# Direct and inverse problems
# for the nonlinear time-harmonic Maxwell equations
# in Kerr-type media

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**Abstract.** In the current paper we consider an inverse boundary value problem of electromagnetism in a nonlinear Kerr medium. We show the unique determination of the electromagnetic material parameters and the nonlinear susceptibility parameters of the medium by making electromagnetic measurements on the boundary. We are interested in the case of the time-harmonic Maxwell equations.

## Contents

## 1. Introduction

Let $(M, g)$ be a compact 3-dimensional Riemannian manifold with smooth boundary and let $\mathcal{E}(\cdot, t)$ and $\mathcal{H}(\cdot, t)$ be the time-dependent 1-forms on $M$ representing electric and magnetic fields. By $d$ and $*$ we denote the exterior derivative and

the Hodge star operator on $(M, g)$, respectively. Consider the time-dependent Maxwell equations on the manifold, with no scalar charge density and with no current density

$$
\begin{cases}
\partial_t \mathcal{B} + *d\mathcal{E} = 0, \\
\partial_t \mathcal{D} - *d\mathcal{H} = 0, \\
*d * \mathcal{D} = 0, \\
*d * \mathcal{B} = 0,
\end{cases}
\tag{1.1}
$$

where $\mathcal{D}$ and $\mathcal{B}$ are 1-forms representing electric displacement and magnetic induction

$$
\mathcal{D} = \varepsilon\mathcal{E} + \mathcal{P}_{\mathrm{NL}}(\mathcal{E}), \quad \mathcal{B} = \mu\mathcal{H} + \mathcal{M}_{\mathrm{NL}}(\mathcal{H}),
$$

with $\mathcal{P}_{\mathrm{NL}}$ and $\mathcal{M}_{\mathrm{NL}}$ being the nonlinear polarization and nonlinear magnetization, respectively. The (time-independent) functions $\varepsilon$ and $\mu$ on $M$, with positive real parts, represent the material parameters (permettivity and permeability, respectively).

The electric and magnetic fields $\mathcal{E}$ and $\mathcal{H}$ are said to be time-harmonic with frequency $\omega > 0$ if

$$
\mathcal{E}(x, t) = E(x)e^{-i\omega t} + \overline{E(x)}e^{i\omega t},
$$
$$
\mathcal{H}(x, t) = H(x)e^{-i\omega t} + \overline{H(x)}e^{i\omega t},
$$

for some complex 1-forms $E$ and $H$ on $M$. Then the time-averages of the intensities of $\mathcal{E}$ and of $\mathcal{H}$ are

$$
\frac{1}{T} \int_0^T |\mathcal{E}(x, t)|_g^2 \, dt = 2|E(x)|_g^2,
$$
$$
\frac{1}{T} \int_0^T |\mathcal{H}(x, t)|_g^2 \, dt = 2|H(x)|_g^2,
$$

where $T = 2\pi/\omega$. In a medium with high intensity electric field, the nonlinear polarization is of the form

$$
\mathcal{P}_{\mathrm{NL}}(x, \mathcal{E}(x, t)) = \chi_e(x, |E|_g^2)\mathcal{E}(x, t),
$$

where $\chi_e$ is the scalar susceptibility depending only on the time-average of the intensity of $\mathcal{E}$. One of the most common nonlinear polarizations appearing in physics and engineering is the Kerr nonlinearity

$$
\chi_e(x, |E|_g^2) = a(x)|E|_g^2.
\tag{1.2}
$$

The reader is refereed to [19, 24] for this and other examples of electric nonlinear phenomenas.

We also assume that the nonlinear magnetization has the similar form

$$\mathcal{M}_{\mathrm{NL}}(x, \mathcal{H}(x,t)) = \chi_m(x, |H|_g^2)\mathcal{H}(x,t),$$

where $\chi_m$ is the scalar susceptibilities depending only on the time-average of the intensity of $\mathcal{H}$. Such nonlinear magnetizations appear in the study of metamaterials built by combining an array of wires and split-ring resonators embedded into a Kerr-type dielectric [32]. These metamaterials have complicated form of nonlinear magnetization. However, if the intensity $|H|_g^2$ is sufficiently small, relatively to the resonant frequency, the nonlinear magnetization can be assumed to be of the Kerr-type [15, 17, 31]

$$\chi_m(x, |H|_g^2) = b(x)|H|_g^2. \tag{1.3}$$

This assumption has successful numerical implementation [17].

For the time-harmonic $\mathcal{E}$ and $\mathcal{H}$, the time-dependent Maxwell's system (1.1) reduces to the nonlinear time-harmonic Maxwell equations for complex 1-forms $E$ and $H$, with a fixed frequency $\omega > 0$, will be

$$\begin{cases} *dE = i\omega\mu H + i\omega b|H|_g^2 H, \\ *dH = -i\omega\varepsilon E - i\omega a|E|_g^2 E. \end{cases} \tag{1.4}$$

The complex functions $\mu$ and $\varepsilon$ represent the material parameters (permettivity and permeability, respectively).

**1.1. Direct problem.** First we consider the boundary value problem for the nonlinear Maxwell equations (1.4). We suppose that $\varepsilon, \mu \in C^1(M)$ are complex functions with positive real parts and $a, b \in C^1(M)$.

The boundary conditions are expressed in terms of *tangential trace*. The latter is defined on $m$-forms by

$$\mathbf{t} \colon C^\infty\Omega^m(M) \longrightarrow C^\infty\Omega^m(\partial M), \quad \mathbf{t}(w) = \iota^*(w), \quad w \in C^\infty\Omega^m(M),$$

where $\iota \colon \partial M \hookrightarrow M$ is the canonical inclusion. Then $\mathbf{t}$ has its extension to a bounded operator $W^{1,p}\Omega^m(M) \to W^{1-1/p,p}\Omega^m(\partial M)$ for $p > 1$. Here and in what follows, $W^{1,p}\Omega^m(M)$ and $W^{1-1/p,p}\Omega^m(\partial M)$ are standard Sobolev spaces of $m$-forms on $M$ and $\partial M$, respectively.

To describe the boundary conditions, we introduce the spaces

$$W_{\mathrm{Div}}^{1,p}(M) = \{u \in W^{1,p}\Omega^1(M) : \mathrm{Div}(\mathbf{t}(u)) \in W^{1-1/p,p}\Omega^1(\partial M)\},$$

$$T W_{\mathrm{Div}}^{1-1/p,p}(\partial M) = \{f \in W^{1-1/p,p}\Omega^1(\partial M) : \mathrm{Div}(f) \in W^{1-1/p,p}\Omega^1(\partial M)\},$$

where Div is the surface divergence on $\partial M$; see Section 2.4 for the exact definition of Div. These spaces are Banach spaces with norms

$$\|u\|_{W_{\mathrm{Div}}^{1,p}(M)} = \|u\|_{W^{1,p}\Omega^1(M)} + \|\mathrm{Div}(\mathbf{t}(u))\|_{W^{1-1/p,p}(\partial M)},$$

$$\|u\|_{T W_{\mathrm{Div}}^{1-1/p,p}(\partial M)} = \|f\|_{W^{1-1/p,p}(\partial M)} + \|\mathrm{Div}(f)\|_{W^{1-1/p,p}(\partial M)}.$$

It is not difficult to see that $\mathbf{t}(W_{\mathrm{Div}}^{1,p}(M)) = T W_{\mathrm{Div}}^{1-1/p,p}(\partial M)$.

Our first main result is the following theorem on well-posedness of the nonlinear Maxwell equations (1.4) with prescribed small $\mathbf{t}(E)$ on $\partial M$.

**Theorem 1.1.** *Let $(M, g)$ be a compact 3-dimensional Riemannian manifold with smooth boundary and let $3 < p \le 6$. Suppose that $\varepsilon, \mu \in C^1(M)$ are complex functions with positive real parts and $a, b \in C^1(M)$. For every $\omega \in \mathbb{C}$, outside a discrete set $\Sigma \subset \mathbb{C}$ of resonant frequencies, there is $\epsilon > 0$ such that for all $f \in T W_{\mathrm{Div}}^{1-1/p,p}(\partial M)$ with $\|f\|_{T W_{\mathrm{Div}}^{1-1/p,p}(\partial M)} < \epsilon$, the Maxwell's equation (1.4) has a unique solution $(E, H) \in W_{\mathrm{Div}}^{1,p}(M) \times W_{\mathrm{Div}}^{1,p}(M)$ satisfying $\mathbf{t}(E) = f$ and*

$$\|E\|_{W_{\mathrm{Div}}^{1,p}(M)} + \|H\|_{W_{\mathrm{Div}}^{1,p}(M)} \le C\|f\|_{T W_{\mathrm{Div}}^{1-1/p,p}(\partial M)},$$

*for some constant $C > 0$ independent of $f$.*

**1.2. Inverse problem.** For $\omega > 0$ with $\omega \notin \Sigma$, we define the *admittance map* $\Lambda_{\varepsilon,\mu,a,b}^{\omega}$ as

$$\Lambda_{\varepsilon,\mu,a,b}^{\omega}(f) = \mathbf{t}(H), \quad f \in T W_{\mathrm{Div}}^{1-1/p,p}(\partial M), \quad \|f\|_{T W_{\mathrm{Div}}^{1-1/p,p}(\partial M)} < \epsilon$$

where $(E, H) \in W_{\mathrm{Div}}^{1,p}(M) \times W_{\mathrm{Div}}^{1,p}(M)$ is the unique solution of the system (1.4) with $\mathbf{t}(E) = f$, guaranteed by Theorem 1.1. Moreover, the estimate provided in Theorem 1.1 implies that the admittance map satisfy

$$\|\Lambda_{\varepsilon,\mu,a,b}^{\omega}(f)\|_{T W_{\mathrm{Div}}^{1-1/p,p}(\partial M)} \le C\|f\|_{T W_{\mathrm{Div}}^{1-1/p,p}(\partial M)} < C\epsilon.$$

The inverse problem is to determine $\varepsilon, \mu, a$ and $b$ from the knowledge of the admittance map $\Lambda_{\varepsilon,\mu,a,b}^{\omega}$.

To state our second main result, let us introduce the notion of admissible manifolds.

**Definition.** A compact Riemannian manifold $(M, g)$ with smooth boundary of dimension $n \geq 3$, is said to be *admissible* if $(M, g) \subset\subset \mathbb{R} \times (M_0, g_0)$, $g = c(e \oplus g_0)$ where $c > 0$ smooth function on $M$, $e$ is the Euclidean metric and $(M_0, g_0)$ is a simple $(n-1)$-dimensional manifold. We say that a compact manifold $(M_0, g_0)$ with boundary is *simple*, if $\partial M_0$ is strictly convex, and for any point $x \in M_0$ the exponential map $\exp_x$ is a diffeomorphism from its maximal domain in $T_x M_0$ onto $M_0$.

Compact submanifolds of Euclidean space, the sphere minus a point and of hyperbolic space are all examples of admissible manifolds.

The notion of admissible manifolds were introduced by Dos Santos Ferreira, Kenig, Salo and Uhlmann [6] as a class of manifolds admitting the existence of *limiting Carleman weights*. In fact, the construction of complex geometrical optics solutions are possible on such manifolds via Carleman estimates approach based on the existence of limiting Carleman weights. Such an approach was introduced by Bukhgeim and Uhlmann [2] and Kenig, Sjöstrand and Uhlmann [13] in the setting of partial data Calderón's inverse conductivity problem in $\mathbb{R}^n$.

Our second main result is as follows.

**Theorem 1.2.** *Let $(M, g)$ be a 3-dimensional admissible manifold and let $4 \leq p < 6$. Suppose that $\varepsilon_j, \mu_j \in C^3(M)$ with positive real parts and that $a_j, b_j \in C^1(M)$, $j = 1, 2$. Fix $\omega > 0$ outside a discrete set of resonant frequencies $\Sigma \subset \mathbb{C}$ and fix sufficiently small $\epsilon > 0$. If*

$$\Lambda^{\omega}_{\varepsilon_1,\mu_1,a_1,b_1}(f) = \Lambda^{\omega}_{\varepsilon_2,\mu_2,a_2,b_2}(f)$$

*for all $f \in T W^{1-1/p,p}_{\mathrm{Div}}(\partial M)$ with $\|f\|_{T W^{1-1/p,p}_{\mathrm{Div}}(\partial M)} < \epsilon$, then $\varepsilon_1 = \varepsilon_2$, $\mu_1 = \mu_2$, $a_1 = a_2$ and $b_1 = b_2$ in $M$.*

Such inverse boundary value problems have been considered for various semilinear and quasilinear elliptic equations and systems (see [7, 8, 9, 10, 11, 25, 26, 27]) based on the linearization approach.

For the type of nonlinearity of Maxwell's equations in a Kerr type medium, after first order linearization, we can recover $\mu$ and $\varepsilon$ by solving corresponding inverse problem for the linear equation (see [12]). The difficulty lies in reconstructing the susceptibility parameters $a$ and $b$. By calculating the next term of the asymptotic expansion for the admittance map, one obtains the tangential trace $\mathbf{t}(H_2)$ of the solution to (7.1). It carries the energy generated by the nonlinear source $(a|E_1|^2 E_1, b|H_1|^2 H_1)$, where $(E_1, H_1)$ is the solution to the linear equation. By polarization, we are able to recover $a$ and $b$ from such energy using

enough proper solutions. The solutions we apply here are complex geometrical optics solutions constructed on an admissible manifold as in [12] or [4] with proper regularity.

The paper is organized as following. In Section 2, we present basic facts on differential forms, trace operators, the type of Sobolev spaces and their properties used in this paper. After proving the well-posedness of the direct problem (Theorem 1.1) in Section 3, we compute the asymptotic expansion of the admittance map $\Lambda^\omega_{\mu,\varepsilon,a,b}$ in Section 4. To solve the inverse problem, the reconstruction of $\mu$ and $\varepsilon$ is given in Section 5, and the reconstruction of $a$ and $b$ is given in Section 7. The CGO solution is constructed in Section 6.

## 2. Preliminaries

In this section we briefly present basic facts on differential forms and trace operators. For more detailed exposition we refer the reader to the manuscript of Schwarz [23].

Let $(M, g)$ be a compact oriented $n$-dimensional Riemannian manifold with smooth boundary. The inner product of tangent vectors with respect to the metric $g$ is denoted by $\langle \cdot, \cdot \rangle_g$, and $|\cdot|_g$ is the notation for the corresponding norm. By $|g|$ we denote the determinant of $g = (g_{ij})$ and $(g^{ij})$ is the inverse matrix of $(g_{ij})$. Finally, there is the induced metric $\iota^* g$ on $\partial M$ which gives a rise to the inner product $\langle \cdot, \cdot \rangle_{\iota^* g}$ of vectors tangent to $\partial M$.

