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Steklov and Robin isospectral manifolds

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Abstract. We use two of the most fruitful methods for constructing isospectral manifolds, the Sunada method and the torus action method, to construct manifolds whose Dirichlet-to-Neumann operators are isospectral at all frequencies. The manifolds are also isospectral for the Robin boundary value problem for all choices of Robin parameter. As in the sloshing problem, we can also impose mixed Dirichlet–Neumann conditions on parts of the boundary. Among the examples we exhibit are Steklov isospectral flat surfaces with boundary, planar domains with isospectral sloshing problems, and Steklov isospectral metrics on balls of any dimension greater than 5. In particular, the latter are the first examples of Steklov isospectral manifolds of dimension greater than 2 that have connected boundaries.

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1. Introduction

Inverse spectral problems on compact Riemannian manifolds ask to what extent geometric and topological data are encoded in the spectra of natural operators. There is an extremely rich literature of both positive and negative results in the case of the Laplace–Beltrami operator on compact manifolds, with Dirichlet or Neumann boundary conditions (or mixed conditions) imposed when the boundary is nonempty. The literature for other natural operators lags behind. The goal of this article is to show that most of the negative results for the Laplace–Beltrami operator in the literature, i.e., the constructions of manifolds whose Laplace–Beltrami operators are isospectral, are equally valid for other natural operators. We were motivated primarily by the surge of interest in Steklov eigenvalue problems and the related "sloshing problem" on compact Riemannian manifolds with boundary, so we will focus primarily on these problems. However, we will also comment on other eigenvalue problems.

1.1. Steklov eigenvalue problems. Let (M, g) be a compact smooth Riemannian manifold with boundary, and let Δ be the associated Laplace–Beltrami operator. For any $\alpha \in \mathbb{R}$ that is not in the spectrum of the Dirichlet Laplacian, and for $\rho \in C^\infty(\partial M)$, the *Steklov spectrum* of M at frequency α with boundary density ρ, denoted by $\mathrm{Stek}_\alpha(M, g, \rho)$ or simply by $\mathrm{Stek}_\alpha(M, g)$ if $\rho \equiv 1$, is the collection of real numbers σ for which there exists a nontrivial solution $u \in C^\infty(M)$ to the eigenvalue problem

$$\begin{cases} \Delta u = \alpha u & \text{on } M \smallsetminus \partial M, \\ \partial_\nu u = \sigma \rho u & \text{on } \partial M, \end{cases} \tag{1}$$

where $\partial_\nu u$ is the normal derivative of u on the boundary. (The problem is well-defined, since it was required that α not be a Dirichlet eigenvalue of Δ.) In two dimensions, $\mathrm{Stek}_0(M, g, \rho)$ corresponds to the collection of squares of eigenfrequencies of a drum all of whose mass is distributed along the boundary according to the density ρ (see [37]). When $\rho \equiv 1$, the Steklov spectrum $\mathrm{Stek}_\alpha(M, g)$ is precisely the eigenvalue spectrum of the *Dirichlet-to-Neumann operator* $\mathcal{D}_\alpha^{(M,g)}\colon C^\infty(\partial M) \to C^\infty(\partial M)$. This operator associates to a function $v \in C^\infty(\partial M)$ the normal derivative of the unique extension $V\colon M \to \mathbb{R}$ of v to M that satisfies $\Delta V = \alpha V$. In particular, when $\alpha = 0$, the extension V is harmonic, so is just the solution of the Dirichlet problem with initial data v. We remark that if the boundary density function ρ is merely L^∞, then (1) is still a well-defined eigenvalue problem, although the eigenfunctions are merely H^1 rather than smooth, and the boundary condition in (1) is interpreted in the sense of the Sobolev trace.

The so-called *sloshing problem*, describing oscillations of a fluid in an open container, is the special case of the Steklov problem (1) in which ρ takes on only the values 0 and 1: $\rho \equiv 0$ on the walls of the container and $\rho \equiv 1$ on the free surface of the fluid.

In dimension two, the Steklov spectrum $\mathrm{Stek}_0(M, g, \rho)$ is invariant under conformal changes of metric away from the boundary; i.e., if $g' = e^f g$ with $f \equiv 0$ on ∂M, then $\mathrm{Stek}_0(M, g, \rho) = \mathrm{Stek}_0(M, g', \rho)$. In fact, we even have $\mathcal{D}_0^{(M,g')} = \mathcal{D}_0^{(M,g)}$. (This is immediate from the fact that the Laplacian of g' is related to that of g by $\Delta' = e^{-f}\Delta$ in dimension two. In higher dimensions, this equality fails.) We will say that (M, g, ρ) and (M', g', ρ') are *trivially Steklov isospectral* for $\alpha = 0$ if there exists a diffeomorphism F from M to M' intertwining ρ and ρ' such that either (i) $F\colon (M, g) \to (M', g')$ is an isometry or (ii) $\dim(M) = 2$ and $F^*g' = e^f g$ with $f|_{\partial M} = 0$. We caution that such conformal changes of metric will in general affect $\mathrm{Stek}_\alpha(M, g, \rho)$ for $\alpha \neq 0$, even in dimension two.

The Steklov spectrum was first introduced by A. Steklov in 1902 and has since found many remarkable applications; see the historical article [36]. For example, by examining the dependence of $\mathrm{Stek}_\alpha(M, g)$ on the parameter α, Friedlander [20] derived an inequality between the Neumann and Dirichlet eigenvalues of bounded C^1-domains in $\mathbb{R}^n$; this inequality was extended to Lipschitz domains by Arendt and Mazzeo [1]. The study of the Steklov spectrum has recently gained impetus; see, for example, [7, 16, 18, 17, 19, 23, 22, 33, 43, 34, 48], and the excellent survey [24]. E.g., $\mathrm{Stek}_0(M, g)$ is known to determine the dimension and volume of ∂M, the geometry of ∂M if $\dim(M) = 2$ [22], whether a domain in $\mathbb{R}^2$ is a disk [22], and whether a domain in $\mathbb{R}^3$ with connected boundary is a ball [43]. Interest in the Steklov problem is also motivated in part by various results suggesting that any sufficiently regular metric g on M can be recovered from the Dirichlet-to-Neumann operator $\mathcal{D}_0^{(M,g)}\colon C^\infty(\partial M) \to C^\infty(\partial M)$; see [39] and [38] for the cases of surfaces and for real analytic manifolds of all dimensions. These are instances of Calderón's inverse problem [15] for electrical impedance tomography, which asks whether a body's conductivity can be determined from current and voltage measurements on its boundary.

