|'^2\leq c(\|(\bar\partial_b\phi)_{E^\perp_{p+1}}\|^2+\|\vartheta_b\phi\|^2+\|\phi\|^2)$$

for any $\phi\in\Gamma(M, E_q^\perp)$.

The proofs of these theorems are higher degree versions of the ones at degree $q=2$ in [2] and $q=1$ in [3] respectively. The righthand side of the estimate in Theorem 1 (resp. in Theorem 2) is the difference of the usual energy form $\|\bar\partial_b\phi\|^2+\|\vartheta_b\phi\|^2+\|\phi\|^2$ and $\|(\vartheta_b\phi)_{E^\perp_{q-1}}\|^2$ (resp. $\|(\bar\partial_b\phi)_{E_{q+1}}\|^2$). We give the expression of $(\vartheta_b\phi)_{E^\perp_{q-1}}$ for $\phi\in\Gamma(M, E_q)$ in §1 and prove Theorem 1 in §2. We, in §2, also prove Theorem 2 by the same commutator calculus as in the proof of Theorem 1, using the expression of $(\bar\partial_b\phi)_{E_{q+1}}$ for $\phi\in\Gamma(M, E_q^\perp)$ given by the duality.

§1. Subcomplexes

Let M be a compact smooth manifold of $\dim_{\mathbf{R}} M=2n-1$ (≥ 7). Let ${}^\circ T''$ be a CR structure on M, that is:

(1) ${}^\circ T'\cap{}^\circ T''=\{0\}$ where ${}^\circ T'={}^\circ\bar T''$,

(2) $\mathbf{C}TM/({}^\circ T'+{}^\circ T'')\simeq\mathbf{C}F$ for some real line bundle F.

We fix a splitting $\mathbf{C}TM={}^\circ T'+{}^\circ T''+\mathbf{C}F$ and denote $T'={}^\circ T'+\mathbf{C}F$. In our case that $M=\partial\Omega$, ${}^\circ T''=\mathbf{C}TM\cap T''N_{|M}$ and $T'\simeq T'N_{|M}$.

We assume that the CR structure is strongly pseudoconvex, that is:

(3) the Levi form ${}^\circ T''+{}^\circ T'\ni(X, Y)\to(1/\sqrt{-1})[X, Y]_{\mathbf{C}F}\in\mathbf{C}F$ is positive definite.

We define vector subbundles E_q and $E_q^\perp$ of $T'\otimes\wedge^q({}^\circ T'')^*$ as in [1] and [2]. For $q\geq 0$, we define E_q by

$$\Gamma(M, E_q)=\{u\in\Gamma(M, T'\otimes\wedge^q({}^\circ T'')^*)\,|\,u\in\Gamma(M, {}^\circ T'\otimes\wedge^q({}^\circ T'')^*)\ \text{and}\ \bar\partial_b u\in\Gamma(M, {}^\circ T'\otimes\wedge^{q+1}({}^\circ T'')^*)\}\ ,$$

and $E_q^\perp$ as the orthogonal complement of E_q with respect to the Levi metric induced from the above Levi form. Then $(\Gamma(M, E_q), \bar{\partial}_b)$ and $(\Gamma(M, E_q^\perp), \vartheta_b)$ form differential complexes.

Let $\{(U_k, h_k)\}_{k\in\Lambda}$ be an atlas of M and $\{\rho_k\}_{k\in\Lambda}$ be a partition of unity subordinate to the covering $\{U_k\}_{k\in\Lambda}$.

If $U\in\{U_k\}_{k\in\Lambda}$, we let $(e_1, \cdots, e_{n-1})$ be a moving frame of ${}^\circ T''_{|U}$ such that

$$[e_i, \bar{e}_j]_{CF} = \sqrt{-1}\,\delta_{ij}\,e_n \tag{1.1}$$

where e_n denotes a real moving frame of $F_{|U}$, and $((e^*)^1, \cdots, (e^*)^{n-1})$ the dual frame of $({}^\circ T'')^*_{|U}$.