### 2.1. Basic notations for differential forms.
In what follows, for $F$ some function space ($C^k$, $L^p$, $W^{k,p}$, etc.), we denote by $F\Omega^m(M)$ the corresponding space of $m$-forms. In particular, the space of smooth $m$-forms is denoted by $C^\infty \Omega^m(M)$. Let $*: C^\infty \Omega^m(M) \to C^\infty \Omega^{n-m}(M)$ be the Hodge star operator. For real valued $\eta, \zeta \in C^\infty \Omega^m(M)$, the inner product with respect to $g$ is defined as

$$\langle \eta, \zeta \rangle_g = *(\eta \wedge *\zeta) = \langle *\eta, *\zeta \rangle_g. \tag{2.1}$$

Its local coordinates expression is $\langle \eta, \zeta \rangle_g = g^{i_1 j_1} \cdots g^{i_m j_m} \eta_{i_1 \ldots i_m} \zeta_{j_1 \ldots j_m}$. This can be extended as a bilinear form on complex valued forms on $M$. We also write $|\eta|^2_g = \langle \eta, \bar{\eta} \rangle_g$.

The inner product on $L^2\Omega^m(M)$ is defined as

$$(\eta \mid \zeta)_{L^2\Omega^m(M)} = \int_M \langle \eta, \bar{\zeta} \rangle_g \, d\,\mathrm{Vol}_g = \int_M \eta \wedge *\bar{\zeta}, \quad \eta, \zeta \in L^2\Omega^m(M),$$

where

$$d\,\mathrm{Vol}_g = *1 = |g|^{1/2} \, dx^1 \wedge \cdots \wedge dx^n$$

is the volume form. The corresponding norm is $\| \cdot \|_{L^2\Omega^m(M)}^2 = (\cdot \mid \cdot)_{L^2\Omega^m(M)}$. Using the definition of the Hodge star operator $*$, it is not difficult to check that

$$(\eta \mid \zeta)_{L^2\Omega^m(M)} = (*\eta \mid *\zeta)_{L^2\Omega^{n-m}(M)}. \tag{2.2}$$

Let $d: C^\infty\Omega^m(M) \to C^\infty\Omega^{m+1}(M)$ be the external differential. Then the codifferential

$$\delta: C^\infty\Omega^m(M) \longrightarrow C^\infty\Omega^{m-1}(M)$$

is defined as

$$(d\eta \mid \zeta)_{L^2\Omega^m(M)} = (\eta \mid \delta\zeta)_{L^2\Omega^{m-1}(M)}$$

for all $\eta \in C_0^\infty\Omega^{m-1}(M^{\mathrm{int}})$, $\zeta \in C^\infty\Omega^m(M)$. The Hodge star operator $*$ and the codifferential $\delta$ have the following properties when acting on $C^\infty\Omega^m(M)$:

$$*^2 = (-1)^{m(n-m)}, \quad \delta = (-1)^{m(n-m)-n+m-1} * (d * \cdot). \tag{2.3}$$

For a given $\xi \in C^\infty\Omega^1(M)$, the interior product

$$i_\xi: C^\infty\Omega^m(M) \longrightarrow C^\infty\Omega^{m-1}(M)$$

is the contraction of differential forms by $\xi$. In local coordinates,

$$i_\xi\eta = g^{ij}\xi_i \, \eta_{ji_1\ldots i_{m-1}}, \quad \eta \in C^\infty\Omega^m(M).$$

It is the formal adjoint of $\xi$, in the inner product $\langle \cdot, \cdot \rangle_g$ on real valued forms, and has the following expression

$$i_\xi\eta = (-1)^{n(m-1)} * (\xi \wedge *\eta), \quad \eta \in C^\infty\Omega^m(M). \tag{2.4}$$

The Hodge Laplacian acting on $C^\infty\Omega^m(M)$ is defined by $-\Delta = d\delta + \delta d$. Finally, the inner product on $L^2\Omega^m(\partial M)$ is given by

$$(u \mid v)_{L^2\Omega^m(\partial M)} = \int_{\partial M} \langle u, \bar{v} \rangle_{\iota^* g} \, d\sigma_{\partial M}, \quad u, v \in L^2\Omega^m(\partial M),$$

where $\langle \cdot, \cdot \rangle_{\iota^* g}$ is extended as a bilinear form on complex forms on $\partial M$, and $d\sigma_{\partial M} = \iota^*(i_\nu d\,\mathrm{Vol}_g)$ is the volume form on $\partial M$ induced by $d\,\mathrm{Vol}_g$.

**2.2. Integration by parts.** The outward unit normal $\nu$ to $\partial M$ can be extended to a vector field near $\partial M$ by parallel transport along normal geodesics, initiating from $\partial M$ in the direction of $-\nu$, and then to a vector field on $M$ via a cutoff function.

The following simple result from [1, Lemma 2.1] will be used in formulating integration by parts formula in appropriate way.

**Lemma 2.1.** *If $\eta \in C^\infty \Omega^m(M)$ and $\zeta \in C^\infty \Omega^{m+1}(M)$, then for an open subset $\Gamma \subset \partial M$ the following holds*

$$(\mathbf{t}(\eta) \mid \mathbf{t}(i_\nu \zeta))_{L^2 \Omega^m(\Gamma)} = \int_\Gamma \mathbf{t}(\eta \wedge *\bar{\zeta}).$$

For $\eta \in C^\infty \Omega^m(M)$ and $\zeta \in C^\infty \Omega^{m+1}(M)$, using Stokes' theorem, Lemma 2.1 (with $\Gamma = \partial M$) and (2.3), we have the following integration by parts formula for $d$ and $\delta$

$$(\mathbf{t}(\eta) \mid \mathbf{t}(i_\nu \zeta))_{L^2 \Omega^m(\partial M)} = (d\eta \mid \zeta)_{L^2 \Omega^{m+1}(M)} - (\eta \mid \delta \zeta)_{L^2 \Omega^m(M)}. \tag{2.5}$$

**2.3. Extensions of trace operators.** The tangential trace operator $\mathbf{t}$ has an extension to a bounded operator from $W^{1,p}\Omega^m(M)$ to $W^{1-1/p,p}\Omega^m(\partial M)$ for $p > 1$. Moreover, for every $f \in W^{1-1/p,p}\Omega^m(\partial M)$, there is $u \in W^{1,p}\Omega^m(M)$ such that $\mathbf{t}(u) = f$ and

$$\|u\|_{W^{1,p}\Omega^m(M)} \le C \|f\|_{W^{1-1/p,p}\Omega^m(\partial M)};$$

see [23, Theorem 1.3.7] and comments.

The operator $\mathbf{t}(i_\nu \cdot)$ is bounded from $W^{1,p}\Omega^m(M)$ to $W^{1-1/p,p}\Omega^{m-1}(\partial M)$. Moreover, for every $h \in W^{1-1/p,p}\Omega^{m-1}(\partial M)$, there is $\zeta \in W^{1,p}\Omega^m(M)$ such that $\mathbf{t}(i_\nu \zeta) = h$ and

$$\|\zeta\|_{W^{1,p}\Omega^m(M)} \le C \|h\|_{W^{1-1/p,p}\Omega^{m-1}(\partial M)}.$$

In fact, we can take $\zeta = \nu \wedge w$, where $w \in H^1\Omega^{m-1}(M)$ such that $\mathbf{t}(w) = h$ and $\|w\|_{W^{1,p}\Omega^{m-1}(M)} \le C \|h\|_{W^{1-1/p,p}\Omega^{m-1}(\partial M)}$.

Finally, if $f \in W^{1-1/p,p}\Omega^m(\partial M)$ and $h \in W^{1-1/p,p}\Omega^{m-1}(\partial M)$, there is $\xi \in W^{1,p}\Omega^m(M)$ such that $\mathbf{t}(\xi) = f$, $\mathbf{t}(i_\nu \xi) = h$ and

$$\|\xi\|_{W^{1,p}\Omega^m(M)} \le C \|f\|_{W^{1-1/p,p}\Omega^m(\partial M)} + C \|h\|_{W^{1-1/p,p}\Omega^{m-1}(\partial M)}.$$

This time, we can take $\xi = (u - \nu \wedge i_\nu u) + \nu \wedge i_\nu \zeta$, where $u \in W^{1,p}\Omega^m(M)$ such that $\mathbf{t}(u) = f$ and $\|u\|_{W^{1,p}\Omega^m(M)} \le C \|f\|_{W^{1-1/p,p}\Omega^m(\partial M)}$ and $\zeta \in W^{1,p}\Omega^m(M)$ such that $\mathbf{t}(i_\nu \zeta) = h$ and $\|\zeta\|_{W^{1,p}\Omega^m(M)} \le C \|h\|_{W^{1-1/p,p}\Omega^{m-1}(\partial M)}$.

**2.4. Surface divergence.** When $n = 3$, we define the surface divergence of $f \in W^{1-1/p,p}\Omega^1(\partial M)$, for $p > 1$, by

$$\mathrm{Div}(f) = \langle d_{\partial M} f, d\sigma_{\partial M} \rangle_{\iota^* g}.$$

If $u \in W^{1,p}\Omega^1(M)$ is arbitrary such that $\mathbf{t}(u) = f$, then for all $h \in C^\infty(M)$

$$\begin{aligned}
(\mathrm{Div}(f) \mid h)_{L^2(\partial M)} &= \int_{\partial M} \langle d_{\partial M} f, \bar{h}\, d\sigma_{\partial M} \rangle_{\iota^* g} \, d\sigma_{\partial M} \\
&= \int_{\partial M} \langle \mathbf{t}(du), \mathbf{t}(\bar{h}\, i_\nu d\, \mathrm{Vol}_g) \rangle_g \, d\sigma_{\partial M} \\
&= \int_{\partial M} \langle \mathbf{t}(du), \mathbf{t}(\bar{h}\, i_\nu * 1) \rangle_g \, d\sigma_{\partial M} \\
&= \int_{\partial M} \langle \mathbf{t}(i_\nu * du), \bar{h} \rangle_g \, d\sigma_{\partial M}.
\end{aligned}$$

In the last step we used Lemma 2.1 twice. Thus, we have

$$\mathrm{Div}(f) = i_\nu * du|_{\partial M} \tag{2.6}$$

for all $u \in W^{1,p}\Omega^1(M)$ with $\mathbf{t}(u) = f$.

**2.5. Technical estimate.** We finish this section with the following lemma which ensures that nonlinear terms in the Maxwell equations (1.4) will be in appropriate functional spaces.

**Lemma 2.2.** *Let $(M, g)$ be a compact n-dimensional Riemannian manifold. If $u \in W^{1,p}\Omega^1(M)$ for $p > n$, then*

$$\||u|_g^2 u\|_{W^{1,p}\Omega^1(M)} \le C \|u\|_{W^{1,p}\Omega^1(M)}^3.$$

*Proof.* To prove the lemma, we first observe that the $W^{1,p}\Omega^m(M)$-norm may be expressed invariantly as

$$\|f\|_{W^{1,p}\Omega^1(M)} = \|f\|_{L^p\Omega^1(M)} + \||\nabla f|_g\|_{L^p(M)},$$

where $\nabla$ is the Levi-Civita connection defined on tensors on $M$ and $|T|_g$ is the norm of a tensor $T$ on $M$ with respect to the metric $g$.

For a given $u \in W^{1,p}\Omega^1(M)$,

$$
\begin{aligned}
\||u|_g^2 u\|_{W^{1,p}\Omega^1(M)} &\leq C \||u|_g^2 u\|_{L^p\Omega^1(M)} + C \| |\nabla(|u|_g^2 u)|_g \|_{L^p(M)} \\
&\leq C \||u|_g^2 u\|_{L^p\Omega^1(M)} + C \||u|_g^2 |\nabla u|_g \|_{L^p(M)} \\
&\leq C \|u\|_{L^\infty\Omega^1(M)}^2 \|u\|_{W^{1,p}\Omega^1(M)}.
\end{aligned}
$$

Since $p > n$ and $M$ is compact, we can use the Sobolev embedding

$$
W^{1,p}\Omega^1(M) \hookrightarrow C\Omega^1(M)
$$

([23, Theorem 1.3.6]), which implies the desired estimate                     $\square$

**2.6. Properties of $W_d^p\Omega^m(M)$ and $W_\delta^p\Omega^m(M)$ spaces, $p > 1$.** Let $(M, g)$ be a compact oriented $n$-dimensional Riemannian manifold with smooth boundary. In this paper we work with the Banach spaces $W_d^p\Omega^m(M)$ and $W_\delta^p\Omega^m(M)$, $p > 1$, which are the largest domains of $d$ and $\delta$, respectively, acting on $m$-forms:

$$
\begin{aligned}
W_d^p\Omega^m(M) &:= \{w \in L^p\Omega^m(M) : dw \in L^p\Omega^{m+1}(M)\}, \\
W_\delta^p\Omega^m(M) &:= \{u \in L^p\Omega^m(M) : \delta u \in L^p\Omega^{m-1}(M)\}
\end{aligned}
$$

endowed with the norms

$$
\begin{aligned}
\|w\|_{W_d^p\Omega^m(M)}^2 &:= \|w\|_{L^p\Omega^m(M)} + \|dw\|_{L^p\Omega^{m+1}(M)}, \\
\|u\|_{W_\delta^p\Omega^m(M)}^2 &:= \|u\|_{L^p\Omega^m(M)} + \|\delta u\|_{L^p\Omega^{m-1}(M)}.
\end{aligned}
$$

We also use the notations

$$
H_d\Omega^m(M) = W_d^2\Omega^m(M) \quad \text{and} \quad H_\delta\Omega^m(M) = W_\delta^2\Omega^m(M),
$$

together with their corresponding traces $TH_d\Omega^m(M)$ and $TH_\delta\Omega^m(M)$.

In the present section we prove some important properties of these spaces, which were proven in [1, Section 3] for the case $p = 2$; see also [5, 14, 18].

First, we show that there are bounded extensions

$$
\mathbf{t} : W_d^p\Omega^m(M) \longrightarrow W^{-1/p,p}\Omega^m(\partial M)
$$

and

$$
\mathbf{t}(i_\nu \cdot) : W_\delta^p\Omega^{m+1}(M) \longrightarrow W^{-1/p,p}\Omega^m(\partial M).
$$

Let $(\cdot \,|\, \cdot)_{\partial M}$ denote the distributional duality on $\partial M$ naturally extending $(\cdot \,|\, \cdot)_{L^2\Omega^m(\partial M)}$.