In this article we adapt to the Steklov setting the two primary techniques for constructing Laplace isospectral manifolds: Sunada's technique [47] and the torus action method (see, e.g., [25, 26, 45, 46]). Both techniques yield pairs of Riemannian manifolds M_1 and M_2 with boundary that are simultaneously Dirichet and Neumann isospectral and that also satisfy $\mathrm{Stek}_\alpha(M_1, g_1) = \mathrm{Stek}_\alpha(M_2, g_2)$ for all α not in the Dirichlet spectrum. Moreover, $\mathrm{Stek}_\alpha(M_1, g_1, \rho_1) = \mathrm{Stek}_\alpha(M_2, g_2, \rho_2)$ for a large family of pairs of densities (ρ_1, ρ_2). The Laplace–Beltrami operators on the boundaries are also isospectral. (In some, but not all cases, the boundaries are isometric.)

We illustrate these techniques with nontrivial examples:

- pairs of (nonplanar) flat Steklov isospectral surfaces embedded in $\mathbb{R}^3$ constructed via the Sunada method;
- continuous families of mutually Steklov isospectral nonflat metrics on a ball in $\mathbb{R}^n$ constructed by the torus action method.

Specializing to the sloshing problem, we obtain, for example,

- pairs of planar domains that are isospectral for the sloshing problem.

Referencing our results, the article [2] gives examples of Steklov isospectral orbifolds using the Sunada and torus action techniques. Example 6.1 in the same article uses direct computation to give examples of orbifold quotients $\Gamma_1\backslash B$ and $\Gamma_2\backslash B$ of Euclidean balls with $\mathrm{Stek}_0(\Gamma_1\backslash B) = \mathrm{Stek}_0(\Gamma_2\backslash B)$. Lemma 6.1

of [17] establishes that cylinders over Laplace–Beltrami isospectral closed manifolds have the same Steklov spectrum, again with $\alpha = 0$. To our knowledge, these examples exhaust the nontrivial examples of Steklov isospectral manifolds in the literature.

There are various notions of Dirichlet-to-Neumann operator acting on the space of p-forms on the boundary of a manifold. The definitions in [44] and [35] (the latter being a modification of a definition in [3]) give operators with discrete spectrum. The Sunada method goes through for these Steklov spectra on p-forms. However, the torus action method does not. (This is not unexpected: the torus action method for the Laplace–Beltrami operator produces manifolds that are isospectral on functions, but it does not establish isospectrality for the Hodge Laplacian on p-forms.)

1.2. Robin eigenvalue problems. The Robin boundary value problem is dual to the Steklov eigenvalue problem in the following sense. Set $\rho \equiv 1$. Fixing a given $\sigma \in \mathbb{R}$ and interpreting (1) as an eigenvalue problem for an unknown α converts (1) into an eigenvalue problem with Robin boundary conditions. Since the Steklov isospectral manifolds that we construct satisfy $\mathrm{Stek}_\alpha(M_1, g_1) = \mathrm{Stek}_\alpha(M_2, g_2)$ for *every* allowable choice of the parameter α, they will also be isospectral for the Robin boundary value problem for every choice of the Robin parameter σ. See [1] for historical comments on this relationship between the Steklov and Robin problems.

The Sunada and torus action methods work equally well for the mixed Robin–Neumann–Dirichlet eigenvalue problem. This problem asks for which $\alpha \in \mathbb{R}$ there exists $u \in C^\infty(M)$, with normal derivative $\partial_\nu u \in C^\infty(\partial M)$, such that

$$\Delta u = \alpha u \quad \text{on } M \smallsetminus \partial M, \tag{2a}$$
$$u = 0 \quad \text{on } D, \tag{2b}$$
$$\partial_\nu u = 0 \quad \text{on } N, \tag{2c}$$
$$\partial_\nu u = \sigma u \quad \text{on } S. \tag{2d}$$

where $\partial M = S \sqcup N \sqcup D$ (set-theoretic disjoint union) and where σ is again a fixed Robin parameter. In case $D = \varnothing$, then the mixed Robin–Neumann problem is dual in the sense above to the Steklov problem with boundary density $\rho \equiv 1$ on S and $\rho \equiv 0$ on N.

1.3. Other eigenvalue problems. Both the Sunada method and the torus action method are very robust. We remark without proof that both methods easily extend, for example, to poly-Laplacians Δ^m with Dirichlet boundary conditions $u = \partial_\nu u = \partial_\nu^2 u = \cdots = \partial_\nu^{m-1} u = 0$ on ∂M, as in the clamped plate problem where $m = 2$.

The paper is organized as follows. In §2 and §3, we adapt the Sunada method and the torus action method, respectively, to the Steklov settings. Examples constructed via the two methods are given in §4. Finally, in §5, we construct Steklov isospectral boundary density functions: more precisely, we adapt both the Sunada method and the torus action method using an idea introduced by R. Brooks in order to construct pairs of boundary density functions ρ_1 and ρ_2 on a compact Riemannian manifold M with boundary such that $\mathrm{Stek}_\alpha(M, \rho_1) = \mathrm{Stek}_\alpha(M, \rho_2)$ for all α not in the Dirichlet spectrum of M.

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2. The Sunada method

We adapt the Sunada method [47] to the context of the Steklov spectra.

Definition 2.1. Let G be a finite group. Two subgroups H and H' of G are called *almost conjugate* or *Gassmann equivalent* if every $g \in G$ has equally many conjugates in H and H'.

Remark 2.2. Gassmann [21] used such almost conjugate subgroups of a finite group to exhibit examples of pairs of nonisomorphic algebraic number fields with the same arithmetic (i.e., the same Dedekind zeta function). The formula for the character of an induced representation shows easily that H and H' are almost conjugate if and only if the representations of G induced from the trivial one-dimensional representations of H and H' are equivalent: i.e., $\mathrm{Ind}_H^G(\mathbf{1}_H) \cong \mathrm{Ind}_{H'}^G(\mathbf{1}_{H'})$, where $\mathbf{1}_H$ and $\mathbf{1}_{H'}$ denote the trivial one-dimensional representations of H and H', respectively.