On U, $\phi\in\Gamma(M, T'\otimes\wedge^q({}^\circ T'')^*)$ can be written in the usual formalism:

$$\phi = \Sigma_\alpha \Sigma'_I\, \phi_{\alpha,I}\, \bar{e}_\alpha \otimes (e^*)^I \tag{1.2}$$

where $I=(i_1, \cdots, i_q)$ with $i_1<\cdots<i_q$, $(e^*)^I=(e^*)^{i_1}\wedge\cdots\wedge(e^*)^{i_q}$ and Σ' is a summation for suffix not including n.

Lemma 1. *For $\phi\in\Gamma(M, T'\otimes\wedge^q({}^\circ T'')^*)$, ϕ is in $\Gamma(M, E_q)$ if and only if*
(1) $\phi_{n,I}=0$ for any I with $|I|=q$ and
(2) $\Sigma'_{j\in K}\varepsilon_j{}^K{}_I\,\phi_{j,I}=0$ for any K with $|K|=q+1$ and $K\not\ni n$, where $\varepsilon_j{}^K{}_I$ is the signe of the permutation changing $(j, I)=(j, i_1, \cdots, i_q)$ into $K=(k_1, \cdots, k_{q+1})$ if $\{j, i_1, \cdots, i_q\}=\{k_1, \cdots, k_{q+1}\}$ as sets and is zero otherwise.

Proof. (1) is clear because ${}^\circ T'$ is generated by $\bar{e}_1, \cdots, \bar{e}_{n-1}$ only.

(2) $$\begin{aligned}(\bar{\partial}_b\phi)_{CF}(e_{k_1}, \cdots, e_{k_{q+1}}) &= \Sigma_{i=1}^{q+1}(-1)^i[e_{k_i}, \phi(e_{k_1}, \cdots \overset{i}{\vee} \cdots, e_{k_{q+1}})]_{CF}\\ &\quad + \Sigma_{i<j}(-1)^{i+j}\,\phi([e_{k_i}, e_{k_j}], e_{k_1}, \cdots \overset{i}{\vee} \cdots \overset{j}{\vee} \cdots, e_{k_{q+1}})_{CF}\\ &= \Sigma_{i=1}^{q+1}\Sigma'_\alpha(-1)^i\,\phi_{\alpha,k_1,\cdots \overset{i}{\vee} \cdots,k_{q+1}}[e_{k_i}, \bar{e}_\alpha]_{CF}\\ &= \sqrt{-1}(\Sigma_{i=1}^{q+1}(-1)^i\,\phi_{k_i,k_1,\cdots \overset{i}{\vee} \cdots,k_{q+1}})\,e_n\end{aligned}$$ Q.E.D.

The following formula is well known (cf. [4]):
For $\phi\in\Gamma(M, T'\otimes\wedge^q({}^\circ T'')^*)$,

$$(\bar{\partial}_b\phi)_{\alpha,K} = \Sigma'_j\,\varepsilon_j{}^K{}_I\,e_j\,\phi_{\alpha,I}+o(\phi)\,, \tag{1.3}$$

$$(\vartheta_b\phi)_{\alpha,H} = -\Sigma'_j\,\varepsilon_j{}^I{}_H\,e_j\,\phi_{\alpha,I}+o(\phi)\,, \tag{1.4}$$

where $o(\phi)$ denotes a term of order zero.

Lemma 2. *For $\phi\in\Gamma(M, E_q)$,*
(1) $((\vartheta_b\phi)_{E_{q-1}})_{\alpha,H} = -\Sigma'_j\,\varepsilon_j{}^I{}_H\,\bar{e}_j\,\phi_{\alpha,I}+(1/q)\,\varepsilon_\alpha{}^J{}_H\,\Sigma'_j\,\bar{e}_j\,\phi_{j,J}+o(\phi)\,,$
(2) $((\vartheta_b\phi)_{E_{q-1}^\perp})_{\alpha,H} = -(1/q)\,\varepsilon_\alpha{}^J{}_H\,\Sigma'_j\,\bar{e}_j\,\phi_{j,J}+o(\phi)\quad(1\le\alpha\le n-1)\,,$

$((\vartheta_b\phi)_{E^{\perp}_{q-1}})_{n,H} = o(\phi)$.