**Proposition 2.3.** (a) *The operator*

$$\mathbf{t} \colon W^{1,p}\Omega^m(M) \longrightarrow W^{1-1/p,p}\Omega^m(\partial M)$$

*has its extension to a bounded operator*

$$\mathbf{t} \colon W_d^p\Omega^m(M) \longrightarrow W^{-1/p,p}\Omega^m(\partial M)$$

*and the following integration by parts formula holds*

$$(\mathbf{t}(\eta) \mid \mathbf{t}(i_\nu\zeta))_{\partial M} = (d\eta \mid \zeta)_{L^2\Omega^{m+1}(M)} - (\eta \mid \delta\zeta)_{L^2\Omega^m(M)}$$

*for all $\eta \in W_d^p\Omega^m(M)$ and $\zeta \in W^{1,p'}\Omega^{m+1}(M)$, where $p' = p/(p-1)$.*

  (b) *The operator*

$$\mathbf{t}(i_\nu \cdot) \colon W^{1,p}\Omega^{m+1}(M) \to W^{1-1/p,p}\Omega^m(\partial M)$$

*has its extension to a bounded operator*

$$\mathbf{t}(i_\nu \cdot) \colon W_\delta^p\Omega^{m+1}(M) \to W^{-1/p,p}\Omega^m(\partial M)$$

*and the following integration by parts formula holds*

$$(\mathbf{t}(i_\nu\zeta) \mid \mathbf{t}(\eta))_{\partial M} = (\zeta \mid d\eta)_{L^2\Omega^{m+1}(M)} - (\delta\zeta \mid \eta)_{L^2\Omega^m(M)}$$

*for all $\zeta \in W_\delta^p\Omega^{m+1}(M)$ and $\eta \in W^{1,p'}\Omega^m(M)$.*

*Proof.* (a) Let $w \in C^\infty\Omega^m(M)$ and $f \in W^{1/p,p'}\Omega^m(\partial M)$, where $p' = p/(p-1)$. Then using integration by parts formula (2.5), we have

$$\begin{aligned}
(\mathbf{t}(w) \mid f)_{L^2\Omega^m(\partial M)} &= (\mathbf{t}(w) \mid \mathbf{t}(i_\nu\zeta))_{L^2\Omega^m(\partial M)} \\
&= (dw \mid \zeta)_{L^2\Omega^{m+1}(M)} - (w \mid \delta\zeta)_{L^2\Omega^m(M)},
\end{aligned}$$

where $\zeta \in W^{1,p'}\Omega^{m+1}(M)$ such that

$$\mathbf{t}(i_\nu\zeta) = f \quad \text{and} \quad \|\zeta\|_{W^{1,p'}\Omega^{m+1}(M)} \le C\|f\|_{W^{1/p,p'}\Omega^m(\partial M)}.$$

Then using Hölder's inequality, we show

$$\begin{aligned}
|(\mathbf{t}(w) \mid f)_{L^2\Omega^m(\partial M)}| &\le C\|w\|_{W_d^p\Omega^m(M)}\|\zeta\|_{W^{1,p'}\Omega^{m+1}(M)} \\
&\le C\|w\|_{W_d^p\Omega^m(M)}\|f\|_{W^{1/p,p'}\Omega^m(\partial M)}.
\end{aligned}$$

Therefore, $\mathbf{t}$ can be extended to a bounded operator

$$W_d^p\Omega^m(M) \longrightarrow W^{-1/p,p}\Omega^m(\partial M).$$

In fact, if $\eta \in W_d^p \Omega^m(M)$, then we define $\mathbf{t}(\eta)$ as

$$(\mathbf{t}(\eta) \mid \mathbf{t}(i_\nu \zeta))_{\partial M} = (d\eta \mid \zeta)_{L^2 \Omega^{m+1}(M)} - (\eta \mid \delta \zeta)_{L^2 \Omega^m(M)},$$

where $\zeta \in W^{1,p'} \Omega^{m+1}(M)$.

Now we prove part (b). Let $w \in C^\infty \Omega^{m+1}(M)$ and $f \in W^{1/p,p'} \Omega^m(\partial M)$. Then using integration parts formula (2.5), we have

$$\begin{aligned}
(\mathbf{t}(i_\nu w) \mid f)_{L^2 \Omega^m(\partial M)} &= (\mathbf{t}(i_\nu w) \mid \mathbf{t}(u))_{L^2 \Omega^m(\partial M)} \\
&= (w \mid du)_{L^2 \Omega^{m+1}(M)} - (\delta w \mid u)_{L^2 \Omega^m(M)},
\end{aligned}$$

where $u \in W^{1,p'} \Omega^m(M)$ such that

$$\mathbf{t}(u) = f \quad \text{and} \quad \|u\|_{W^{1,p'} \Omega^m(M)} \le C \|f\|_{W^{1/p,p'} \Omega^m(\partial M)}.$$

Therefore, using Hölder's inequality, we can estimate

$$\begin{aligned}
|(\mathbf{t}(i_\nu w) \mid f)_{L^2 \Omega^m(\partial M)}| &\le C \|w\|_{W_\delta^p \Omega^{m+1}(M)} \|u\|_{W^{1,p'} \Omega^m(M)} \\
&\le C \|w\|_{W_\delta^p \Omega^{m+1}(M)} \|f\|_{W^{1/p,p'} \Omega^m(\partial M)}.
\end{aligned}$$

Thus, $\mathbf{t}(i_\nu \cdot)$ can be extended to a bounded operator

$$W_\delta^p \Omega^{m+1}(M) \longrightarrow W^{-1/p,p} \Omega^m(\partial M).$$

In fact, if $\zeta \in W_\delta^p \Omega^{m+1}(M)$ we define $\mathbf{t}(i_\nu \zeta)$ as

$$(\mathbf{t}(i_\nu \zeta) \mid \mathbf{t}(\eta))_{\partial M} = (\zeta \mid d\eta)_{L^2 \Omega^{m+1}(M)} - (\delta \zeta \mid \eta)_{L^2 \Omega^m(M)},$$

where $\eta \in W^{1,p'} \Omega^m(M)$. $\qquad \square$

We will also need the following embedding results. For $p = 2$, these were proven in Euclidean and Riemannian settings [1, 14, 18].

**Proposition 2.4.** *Suppose that $p > 1$, $u \in W_d^p \Omega^m(M) \cap W_\delta^p \Omega^m(M)$ and $\mathbf{t}(u) \in W^{1-1/p} \Omega^m(\partial M)$. Then $u \in W^{1,p} \Omega^m(M)$ and*

$$\|u\|_{W^{1,p} \Omega^m(M)} \le C(\|u\|_{W_d^p \Omega^m(M)} + \|\delta u\|_{L^p \Omega^{m-1}(M)} + \|\mathbf{t}(u)\|_{W^{1-1/p,p} \Omega^m(\partial M)})$$

*for some constant $C > 0$ independent of $u$.*

In Euclidean setting, this was proven in the case $m = 1$ and $p = 2$ by Costabel [5]; see also [14, 18]. On manifolds, for the case $p = 2$ and for arbitrary $m$, this was proved in [1].

Write

$$\mathcal{H}_D^m(M) := \{u \in W^{1,2} \Omega^m(M) : du = 0, \, \delta u = 0, \, \mathbf{t}(u) = 0\}.$$

Proposition 2.4 is based on the following result from [23].

**Lemma 2.5.** *Let $k \geq 0$ be an integer and let $p > 1$. Given $w \in W^{k,p}\Omega^{m+1}(M)$, $v \in W^{k,p}\Omega^{m-1}(M)$ and $h \in W^{k+1,p}\Omega^m(M)$, there is a unique*

$$\psi \in W^{k+1,p}\Omega^m(M),$$

*up to a form in $\mathcal{H}_D^m(M)$, that solves*

$$d\psi = w, \quad \delta\psi = v, \quad \mathbf{t}(\psi) = \mathbf{t}(h)$$

*if and only if*

$$dw = 0, \quad \mathbf{t}(w) = \mathbf{t}(dh), \quad \delta v = 0$$

*and*

$$(w \mid \chi)_{L^2\Omega^{m+1}(M)} = (\mathbf{t}(h) \mid \mathbf{t}(i_\nu\chi))_{L^2\Omega^m(\partial M)}, \quad (v \mid \lambda)_{L^2\Omega^{m-1}(M)} = 0$$

*for all $\chi \in \mathcal{H}_D^{m+1}(M)$, $\lambda \in \mathcal{H}_D^{m-1}(M)$. Moreover, $\psi$ satisfies the estimate*

$$\begin{aligned}
&\|\psi\|_{W^{k+1,p}\Omega^m(M)} \\
&\leq C(\|w\|_{W^{k,p}\Omega^{m+1}(M)} + \|v\|_{W^{k,p}\Omega^{m-1}(M)}) \\
&\quad + C(\|\mathbf{t}(h)\|_{W^{k+1-1/p,p}\Omega^m(\partial M)} + \|\mathbf{t}(*h)\|_{W^{k+1-1/p,p}\Omega^{n-m}(\partial M)}).
\end{aligned}$$

*Proof.* Follows from [23, Theorem 3.2.5]. $\qquad\square$

The proof of Proposition 2.4 is identical to the proof of [1, Proposition 3.2] (case $p = 2$), but for different integrability spaces. Therefore, we do not include it here. We only mention that the use of Lemma 2.5 is crucial and similar ideas were used in the next proposition, after certain modifications.

**Proposition 2.6.** *Suppose that $p > 1$, $u \in W_d^p\Omega^m(M) \cap W_\delta^p\Omega^m(M)$ and $\mathbf{t}(i_\nu u) \in W^{1-1/p,p}\Omega^{m-1}(\partial M)$. Then $u \in W^{1,p}\Omega^m(M)$ and*

$$\begin{aligned}
&\|u\|_{W^{1,p}\Omega^m(M)} \\
&\leq C(\|u\|_{W_d^p\Omega^m(M)} + \|\delta u\|_{L^p\Omega^{m-1}(M)} + \|\mathbf{t}(i_\nu u)\|_{W^{1-1/p,p}\Omega^{m-1}(\partial M)})
\end{aligned}$$

*for some constant $C > 0$ independent of $u$.*

*Proof.* Since $\mathbf{t}(i_\nu u) \in W^{1-1/p,p}\Omega^{m-1}(\partial M)$, by discussion in Section 2.3 there is $\eta \in W^{1,p}\Omega^m(M)$ such that $\mathbf{t}(\eta) = 0$, $\mathbf{t}(i_\nu\eta) = \mathbf{t}(i_\nu u)$ and

$$\|\eta\|_{W^{1,p}\Omega^m(M)} \leq C\|\mathbf{t}(i_\nu\eta)\|_{W^{1-1/p,p}\Omega^{m-1}(\partial M)} = C\|\mathbf{t}(i_\nu u)\|_{W^{1-1/p,p}\Omega^{m-1}(\partial M)}.$$

Set $h = *\eta$, then $h \in W^{1,p}\Omega^{n-m}(M)$. Using boundedness of

$$\mathbf{t}: W^{1,p}\Omega^{n-m}(M) \longrightarrow W^{1-1/p,p}\Omega^{n-m}(\partial M)$$

and the above estimate,

$$
\begin{aligned}
\|\mathbf{t}(h)\|_{W^{1-1/p,p}\Omega^{n-m}(\partial M)} &\leq C \|h\|_{W^{1,p}\Omega^{n-m}(M)} \\
&\leq C \|\eta\|_{W^{1,p}\Omega^m(M)} \\
&\leq C \|\mathbf{t}(i_\nu u)\|_{W^{1-1/p,p}\Omega^{m-1}(\partial M)}.
\end{aligned}
\tag{2.7}
$$

Set

$$\tilde{u} := *u,$$

then it is clear that $\tilde{u} \in W_d^p\Omega^{n-m}(M) \cap W_\delta^p\Omega^{n-m}(M)$. Write

$$w = d\tilde{u} \in L^p\Omega^{n-m+1}(M)$$

and

$$v = \delta\tilde{u} \in L^p\Omega^{n-m-1}(M).$$

An important fact is that

$$\mathbf{t}(\tilde{u}) = \mathbf{t}(h).$$

Indeed, for arbitrary $\varphi \in W^{1/p,p/(p-1)}\Omega^{n-m}(\partial M)$, as discussed in Section 2.3, there is $\zeta \in W^{1,p/(p-1)}\Omega^{n-m+1}(M)$ such that $\mathbf{t}(i_\nu\zeta) = \varphi$. Then, using integration by parts formulas in Proposition 2.3, we get

$$
\begin{aligned}
(\mathbf{t}(\tilde{u} - h) \mid \varphi)_{\partial M} &= (\mathbf{t}(*(u - \eta)) \mid \mathbf{t}(i_\nu\zeta))_{\partial M} \\
&= (d*(u - \eta) \mid \zeta)_{L^2\Omega^{n-m+1}(M)} - (*(u - \eta) \mid \delta\zeta)_{L^2\Omega^{n-m}(M)} \\
&= (\delta(u - \eta) \mid *\zeta)_{L^2\Omega^{m-1}(M)} - (u - \eta \mid d*\zeta)_{L^2\Omega^{m-1}(M)} \\
&= -(\mathbf{t}(i_\nu(u - \eta)) \mid \mathbf{t}(*\zeta))_{\partial M} = 0,
\end{aligned}
$$

since $\mathbf{t}(i_\nu\eta) = \mathbf{t}(i_\nu u)$. Therefore, $\mathbf{t}(\tilde{u}) = \mathbf{t}(h)$.

We wish to use Lemma 2.5, and hence we need to show that $w$, $v$ and $h$ satisfy the hypothesis of Lemma 2.5. Obviously, we have $dw = 0$ and $\delta v = 0$. Integrating by parts and using that $\mathbf{t}(\tilde{u}) = \mathbf{t}(h)$, we can show that for all $\chi \in \mathcal{H}_D^{n-m+1}(M)$

$$(w \mid \chi)_{L^2\Omega^{n-m+1}(M)} = (d\tilde{u} \mid \chi)_{L^2\Omega^{n-m+1}(M)} = (\mathbf{t}(h) \mid \mathbf{t}(i_\nu\chi))_{L^2\Omega^{n-m}(\partial M)}.$$

Similary for all $\lambda \in \mathcal{H}_D^{n-m-1}(M)$, using the integration by parts formula in part (b) of Proposition 2.3, we can show that

$$(v \mid \lambda)_{L^2\Omega^{n-m-1}(M)} = (\delta\tilde{u} \mid \lambda)_{L^2\Omega^{n-m-1}(M)} = -(\mathbf{t}(i_\nu\tilde{u}) \mid \mathbf{t}(\lambda))_{\partial M} = 0.$$

Next, we show that $\mathbf{t}(w) = \mathbf{t}(dh)$. For arbitrary $\varphi \in W^{1/p,p/(p-1)}\Omega^{n-m+1}(\partial M)$, as discussed in Section 2.3, there is $\zeta \in W^{1,p/(p-1)}\Omega^{n-m+2}(M)$ such that $\mathbf{t}(i_\nu\zeta) = \varphi$. Then, using integration by parts formulas in Proposition 2.3, we get

$$
\begin{aligned}
(\mathbf{t}(w) \mid \varphi)_{\partial M} &= (\mathbf{t}(d\tilde{u}) \mid \mathbf{t}(i_\nu\zeta))_{\partial M} \\
&= -(d\tilde{u} \mid \delta\zeta)_{L^2\Omega^{n-m+1}(M)} \\
&= -(\mathbf{t}(\tilde{u}) \mid \mathbf{t}(i_\nu\delta\zeta))_{\partial M}.
\end{aligned}
$$

Since $\mathbf{t}(\tilde{u}) = \mathbf{t}(h)$, using integration by parts formulas in Proposition 2.3, gives

$$
\begin{aligned}
(\mathbf{t}(w) \mid \varphi)_{\partial M} &= -(\mathbf{t}(h) \mid \mathbf{t}(i_\nu\delta\zeta))_{\partial M} \\
&= -(dh \mid \delta\zeta)_{L^2\Omega^{n-m+1}(M)} \\
&= (\mathbf{t}(dh) \mid \varphi)_{\partial M},
\end{aligned}
$$

which implies $\mathbf{t}(w) = \mathbf{t}(dh)$.