Theorem 2.3 (Sunada's Theorem adapted to the Steklov setting). *Let H and H' be almost conjugate subgroups of a finite group G. Assume that G acts by isometries on a compact Riemannian manifold M with boundary and that the restrictions of the action to the subgroups H and H' are free. Let ρ be an L^∞, nonnegative, G-invariant function on ∂M. Continue to denote by g and ρ the Riemannian metric and the function induced on each of the orbit spaces $H\backslash M$ and $H'\backslash M$ by g and ρ. Then,*

$$\mathrm{Stek}_\alpha(H\backslash M, g, \rho) = \mathrm{Stek}_\alpha(H'\backslash M, g, \rho)$$

for all α not in the Dirichlet spectrum of $H\backslash M$ and $H'\backslash M$. (Sunada's original theorem guarantees that the two quotient manifolds are both Dirichlet and Neumann isospectral, so the allowable choices of α are the same in both cases.)

Proof. Fix α and ρ as in the theorem. We will abuse language and refer to solutions u of equation (1) in the Introduction as *σ-eigenfunctions* for the (α, ρ)-Steklov problem on M.

In what follows, if Γ is any group acting linearly on a vector space V, we denote by V^Γ the subspace of Γ-fixed vectors in V.

There are numerous simple and elegant proofs of Sunada's original theorem, some of which compare the dimension of each eigenspace in the two manifolds. These proofs go through without change in our setting. The σ-eigenfunctions for the (α, ρ)-Steklov problem on each of the quotient manifolds $H\backslash M$ and $H'\backslash M$ pull back to H-invariant, respectively H'-invariant, σ-eigenfunctions for the (α, ρ)-Steklov problem on M. Thus letting $E_\sigma \subseteq C^\infty(M)$ be the σ-eigenspace for the (α, ρ)-Steklov problem on M, we need only show that the subspaces E_σ^H and $E_\sigma^{H'}$ of H-invariant and H'-invariant functions, respectively, have the same dimension. Hence the proof of Theorem 2.3 reduces to the following lemma.

Lemma 2.4. *Let H and H' be almost conjugate subgroups of a finite group G and let V be any vector space on which G acts. Then* $\dim(V^H) = \dim(V^{H'})$.

T. Sunada [47] gave an elementary proof of this lemma by a trace formula; see also [14], p. 295. H. Pesce [42] gave a representation theoretic proof by applying Remark 2.2 along with Frobenius reciprocity to obtain

$$\dim(V^H) = [\mathbf{1}_H : \mathrm{Res}_H^G(V)] = [\mathrm{Ind}_H^G(\mathbf{1}_H) : V],$$

where $[U : W]$ denotes the multiplicity of the representation U in W. Since $\mathrm{Ind}_H^G(\mathbf{1}_H)$ and $\mathrm{Ind}_{H'}^G(\mathbf{1}_{H'})$ are equivalent, it follows that $\dim(V^H) = \dim(V^{H'})$. $\square$

Remarks 2.5. We note a couple of features of the Sunada construction.

(1) Lemma 2.4 says that the vector spaces V^H and $V^{H'}$ are isomorphic. In fact, the equivalence τ between the induced representations $\mathrm{Ind}_H^G(\mathbf{1}_H)$ and $\mathrm{Ind}_{H'}^G(\mathbf{1}_{H'})$ actually yields an explicit and natural isomorphism $\tau^\sharp\colon V^{H'} \to V^H$, which Peter Buser and Pierre Bérard [12, 4] called *transplantation*. See also [49], [10], [28].

(2) If H and H' are conjugate subgroups of G, then the resulting quotient manifolds $H\backslash M$ and $H'\backslash M$ are isometric. Even when H and H' are not conjugate, the quotient manifolds may be accidentally isometric. Thus one must always verify nontriviality when using Sunada's technique (in fact, when using any of the known techniques for constructing isospectral manifolds).

More important for our purposes is:

Remark 2.6. One may drop the hypothesis that H and H' act freely. The resulting quotients $H\backslash M$ and $H'\backslash M$ will then be Steklov isospectral good Riemannian orbifolds. (A *good orbifold* is the orbit space $\mathcal{O} = \Gamma\backslash M$ of a manifold by a smooth discrete group action satisfying the condition that the isotropy group at any point is finite. A function on $\mathcal{O}$ is said to be *smooth* if its pullback to M is smooth. If g is a Riemannian metric on M and Γ acts by isometries, then g gives $\mathcal{O}$ the structure of a Riemannian orbifold. The associated Laplacian $\Delta_{\mathcal{O}}: C^\infty(\mathcal{O}) \to C^\infty(\mathcal{O})$ is defined by $\pi^* \circ \Delta_{\mathcal{O}} = \Delta_M \circ \pi^*$ where $\pi: M \to \mathcal{O}$ is the projection.) We will apply the orbifold version in Example 4.1.2 when we construct planar domains that are isospectral for the sloshing problem.

Other eigenvalue problems 2.7. (i) There are various notions in the literature of a Dirichlet-to-Neumann operator acting on the space of smooth differential p-forms on ∂M where M is a smooth compact Riemannian manifold with smooth boundary. The notions of Dirichlet-to-Neumann operator on forms defined by S. Raulot and A. Savo [44] and by Karpukhin [35] have discrete spectra. Using either of these definitions of Steklov spectrum on p-forms, the hypotheses of Theorem 2.3 (with $\rho \equiv 1$) guarantee that the manifolds $H\backslash M$ and $H'\backslash M$ have the same Steklov spectra on p-forms, for all p.

(ii) As noted in the introduction, taking $\rho \equiv 1$ in Theorem 2.3 immediately yields isospectrality of the Robin problems on $H\backslash M$ and $H'\backslash M$ for every choice of Robin parameter. Alternatively, one can prove the Robin isospectrality directly using the same method as in the proof of Theorem 2.3.

Moreover, one can easily modify Theorem 2.3 to address mixed Robin–Neumann–Dirichlet problems. One assumes that $\partial M = \partial_R M \sqcup \partial_N M \sqcup \partial_D M$, where each of the three subsets is G-invariant and where the decomposition is sufficiently nice so that the mixed Robin–Neumann–Dirichlet problem, in which Robin, Neumann, and Dirichlet conditions are imposed on $\partial_R M$, $\partial_N M$ and $\partial_D M$, respectively, is well-defined with discrete spectrum. Then the mixed problems on $H\backslash M$ and $H'\backslash M$ are isospectral, where the respective boundary conditions are imposed on $H\backslash(\partial_R M)$, $H\backslash(\partial_N M)$, and $H\backslash(\partial_D M)$ and similarly for H'.

3. The torus action method

The torus action method was developed to construct Riemannian manifolds that have the same Laplace spectrum but that are not even locally isometric. There are several versions, e.g., [25, 26, 45, 46]. We first state the version in [46] and then adapt it to the Steklov setting.