Proof. Let $U\in\{U_k\}_{K\in\Lambda}$ and λ, $\mu\in\Gamma(U, {}^{\circ}T'\otimes\wedge^{q-1}({}^{\circ}T'')^*)$ defined by

$$\lambda_{\alpha,H} = -\Sigma'_j\, \varepsilon_j{}^I{}_H\, \bar{e}_j\, \phi_{j,I} + (1/q)\, \varepsilon_\alpha{}^J{}_H\, \Sigma'_j\, \bar{e}_j\, \phi_{j,J} \quad (1\leq\alpha\leq n-1)$$

and $$\mu_{\alpha,H} = -(1/q)\, \Sigma'_j\, \varepsilon_\alpha{}^J{}_H\, \bar{e}_j\, \phi_{j,J} \quad (1\leq\alpha\leq n-1)\,.$$

For each fixed J, $\Sigma'_\alpha\, \varepsilon_\alpha{}^J{}_H\, \lambda_{\alpha,H}$

$$= -\Sigma'_{\alpha,j}\, \varepsilon_\alpha{}^J{}_H\, \varepsilon_j{}^I{}_H\, \bar{e}_j\, \phi_{\alpha,I} + (1/q)\, \Sigma'_\alpha\, \varepsilon_\alpha{}^J{}_H\, \varepsilon_\alpha{}^J{}_H\, \Sigma'_j\, \bar{e}_j\, \phi_{j,J}$$
$$= -\Sigma'_j\, \bar{e}_j(\Sigma'_\alpha\, \varepsilon_\alpha{}^J{}_H\, \varepsilon_j{}^I{}_H\, \phi_{\alpha,I} - \phi_{j,J})\,.$$

Now, if $j\in J$ then $\Sigma'_\alpha\, \varepsilon_\alpha{}^J{}_H\, \varepsilon_j{}^I{}_H\, \phi_{\alpha,I} - \phi_{j,J} = 0$, and
if $j\notin J$ then $\Sigma'_\alpha\, \varepsilon_\alpha{}^J{}_H\, \varepsilon_j{}^I{}_H\, \phi_{\alpha,I} - \phi_{j,J}$

$$= -\Sigma'_{\alpha\in K,\alpha\neq j}\, \varepsilon_\alpha{}^K{}_I\, \varepsilon_j{}^K{}_J\, \phi_{\alpha,I} - \phi_{j,J}$$
$$= -\varepsilon_j{}^K{}_J(\Sigma'_{\alpha\in K,\alpha\neq j}\, \varepsilon_\alpha{}^K{}_I\, \phi_{\alpha,I} + \varepsilon_j{}^K{}_J\, \phi_{j,J})$$
$$= 0, \text{ because } \phi\in\Gamma(M, E_q)\,.$$

Hence $\lambda\in\Gamma(U, E_{q-1})$ by Lemma 1.

For $\psi\in\Gamma(U, E_{q-1})$,

$$\langle\psi, \mu\rangle = -(1/q)\, \Sigma'_\alpha\, \Sigma'_H\, \Sigma'_j\, \varepsilon_\alpha{}^J{}_H \langle\psi_{\alpha,H}, \bar{e}_j\, \phi_{j,J}\rangle$$
$$= -(1/q)\, \Sigma'_j\, \Sigma'_J \langle\Sigma'_\alpha\, \varepsilon_\alpha{}^J{}_H\, \psi_{\alpha,H}, \bar{e}_j\, \phi_{j,J}\rangle = 0\,.$$

Hence $\mu\in\Gamma(U, E^{\perp}_{q-1})$.

Therefore, by (1.4), we have our lemma.

§ 2. A Priori Estimates

With the expression (1.2) we introduce the norm $||\ \ ||'$ as follows:

$$||\phi||'^2 = \Sigma_{k\in\Lambda}\, \Sigma_\alpha\, \Sigma'_I\, \Sigma'_i \{||e_i\, \rho_k\, \phi_{\alpha,I}||^2 + ||\bar{e}_i\, \rho_k\, \phi_{\alpha,I}||^2\} + ||\phi||^2\,.$$

The main pourpose of this section is to prove Theorems 1 and 2 which are proven at $q=2$ in [1] and at $q=1$ in [2] respectively.