Thus, all hypotheses of Lemma 2.5 are satisfied for $w$, $v$ and $h$. Hence, we find $\psi \in W^{1,p}\Omega^{n-m}(M)$ such that $d\psi = w$, $\delta\psi = v$ and $\mathbf{t}(\psi) = \mathbf{t}(h) = \mathbf{t}(\tilde{u})$ and satisfying

$$
\begin{aligned}
\|\psi\|_{W^{1,p}\Omega^{n-m}(M)} \leq &C(\|w\|_{L^p\Omega^{n-m+1}(M)} + \|v\|_{L^p\Omega^{n-m-1}(M)}) \\
&+ C(\|\mathbf{t}(h)\|_{W^{1-1/p,p}\Omega^{n-m}(\partial M)} + \|\mathbf{t}(*h)\|_{W^{1-1/p,p}\Omega^{m}(\partial M)}).
\end{aligned}
$$

Using (2.7), $\mathbf{t}(*h) = \mathbf{t}(\eta) = 0$, $w = d * u$ and $v = \delta * u$, we get

$$
\begin{aligned}
\|\psi\|_{W^{1,p}\Omega^{n-m}(M)} \\
\leq C(\|u\|_{W_d^p\Omega^m(M)} + \|\delta u\|_{L^p\Omega^{m-1}(M)} + \|\mathbf{t}(i_\nu u)\|_{W^{1-1/p,p}\Omega^{m-1}(\partial M)}).
\end{aligned}
$$

Write $\rho = \tilde{u} - \psi$, then $d\rho = 0$ and $\delta\rho = 0$. Therefore, $\rho$ solves $-\Delta\rho = 0$ with $\mathbf{t}(\rho) = 0$, $\mathbf{t}(\delta\rho) = 0$. By [23, Theorem 2.2.4], it follows that $\rho = 0$. Since $\tilde{u} = *u$, the last estimate together with (2.7) clearly implies the result. $\square$

## 3. Well-posedness of the direct problem

**3.1. Direct problem for linear equations.** To prove existence and uniqueness result for nonlinear equations, we first need to study the direct problem for linear equations.

**Theorem 3.1.** *Let $2 \leq p \leq 6$ and let $\varepsilon, \mu \in C^1(M)$ be complex functions with positive real parts. There is a discrete subset $\Sigma$ of $\mathbb{C}$ such that for all $\omega \notin \Sigma$ and for a given $f \in TW_{\mathrm{Div}}^{1-1/p,p}(\partial M)$ the Maxwell's equation*

$$* dE = i\omega\mu H, \quad *dH = -i\omega\varepsilon E \tag{3.1}$$

*has a unique solution $(E, H) \in W_{\mathrm{Div}}^{1,p}(M) \times W_{\mathrm{Div}}^{1,p}(M)$ satisfying $\mathbf{t}(E) = f$ and*

$$\|E\|_{W_{\mathrm{Div}}^{1,p}(M)} + \|H\|_{W_{\mathrm{Div}}^{1,p}(M)} \leq C\|f\|_{TW_{\mathrm{Div}}^{1-1/p,p}(\partial M)},$$

*for some constant $C > 0$ independent of $f$.*

*Proof.* Since $p > 1$, the inclusion $TW_{\mathrm{Div}}^{1-1/p,p}(\partial M) \hookrightarrow TH_d\Omega^1(\partial M)$ is bounded. Then by [1, Theorem 1.1] there is a unique solution

$$(E, H) \in H_d\Omega^1(M) \times H_d\Omega^1(M)$$

of (3.1) such that $\mathbf{t}(E) = f$. By Theorem A.1, $(E, H) \in H_{\mathrm{Div}}^1(M) \times H_{\mathrm{Div}}^1(M)$. By Sobolev embedding, inclusion $W^{1,2}\Omega^1(M) \hookrightarrow L^p\Omega^1(M)$ is bounded for $2 \leq p \leq 6$; see [23, Theorem 1.3.6 (a)]. Using this together with (3.1), we get $(E, H) \in W_d^p\Omega^1(M) \times W_d^p\Omega^1(M)$. Recall that $\mathbf{t}(E) = f \in TW_{\mathrm{Div}}^{1-1/p,p}(\partial M)$. Then an application of Theorem A.1 implies that $(E, H) \in W_{\mathrm{Div}}^{1,p}(M) \times W_{\mathrm{Div}}^{1,p}(M)$ and satisfies the estimate stated in the theorem. The proof is complete. $\square$

We also consider the linear non-homogeneous problem. The following well-posedness result will be used in dealing with nonlinear terms of (1.4). We define

$$W_D^{1,p}\Omega^1(M) := \{u \in W^{1,p}\Omega^1(M) : \mathbf{t}(u) = 0\}.$$

**Theorem 3.2.** *Let $2 \leq p \leq 6$ and let $\varepsilon, \mu \in C^1(M)$ be complex functions with positive real parts. Suppose that $J_e, J_m \in W_\delta^p\Omega^1(M)$ and $i_\nu J_e|_{\partial M}, i_\nu J_m|_{\partial M} \in W^{1-1/p,p}(\partial M)$. There is a discrete subset $\Sigma$ of $\mathbb{C}$ such that for all $\omega \notin \Sigma$ the Maxwell's system*

$$* dE = i\omega\mu H + J_m, \quad *dH = -i\omega\varepsilon E - J_e \tag{3.2}$$

*has a unique solution $(E, H) \in W_D^{1,p}\Omega^1(M) \times W_{\mathrm{Div}}^{1,p}(M)$ satisfying*

$$\|E\|_{W_{\mathrm{Div}}^{1,p}(M)} + \|H\|_{W_{\mathrm{Div}}^{1,p}(M)}$$
$$\leq C(\|i_\nu J_e|_{\partial M}\|_{W^{1-1/p,p}(\partial M)} + \|i_\nu J_m|_{\partial M}\|_{W^{1-1/p,p}(\partial M)})$$
$$+ C(\|J_e\|_{W_\delta^p\Omega^1(M)} + \|J_m\|_{W_\delta^p\Omega^1(M)})$$

*for some constant $C > 0$ independent of $J_e$ and $J_m$.*

*Proof.* We follow the similar approach as in the proof of Theorem 3.1. Since $p \geq 2$, the inclusion $W_\delta^p \Omega^1(M) \hookrightarrow L^2 \Omega^1(M)$ is bounded. Then by [1, Theorem 1.2] there is a unique solution $(E, H) \in H_d \Omega^1(M) \times H_d \Omega^1(M)$ of (3.2) such that $\mathbf{t}(E) = 0$. According to Theorem A.1, we have $(E, H) \in H_D^1 \Omega^1(M) \times H_{\mathrm{Div}}^1(M)$. Using Sobolev embedding $H^1 \Omega^1(M) \hookrightarrow L^p \Omega^1(M)$ for $2 \leq p \leq 6$ together with (3.2), we get $(E, H) \in W_d^p \Omega^1(M) \times W_d^p \Omega^1(M)$. Since $\mathbf{t}(E) = 0$, Theorem A.1 implies that $(E, H) \in W_D^{1,p} \Omega^1(M) \times W_{\mathrm{Div}}^{1,p}(M)$ and satisfies the estimate stated in the theorem. The proof is complete.    $\square$

**3.2. Proof of Theorem 1.1.** Suppose $f \in TW_{\mathrm{Div}}^{1-1/p,p}(\partial M)$ such that one has $\|f\|_{TW_{\mathrm{Div}}^{1-1/p,p}(\partial M)} < \epsilon$, where $\epsilon > 0$ to be determined. By Theorem 3.1, when $2 \leq p \leq 6$, there is a unique $(E_0, H_0) \in W_{\mathrm{Div}}^{1,p}(M) \times W_{\mathrm{Div}}^{1,p}(M)$ solving

$$*dE_0 = i\omega\mu H_0, \quad *dH_0 = -i\omega\varepsilon E_0, \quad \mathbf{t}(E_0) = f$$

and satisfying

$$\|E_0\|_{W_{\mathrm{Div}}^{1,p}(M)} + \|H_0\|_{W_{\mathrm{Div}}^{1,p}(M)} \leq C\|f\|_{TW_{\mathrm{Div}}^{1-1/p,p}(\partial M)}.$$

Then $(E, H)$ is a solution of (1.4) if and only if $(E', H')$ defined by

$$(E, H) = (E_0, H_0) + (E', H')$$

satisfies

$$\begin{cases} *dE' = i\omega\mu H' + i\omega b|H_0 + H'|_g^2 (H_0 + H'), \\ *dH' = -i\omega\varepsilon E' - i\omega a|E_0 + E'|_g^2 (E_0 + E'), \\ \mathbf{t}(E') = 0. \end{cases} \tag{3.3}$$

By Theorem 3.2, there is a bounded and linear operator

$$\mathcal{G}_\omega^{\varepsilon,\mu} \colon W^{1,p}\Omega^1(M) \times W^{1,p}\Omega^1(M) \longrightarrow W_D^{1,p}\Omega^1(M) \times W_{\mathrm{Div}}^{1,p}(M)$$

mapping $(J_e, J_m) \in W^{1,p}\Omega^1(M) \times W^{1,p}\Omega^1(M)$ to the unique solution $(\widetilde{E}, \widetilde{H})$ of the problem

$$*d\widetilde{E} = i\omega\mu\widetilde{H} + J_m, \quad *d\widetilde{H} = -i\omega\varepsilon\widetilde{E} - J_e, \quad \mathbf{t}(\widetilde{E}) = 0.$$

Define $X_\delta$ to be the set of $(e, h) \in W_D^{1,p}\Omega^1(M) \times W_{\mathrm{Div}}^{1,p}(M)$ such that

$$\|(e, h)\|_{W^{1,p}\Omega^1(M) \times W_{\mathrm{Div}}^{1,p}(M)} := \|e\|_{W^{1,p}\Omega^1(M)} + \|h\|_{W_{\mathrm{Div}}^{1,p}(M)} \leq \delta,$$

where $\delta > 0$ will be determined later. Define an operator $A$ on $X_\delta$ as

$$A(e, h) := \mathcal{G}_\omega^{\varepsilon,\mu}\left(i\omega a|E_0 + e|_g^2 (E_0 + e), \, i\omega b|H_0 + h|_g^2 (H_0 + h)\right).$$

We wish to show that for sufficiently small $\epsilon > 0$ and $\delta > 0$, depending on the frequency $\omega$, the operator $A$ is a contraction on $X_\delta$.

First, we show that $A$ maps $X_\delta$ into itself. Using Lemma 2.2, we can show that when $p > n = 3$, for all $(e, h) \in X_\delta$,

$$
\begin{aligned}
\|A(e,h)&\|_{W^{1,p}\Omega^1(M) \times W^{1,p}_{\mathrm{Div}}(M)} \\
&\leq C\omega(\| \, |E_0 + e|^2_g (E_0 + e)\|_{W^{1,p}\Omega^1(M)} + \| \, |H_0 + h|^2_g (H_0 + h)\|_{W^{1,p}\Omega^1(M)}) \\
&\leq C\omega(\|E_0 + e\|^3_{W^{1,p}\Omega^1(M)} + \|H_0 + h\|^3_{W^{1,p}\Omega^1(M)}) \\
&\leq C\omega(\|E_0\|^3_{W^{1,p}\Omega^1(M)} + \|e\|^3_{W^{1,p}\Omega^1(M)} + \|H_0\|^3_{W^{1,p}\Omega^1(M)} + \|h\|^3_{W^{1,p}\Omega^1(M)}).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|A(e,h)&\|_{W^{1,p}\Omega^1(M) \times W^{1,p}_{\mathrm{Div}}(M)} \\
&\leq C\omega\epsilon^2 \|f\|_{TW^{1-1/p,p}_{\mathrm{Div}}(\partial M)} + C\omega\delta^2 \|(e,h)\|_{W^{1,p}\Omega^1(M) \times W^{1,p}_{\mathrm{Div}}(M)}.
\end{aligned}
\tag{3.4}
$$

In particular, this gives

$$
\|A(e,h)\|_{W^{1,p}\Omega^1(M) \times W^{1,p}_{\mathrm{Div}}(M)} \leq C\omega(\epsilon^3 + \delta^3).
$$

Taking $\epsilon > 0$ and $\delta > 0$ sufficently small, below we will ensure that $A$ maps $X_\delta$ into itself.

Next, we show that $A$ is contraction on $X_\delta$. For this we need the following technical lemma.

**Lemma 3.3.** *Let $(M, g)$ be a compact $n$-dimensional Riemannian manifold and let $p > n$. If $u, v \in W^{1,p}\Omega^1(M)$, then*

$$
\begin{aligned}
\|(|u|^2_g u &- |v|^2_g v)\|_{W^{1,p}\Omega^1(M)} \\
&\leq C(\|u\|^2_{W^{1,p}\Omega^1(M)} + \|v\|^2_{W^{1,p}\Omega^1(M)})\|u - v\|_{W^{1,p}\Omega^1(M)}.
\end{aligned}
$$

Assuming this result, we continue the proof of Theorem 1.1. Using Lemma 3.3, we also can show that for all $(e_1, h_1), (e_2, h_2) \in X_\delta$

$$
\begin{aligned}
\|A(e_1,& h_1) - A(e_2, h_2)\|_{W^{1,p}\Omega^1(M) \times W^{1,p}_{\mathrm{Div}}(M)} \\
&\leq C\omega\| \, |E_0 + e_1|^2_g (E_0 + e_1) - |E_0 + e_2|^2_g (E_0 + e_2)\|_{W^{1,p}\Omega^1(M)} \\
&\quad + C\omega\| \, |H_0 + h_1|^2_g (H_0 + h_1) - |H_0 + h_2|^2_g (H_0 + h_2)\|_{W^{1,p}\Omega^1(M)} \\
&\leq C\omega(\|E_0 + e_1\|^2_{W^{1,p}\Omega^1(M)} + \|E_0 + e_2\|^2_{W^{1,p}\Omega^1(M)})\|e_1 - e_2\|_{W^{1,p}\Omega^1(M)} \\
&\quad + C\omega(\|H_0 + h_1\|^2_{W^{1,p}\Omega^1(M)} + \|H_0 + h_2\|^2_{W^{1,p}\Omega^1(M)})\|h_1 - h_2\|_{W^{1,p}\Omega^1(M)}
\end{aligned}
$$

$$\leq C\omega(\|E_0\|^2_{W^{1,p}\Omega^1(M)} + \|e_1\|^2_{W^{1,p}\Omega^1(M)}$$
$$+ \|e_2\|^2_{W^{1,p}\Omega^1(M)})\|e_1 - e_2\|_{W^{1,p}\Omega^1(M)}$$
$$+ C\omega(\|H_0\|^2_{W^{1,p}\Omega^1(M)} + \|h_1\|^2_{W^{1,p}\Omega^1(M)}$$
$$+ \|h_2\|^2_{W^{1,p}\Omega^1(M)})\|h_1 - h_2\|_{W^{1,p}\Omega^1(M)}$$
$$\leq C\omega(\epsilon^2 + \delta^2)(\|e_1 - e_2\|_{W^{1,p}\Omega^1(M)} + \|h_1 - h_2\|_{W^{1,p}\Omega^1(M)}).$$

These imply that $A$ is contraction on $X_\delta$, if $C\omega(\epsilon^3 + \delta^3) \leq \delta$ and $C\omega(\epsilon^2 + \delta^2) < 1$. Now, using the contraction mapping theorem, we find a unique $(E', H') \in X_\delta$ such that $A(E', H') = (E', H')$ and hence solving (3.3). Using $A(E', H') = (E', H')$ in (3.4) and taking $\delta > 0$ sufficently small, one can see that $(E', H')$ satisfies the estimate

$$\|E'\|_{W^{1,p}\Omega^1(M)} + \|H'\|_{W^{1,p}_{\mathrm{Div}}(M)} \leq C\|f\|_{TW^{1-1/p,p}_{\mathrm{Div}}(\partial M)}.$$

Finally, $(E, H) = (E_0, H_0) + (E', H')$ solves (1.4) with $\mathbf{t}(E) = f$ and satisfies the estimate

$$\|E\|_{W^{1,p}_{\mathrm{Div}}(M)} + \|H\|_{W^{1,p}_{\mathrm{Div}}(M)} \leq C\|f\|_{TW^{1-1/p,p}_{\mathrm{Div}}(\partial M)}.$$

The proof of Theorem 1.1 is thus complete.