In the following, a *torus* always means a nontrivial, compact, connected, abelian Lie group. Let T be a torus acting effectively by isometries on a compact, connected Riemannian manifold M. The union of those orbits on which T acts freely is an open, dense submanifold of M (see [6]) that we will denote by $\widehat{M}$; it carries the structure of a principal T-bundle.

Theorem 3.1 ([46]). *Let T be a torus which acts effectively on two compact, connected Riemannian manifolds (M, g) and (M', g') by isometries. For each subtorus $W \subset T$ of codimension one, suppose that there exists a T-equivariant diffeomorphism $F_W\colon M \to M'$ such that*

(1) *$F_W\colon M \to M'$ is volume-preserving; i.e., $F_W^* \,\mathrm{dvol}_{M'} = \mathrm{dvol}_M$ where dvol_M and $\mathrm{dvol}_{M'}$ are the Riemannian volume densities of M and M';*

(2) *F_W induces an isometry $\overline{F}_W\colon (W\backslash\widehat{M}, g_W) \to (W\backslash\widehat{M'}, g'_W)$ where g_W and g'_W are the metrics induced by g and g' on the quotients.*

Then (M, g) and (M', g') are isospectral. Moreover, if the manifolds have boundary, then they are both Dirichlet and Neumann isospectral.

We now adapt this method to the Steklov setting.

Theorem 3.2. *Let T be a torus which acts isometrically and effectively on two compact, connected Riemannian manifolds (M, g) and (M', g') with boundary. Let $\rho \in L^\infty(\partial M)$ and $\rho' \in L^\infty(\partial M')$ be T-invariant. For each subtorus $W \subset T$ of codimension one, suppose that there exists a T-equivariant diffeomorphism $F_W\colon M \to M'$ such that*

(1) *$F_W\colon M \to M'$ is volume-preserving;*

(2) *$F_W|_{\partial M}\colon \partial M \to \partial M'$ is volume-preserving, i.e., $F_W^* \,\mathrm{dvol}_{\partial M'} = \mathrm{dvol}_{\partial M}$;*

(3) *$F_W^* \rho' = \rho$;*

(4) *F_W induces an isometry $\overline{F}_W\colon (W\backslash\widehat{M}, g_W) \to (W\backslash\widehat{M'}, g'_W)$, where g_W and g'_W are the metrics induced by g and g' on the quotients.*

Then for each α not in the Dirichlet spectrum of (M, g), we have

$$\mathrm{Stek}_\alpha(M, g, \rho) = \mathrm{Stek}_\alpha(M', g', \rho'). \tag{3}$$

(*Theorem* 3.1 *guarantees that the two quotient manifolds are Dirichlet isospectral, so the allowable choices of* α *are the same in both cases.*)

Before proving Theorem 3.2, we recall the variational characterization of the eigenvalues in $\mathrm{Stek}_\alpha(M, g, \rho)$. First recall that the boundary restriction map that takes $u \in H^1(M) \cap C^0(M)$ to $u|_{\partial M}$ extends to the compact trace operator $\mathrm{Tr}\colon H^1(M) \to L^2(\partial M)$. We write $u|_{\partial M} = \mathrm{Tr}(u)$. Define

$$R_{M,\alpha,\rho}(u) = \frac{\int_M \|\nabla u\|^2 \,\mathrm{dvol}_M - \alpha \int_M u^2 \,\mathrm{dvol}_M}{\int_{\partial M} u|_{\partial M}^2 \,\rho \,\mathrm{dvol}_{\partial M}}. \tag{4}$$

Denoting the eigenvalues in $\mathrm{Stek}_\alpha(M, g, \rho)$ as

$$0 = \sigma_0 < \sigma_1 \le \sigma_2 \le \cdots ,$$

we have

$$\sigma_k = \inf_{E_k(M,\rho)} \sup_{0 \ne u \in E_k} R_{M,\alpha,\rho}(u) \tag{5}$$

where the infimum is over all k-dimensional subspaces $E_k(M, \rho)$ of $H^1(M)$ consisting of functions whose restrictions to the boundary are ρ-orthogonal to the constant functions, i.e. $\int_{\partial M} u|_{\partial M} \,\rho \,\mathrm{dvol}_{\partial M} = 0$.

Proof of Theorem 3.2. We adapt the proof of [46, Theorem 1.4]. For $W < T$ any subtorus, let $H^1(M)^W \subset H^1(M)$, $L^2(M)^W \subset L^2(M)$, $H^1(M')^W \subset H^1(M')$, and $L^2(M')^W \subset L^2(M')$ denote the subspaces of W-invariant functions. By Fourier decomposition with respect to the isometric action of T, we have

$$H^1(M) = H^1(M)^T \oplus \bigoplus_W (H^1(M)^W \ominus H^1(M)^T) \tag{6}$$

and

$$L^2(\partial M) = L^2(\partial M)^T \oplus \bigoplus_W (L^2(\partial M)^W \ominus L^2(\partial M)^T) \tag{7}$$

where the sum is over all subtori W of T of codimension one. Multiplication by the T-invariant density ρ preserves each of the subspaces $L^2(M)^T$ and $L^2(M)^W$. Moreover the trace operator $\mathrm{Tr}\colon H^1(M) \to L^2(\partial M)$ respects these decompositions. Analogous statements hold with M replaced by M'.

As shown in [46], conditions (1) and (4) of Theorem 3.2 imply that if W is a subtorus of T of codimension at most one and $u \in H^1(M')^W$, then

$$\|F_W^* u\|_{H^1(M)} = \|u\|_{H^1(M')} \quad \text{and also} \quad \|F_W^* u\|_{L^2(M)} = \|u\|_{L^2(M')}. \tag{8}$$

The first of the equations in (8), the T-equivariance of the maps F_W, and equation (6) yield an isomorphism

$$\tau\colon H^1(M') \longrightarrow H^1(M)$$

given by

$$\tau = F_T^* \oplus \bigoplus_W F_W^*.$$

Hypothesis (2) of the theorem and equation (7) similarly yield an isomorphism

$$\tau_\partial := F_T^* \oplus \bigoplus_W F_W^*\colon L^2(\partial M') \longrightarrow L^2(\partial M)$$

and the diagram

$$\begin{array}{ccc} H^1(M') & \xrightarrow{\tau} & H^1(M) \\ {\scriptstyle \mathrm{Tr}}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}} \\ L^2(\partial M') & \xrightarrow[\tau_\partial]{} & L^2(\partial M) \end{array}$$

commutes.