Proof of Theorem 1.

Let $U\in\{U_k\}_{k\in\Lambda}$, and we may assume that Supp $\phi\subset U$.

$$||\bar{\partial}_b\phi||^2 + ||(\vartheta_b\phi)_{E_{q-1}}||^2 = ||\bar{\partial}_b\phi||^2 + ||\vartheta_b\phi||^2 - ||(\vartheta_b\phi)_{F^{\perp}_{q-1}}||^2\,.$$

By Lemma 2 (2),

$$||(\vartheta_b\phi)_{E^{\perp}_{q-1}}||^2 = (1/q^2)\, \Sigma'_\alpha\, \Sigma'_H (\varepsilon_\alpha{}^A{}_H)^2 ||\Sigma'_k\, \bar{e}_k\, \phi_{k,A}||^2 + o(||\phi||'||\phi||)$$
$$= (1/q)\, \Sigma'_I\, \Sigma'_{i,j} \langle\bar{e}_i\, \phi_{i,I}, \bar{e}_j\, \phi_{j,I}\rangle + o(||\phi||'||\phi||)\,.$$

By a standard calculation (cf. [4]), we have

(2.1)
$$\begin{aligned}&\|\bar\partial_b\phi\|^2+\|(\vartheta_b\phi)_{E_{q-1}}\|^2\\&=\Sigma'_\alpha\,\Sigma'_I\,\Sigma'_{j\notin I}\|e_j\,\phi_{\alpha,I}\|^2+\Sigma'_\alpha\,\Sigma'_I\,\Sigma'_{i\in I}\|\bar e_i\,\phi_{\alpha,I}\|^2\\&\quad-(1/q)\,\Sigma'_I\,\Sigma'_{i,j}\langle\bar e_i\,\phi_{i\ I},\,\bar e_j\,\phi_{j,I}\rangle+o(\|\phi\|'\|\phi\|)\,.\end{aligned}$$

Since
$$\begin{aligned}&\Sigma'_I\,\Sigma'_{i\neq j}\langle\bar e_i\,\phi_{i,I},\,\bar e_j\,\phi_{j,I}\rangle\\&=\Sigma'_I\,\Sigma'_{i\neq j}\langle e_j\,\phi_{j,I},\,e_i\,\phi_{j,I}\rangle+o(\|\phi\|'\|\phi\|)\\&\leq\Sigma'_I\,\Sigma'_{i\neq j}\|e_j\,\phi_{i,I}\|^2+o(\|\phi\|'\|\phi\|)\,,\end{aligned}$$

we have
$$\begin{aligned}&\|\bar\partial_b\phi\|^2+\|(\vartheta_b\phi)_{E_{q-1}}\|^2\\&=\Sigma'_I\,\{\Sigma'_{\alpha\notin I}(\|e_\alpha\,\phi_{\alpha,I}\|^2+\Sigma'_{j\notin I,j\neq\alpha}\|e_j\,\phi_{\alpha,I}\|^2+\Sigma'_{i\in I}\|\bar e_i\,\phi_{\alpha,I}\|^2\\&\quad-(1/q)\|\bar e_\alpha\,\phi_{\alpha,I}\|^2-(1/q)\,\Sigma'_{j\notin I,j\neq\alpha}\|e_j\,\phi_{\alpha,I}\|^2\\&\quad-(1/q)\,\Sigma'_{i\in I}\|e_i\,\phi_{\alpha,I}\|^2)\\&\quad+\Sigma'_{\alpha\in I}(\|\bar e_\alpha\,\phi_{\alpha,I}\|^2+\Sigma'_{i\in I,i\neq\alpha}\|\bar e_i\,\phi_{\alpha,I}\|^2+\Sigma'_{j\notin I}\|e_j\,\phi_{\alpha,I}\|^2\\&\quad-(1/q)\|\bar e_\alpha\,\phi_{\alpha,I}\|^2-(1/q)\,\Sigma'_{i\in I,i\neq\alpha}\|e_i\,\phi_{\alpha,I}\|^2\\&\quad-(1/q)\,\Sigma'_{j\notin I}\|e_j\,\phi_{\alpha,I}\|^2)+o(\|\phi\|'\|\phi\|)\,,\end{aligned}$$