We emphasize that the requirement $3 < p \leq 6$ is due to the Sobolev embedding theorem (see [23, Theorem 1.3.6 (a)]) used in Lemma 2.2, Lemma 3.3 and Theorem 3.1.

*Proof of Lemma* 3.3. Recall that the $W^{1,p}\Omega^m(M)$-norm may be expressed invariantly as

$$\|f\|_{W^{1,p}\Omega^1(M)} = \|f\|_{L^p\Omega^1(M)} + \| |\nabla f|_g \|_{L^p(M)},$$

where $\nabla$ is the Levi-Civita connection defined on tensors on $M$ and $|T|_g$ is the norm of a tensor $T$ on $M$ with respect to the metric $g$.

By density of $C^\infty\Omega^1(M)$ in $W^{1,p}\Omega^1(M)$, it is enough to assume that $u, v \in C^\infty\Omega^1(M)$. Recall that

$$\|(|u|^2_g u - |v|^2_g v)\|_{W^{1,p}\Omega^1(M)}$$
$$= \||u|^2_g u - |v|^2_g v\|_{L^p\Omega^1(M)} + \| |\nabla(|u|^2_g u - |v|^2_g v)|_g \|_{L^p(M)}.$$

We can write

$$\nabla(|u|^2_g u - |v|^2_g v) = |u|^2_g \nabla u - |v|^2_g \nabla v + 2\operatorname{Re}\langle u, \nabla\bar{u}\rangle_g u - 2\operatorname{Re}\langle v, \nabla\bar{v}\rangle_g v.$$

Therefore,

$$\| |u|_g^2 u - |v|_g^2 v \|_{W^{1,p}\Omega^1(M)}$$
$$\leq C \| |u|_g^2 u - |v|_g^2 v \|_{L^p\Omega^1(M)} + C \| |(|u|_g^2 \nabla u - |v|_g^2 \nabla v)|_g \|_{L^p(M)} \qquad (3.5)$$
$$+ C \| |(\operatorname{Re}\langle u, \nabla \bar{u}\rangle_g u - \operatorname{Re}\langle v, \nabla \bar{v}\rangle_g v)|_g \|_{L^p(M)}.$$

Write $w_\theta = u + \theta(v - u)$. Let us estimate the first term on the right hand-side of (3.5). Then

$$|v|_g^2 v - |u|_g^2 u = \int_0^1 \frac{\partial}{\partial\theta}\{|w_\theta|_g^2 w_\theta\} \, d\theta$$

$$= \int_0^1 \{2\operatorname{Re}\langle w_\theta, v - u\rangle_g \, w_\theta + |w_\theta|_g^2(v - u)\} \, d\theta$$

and hence

$$|(|u|_g^2 u - |v|_g^2 v)|_g \leq C(\|u\|_{L^\infty\Omega^1(M)}^2 + \|v\|_{L^\infty\Omega^1(M)}^2)|u - v|_g.$$

Using the Sobolev embedding $W^{1,p}\Omega^1(M) \hookrightarrow C\Omega^1(M)$ as in Lemma 2.2, we get for $p > n$

$$\| |u|_g^2 u - |v|_g^2 v \|_{L^p\Omega^1(M)}$$
$$\leq C(\|u\|_{L^\infty\Omega^1(M)}^2 + \|v\|_{L^\infty\Omega^1(M)}^2)\|u - v\|_{L^p\Omega^1(M)}$$
$$\leq C(\|u\|_{W^{1,p}\Omega^1(M)}^2 + \|v\|_{W^{1,p}\Omega^1(M)}^2)\|u - v\|_{L^p\Omega^1(M)}.$$

Now, we estimate the second term on the right-hand-side of (3.5). Similarly as before, we can show

$$|v|_g^2 \nabla v - |u|_g^2 \nabla u = \int_0^1 \frac{\partial}{\partial\theta}\{|w_\theta|_g^2 \, \nabla w_\theta\} \, d\theta$$

$$= \int_0^1 \{2\operatorname{Re}\langle w_\theta, \bar{v} - \bar{u}\rangle_g \, \nabla w_\theta + |w_\theta|_g^2 \nabla(v - u)\} \, d\theta.$$

Then

$$| |u|_g^2 \nabla u - |v|_g^2 \nabla v|_g$$
$$\leq C(\|u\|_{L^\infty\Omega^1(M)} + \|v\|_{L^\infty\Omega^1(M)})\|u - v\|_{L^\infty\Omega^1(M)}(|\nabla u|_g + |\nabla v|_g)$$
$$+ C(\|u\|_{L^\infty\Omega^1(M)}^2 + \|v\|_{L^\infty\Omega^1(M)}^2)|\nabla(u - v)|_g.$$

Therefore, using the Sobolev embedding $W^{1,p}\Omega^1(M) \hookrightarrow C\Omega^1(M)$, we get

$$\| \, | \, |u|_g^2 \nabla u - |v|_g^2 \nabla v \, |_g \|_{L^p(M)}$$
$$\leq C(\|u\|_{W^{1,p}\Omega^1(M)}^2 + \|v\|_{W^{1,p}\Omega^1(M)}^2)\|u - v\|_{W^{1,p}\Omega^1(M)}.$$

Finally, we estimate the last term on the right hand-side of (3.5). For this, we write

$$\operatorname{Re}\langle v, \nabla \bar{v}\rangle_g v - \operatorname{Re}\langle u, \nabla \bar{u}\rangle_g u$$
$$= \int_0^1 \frac{\partial}{\partial\theta}\{\operatorname{Re}\langle w_\theta, \nabla \bar{w}_\theta\rangle_g \, w_\theta\} \, d\theta$$
$$= \int_0^1 \{(\operatorname{Re}\langle v - u, \nabla \bar{w}_\theta\rangle_g + \operatorname{Re}\langle w_\theta, \nabla(\bar{v} - \bar{u})\rangle_g) \, w_\theta$$
$$+ \operatorname{Re}\langle w_\theta, \nabla \bar{w}_\theta\rangle_g (v - u)\} \, d\theta.$$

Therefore,

$$| \operatorname{Re}\langle u, \nabla \bar{u}\rangle_g u - \operatorname{Re}\langle v, \nabla \bar{v}\rangle_g v|_g$$
$$\leq C(\|u\|_{L^\infty\Omega^1(M)} + \|v\|_{L^\infty\Omega^1(M)})\|u - v\|_{L^\infty\Omega^1(M)}(|\nabla u|_g + |\nabla v|_g)$$
$$+ C(\|u\|_{L^\infty\Omega^1(M)}^2 + \|v\|_{L^\infty\Omega^1(M)}^2)|\nabla(u - v)|_g.$$

Using Sobolev embedding $W^{1,p}\Omega^1(M) \hookrightarrow C\Omega^1(M)$, this implies that

$$\| \, | \operatorname{Re}\langle u, \nabla \bar{u}\rangle_g u - \operatorname{Re}\langle v, \nabla \bar{v}\rangle_g v|_g \|_{L^p(M)}$$
$$\leq C(\|u\|_{W^{1,p}\Omega^1(M)}^2 + \|v\|_{W^{1,p}\Omega^1(M)}^2)\|u - v\|_{W^{1,p}\Omega^1(M)}.$$

Combining all these three estimates with (3.5), we finish the proof. $\qquad\square$

## 4. Asymptotics of the admittance map

Let $(M, g)$ be a compact 3-dimensional Riemannian manifold with smooth boundary. Suppose that $\varepsilon, \mu \in C^1(M)$ are complex functions with positive real parts and $a, b \in C^1(M)$. Let $m \geq 1$ be an integer and let $3 < p \leq 6$. Fix $\omega > 0$ outside a discrete set of resonant frequencies. Suppose that $f \in T W_{\mathrm{Div}}^{1-1/p,p}(\partial M)$ and $s \in \mathbb{R}$ is a small parameter. By Theorem 1.1, there is a unique solution $(E^{(s)}, H^{(s)}) \in W_{\mathrm{Div}}^{1,p}(M) \times W_{\mathrm{Div}}^{1,p}(M)$ of (1.4) such that $\mathbf{t}(E^{(s)}) = sf$ and

$$\|E^{(s)}\|_{W_{\mathrm{Div}}^{1,p}(M)} + \|H^{(s)}\|_{W_{\mathrm{Div}}^{1,p}(M)} \leq C|s| \, \|f\|_{T W_{\mathrm{Div}}^{1-1/p,p}(\partial M)}. \qquad (4.1)$$

By Theorem 3.1, there is a unique $(E_1, H_1) \in W^{1,p}_{\mathrm{Div}}(M) \times W^{1,p}_{\mathrm{Div}}(M)$ solving (3.1) with $\mathbf{t}(E_1) = f$ such that

$$\|E_1\|_{W^{1,p}_{\mathrm{Div}}(M)} + \|H_1\|_{W^{1,p}_{\mathrm{Div}}(M)} \le C \|f\|_{TW^{1-1/p,p}_{\mathrm{Div}}(\partial M)}. \tag{4.2}$$

Also, by Theorem 3.2 there is a unique solution $(E_2, H_2) \in W^{1,p}_D(M) \times W^{1,p}_{\mathrm{Div}}(M)$ for

$$*dE_2 = i\omega\mu H_2 + i\omega b |H_1|^2_g H_1,$$

$$*dH_2 = -i\omega\varepsilon E_2 - i\omega a |E_1|^2_g E_1$$

and satisfying

$$\|E_2\|_{W^{1,p}_{\mathrm{Div}}(M)} + \|H_2\|_{W^{1,p}_{\mathrm{Div}}(M)}$$
$$\le C \| |E_1|^2_g E_1 \|_{W^{1,p}\Omega^1(M)} + C \| |H_1|^2_g H_1 \|_{W^{1,p}\Omega^1(M)}.$$

Then by Lemma 2.2,

$$\|E_2\|_{W^{1,p}_{\mathrm{Div}}(M)} + \|H_2\|_{W^{1,p}_{\mathrm{Div}}(M)} \le C \|E_1\|^3_{W^{1,p}\Omega^1(M)} + C \|H_1\|^3_{W^{1,p}\Omega^1(M)}$$
$$\le C \|E_1\|^3_{W^{1,p}_{\mathrm{Div}}(M)} + C \|H_1\|^3_{W^{1,p}_{\mathrm{Div}}(M)} \tag{4.3}$$
$$\le C \|f\|^3_{TW^{1-1/p,p}_{\mathrm{Div}}(\partial M)}.$$

Now we define $(F^{(s)}, G^{(s)})$ by

$$(E^{(s)}, H^{(s)}) = s(E_1 + s^2 F^{(s)}, H_1 + s^2 G^{(s)}). \tag{4.4}$$

Then by (4.1) and (4.2), $(F^{(s)}, G^{(s)})$ satisfies

$$|s|^3 \|F^{(s)}\|_{W^{1,p}_{\mathrm{Div}}(M)} + |s|^3 \|G^{(s)}\|_{W^{1,p}_{\mathrm{Div}}(M)}$$
$$\le \|E^{(s)}\|_{W^{1,p}_{\mathrm{Div}}(M)} + \|H^{(s)}\|_{W^{1,p}_{\mathrm{Div}}(M)} + |s| \|E_1\|_{W^{1,p}_{\mathrm{Div}}(M)} + |s| \|H_1\|_{W^{1,p}_{\mathrm{Div}}(M)}$$
$$\le C |s| \|f\|_{TW^{1-1/p,p}_{\mathrm{Div}}(\partial M)}.$$

Therefore,

$$|s|^2 \|F^{(s)}\|_{W^{1,p}_{\mathrm{Div}}(M)} + |s|^2 \|G^{(s)}\|_{W^{1,p}_{\mathrm{Div}}(M)} \le C \|f\|_{TW^{1-1/p,p}_{\mathrm{Div}}(\partial M)}. \tag{4.5}$$

**Lemma 4.1.** *Suppose that $f \in TW^{1-1/p,p}_{\mathrm{Div}}(\partial M)$. There is $s_0 > 0$ and there is $C_f > 0$ depending on $f$, $\omega$ and $s_0$, but independent of $s$, such that for all $s \in \mathbb{R}$ with $|s| < s_0$,*

$$\|F^{(s)} - E_2\|_{W^{1,p}_{\mathrm{Div}}(M)} + \|G^{(s)} - H_2\|_{W^{1,p}_{\mathrm{Div}}(M)} \le C_f |s|^2.$$

*In particular,*

$$\|F^{(s)}\|_{W^{1,p}_{\mathrm{Div}}(M)} + \|G^{(s)}\|_{W^{1,p}_{\mathrm{Div}}(M)} \leq C_f. \tag{4.6}$$

*Proof.* Set $(P^{(s)}, Q^{(s)}) = (F^{(s)}, G^{(s)}) - (E_2, H_2)$. Then it is easy to see that

$$*dP^{(s)} = i\omega\mu Q^{(s)} + i\omega s h^{(s)}, \quad *dQ^{(s)} = -i\omega\varepsilon P^{(s)} - i\omega s e^{(s)}, \quad \mathbf{t}(P^{(s)}) = 0, \tag{4.7}$$

where

$$e^{(s)} = s^{-1}a(|E_1 + s^2 F^{(s)}|_g^2 (E_1 + s^2 F^{(s)}) - |E_1|_g^2 E_1),$$

$$h^{(s)} = s^{-1}b(|H_1 + s^2 G^{(s)}|_g^2 (H_1 + s^2 G^{(s)}) - |H_1|_g^2 H_1).$$

Using Lemma 3.3,

$$\|e^{(s)}\|_{W^{1,p}\Omega^1(M)}$$
$$\leq C|s|[\|E_1\|_{W^{1,p}\Omega^1(M)}^2 + (|s|^2\|F^{(s)}\|_{W^{1,p}\Omega^1(M)})^2]\|F^{(s)}\|_{W^{1,p}\Omega^1(M)},$$

$$\|h^{(s)}\|_{W^{1,p}\Omega^1(M)}$$
$$\leq C|s|[\|H_1\|_{W^{1,p}\Omega^1(M)}^2 + (|s|^2\|G^{(s)}\|_{W^{1,p}\Omega^1(M)})^2]\|G^{(s)}\|_{W^{1,p}\Omega^1(M)}.$$

Then by (4.2) and (4.5),

$$\|e^{(s)}\|_{W^{1,p}\Omega^1(M)} + \|h^{(s)}\|_{W^{1,p}\Omega^1(M)}$$
$$\leq C|s|\|f\|_{TW^{1-1/p,p}_{\mathrm{Div}}(\partial M)}^2 (\|F^{(s)}\|_{W^{1,p}_{\mathrm{Div}}(M)} + \|G^{(s)}\|_{W^{1,p}_{\mathrm{Div}}(M)}).$$