Hypotheses (2) and (3) of the theorem guarantee for each $k = 1, 2, \dots$ that τ maps $E_k(M', \rho')$ to $E_k(M, \rho)$ and that the denominators in the Rayleigh quotients $R_{M,\alpha,\rho}(\tau(u))$ and $R_{M',\alpha,\rho'}(u)$ coincide for each $u \in E_k(M', \rho')$. The pair of equalities (8) imply that the numerators in $R_{M,\alpha,\rho}(\tau(u))$ and $R_{M',\alpha,\rho'}(u)$ also agree, and the theorem follows from equation (5). □

Although condition (2) in Theorem 3.2 does not appear in Theorem 3.1 or in any of the other versions of the torus action method, it is actually satisfied in all of the examples that have been constructed thus far by these methods, as will be explained in §4. Moreover, the version of the torus action method in [26, Theorem 1.2] includes a hypothesis that the principal T-orbits be dense in ∂M and $\partial M'$ in order to produce Neumann isospectral manifolds; this condition is stronger than condition (2) in the following sense.

Proposition 3.3. *Let M and M' be compact, connected, orientable Riemannian manifolds with a faithful isometric action by a torus T satisfying conditions (1) and (4) of Theorem 3.2. If $\widehat{M} \cap \partial M$ is dense in ∂M, then condition (2) of Theorem 3.2 is satisfied as well.*

Proof. Choose orientations on M and M' and give ∂M and $\partial M'$ the induced orientations. Let Ω_M, $\Omega_{M'}$, $\Omega_{\partial M}$ and $\Omega_{\partial M'}$ be the associated Riemannian volume forms. Condition (1) says that $F_W^* \Omega_{M'} = \pm\Omega_M$. Since $\widehat{M} \cap \partial M$ is dense in ∂M, it suffices to show that condition (2) holds at each point $p \in \widehat{M} \cap \partial M$. Let $p \in \widehat{M} \cap \partial M$ and let $p' = F_W(p)$. The T-equivariance of F_W guarantees that $p' \in \widehat{M}'$. Let ν and ν' denote the outward unit normals to ∂M and $\partial M'$ at p and p', respectively, and let $i\colon \partial M \to M$ and $i'\colon \partial M' \to M'$ be the inclusion maps. The facts that F_W is an isometry and that the action of W on M and M' preserves the boundaries imply that $\nu' - F_{W*}(\nu)$ is tangent to $\partial M'$ and hence

$$(i')^*(F_{W*}(\nu) \lrcorner \Omega_{M'}) = (i')^*(\nu' \lrcorner \Omega_{M'}) = \Omega_{\partial M'}. \tag{9}$$

By condition (1) we have

$$F_{W*}(\nu) \lrcorner \Omega_{M'} = \pm F_{W*}(\nu) \lrcorner (F_W^{-1})^*\Omega_M = \pm (F_W^{-1})^*(\nu \lrcorner \Omega_M). \tag{10}$$

Since $F_W \circ i = i' \circ F_W$, equations 9 and 10 yield

$$F_W^*(\Omega_{\partial M'}) = \pm F_W^* \circ (i')^* \circ (F_W^{-1})^*(\nu \lrcorner \Omega_M) = \pm i^*(\nu \lrcorner \Omega_M) = \pm \Omega_{\partial M}$$

and thus $F_W^* \operatorname{dvol}_{\partial M'} = \operatorname{dvol}_{\partial M}$. □

4. Examples

4.1. Examples using the Sunada technique. There is a wealth of examples of Dirichlet or Neumann isospectral manifolds that have been constructed by the Sunada method and its various generalizations; see [27] and references therein. The original Sunada technique has yielded, for example, isospectral flat surfaces embedded in $\mathbb{R}^3$ [13] and large finite families of mutually isospectral Riemann surfaces [9], which can be easily modified to produce families of mutually isospectral hyperbolic surfaces with boundary. All examples of isospectral manifolds with boundary constructed by the original Sunada technique are also Steklov isospectral.

There are various generalizations of Sunada's theorem, surveyed in [27], not all of which go through directly for the Steklov spectrum. For example, the pair of Neumann isospectral flat surfaces with boundary constructed in [5] (one orientable, the other nonorientable) using the orbifold version of Sunada's Theorem are not Steklov isospectral, since one of the manifolds has four boundary components while its isospectral companion has only three boundary components. Yet the number of boundary components of a surface is determined by the Steklov spectrum (see [22]). See §4.1.2 for some further comments.

In this subsection we illustrate the Sunada method with just a sampling of the many examples.

4.1.1. Steklov isospectral flat surfaces embedded in $\mathbb{R}^3$. In [13], Peter Buser introduced the use of Schreier graphs to construct isospectral manifolds via Sunada's technique and illustrated the method by constructing a pair of Dirichlet and Neumann isospectral flat surfaces with boundary in $\mathbb{R}^3$.

For the reader's convenience, we briefly review Buser's construction before addressing the Steklov setting. Recall that if G is a finite group and $S = \{s_1, s_2, \dots, s_n\}$ is a set of nonidentity elements generating G, the *Cayley graph* $\Gamma(G, S)$ is the n-regular edge-colored directed graph whose vertices are the elements of G, and whose i-colored edges encode right multiplication by the generators s_i. More precisely, there is an i-colored edge from g to g' if and only if $gs_i = g'$. The group G acts transitively and faithfully on $\Gamma(G, S)$ by left multiplication. If H is a subgroup of G, then the *Schreier graph* $\Gamma(H\backslash G, S)$ is the quotient of $\Gamma(G, S)$ by the action of H. Equivalently, the vertices of the Schreier graph correspond to the elements of the space of right-cosets $H\backslash G$ and the edges indicate the right action of the elements of S on $H\backslash G$. The graph theoretic version of Sunada's Theorem says that if H_1 and H_2 are almost conjugate subgroups of G, then for any fixed choice of generating set S, the adjacency operators (or Laplacians or other natural operators) associated with the Schreier graphs $\Gamma(H_1\backslash G, S)$ and $\Gamma(H_2\backslash G, S)$ are isospectral.