where, if $q=n-2$, the terms $\Sigma'_{j\notin I,j\neq\alpha}\|e_j\,\phi_{\alpha,I}\|^2$ and $-(1/q)\,\Sigma'_{j\notin I,j\neq\alpha}\|e_j\,\phi_{\alpha,I}\|^2$ do not appear.

We devide the proof into two parts.

The following fact plays an essential role in the proof.

Lemma 3. $\|e_i\,\phi_{\alpha,I}\|^2+\|\bar e_j\,\phi_{\alpha,I}\|^2$
$$=\|\bar e_i\,\phi_{\alpha,I}\|^2+\|e_j\,\phi_{\alpha,I}\|^2+o(\|\phi\|'\|\phi\|)\,.$$

Proof. By (1.1),

$$\begin{aligned}\|e_i\,\phi_{\alpha,I}\|^2&=\|\bar e_i\,\phi_{\alpha,I}\|^2+\sqrt{-1}\langle e_n\,\phi_{\alpha,I},\,\phi_{\alpha,I}\rangle+o(\|\phi\|'\|\phi\|)\,,\\\|\bar e_j\,\phi_{\alpha,I}\|^2&=\|e_j\,\phi_{\alpha,I}\|^2-\sqrt{-1}\langle e_n\,\phi_{\alpha,I},\,\phi_{\alpha,I}\rangle+o(\|\phi\|'\|\phi\|)\,.\end{aligned}$$

Q.E.D.

(I) The case $2\leq q\leq n-3$ ($n\geq 5$).

By Lemma 3, if $a+(n-q-2)\,b=qc$, we have

$$\begin{aligned}&\|e_\alpha\,\phi_{\alpha,I}\|^2+\Sigma'_{j\notin I,j\neq\alpha}\|e_j\,\phi_{\alpha,I}\|^2+\Sigma'_{i\in I}\|\bar e_i\,\phi_{\alpha,I}\|^2\\&\quad-(1/q)\|\bar e_\alpha\,\phi_{\alpha,I}\|^2-(1/q)\,\Sigma'_{j\notin I,j\neq\alpha}\|e_j\,\phi_{\alpha,I}\|^2\\&\quad-(1/q)\,\Sigma'_{i\in I}\|e_i\,\phi_{\alpha,I}\|^2\\&=(1-a)\|e_\alpha\,\phi_{\alpha,I}\|^2+(1-b-(1/q))\,\Sigma'_{j\notin I,j\neq\alpha}\|\bar e_j\,\phi_{\alpha,I}\|^2\\&\quad+(1-c)\,\Sigma'_{i\in I}\|\bar e_i\,\phi_{\alpha,I}\|^2\\&\quad+(a-(1/q))|\bar e|_\alpha\,\phi_{\alpha,I}\|^2+b\Sigma'_{j\notin I,j\neq\alpha}\|e_j\,\phi_{\alpha,I}\|^2\\&\quad+(c-(1/q))\,\Sigma'_{i\in I}\|e_i\,\phi_{\alpha,I}\|^2+o(\|\phi\|'\|\phi\|)\,.\end{aligned}$$

Hence, if we can choose a, b and c satisfying:

(2.2) $1/q<a<1$, $0<b<1-(1/q)$, $1/q<c<1$ and $a+(n-q-2)b=qc$,

we have

$$\begin{aligned}&\|e_\alpha\,\phi_{\alpha,I}\|^2+\Sigma'_{j\neq I,j\neq\alpha}\|e_j\,\phi_{\alpha,I}\|^2+\Sigma'_{i\in I}\|\bar{e}_i\,\phi_{\alpha,I}\|^2\\&\quad-(1/q)\|\bar{e}_\alpha\,\phi_{\alpha,I}\|^2-(1/q)\,\Sigma'_{j\notin I,j\neq\alpha}\|e_j\,\phi_{\alpha,I}\|^2\\&\quad-(1/q)\,\Sigma'_{i\in I}\|\bar{e}_i\,\phi_{\alpha,I}\|^2\\&\geq C_1\|\phi\|'^2+o(\|\phi\|'\|\phi\|)\,,\end{aligned}$$

where C_1 denotes a positive constant.