By Theorem 3.2, $(P^{(s)}, Q^{(s)}) = (F^{(s)}, G^{(s)}) - (E_2, H_2)$ is the unique solution of (4.7) and satisfies the estimate

$$\|F^{(s)} - E_2\|_{W^{1,p}_{\mathrm{Div}}(M)} + \|G^{(s)} - H_2\|_{W^{1,p}_{\mathrm{Div}}(M)}$$
$$= \|P^{(s)}\|_{W^{1,p}_{\mathrm{Div}}(M)} + \|Q^{(s)}\|_{W^{1,p}_{\mathrm{Div}}(M)}$$
$$\leq C\omega|s|(\|e^{(s)}\|_{W^{1,p}\Omega^1(M)} + \|h^{(s)}\|_{W^{1,p}\Omega^1(M)}).$$

Therefore,

$$\|F^{(s)} - E_2\|_{W^{1,p}_{\mathrm{Div}}(M)} + \|G^{(s)} - H_2\|_{W^{1,p}_{\mathrm{Div}}(M)}$$
$$\leq C\|f\|_{TW^{1-1/p,p}_{\mathrm{Div}}(\partial M)}^2 \omega|s|^2 (\|F^{(s)}\|_{W^{1,p}_{\mathrm{Div}}(M)} + \|G^{(s)}\|_{W^{1,p}_{\mathrm{Div}}(M)}). \tag{4.8}$$

Using the reverse triangle inequality to the left hand-side, we get

$$
\begin{aligned}
\|F^{(s)}\|_{W_{\mathrm{Div}}^{1,p}(M)} &+ \|G^{(s)}\|_{W_{\mathrm{Div}}^{1,p}(M)} \\
&\leq C \|f\|^2_{TW_{\mathrm{Div}}^{1-1/p,p}(\partial M)} \omega|s|^2 \left( \|F^{(s)}\|_{W_{\mathrm{Div}}^{1,p}(M)} + \|G^{(s)}\|_{W_{\mathrm{Div}}^{1,p}(M)} \right) \\
&\quad + \|E_2\|_{W_{\mathrm{Div}}^{1,p}(M)} + \|H_2\|_{W_{\mathrm{Div}}^{1,p}(M)}.
\end{aligned}
$$

Using (4.3), this gives

$$
\begin{aligned}
\|F^{(s)}\|_{W_{\mathrm{Div}}^{1,p}(M)} &+ \|G^{(s)}\|_{W_{\mathrm{Div}}^{1,p}(M)} \\
&\leq C \|f\|^2_{TW_{\mathrm{Div}}^{1-1/p,p}(\partial M)} \omega|s|^2 \left( \|F^{(s)}\|_{W_{\mathrm{Div}}^{1,p}(M)} + \|G^{(s)}\|_{W_{\mathrm{Div}}^{1,p}(M)} \right) \\
&\quad + C \|f\|^3_{TW_{\mathrm{Div}}^{1-1/p,p}(\partial M)}.
\end{aligned}
$$

The first term in the last line can be absorbed into the left hand-side by taking sufficiently small $s_0 > 0$ so that

$$
C \|f\|^2_{TW_{\mathrm{Div}}^{1-1/p,p}(\partial M)} \omega|s_0|^2 < 1/2.
$$

Then we obtain

$$
\|F^{(s)}\|_{W_{\mathrm{Div}}^{1,p}(M)} + \|G^{(s)}\|_{W_{\mathrm{Div}}^{1,p}(M)} \leq C \|f\|^3_{TW_{\mathrm{Div}}^{1-1/p,p}(\partial M)}.
$$

Substituting this into (4.8), we arrive to the desired estimate.                    □

Denote by $\Lambda^{\omega}_{\varepsilon,\mu}$ the admittance map $\Lambda^{\omega}_{\varepsilon,\mu,0,0}$ for linear Maxwell's equations. We obtain the following asymptotic expansion of the admittance map.

**Proposition 4.2.** *Suppose that $f \in TW_{\mathrm{Div}}^{1-1/p,p}(\partial M)$ with $3 < p \leq 6$. Then*

$$
s^{-1}[\Lambda^{\omega}_{\varepsilon,\mu,a,b}(sf) - s\Lambda^{\omega}_{\varepsilon,\mu}(f)] \longrightarrow 0 \qquad in \ W_{\mathrm{Div}}^{1-1/p,p}(\partial M) \ as \ s \to 0,
\tag{4.9}
$$

$$
s^{-3}[\Lambda^{\omega}_{\varepsilon,\mu,a,b}(sf) - s\Lambda^{\omega}_{\varepsilon,\mu}(f)] \longrightarrow \mathbf{t}(H_2) \quad in \ W_{\mathrm{Div}}^{1-1/p,p}(\partial M) \ as \ s \to 0.
\tag{4.10}
$$

*Proof.* From (4.4) we have

$$
\Lambda^{\omega}_{\varepsilon,\mu,a,b}(sf) - s\Lambda^{\omega}_{\varepsilon,\mu}(f) = \mathbf{t}(H^{(s)}) - s\mathbf{t}(H_1) = s^3 \mathbf{t}(G^{(s)}).
$$

Then by boundedness of **t** from $W_{\mathrm{Div}}^{1,p}(M)$ onto $TW_{\mathrm{Div}}^{1-1/p,p}(\partial M)$ and by (4.6),

$$\|s^{-1}[\Lambda_{\varepsilon,\mu,a,b}^{\omega}(sf) - s\Lambda_{\varepsilon,\mu}^{\omega}(f)]\|_{TW_{\mathrm{Div}}^{1-1/p,p}(\partial M)} \le C|s|^2 \|G^{(s)}\|_{W_{\mathrm{Div}}^{1,p}(M)}$$
$$\le C_f|s|^2.$$

Taking $s \to 0$, this implies (4.9).

Now, by boundedness of **t** from $W_{\mathrm{Div}}^{1,p}(M)$ onto $TW_{\mathrm{Div}}^{1-1/p,p}(\partial M)$ and by Lemma 4.1,

$$\|s^{-3}[\Lambda_{\varepsilon,\mu,a,b}^{\omega}(sf) - s\Lambda_{\varepsilon,\mu}^{\omega}(f)] - \mathbf{t}(H_2)\|_{TW_{\mathrm{Div}}^{1-1/p,p}(\partial M)}$$
$$\le C\|G^{(s)} - H_2\|_{W_{\mathrm{Div}}^{1,p}(M)}$$
$$\le C_f|s|^2.$$

Taking $s \to 0$, this implies (4.10). $\qquad\square$

## 5. Proof of Theorem 1.2: Part I

In this section we show that the material parameters and electric and magnetic susceptibilities of the nonlinear time-harmonic Maxwell equation (1.4) can be uniquely determined from the knowledge of admittance map.

Let $(M, g)$ be a 3-dimensional admissible manifold, that is $(M, g) \subset\subset \mathbb{R} \times (M_0, g_0)$ with $g = c(e \oplus g_0)$, where $c > 0$ is a smooth function on $M$ and $(M_0, g_0)$ is a simple manifold of dimension two.

The first ingredient in the proof of Theorem 1.2 is the reduction to the case $c = 1$.

**Lemma 5.1.** *Let $(M, g)$ be a compact Riemannian 3-dimensional manifold with boundary and let $c > 0$ be a smooth function on $M$. Suppose that $\varepsilon, \mu \in C^\infty(M)$ with positive real parts and $a, b \in C^\infty(M)$. Then*

$$\Lambda_{cg,\varepsilon,\mu,a,b}^{\omega} = \Lambda_{g,c^{1/2}\varepsilon,c^{1/2}\mu,c^{3/2}a,c^{3/2}b}^{\omega}.$$

*Proof.* Let $*_{cg}$ and $*_g$ denote the Hodge star operators corresponding to the metrics $cg$ and $g$, respectively. Following [12, Lemma 7.1], we note that

$$*_{cg}u = c^{3/2-k} *_g u$$

for a $k$-form $u$. Therefore, $(E, H)$ solves

$$*_{cg}dE = i\omega\mu H + i\omega b|H|_{cg}^2 H,$$
$$*_{cg}dH = -i\omega\varepsilon E - i\omega a|E|_{cg}^2 E$$

if and only if it solves

$$*_g dE = i\omega c^{1/2} \mu H + i\omega c^{3/2} b |H|_g^2 H,$$
$$*_g dH = -i\omega c^{1/2} \varepsilon E - i\omega c^{3/2} a |E|_g^2 E.$$

Therefore, $\Lambda_{cg,\varepsilon,\mu,a,b}^{\omega} = \Lambda_{g,c^{1/2}\varepsilon,c^{1/2}\mu,c^{3/2}a,c^{3/2}b}^{\omega}$. $\qquad\square$

Therefore, it is enough to prove Theorem 1.2 in the case $c = 1$. Thus, in the rest of this section we assume that $(M, g) \subset\subset \mathbb{R} \times (M_0, g_0)$ with $g = e \oplus g_0$, where $(M_0, g_0)$ is a simple manifold of dimension two.

By (4.9) in Proposition 4.2, we obtain $\Lambda_{\varepsilon_1,\mu_1}^{\omega} = \Lambda_{\varepsilon_2,\mu_2}^{\omega}$. Then by [12, Theorem 1.1], we get $\varepsilon_1 = \varepsilon_2$ and $\mu_1 = \mu_2$ in $M$. In what follows, we write $\varepsilon = \varepsilon_1 = \varepsilon_2$ and $\mu = \mu_1 = \mu_2$.

## 6. Construction of CGO solutions

Our aim is to very briefly review the construction of CGO solutions; see [12] for details. In Section 6.1, we recall the reduction of the Maxwell equations to the Hodge-Dirac and Schrödinger type equations, introduced in [20, 12]. Then, in Section 6.2, we restate the form of existence and basic properties of CGO solutions for Maxwell's equations using the reduction in Section 6.1.

**6.1. Reduction to the Hodge–Schrödinger equation.** Let $(M, g)$ be a smooth compact Riemannian 3-dimensional manifold with boundary. The arguments require $\varepsilon, \mu \in C^2(M)$ to be complex functions with positive real parts. If $\Phi$, $\Psi$ are complex scalar functions on $M$ and $E$, $H$ are complex 1-forms on $M$, we consider the graded forms $X = \Phi + E + *H + *\Psi$ and we denote them in vector notation

$$X = (\ \Phi \ \ *H \ \big| \ *\Psi \ \ E\ )^t.$$

We define the following matrix operators acting on graded forms on $M$

$$P = \frac{1}{i}(d - \delta) = \left( \begin{array}{cc|cc} & & & -\delta \\ & & -\delta & d \\ \hline & d & & \\ d & -\delta & & \end{array} \right),$$

$$V = \left( \begin{array}{cc|cc} -\omega\mu & & & *(D\alpha \wedge *\cdot) \\ & -\omega\mu & *(D\alpha \wedge *\cdot) & \\ \hline & D\beta\wedge & -\omega\varepsilon & \\ D\beta\wedge & & & -\omega\varepsilon \end{array} \right),$$

where $D = -\,\mathrm{id}$, $\alpha = \log \varepsilon$ and $\beta = \log \mu$. Note that $P$ is the self-adjoint Hodge-Dirac operator. It was shown in [12, Section 3] that $(E, H)$ is a solution of the original Maxwell's equations

$$*dE = i\omega\mu H, \quad *dH = -i\omega\varepsilon E$$

if and only if $X = (\ \Phi\ \ *H\ |\ *\Psi\ \ E\ )^t$ is a solution of $(P + V)X = 0$ with $\Phi = \Psi = 0$.

To reduce the Maxwell equations to the Schrödinger type equation, we consider the rescaling

$$Y = \left( \begin{array}{c|c} \mu^{1/2}\,\mathrm{Id}_2 & \\ \hline & \varepsilon^{1/2}\,\mathrm{Id}_2 \end{array} \right) X,$$

where $\mathrm{Id}_2$ is the $2 \times 2$ identity matrix. We always assume that $X$ and $Y$ are related via this rescaling. We write the graded form $Y$ as

$$Y = (\ Y^0\ \ Y^2\ |\ Y^3\ \ Y^1\ )^t,$$

with $Y^k$ being the $k$-form part of $Y$. One can check that $(P + V)X = 0$ if and only if $(P + W)Y = 0$. Here

$$W = -\kappa + \frac{1}{2} \left( \begin{array}{cc|cc} & & & *(D\alpha \wedge *\cdot) \\ & & *(D\alpha \wedge *\cdot) & -D\alpha\wedge \\ \hline & D\beta\wedge & & \\ D\beta\wedge & *(D\beta \wedge *\cdot) & & \end{array} \right),$$

for $\kappa = \omega(\varepsilon\mu)^{1/2}$. Then

$$(P + W)(P - W^t) = -\Delta + Q,$$

where $Q$ is $L^\infty$ potential. For the exact expression of $Q$, see [12, Lemma 3.1].

## 6.2. CGO solutions for Maxwell's equations.
Let $(M, g)$ be a 3-dimensional admissible manifold. Throughout this section, we assume that $M \subset \mathbb{R} \times M_0^{\mathrm{int}}$ and the metric $g$ has the form $g = e \oplus g_0$ and $(M_0, g_0)$ is simple. Choose another simple manifold $(\tilde{M}_0, g_0)$ such that $M_0 \subset\subset \tilde{M}_0$ and choose $p \in \tilde{M}_0 \setminus M_0$. Simplicity of $(\tilde{M}_0, g_0)$ implies that there are globally defined polar coordinates $(r, \theta)$ centered at $p$. In these coordinates, the metric $g$ has the form

$$g = e \oplus \begin{pmatrix} 1 & 0 \\ 0 & m(r, \theta) \end{pmatrix}, \tag{6.1}$$

where $m$ is a smooth positive function.

The following result states the existence and basic properties of CGO solutions.

**Proposition 6.1.** *Let $(M, g)$ be a 3-dimensional admissible manifold with $g = e \oplus g_0$ and let $2 \leq p \leq 6$. Suppose that $\varepsilon, \mu \in C^3(M)$ with $\mathrm{Re}(\varepsilon), \mathrm{Re}(\mu) > 0$ in $M$. Let $s_0, t_0 \in \mathbb{R}$ and $\lambda \in \mathbb{R} \setminus \{0\}$ be constants and let $\chi \in C^\infty(S^1)$. Then for $\tau \in \mathbb{R}$ with sufficiently large $|\tau| > 0$ and outside a countable subset of $\mathbb{R}$, the Maxwell's equations*

$$* dE = i\omega\mu H, \quad * dH = -i\omega\varepsilon E \tag{6.2}$$

*has a solution $(E, H) \in W^{1,p}_{\mathrm{Div}}(M) \times W^{1,p}_{\mathrm{Div}}(M)$ of the form*

$$E = e^{-\tau(x_1 + ir)}[t_0 \varepsilon^{-1/2} |g|^{-1/4} e^{i\lambda(x_1 + ir)} \chi(\theta)(dx_1 + idr) + R],$$
$$H = e^{-\tau(x_1 + ir)}[s_0 \mu^{-1/2} |g|^{-1/4} e^{i\lambda(x_1 + ir)} \chi(\theta)(dx_1 + idr) + R'],$$

*where $R, R' \in W^{1,p}_{\mathrm{Div}}(M)$ are correction terms satisfying the estimates*

$$\|R\|_{L^p \Omega^1(M)}, \|R'\|_{L^p \Omega^1(M)} \leq C \frac{1}{|\tau|^{\frac{6-p}{2p}}}, \tag{6.3}$$

*with $C > 0$ constant independent of $\tau$. Note that when $p < 6$ the remainders decay as $\tau$ increases.*

**Remark 6.2.** The proof follows by carefully rewriting the result [4, Theorem 3.1] (see also [12, Theorem 6.1(a)] for the original result) which requires $\varepsilon, \mu \in C^3(M)$. We need the former result in order to get (6.3) for all $2 \geq p \leq 6$ rather than just $p = 2$. This will be very important in the next section.