To construct a manifold from a Schreier graph, Buser chooses a basic tile T, whose piecewise-smooth boundary contains $2n$ disjoint line segments called sides, labelled s_1, s_1^{-1}, s_2, $s_2^{-1}, \dots, s_n, s_n^{-1}$. Sides s_i and s_i^{-1} are required to have the same length. The sides need not exhaust the entire boundary of the tile. To construct a manifold $M(H\backslash G, S)$, consider a collection of $[G : H]$ identical tiles, labelled by the elements of $H\backslash G$, whose sides are glued together in pairs according to the pattern encoded by the Schreier graph. More precisely, side s_i of tile Hg is glued to side s_i^{-1} of tile Hgs_i. Similarly, one uses the Cayley graph $\Gamma(G, S)$ to construct a manifold $M(G, S)$. Observe that G acts on $M(G, S)$ on the left, and that $M(H\backslash G, S) = H\backslash M(G, S)$. Let $\partial_0(T)$ denote the complement in ∂T of the union of the sides $s_1, s_1^{-1}, s_2, s_2^{-1}, \dots, s_n, s_n^{-1}$. Buser arbitrarily chooses boundary conditions on $\partial_0 T$. The boundary conditions chosen on $\partial_0 T$ then determine the boundary conditions on the manifold $M(G, S)$ and on $M(H\backslash G, S)$ for any subgroup $H < G$.

Now suppose that H_1 and H_2 are almost conjugate subgroups of G. Then, as observed by Buser, Sunada's Theorem immediately yields isospectrality of $M_1 := M(H_1\backslash G, S)$ and $M_2 := M(H_2\backslash G, S)$ with respect to the given boundary conditions.

Moving to our setting, we instead choose arbitrarily an L^∞ density function ρ on $\partial_0 T$, thus giving rise to a density function, still denoted ρ, on the boundaries of $M := M(G, S)$ and M_i, $i = 1, 2$. The density on $\partial M(G, S)$ is G-invariant and Theorem 2.3 yields

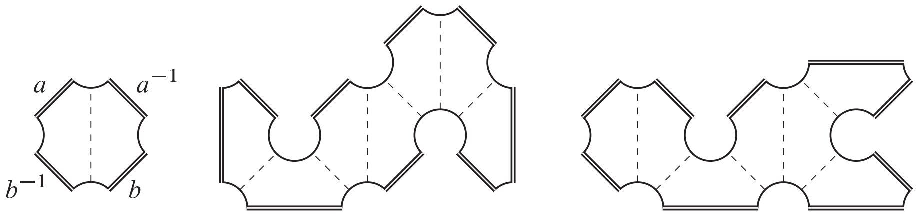
$$\mathrm{Stek}_\alpha(M_1, \rho) = \mathrm{Stek}_\alpha(M_2, \rho).$$

It is easy to construct an abundance of examples this way. For a concrete example, we consider the pair of flat surfaces in $\mathbb{R}^3$ given by Buser in [13]. In this example, $G = \mathrm{GL}(3, \mathbb{Z}_2)$, H_1 is the subset of matrices with first row $(1, 0, 0)$, and $H_2 = H_1^t$ is the subset consisting of transposes of elements of H_1. The two subgroups H_1 and H_2 are almost conjugate in G (each element of H_1 is similar to its transpose in H_2) and have index 7 in G. Buser's surfaces are obtained by using a particular generating set $S = \{a, b\}$ of order 2 and the basic tile shown in Figure 1(A). (Ignore for now the dashed line in Figure 1(A); it will be used in the next example.) Buser actually used a cross-shaped tile; we have smoothed out the corners of the tile so that the resulting isospectral surfaces M_1 and M_2 are smooth.

We have not included a picture of the two surfaces here. However, Figure 1(B) shows the quotient of each of the two surfaces by a reflection. To visualize the original surfaces, simply double the two domains in the figure across the part of the boundary indicated by double lines. Alternatively, see [13], where the surfaces constructed with a cross-shaped tile are drawn.

The surfaces are easily seen to be nonisometric; in fact they have different diameter. Since we are in dimension two, we also verify that they are not trivially Steklov isospectral when $\alpha = 0$ by showing that M_2 is not isometric to the surface M_1 endowed with a metric $e^f g_E$, where g_E is the Euclidean metric and where the conformal factor f vanishes on the boundary. Recall that the scalar curvature of $e^f g_E$ is $4e^{-f}\Delta f$, where Δ denotes the Euclidean Laplacian. Noting that M_2 is flat, we conclude that f must be a harmonic function. Since f vanishes on the boundary, f must be identically zero. Thus no such conformal equivalence exists and the surfaces are nontrivially Steklov isospectral.

4.1.2. Planar domains with isospectral sloshing problems. The first examples of isospectral planar domains [30] arose from the observation that each of the two isospectral flat surfaces M_i, $i = 1, 2$, described in the previous example admits an isometric involution β_i covering the symmetry β_0 of the basic tile in Figure 1(A) given by reflection across the dashed line. The quotients of the surfaces by the involutions, shown in Figure 1(B), are both Dirichlet and Neumann isospectral. As we will explain below, the version of Sunada's technique used to prove isospectrality does not yield Steklov isospectrality of these domains except



(A) Buser's tile (B) Steklov–Neumann and Robin–Neumann isospectral planar domains

Figure 1. Neumann conditions are imposed on all straight boundary parts (double-lined). The domains arise from [30, Figure 7] by using tiles as in (A).

in the special case that the density ρ is identically zero on the part of the boundary indicated by double lines (the straight segments of the boundary) in Figure 1(B). However, if we choose ρ to be zero on this part of the boundary and $\rho \equiv 1$ on the curved edges, then we do obtain isospectrality for the mixed Neumann-Steklov problem (the sloshing problem). One can also make a more general choice of ρ on the curved parts of the boundary as long as consistency is maintained among the various tiles.

The proof in [30] of Neumann isospectrality of the planar domains goes as follows. The involutive isometries β_i, $i = 1, 2$, lift to an involutive isometry β of the covering manifold $M = M(G, S)$. The isometry β normalizes the group G and each of the subgroups H_i, $i = 1, 2$. The groups $\widetilde{H_1} := H_1 \rtimes \langle\beta\rangle$ and $\widetilde{H_2} := H_2 \rtimes \langle\beta\rangle$ are almost conjugate subgroups of $\widetilde{G} := G \rtimes \langle\beta\rangle$. The group $\widetilde{G}$ does not act freely on M. However, we may apply the orbifold version of Sunada's Theorem as in Remark 2.6 to conclude that the quotients $\widetilde{H_1}\backslash M$ and $\widetilde{H_2}\backslash M$ are isospectral orbifolds. The underlying spaces of these orbifolds are the domains in Figure 1(B). The singular sets of these orbifolds consist of the doubled line segments in Figure 1(B), which are reflector edges where the isotropy group has order 2. (Note that these line segments lift to interior segments of M, not to boundary edges.) By the definition of smooth functions and of the Laplacian on these orbifolds (see Remark 2.6), the isospectrality of the two orbifolds is equivalent to isospectrality of the underlying planar domains with Neumann boundary conditions placed on the doubled line segments of the boundary and whatever boundary conditions on the curved edges were chosen on the curved edges of the basic tile T used to construct M.