Lemma 4. *If $n\geq 5$ and $2\leq q\leq n-3$ then it is possible to choose a, b and c such that* (2.2) *is saiisfied.*

Proof. If $(1/q)<a<1$ and $0<b<1-(1/q)$ then $1/q<a+(n-q-2)\,b<(-q^2+nq-n+2)/q$. Since $n>4$ and $1<q<n-2$, $(-q^2+nq-n+2)/q^2>(1/q)$ holds. Q.E.D.

Similarly we have

$$\begin{aligned}&\|\bar{e}_\alpha\,\phi_{\alpha,I}\|^2+\Sigma'_{i\in I,i\neq\alpha}\|\bar{e}_i\,\phi_{\alpha,I}\|^2+\Sigma'_{j\notin I}\|e_j\,\phi_{\alpha,I}\|^2\\&\quad-(1/q)\|\bar{e}_\alpha\,\phi_{\alpha,I}\|^2\;-(1/q)\,\Sigma'_{i\in I,i\neq\alpha}\|e_i\,\phi_{\alpha,I}\|^2\\&\quad-(1/q)\,\Sigma'_{j\notin I}\|e_j\,\phi_{\alpha,I}\|^2\\&\geq C_2\|\phi\|'^2+o(\|\phi\|'\|\phi\|)\,,\end{aligned}$$

if we can choose a, b and c satisfying:

(2.3) $$\begin{aligned}&0<a<1-(1/q),\ (1/q)<b<1,\ 0<c<1-(1/q)\\&\text{and}\quad a+(q-1)\,b=(n-q-1)\,c\,,\end{aligned}$$

where C_2 denotes a positive constant.

Lemma 5. *If $q\leq n-3$, then it is possible to choose a, b and c such that* (2.3) *is satisfied.*

Proof. If $0<a<1-(1/q)$ and $(1/q)<b<1$ then $(q-1)/q<a+(q-1)\,b<(q^2-1)/q$. Since $q<n-2$, $(q-1)/q(n-q-1)<(q-1)/q$ holds. Q.E.D.

Therefore we have

$$\begin{aligned}\|\bar{\partial}_b\phi\|^2+\|(\vartheta_b\phi)_{E_{q-1}}\|^2&\geq C_3\|\phi\|'^2+o(\|\phi\|'\|\phi\|)\\&\geq C_3\|\phi\|'^2-\varepsilon\|\phi\|'^2-(K/\varepsilon)\|\phi\|^2\,,\end{aligned}$$

where C_3 and K denote positive constants.

This follows

$$||\bar{\partial}_b\phi||^2+||(\vartheta_b\phi)_{E_{q-1}}||^2+||\phi||^2\geq C||\phi||'^2 .$$

(II) The case $q=n-2$ $(n\geq 4)$.

By Lemma 3,

$$\begin{aligned}&||e_\alpha \phi_{\alpha,I}||^2+\Sigma'_{i\in I}||\bar{e}_i \phi_{\alpha,I}||^2\\ &\quad -(1/q)||\bar{e}_\alpha \phi_{\alpha,I}||^2-(1/q)\,\Sigma'_{i\in I}||e_i \phi_{\alpha,I}||^2\\ &\quad =(1-(1/q))\{||\bar{e}_\alpha \phi_{\alpha,I}||^2+\Sigma'_{i\in I}||\bar{e}_i \phi_{\alpha,I}||^2+o(||\phi||'||\phi||)\,,\end{aligned}$$