*Proof.* By [4, Theorem 3.1], for $\tau \in \mathbb{R}$ with sufficiently large $|\tau| > 0$ and outside a countable subset of $\mathbb{R}$, there is a solution for $(-\Delta + Q)Z = 0$ such that $Z \in H^3 \Omega(M)$ and

$$(P + W)Y = 0, \quad Y = (P - W^t)Z, \quad Y^0 = Y^3 = 0 \quad \text{in } M,$$

and having the form

$$Z = e^{-\tau(x_1 + ir)}(A + R_0), \quad A = -i|g|^{-1/4} e^{i\lambda(x_1 + ir)} \chi(\theta)( \begin{array}{cc|cc} s_0 & 0 & t_0 * 1 & 0 \end{array} )^t$$

and $\|R_0\|_{H^s \Omega(M)} \leq C|\tau|^{1-s}$, $0 \leq s \leq 2$, where $C > 0$ is a constant independent of $\tau$.

Let us compute $Y^1$ and $Y^3$. Writing $\rho = x_1 + ir$ and using the fact that $A^1 = A^2 = 0$, one can see that

$$Y^1 = [(P - W^t)Z]^1 = e^{-\tau\rho}(y_1 + r_1),$$
$$Y^2 = [(P - W^t)Z]^2 = e^{-\tau\rho}(y_2 + r_2),$$

where

$$y_1 = -\frac{\tau}{i} i_{d\rho} A^2,$$

$$r_1 = -[W^t(A + R_0)]^1 - \frac{1}{i}\delta A^2 + \frac{1}{i}dR_0^0 - \frac{\tau}{i}R_0^0 \, d\rho - \frac{1}{i}\delta R_0^2 - \frac{\tau}{i}i_{d\rho}R_0^2,$$

$$y_2 = -\frac{\tau}{i} i_{d\rho} A^3,$$

$$r_2 = -[W^t(A + R_0)]^2 - \frac{1}{i}\delta A^3 + \frac{1}{i}dR_0^1 - \frac{\tau}{i}d\rho \wedge R_0^1 - \frac{1}{i}\delta R_0^3 - \frac{\tau}{i}i_{d\rho}R_0^3.$$

It is easy to see that

$$\|r_1\|_{H^s\Omega^1(M)}, \|r_2\|_{H^s\Omega^2(M)} \le C|\tau|^{s-1}, \quad 0 \le s \le 1.$$

Using (2.4), one can show that

$$y_1 = s_0\tau|g|^{-1/4}e^{i\lambda\rho}\chi(\theta)\, d\rho, \quad y_2 = t_0\tau|g|^{-1/4}e^{i\lambda\rho}\chi(\theta) * d\rho.$$

Note that $Y_0 := \tau^{-1}Y$ will solve $(P + W)Y_0 = 0$ with $Y_0^0 = Y_0^3 = 0$. If we define

$$E := \varepsilon^{-1/2}Y_0^1$$
$$= e^{-\tau(x_1+ir)}[s_0\varepsilon^{-1/2}|g|^{-1/4}e^{i\lambda(x_1+ir)}\chi(\theta)(dx_1 + i\,dr) + \underbrace{\varepsilon^{-1/2}r_1}_{R}],$$

$$H := \mu^{-1/2} * Y_0^2$$
$$= e^{-\tau(x_1+ir)}[t_0\mu^{-1/2}|g|^{-1/4}e^{i\lambda(x_1+ir)}\chi(\theta)(dx_1 + i\,dr) + \underbrace{\mu^{-1/2}r_2}_{R'}].$$

Then $(E, H) \in H^2\Omega^1(M) \times H^2\Omega^1(M)$ will be a solution of the Maxwell's equations (6.2) and the correction terms $R, R'$ satisfy the estimates

$$\|R\|_{H^s\Omega^1(M)}, \|R'\|_{H^s\Omega^1(M)} \le C|\tau|^{s-1}, \quad 0 \le s \le 1, \tag{6.4}$$

with $C > 0$ constant independent of $\tau$.

By Sobolev embedding, we have $(E, H) \in W^{1,p}\Omega^1(M) \times W^{1,p}\Omega^1(M)$ for $2 \le p \le 6$. Then, using (6.2) and (2.6), it is straightforward to check that $\mathbf{t}(E), \mathbf{t}(H) \in TW_{\text{Div}}^{1-1/p,p}(\partial M)$. Thus, $(E, H) \in W_{\text{Div}}^{1,p}(M) \times W_{\text{Div}}^{1,p}(M)$ for $2 \le p \le 6$.

Finally, one can obtain the estimates in (6.3), using the inequality $0 \le \frac{3p-6}{2p} \le 1$, Sobolev embedding and (6.4). $\qquad \square$

## 7. Proof of Theorem 1.2: Part II

In this section we continue proof of Theorem 1.2. Our aim is to show that $a_1 = a_2$ and $b_1 = b_2$. To that end, we shall use complex geometrical optics solutions, constructed in the previous section, in the following integral identity (7.6).

**7.1. An important energy integral identity.** Now, by (4.10) in Proposition 4.2, we obtain $\mathbf{t}(H_2^1) = \mathbf{t}(H_2^2)$, where $(E_2^j, H_2^j) \in W_D^{1,p}(M) \times W_{\mathrm{Div}}^{1,p}(M)$, $j = 1, 2$, is the unique solution of

$$* dE_2^j = i\omega\mu H_2^j + i\omega b_j |H_1|_g^2 H_1, \quad *dH_2^j = -i\omega\varepsilon E_2^j - i\omega a_j |E_1|_g^2 E_1 \quad (7.1)$$

with $\mathbf{t}(E_2^j) = 0$ and $(E_1, H_1) \in W_{\mathrm{Div}}^{1,p}(M) \times W_{\mathrm{Div}}^{1,p}(M)$ is a solution of

$$* dE_1 = i\omega\mu H_1, \quad *dH_1 = -i\omega\varepsilon E_1 \quad (7.2)$$

satisfying $\mathbf{t}(E_1) = f$. Let $(E, H) \in W_{\mathrm{Div}}^{1,p}(M) \times W_{\mathrm{Div}}^{1,p}(M)$ be a solution of

$$* dE = i\omega\bar{\mu}H, \quad *dH = -i\omega\bar{\varepsilon}E. \quad (7.3)$$

Using integration by parts,

$$(\mathbf{t}(H_2^j)|\mathbf{t}(i_\nu * E))_{L^2\Omega^1(\partial M)} = (dH_2^j | * E)_{L^2\Omega^2(M)} - (H_2^j|\delta(*E))_{L^2\Omega^1(M)}$$
$$= (*dH_2^j|E)_{L^2\Omega^1(M)} - (H_2^j | * dE)_{L^2\Omega^1(M)}.$$

Since $(E_2^j, H_2^j)$ satisfy (7.1) and $(E, H)$ satisfy (7.3), we can show

$$(\mathbf{t}(H_2^j)|\mathbf{t}(i_\nu * E))_{L^2\Omega^1(\partial M)}$$
$$= -(i\omega\varepsilon E_2^j|E)_{L^2\Omega^1(M)} - (i\omega a_j |E_1|_g^2 E_1|E)_{L^2\Omega^1(M)} - (H_2^j|i\omega\bar{\mu}H)_{L^2\Omega^1(M)}$$
$$= (E_2^j|i\omega\bar{\varepsilon}E)_{L^2\Omega^1(M)} - (i\omega a_j |E_1|_g^2 E_1|E)_{L^2\Omega^1(M)} + (i\omega\mu H_2^j|H)_{L^2\Omega^1(M)}.$$

Here and in what follows, all integrals make sense because of the assumption $p \geq 4$. Since $(E_2^j, H_2^j)$ satisfy (7.1) and $(E, H)$ satisfy (7.3), this can be rewritten as

$$(\mathbf{t}(H_2^j)|\mathbf{t}(i_\nu * E))_{L^2\Omega^1(\partial M)}$$
$$= (*dE_2^j|H)_{L^2\Omega^1(M)} - (E_2^j | * dH)_{L^2\Omega^1(M)}$$
$$\quad - (i\omega a_j |E_1|_g^2 E_1|E)_{L^2\Omega^1(M)} - (i\omega b_j |H_1|_g^2 H_1|H)_{L^2\Omega^1(M)}$$
$$= (dE_2^j | * H)_{L^2\Omega^2(M)} - (E_2^j|\delta(*H))_{L^2\Omega^1(M)}$$
$$\quad - (i\omega a_j |E_1|_g^2 E_1|E)_{L^2\Omega^1(M)} - (i\omega b_j |H_1|_g^2 H_1|H)_{L^2\Omega^1(M)}.$$

Using integration by parts and the fact that $\mathbf{t}(E_2^j) = 0$, we can show that

$$(dE_2^j | * H)_{L^2\Omega^2(M)} - (E_2^j|\delta(*H))_{L^2\Omega^1(M)} = 0.$$

Therefore,

$$-\frac{1}{i\omega}(\mathbf{t}(H_2^j)|\mathbf{t}(i_\nu * E))_{L^2\Omega^1(\partial M)}$$
$$= (a_j|E_1|_g^2 E_1|E)_{L^2\Omega^1(M)} + (b_j|H_1|_g^2 H_1|H)_{L^2\Omega^1(M)}.$$

Since $\mathbf{t}(H_2^1) = \mathbf{t}(H_2^2)$, this implies that

$$((a_1 - a_2)|E_1|_g^2 E_1|E)_{L^2\Omega^1(M)} + ((b_1 - b_2)|H_1|_g^2 H_1|H)_{L^2\Omega^1(M)} = 0 \quad (7.4)$$

for all $(E_1, H_1) \in W_{\text{Div}}^{1,p}(M) \times W_{\text{Div}}^{1,p}(M)$ solving (7.2) and for all $(E, H) \in W_{\text{Div}}^{1,p}(M) \times W_{\text{Div}}^{1,p}(M)$ solving (7.3).

Note that if $(\widetilde{E}, \widetilde{H}), (E', H') \in W_{\text{Div}}^{1,p}(M) \times W_{\text{Div}}^{1,p}(M)$ solve (7.2), then $(\widetilde{E} + E', \widetilde{H} + H')$ also solves (7.2). Therefore, polarizing (7.4) by setting

$$(E_1, H_1) = (\widetilde{E} + E', \widetilde{H} + H'),$$

we obtain

$$\begin{aligned}
0 = &((a_1 - a_2)[|\widetilde{E}|_g^2 E' + 2\operatorname{Re}\langle E', \bar{\widetilde{E}}\rangle_g E' + |E'|_g^2 \widetilde{E} \\
&+ 2\operatorname{Re}\langle E', \bar{\widetilde{E}}\rangle_g \widetilde{E}]|E)_{L^2\Omega^1(M)} \\
&+ ((b_1 - b_2)[|\widetilde{H}|_g^2 H' + 2\operatorname{Re}\langle H', \bar{\widetilde{H}}\rangle_g H' + |H'|_g^2 \widetilde{H} \\
&+ 2\operatorname{Re}\langle H', \bar{\widetilde{H}}\rangle_g \widetilde{H}]|H)_{L^2\Omega^1(M)}
\end{aligned} \quad (7.5)$$

for all $(\widetilde{E}, \widetilde{H}), (E', H') \in W_{\text{Div}}^{1,p}(M) \times W_{\text{Div}}^{1,p}(M)$ solving (7.2).

Now, take $(E_{(j)}, H_{(j)}) \in W_{\text{Div}}^{1,p}(M) \times W_{\text{Div}}^{1,p}(M)$, $j = 1, 2, 3$, which solve (7.2). Setting $(E_1, H_1) = (E_{(1)} + E_{(2)} + E_{(3)}, H_{(1)} + H_{(2)} + H_{(3)})$ in (7.4) and using (7.5), we get

$$\begin{aligned}
0 = &((a_1 - a_2)\operatorname{Re}\langle E_{(3)}, \bar{E}_{(2)}\rangle_g E_{(1)}|E)_{L^2\Omega^1(M)} \\
&+ ((a_1 - a_2)\operatorname{Re}\langle E_{(3)}, \bar{E}_{(1)}\rangle_g E_{(2)}|E)_{L^2\Omega^1(M)} \\
&+ ((a_1 - a_2)\operatorname{Re}\langle E_{(1)}, \bar{E}_{(2)}\rangle_g E_{(3)}|E)_{L^2\Omega^1(M)} \\
&+ ((b_1 - b_2)\operatorname{Re}\langle H_{(3)}, \bar{H}_{(2)}\rangle_g H_{(1)}|H)_{L^2\Omega^1(M)} \\
&+ ((b_1 - b_2)\operatorname{Re}\langle H_{(3)}, \bar{H}_{(1)}\rangle_g H_{(2)}|H)_{L^2\Omega^1(M)} \\
&+ ((b_1 - b_2)\operatorname{Re}\langle H_{(1)}, \bar{H}_{(2)}\rangle_g H_{(3)}|H)_{L^2\Omega^1(M)}
\end{aligned} \quad (7.6)$$

for all $(E_{(j)}, H_{(j)}) \in W_{\text{Div}}^{1,p}(M) \times W_{\text{Div}}^{1,p}(M)$, $j = 1, 2, 3$, solving (7.2).

**7.2. Proof of Theorem 1.2: Part II.** Recall that we assume $M \subset \mathbb{R} \times M_0^{\text{int}}$ and the metric has the form $g = e \oplus g_0$, where $e$ is Euclidean metric on $\mathbb{R}$ and $(M_0, g_0)$ is a simple 2-dimensional manifold.