If we choose the boundary density function $\rho \equiv 1$ on the boundary of the basic tile, the same argument yields the Steklov isospectrality of the two orbifolds, which in turn corresponds to isospectrality for the sloshing problem on the two underlying planar domains.

Remark 4.1. We have summarized the original proof of the isospectrality of the planar domains in order to clarify why we only get sloshing isospectrality rather than more general Steklov isospectrality of the planar domains. However, transplantation as in Remark 1 yields a very simple proof by picture of the sloshing isospectrality.

4.1.3. Isospectral domains with mixed boundary conditions. In [40], Levitin, Parnovski, and Polterovich constructed examples of pairs of domains that are isospectral with mixed boundary conditions, including a pair consisting of a triangle and a square, whose isospectrality cannot be explained directly by Sunada's technique but can be shown by an explicit transplantation of eigenfunctions. Later Band and Parzanchevsky [41] gave a representation theoretic explanation, which was further developed and applied systematically by Herbrich [32].

One can similarly use transplantation directly to obtain domains that are isospectral for both the mixed Robin–Neumann–Dirichlet and the mixed Steklov–Neumann–Dirichlet problems. We give two examples here, both obtained by modifying the construction of the isospectral triangle and square in [40]. The triangle and square in [40] are each constructed by gluing together two copies of an isosceles right triangle (the basic tile); they are glued along the hypotenuse to obtain the square and along one of the legs to obtain the triangle in the isospectral pair. Figure 2 shows two modifications of their construction, both obtained by cutting out a half disk from the basic tile.

For the mixed Robin–Neumann–Dirichlet problem, we impose Robin boundary conditions—with the same Robin parameter on both domains in each pair—on the curved part of the boundary indicated by a solid line in the figures, Neumann conditions on the part of the boundary indicated by doubled lines, and Dirichlet conditions on the part indicated by dashed lines. With these boundary conditions we claim that M is isospectral to M' and P is isospectral to P'.

Let u be an eigenfunction for the mixed problem on M, say with eigenvalue λ, and denote by u_1 and u_2 the restrictions of u to the two tiles making up M as in Figure 2. We transplant u to an eigenfunction $u' = T(u)$ on M' whose restrictions u'_1 and u'_2 to the two tiles of M' as in Figure 2 are given by

$$\begin{pmatrix} u'_1 \\ u'_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{11}$$

In writing $u_1 \pm u_2$, we implicitly identify the tiles underlying u_1 and u_2, which involves a reflection in the dotted diagonal of M. To see that u' is smooth on the dotted interior segment, we observe that u_1 extends smoothly by reflection across this segment (since the segment corresponds to an edge in M where u_1 satisfies

Neumann conditions) and, similarly, u_2 smoothly extends by negative reflection across this segment (which corresponds to an edge of M where u_2 satisfies Dirichlet conditions). It is then straightforward to verify that u' is an eigenfunction with eigenvalue λ for the mixed Robin–Neumann–Dirichlet problem. The transplantation map T is invertible and isospectrality follows. The same transplantation map yields the mixed Robin–Neumann–Dirichlet isospectrality of P and P'.

To prove the Steklov–Neumann–Dirichlet isospectrality of M and M' and of P and P', one uses the same expression for the transplantation map T, but now acting on Steklov–Neumann–Dirichlet eigenfunctions. Alternatively, the isospectrality is immediate from the duality between the Steklov–Neumann–Dirichlet and the Robin–Neumann–Dirichlet problem.

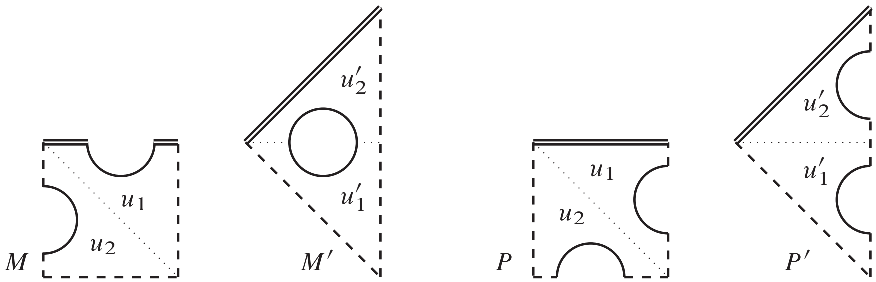


Figure 2. Mixed Robin–Neumann–Dirichlet and Steklov–Neumann–Dirichlet isospectral pairs. They are based on the main example in [40]. Isospectrality follows from the transplantation (11). Robin boundary conditions are imposed on the solid boundary edges, Neumann conditions on the doubled boundary edges, and Dirichlet conditions on the dashed boundary edges.

4.2. Examples using the torus action method. The torus action method, e.g., Theorem 3.1, has led to numerous pairs and families of Dirichlet and Neumann isospectral manifolds as well as isospectral closed manifolds. All known examples satisfy the additional condition (2) of Theorem 3.2 and therefore have isospectral Dirichlet-to-Neumann operators at all frequencies. In fact, Proposition 3.3 applies to all of them, yielding condition (2) in Theorem 3.2. Letting B^n and T^n denote the n-dimensional ball and torus, respectively, the examples include:

(1) continuous families of nonisometric metrics on B^n for $n \geq 8$ [26, 46], and pairs of such metrics on B^6 and B^7 [46] (These metrics can be chosen as Euclidean outside of a smaller concentric ball [46].);

(2) continuous families of metrics on $B^n \times T^k$ for $n \geq 5$ and $k \geq 2$ that are the restrictions of locally nonisometric homogeneous metrics on $\mathbb{R}^n \times T^k$ [31];

(3) for $n \geq 6$, if one removes a concentric ball from B^n to obtain an annulus M and takes $\rho \equiv 1$ on one of the boundary spheres and $\rho \equiv 0$ on the other, then the metrics in (1) and (2) restrict to metrics with isospectral sloshing problems on M.