and

$$\begin{aligned}&||\bar{e}_\alpha \phi_{\alpha,I}||^2+\Sigma'_{i\in I,i\neq\alpha}||\bar{e}_i \phi_{\alpha,I}||^2+\Sigma'_{j\notin I}||e_j \phi_{\alpha,I}||^2\\ &\quad -(1/q)||\bar{e}_\alpha \phi_{\alpha,I}||^2-(1/q)\,\Sigma'_{i\in I,i\neq\alpha}||e_i \phi_{\alpha,I}||^2\\ &\qquad -(1/q)\,\Sigma'_{j\notin I}||e_j \phi_{\alpha,I}||^2\\ &\geq(1-1/q)\{||e_\alpha \phi_{\alpha,I}||^2+\Sigma'_{i\in I,i\neq\alpha}||\bar{e}_i \phi_{\alpha,I}||^2+\Sigma'_{j\notin I}||\bar{e}_j \phi_{\alpha,I}||^2\}\\ &\quad +o(||\phi||'||\phi||)\,.\end{aligned}$$

Hence we have

$$(2.4)\qquad ||\bar{\partial}_b\phi||^2+||(\vartheta_b\phi)_{E_{q-1}}||^2\geq(1-(1/q))\,\Sigma'_\alpha\,\Sigma'_I\,\Sigma'_i||\bar{e}_i \phi_{\alpha,I}||^2 +o(||\phi||'||\phi||)\,.$$

Substituting (2.4) into (2.1), we have

$$\begin{aligned}&||\bar{\partial}_b\phi||^2+||(\vartheta_b\phi)_{E_{q-1}}||^2\\ &\quad\geq\Sigma'_\alpha\,\Sigma'_I\,\Sigma'_{j\notin I}||e_j \phi_{\alpha,I}||^2+\Sigma'_\alpha\,\Sigma'_I\,\Sigma'_{i\in I}||\bar{e}_i \phi_{\alpha,I}||^2\\ &\qquad -(1/q)\,\Sigma'_I\,\Sigma'_{i,j}(||\bar{\partial}_b\phi||^2+||(\vartheta_b\phi)_{E_{q-1}}||^2)+o(||\phi||'||\phi||)\,.\end{aligned}$$

Hence

$$\begin{aligned}&||\bar{\partial}_b\phi||^2+||(\vartheta_b\phi)_{E_{q-1}}||^2\\ &\quad\geq C_4(\Sigma'_\alpha\,\Sigma'_I\,\Sigma'_{j\notin I}||e_j \phi_{\alpha,I}||^2+\Sigma'_\alpha\,\Sigma'_I\,\Sigma'_{i\in I}||\bar{e}_i \phi_{\alpha,I}||^2)+o(||\phi||'||\phi||),\end{aligned}$$

where C_4 is a positive constant.

Thus, by the same calculus as above, we have

$$||\bar{\partial}_b\phi||^2+||(\vartheta_b\phi)_{E_{q-1}}||^2\geq C_5||\phi||'^2+o(||\phi||'||\phi||)\,.$$

This completes the proof of Theorem 1.

Proof of Theorem 2.

We may assume Supp $\phi\subset U$ for some $U\in\{U_k\}_{k\in A}$ as in the proof of Theorem 1.

We first prove some lemmas about $(\bar{\partial}_b\phi)_{E_{q+1}}$.

Lemma 6. *Let $\phi\in\Gamma(U, E_q^+)$ and $\lambda\in\Gamma(U, T'\otimes\wedge^{q+1}({}^\circ T'')^*)$ defined by*

$$\lambda_{\alpha,K}=(1/(q+1))\,\Sigma'_{j\in K}\,\varepsilon_j{}^K{}_H\,e_\alpha\,\phi_{j,H}\quad(1\leq\alpha\leq n-1)\,,$$

$$\lambda_{n,K} = 0\,.$$

Then $(\bar\partial_b\phi)_{E_{q+1}} = (\lambda)_{E_{q+1}} + o(\phi)$.