Recall that we assume $3 < p < 6$. Using Proposition 6.1, for $\tau \in \mathbb{R}$ with sufficiently large $|\tau|$, for arbitrary $\chi \in C^\infty(S^1)$, $s_0, t_0 \in \mathbb{R}$ and $\lambda \in \mathbb{R} \setminus \{0\}$, there are $(E_{(j)}, H_{(j)}) \in W_{\text{Div}}^{1,p}(M) \times W_{\text{Div}}^{1,p}(M)$, $j = 1, 2, 3$, solving (7.2) and there is $(E, H) \in W_{\text{Div}}^{1,p}(M) \times W_{\text{Div}}^{1,p}(M)$ solving (7.3) of the forms

$$E_{(1)} = e^{-\tau(x_1+ir)}[t_0 \, \varepsilon^{-1/2}|g|^{-1/4}e^{i\lambda(x_1+ir)}\chi(\theta)(dx_1 + i\,dr) + R_{(1)}]$$
$$= e^{-\tau(x_1+ir)}(A_{(1)} + R_{(1)}),$$

$$H_{(1)} = e^{-\tau(x_1+ir)}[s_0 \, \mu^{-1/2}|g|^{-1/4}e^{i\lambda(x_1+ir)}\chi(\theta)(dx_1 + i\,dr) + R'_{(1)}]$$
$$= e^{-\tau(x_1+ir)}(A'_{(1)} + R'_{(1)}),$$

$$E_{(2)} = e^{\tau(x_1-ir)}[\varepsilon^{-1/2}|g|^{-1/4}e^{i\lambda(x_1-ir)}(dx_1 - i\,dr) + R_{(2)}]$$
$$= e^{\tau(x_1-ir)}(A_{(2)} + R_{(2)}),$$

$$H_{(2)} = e^{\tau(x_1-ir)}[\mu^{-1/2}|g|^{-1/4}e^{i\lambda(x_1-ir)}(dx_1 - i\,dr) + R'_{(2)}]$$
$$= e^{\tau(x_1-ir)}(A'_{(2)} + R'_{(2)}),$$

$$E_{(3)} = e^{-\tau(x_1-ir)}[\varepsilon^{-1/2}|g|^{-1/4}e^{-i\lambda(x_1-ir)}(dx_1 - i\,dr) + R_{(3)}]$$
$$= e^{-\tau(x_1-ir)}(A_{(3)} + R_{(3)}),$$

$$H_{(3)} = e^{-\tau(x_1-ir)}[\mu^{-1/2}|g|^{-1/4}e^{-i\lambda(x_1-ir)}(dx_1 - i\,dr) + R'_{(2)}]$$
$$= e^{-\tau(x_1-ir)}(A'_{(3)} + R'_{(3)}),$$

$$E = e^{\tau(x_1+ir)}[\varepsilon^{-1/2}|g|^{-1/4}e^{i\lambda(x_1+ir)}(dx_1 + i\,dr) + R]$$
$$= e^{-\tau(x_1+ir)}(A + R),$$

$$H = e^{\tau(x_1+ir)}[\mu^{-1/2}|g|^{-1/4}e^{i\lambda(x_1+ir)}(dx_1 + i\,dr) + R']$$
$$= e^{-\tau(x_1+ir)}(A' + R'),$$

with, for $j = 1, 2, 3$,

$$\|R_{(j)}\|_{L^p\Omega^1(M)}, \|R'_{(j)}\|_{L^p\Omega^1(M)}, \|R\|_{L^p\Omega^1(M)}, \|R'\|_{L^p\Omega^1(M)} \leq C \frac{1}{|\tau|^{\frac{6-p}{2p}}},$$

Since we assume $p < 6$, these imply, as $\tau \to \infty$, for $j = 1, 2, 3$,

$$\|R_{(j)}\|_{L^p\Omega^1(M)}, \quad \|R'_{(j)}\|_{L^p\Omega^1(M)}, \quad \|R\|_{L^p\Omega^1(M)}, \quad \|R'\|_{L^p\Omega^1(M)} \leq o(1). \tag{7.7}$$

Substituting these solutions into (7.6), we get

$$
\begin{aligned}
0 = {}& ((a_1 - a_2)\operatorname{Re}\langle A_{(3)} + R_{(3)}, \bar{A}_{(2)} + \bar{R}_{(2)}\rangle_g (A_{(1)} + R_{(1)})|(A+R))_{L^2\Omega^1(M)} \\
& + ((a_1 - a_2)\operatorname{Re}\langle A_{(3)} + R_{(3)}, \bar{A}_{(1)} + \bar{R}_{(1)}\rangle_g (A_{(2)} + R_{(2)})|(A+R))_{L^2\Omega^1(M)} \\
& + ((a_1 - a_2)\operatorname{Re}\langle A_{(1)} + R_{(1)}, \bar{A}_{(2)} + \bar{R}_{(2)}\rangle_g (A_{(3)} + R_{(3)})|(A+R))_{L^2\Omega^1(M)} \\
& + ((b_1 - b_2)\operatorname{Re}\langle A'_{(3)} + R'_{(3)}, \bar{A}'_{(2)} + \bar{R}'_{(2)}\rangle_g (A'_{(1)} + R'_{(1)})|(A'+R'))_{L^2\Omega^1(M)} \\
& + ((b_1 - b_2)\operatorname{Re}\langle A'_{(3)} + R'_{(3)}, \bar{A}'_{(1)} + \bar{R}'_{(1)}\rangle_g (A'_{(2)} + R'_{(2)})|(A'+R'))_{L^2\Omega^1(M)} \\
& + ((b_1 - b_2)\operatorname{Re}\langle A'_{(1)} + R'_{(1)}, \bar{A}'_{(2)} + \bar{R}'_{(2)}\rangle_g (A'_{(3)} + R'_{(3)})|(A'+R'))_{L^2\Omega^1(M)}.
\end{aligned}
\tag{7.8}
$$

Letting $\tau \to \infty$, we come to

$$
\begin{aligned}
0 = {}& ((a_1 - a_2)\operatorname{Re}\langle A_{(3)}, \bar{A}_{(2)}\rangle_g A_{(1)}|A)_{L^2\Omega^1(M)} \\
& + ((a_1 - a_2)\operatorname{Re}\langle A_{(3)}, \bar{A}_{(1)}\rangle_g A_{(2)}|A)_{L^2\Omega^1(M)} \\
& + ((a_1 - a_2)\operatorname{Re}\langle A_{(1)}, \bar{A}_{(2)}\rangle_g A_{(3)}|A)_{L^2\Omega^1(M)} \\
& + ((b_1 - b_2)\operatorname{Re}\langle A'_{(3)}, \bar{A}'_{(2)}\rangle_g A'_{(1)}|A')_{L^2\Omega^1(M)} \\
& + ((b_1 - b_2)\operatorname{Re}\langle A'_{(3)}, \bar{A}'_{(1)}\rangle_g A'_{(2)}|A')_{L^2\Omega^1(M)} \\
& + ((b_1 - b_2)\operatorname{Re}\langle A'_{(1)}, \bar{A}'_{(2)}\rangle_g A'_{(3)}|A')_{L^2\Omega^1(M)}.
\end{aligned}
\tag{7.9}
$$

To see this, one expands every term in (7.8) and uses generalized Hölder's inequality together with (7.7). Then all terms in (7.8) go to zero as $\tau \to \infty$ except the terms written in (7.9).

Recall that the amplitudes $A_{(j)}$, $A'_{(j)}$, $j = 1, 2, 3$, and $A$, $A'$ are of the form

$$
\begin{aligned}
A_{(1)} &= t_0\, \varepsilon^{-1/2}|g|^{-1/4}e^{i\lambda(x_1+ir)}\chi(\theta)(dx_1 + i\,dr), \\
A'_{(1)} &= s_0\, \mu^{-1/2}|g|^{-1/4}e^{i\lambda(x_1+ir)}\chi(\theta)(dx_1 + i\,dr), \\[4pt]
A_{(2)} &= \varepsilon^{-1/2}|g|^{-1/4}e^{i\lambda(x_1-ir)}(dx_1 - i\,dr), \\
A'_{(2)} &= \mu^{-1/2}|g|^{-1/4}e^{i\lambda(x_1-ir)}(dx_1 - i\,dr), \\[4pt]
A_{(3)} &= \varepsilon^{-1/2}|g|^{-1/4}e^{-i\lambda(x_1-ir)}(dx_1 - i\,dr), \\
A'_{(3)} &= \mu^{-1/2}|g|^{-1/4}e^{-i\lambda(x_1-ir)}(dx_1 - i\,dr), \\[4pt]
A &= \varepsilon^{-1/2}|g|^{-1/4}e^{i\lambda(x_1+ir)}(dx_1 + i\,dr), \\
A' &= \mu^{-1/2}|g|^{-1/4}e^{i\lambda(x_1+ir)}(dx_1 + i\,dr).
\end{aligned}
$$

Then the second, third, fifth and last terms in (7.9) will vanish. Considering the cases $t_0 = 1$, $s_0 = 0$ and $t_0 = 0$, $s_0 = 1$ separately, we obtain

$$\int_M f e^{-i2\lambda(x_1 - ir)} \chi(\theta) |g|^{-1/2} \, d\operatorname{Vol}_g = 0$$

and

$$\int_M h e^{-i2\lambda(x_1 - ir)} \chi(\theta) |g|^{-1/2} \, d\operatorname{Vol}_g = 0,$$

where

$$f := \frac{a_1 - a_2}{|\varepsilon|^2 |g|^{1/2}} \quad \text{and} \quad h := \frac{b_1 - b_2}{|\mu|^2 |g|^{1/2}}.$$

Now, we extend $f$ and $h$ as zero to $\mathbb{R} \times M_0$. Since $d\operatorname{Vol}_g = |g|^{1/2} \, dx_1 dr d\theta$, we get

$$\int_{S^1} \chi(\theta) \int_0^\infty e^{-2\lambda r} \left( \int_{-\infty}^\infty f e^{-i2\lambda x_1} \, dx_1 \right) dr \, d\theta = 0$$

and

$$\int_{S^1} \chi(\theta) \int_0^\infty e^{-2\lambda r} \left( \int_{-\infty}^\infty h e^{-i2\lambda x_1} \, dx_1 \right) dr \, d\theta = 0.$$

Varying $\chi \in C^\infty(S^1)$ and noting that the terms in the brackets are the one-dimensional Fourier transforms of $f$ and $h$ with respect to the $x_1$-variable, which we denote by $\hat{f}$ and $\hat{h}$, respectively, we get

$$\int_0^\infty e^{-2\lambda r} \hat{f}(2\lambda, r, \theta) \, dr = \int_0^\infty e^{-2\lambda r} \hat{h}(2\lambda, r, \theta) \, dr = 0, \quad \theta \in S^1.$$

Recall that $(r, \theta)$ are polar coordinates in $M_0$. Therefore, $r \mapsto (r, \theta)$ is a geodesic in $M_0$ and the integrals above are the attenuated geodesic ray transforms of $\hat{f}$ and $\hat{h}$ on $M_0$ with constant attenuation $-2\lambda$. Then injectivity of this transform on simple manifolds of dimension two [22, Theorem 1.1] implies that $\hat{f}(2\lambda, \cdot) = \hat{h}(2\lambda, \cdot) = 0$ in $M_0$ for all $\lambda \in \mathbb{R} \setminus \{0\}$. Now, using the uniqueness result for the Fourier transform, we show that $f = h = 0$ and hence $a_1 = a_2$ and $b_1 = b_2$ in $M$, finishing the proof of Theorem 1.2.

## Appendix A. Regularity of solutions of linear Maxwell equations

In this section we prove the following regularity result for linear, inhomogeneous, time-harmonic Maxwell equations.

**Theorem A.1.** *Let $2 \le p \le 6$ and let $\varepsilon, \mu \in C^1(M)$ be complex functions with positive real parts. Suppose that $J_e, J_m \in W_\delta^p \Omega^1(M)$ and $i_\nu J_e|_{\partial M}, i_\nu J_m|_{\partial M} \in W^{1-1/p,p}(\partial M)$. If $(E, H) \in W_d^p \Omega^1(M) \times W_d^p \Omega^1(M)$ is a solution of the Maxwell's system*

$$\begin{cases} *dE = i\omega\mu H + J_m, \\ *dH = -i\omega\varepsilon E + J_e \end{cases} \tag{A.1}$$

*such that $\mathbf{t}(E) \in TW_{\text{Div}}^{1-1/p,p}(\partial M)$, then $(E, H) \in W_{\text{Div}}^{1,p}(M) \times W_{\text{Div}}^{1,p}(M)$ and*

$$\|E\|_{W_{\text{Div}}^{1,p}(M)} + \|H\|_{W_{\text{Div}}^{1,p}(M)}$$
$$\le C(\|\mathbf{t}(E)\|_{TW_{\text{Div}}^{1-1/p,p}(\partial M)} + \|J_e\|_{W_\delta^p \Omega^1(M)} + \|J_m\|_{W_\delta^p \Omega^1(M)})$$
$$+ C(\|i_\nu J_e|_{\partial M}\|_{W^{1-1/p,p}(\partial M)} + \|i_\nu J_m|_{\partial M}\|_{W^{1-1/p,p}(\partial M)})$$

*for some constant $C > 0$ independent of $E, H, J_e$ and $J_m$.*

*Proof.* We apply $\delta$ to the Maxwell equations (A.1) to obtain

$$\begin{cases} \delta H = i_{d\log\mu} H - (i\omega\mu)^{-1}\delta J_m, \\ \delta E = i_{d\log\varepsilon} E + (i\omega\varepsilon)^{-1}\delta J_e. \end{cases} \tag{A.2}$$

Since $E, H \in W_d^p \Omega^1(M)$ and $J_e, J_m \in W_\delta^p \Omega^1(M)$, this clearly implies that $E, H \in W_d^p \Omega^1(M) \cap W_\delta^p \Omega^1(M)$. Now, by Proposition 2.4, we get $E \in W_{\text{Div}}^{1,p}(M)$ and

$$\|E\|_{W_{\text{Div}}^{1,p}(M)} \le C(\|E\|_{W_d^p \Omega^1(M)} + \|\mathbf{t}(E)\|_{TW_{\text{Div}}^{1-1/p,p}(\partial M)}$$
$$+ \|J_e\|_{W_\delta^p \Omega^1(M)} + \|J_m\|_{W_\delta^p \Omega^1(M)}),$$

since $\mathbf{t}(E) \in TW_{\text{Div}}^{1-1/p,p}(\partial M)$.

To show that $H \in W^{1,p}\Omega^1(M)$, we use similar reasonings. Using (A.1) and (2.6) we get

$$i_\nu H|_{\partial M} = \frac{1}{i\omega\mu} i_\nu * dE|_{\partial M} - \frac{1}{i\omega\mu} i_\nu J_m|_{\partial M} \in W^{1-1/p,p}(\partial M),$$

since $\mathbf{t}(E) \in TW_{\text{Div}}^{1-1/p,p}(\partial M)$ and $i_\nu J_m|_{\partial M} \in W^{1-1/p,p}(\partial M)$. Then by Proposition 2.6, we have $H \in W^{1,p}\Omega^1(M)$ and

$$\|H\|_{W^{1,p}\Omega^1(M)} \le C(\|H\|_{W_d^p \Omega^1(M)} + \|\mathbf{t}(E)\|_{TW_{\text{Div}}^{1-1/p,p}(\partial M)})$$
$$+ C(\|J_m\|_{W_\delta^p \Omega^1(M)} + \|i_\nu J_m|_{\partial M}\|_{W^{1-1/p,p}(\partial M)}).$$

We also have $H \in W_{\mathrm{Div}}^{1,p}(M)$, since $\mathbf{t}(H) \in W^{1-1/p,p}(\partial M)$ and

$$\mathrm{Div}(\mathbf{t}(H)) = i_\nu * dH|_{\partial M} = -i\omega\varepsilon\, i_\nu E|_{\partial M} + i_\nu J_e|_{\partial M} \in W^{1-1/p,p}(\partial M),$$

where we have used (2.6) and the hypothesis $i_\nu J_e|_{\partial M} \in W^{1-1/p,p}(\partial M)$. Finally, the estimate in the statement of the theorem follows by combining all the above estimates. The proof is complete. $\qquad\square$

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