5. Isospectral density functions

In [8], R. Brooks modified Sunada's theorem in order to construct isospectral potentials for the Schrödinger operator. Shortly thereafter, a similar method was used to construct isospectral conformally equivalent Riemannian metrics, see [11]. The technique became standard and produced many new examples. Later D. Schueth [45] analogously modified the torus action method in order to produce isospectral potentials and isospectral conformally equivalent Riemannian metrics. In this section, we observe that similar modifications of Theorem 2.3 and Theorem 3.2 allow us to produce isospectral boundary density functions for the Steklov spectrum. Here we carry out the modification of Theorem 2.3; the modification of Theorem 3.2 is similar.

Theorem 5.1. *Let M, G, H, H', g and ρ satisfy the hypotheses of Theorem 2.3. Assume in addition that there exists an isometry τ of (M, g), not in G, such that $\tau H \tau^{-1} = H'$. Then for all α not in the Dirichlet spectrum of $(H\backslash M, g)$, we have*

$$\mathrm{Stek}_\alpha(H\backslash M, g, \rho) = \mathrm{Stek}_\alpha(H\backslash M, g, \tau^*\rho)$$

where we continue to denote by ρ and $\tau^*\rho$ the boundary density functions on $H\backslash M$ induced by those on M.

Proof. By Theorem 2.3, $\mathrm{Stek}_\alpha(H\backslash M, g, \rho) = \mathrm{Stek}_\alpha(H'\backslash M, g, \rho)$ for all α not in the Dirichlet spectrum of $H\backslash M$. By the additional hypothesis of Theorem 5.1, τ induces an isometry $\tau\colon (H\backslash M, g) \to (H'\backslash M, g)$, so we have $\mathrm{Stek}_\alpha(H'\backslash M, g, \rho) = \mathrm{Stek}_\alpha(H\backslash M, g, \tau^*\rho)$. □

Example 5.2 (flat surfaces and planar domains). In Example 5.6 in [30], the tile in Figure 1(A) is replaced by a tile T that has not only a reflection symmetry β_0 as in 4.1.2 but also a rotational symmetry τ_0 that commutes with β_0. The tile is pictured in Figure 15 of [30]. Construct $M = M(G, S)$ and $M_i = M(H_i\backslash M, S)$, $i = 1, 2$ exactly as in 4.1.1 but using the more symmetric tile. The isometry τ_0 of the basic tile lifts to an isometry τ of M. The isometry τ normalizes the group G and $\tau A \tau^{-1} = (A^t)^{-1}$ for all $A \in G$. In particular, $\tau H_1 \tau^{-1} = H_2$. Define

$\partial_0 T$ as in 4.1.1 and let $\rho_0 \colon \partial_0 T \to \mathbb{R}$ be a boundary density function that is *not* invariant under the restriction to $\partial_0 T$ of the rotational symmetry τ_0. Denote by ρ the resulting boundary density on M. Then the hypotheses of Theorem 5.1 are satisfied with H_1 and H_2 playing the roles of H and H'. Thus we have $\mathrm{Stek}_\alpha(M_1, \rho) = \mathrm{Stek}_\alpha(M_1, \tau^*\rho)$ for all α not in the Dirichlet spectrum of M_1.

Next we construct planar domains. In the construction in the previous paragraph, impose the additional requirement that ρ_0 be invariant under the reflection symmetry β_0. As in §4.1.2, the symmetry β_0 of the new basic tile lifts to isometric involutions of M, M_1, and M_2. Let $\mathcal{O}_i$ be the orbifold quotient of M_i by the involution β_i. As before, the underlying space of $\mathcal{O}_i$ is a planar domain D_i whose boundary consists of the projection to $\mathcal{O}_i$ of the boundary of M_i (this part is the boundary of the orbifold) together with a collection of straight line segments corresponding to the singular set of the orbifold. The boundary density ρ on M_i projects to a density function, still denoted ρ, on the first part of the bounary of D_i; we extend ρ to the full boundary by setting it to be zero on the orbifold singular set. Because β_0 and τ_0 commute, the isometry $\tau \colon M_1 \to M_2$ satisfies $\tau \circ \beta_1 = \beta_2 \circ \tau$, and thus τ induces an isometry between the planar domains D_1 to D_2. We then have $\mathrm{Stek}_\alpha(D_1, \rho) = \mathrm{Stek}_\alpha(D_1, \tau^*\rho)$ for all α not in the Dirichlet spectrum of D_1.

The modification of the torus action method is similar:

Theorem 5.3. *Let T be a torus which acts isometrically and effectively on two compact, connected Riemannian manifolds (M, g) and (M', g') with boundary. Let $\rho \in L^\infty(\partial M)$ and $\rho' \in L^\infty(\partial M')$ be T-invariant. Assume that all the hypotheses of Theorem 3.2 are satisfied and, in addition, that there exists an isometry $\tau \colon (M, g) \to (M', g')$. Then for all α not in the Dirichlet spectrum of (M, g), we have*

$$\mathrm{Stek}_\alpha(M, g, \rho) = \mathrm{Stek}_\alpha(M, g, \tau^*\rho').$$

Proof.

$$\mathrm{Stek}_\alpha(M, g, \rho) = \mathrm{Stek}_\alpha(M', g', \rho') = \mathrm{Stek}_\alpha(M, g, \tau^*\rho')$$

where the first equality follows from Theorem 3.2. ☐

Example 5.4. Let B^n be the n-dimensional ball. For various values of $n \geq 10$, D. Schueth and the first author [29] constructed pairs of conformally equivalent metrics on B^n that are isospectral but not isometric and whose restrictions to the boundary spheres are also isospectral but non-isometric. The construction begins with (i) a pair of metrics g, g' on B^n that admit a torus action satisfying the hypotheses of Theorem 3.1 and Proposition 3.3, (ii) an isometry $\tau\colon (B^n, g) \to (B^n, g')$, and (iii) a smooth positive function φ on B^n such that $\varphi = F_W^*\varphi$ for all the functions F_W in Theorem 3.1 but such that $\tau^*\varphi \neq \varphi$. The conformally equivalent metrics are given by φg and $\tau^*(\varphi)g$. To adapt this construction to our setting, let $\rho = \rho'$ be the restriction of φ to the boundary sphere. The hypotheses of Theorem 5.3 are satisfied and we have $\mathrm{Stek}_\alpha(B^n, g, \rho) = \mathrm{Stek}_\alpha(B^n, g, \tau^*\rho)$ for all α not in the Dirichlet spectrum of (B^n, g), but (as shown in [29]) the densities ρ and $\tau^*\rho$ are not congruent under any isometry of the boundary sphere.

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