Proof. For $\psi \in \Gamma(U, E_{q+1})$,

$$\begin{aligned}\langle(\bar\partial_b\phi)_{E_{q+1}}, \psi\rangle &= \langle\bar\partial_b\phi, \psi\rangle = \langle\phi, \vartheta_b\psi\rangle = \langle\phi, (\vartheta_b\psi)_{E_q^\perp}\rangle \\ &= -(1/(q+1))\, \Sigma'_\alpha\, \Sigma'_j\, \Sigma'_K\, \varepsilon_\alpha{}^K{}_I\langle\phi_{\alpha,I}, \bar e_j\, \psi_{j,K}\rangle + o(\phi, \bar\psi) \quad \text{(by Lemma 2 (2))} \\ &= \Sigma'_j\, \Sigma'_K\langle\lambda_{j,K}, \psi_{j,K}\rangle + o(\phi, \bar\psi).\end{aligned}$$

Q.E.D.

Lemma 7. *If* $\phi \in \Gamma(U, E_q^\perp)$ *then we have*

$$\|(\bar\partial_b\phi)_{E_{q+1}}\|^2 \le (1/(q+1))\, \Sigma'_\alpha\, \Sigma'_J\, \Sigma'_{i\notin J}\|e_\alpha\, \phi_{j,J}\|^2 + o(\|\phi\|'\|\phi\|)\,.$$

Proof. By Lemma 6, $\|(\bar\partial_b\phi)_{E_{q+1}}\|^2 \le \|\phi\|^2$

$$\begin{aligned}&= (1/(q+1)^2)\, \Sigma'_\alpha\, \Sigma'_K\, \Sigma'_{i\in K, j\in K}\, \varepsilon_j{}^K{}_J\langle e_\alpha\, \phi_{i,J}, e_\alpha\, \phi_{j,I}\rangle \\ &\quad + o(\|\phi\|'\|\phi\|) \\ &\le (1/(q+1))\, \Sigma'_\alpha\, \Sigma'_J\, \Sigma'_{i\notin J}\|e_\alpha\, \phi_{i,J}\|^2 + o(\|\phi\|'\|\phi\|)\,.\end{aligned}$$

Q.E.D.

By Lemma 7,

$$\begin{aligned}&\|(\bar\partial_b\phi)_{E_{q+1}^\perp}\|^2 + \|\vartheta_b\phi\|^2 \\ &= \|\bar\partial_b\phi\|^2 + \|\vartheta_b\phi\|^2 - \|(\bar\partial_b\phi)_{E_{q+1}}\|^2 \\ &\ge \|\bar\partial_b\phi\|^2 + \|\vartheta_b\phi\|^2 \\ &\quad -(1/(q+1))\, \Sigma'_\alpha\, \Sigma'_J\, \Sigma'_{i\notin J}\|e_\alpha\, \phi_{i,J}\|^2 + o(\|\phi\|'\|\phi\|) \\ &= \Sigma'_\alpha\, \Sigma'_J\, \Sigma'_{i\notin J}\|e_i\, \phi_{\alpha,J}\|^2 + \Sigma'_\alpha\, \Sigma'_I\, \Sigma'_{i\in I}\|\bar e_i\, \phi_{\alpha,I}\|^2 \\ &\quad -(1/(q+1))\, \Sigma'_\alpha\, \Sigma'_J\, \Sigma'_{i\notin J}\|e_\alpha\, \phi_{i,J}\|^2 + o(\|\phi\|'\|\phi\|)\,, \quad \text{(by (1.4))}\,.\end{aligned}$$

By the same argument in the proof of Theorem 1, if $1 \le q \le n-2$, we have

$$\begin{aligned}&\|(\bar\partial_b\phi)_{E_{q+1}^\perp}\|^2 + \|\vartheta_b\phi\|^2 \\ &\quad \ge c(\Sigma'_\alpha\, \Sigma'_I\, \Sigma'_i\|e_i\, \phi_{\alpha,I}\|^2 + \Sigma'_\alpha\, \Sigma'_I\, \Sigma'_i\|\bar e_i\, \phi_{\alpha,I}\|^2) + o(\|\phi\|'\|\phi\|)\,,\end{aligned}$$

where c denotes a positive constant.

Therefore Theorem 2 follows.

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