The density of states of 1D random band matrices via a supersymmetric transfer operator

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Abstract. Recently, M. and T. Shcherbina proved a pointwise semicircle law for the density of states of one-dimensional Gaussian band matrices of large bandwidth. The main step of their proof is a new method to study the spectral properties of non-self-adjoint operators in the semiclassical regime. The method is applied to a transfer operator constructed from the supersymmetric integral representation for the density of states.

We present a simpler proof of a slightly upgraded version of the semicircle law, which requires only standard semiclassical arguments and some peculiar elementary computations. The simplification is due to the use of supersymmetry, which manifests itself in the commutation between the transfer operator and a family of transformations of superspace, and was applied earlier in the context of band matrices by Constantinescu. Other versions of this supersymmetry have been a crucial ingredient in the study of the localization–delocalization transition by theoretical physicists.

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Contents

1. Introduction

Band operators and band matrices   Random band operators are popular toy models of disordered systems in theoretical physics. Their properties depend on a large parameter, called the *bandwidth* and denoted W. Informally, the large matrix elements lie in a band of width W about the main diagonal. One natural example is an Hermitian random operator H, represented by the biinfinite Gaussian random matrix with covariance

$$\mathbb{E}H_{x,y}\overline{H_{x',y'}} = \frac{1}{W}\delta_{x,x'}\delta_{y,y'}\mathbb{1}_{|x-y|\leq W}, \quad x,y \in \mathbb{Z}. \tag{1.1}$$

In this paper we mostly focus on a different example, the Hermitian Gaussian operator with covariance

$$\mathbb{E}H_{x,y}\overline{H_{x',y'}} = \delta_{x,x'}\delta_{y,y'}(-W^2\Delta + \mathbb{1})^{-1}_{xy}, \tag{1.2}$$

where Δ is the one-dimensional discrete Laplacian and $\mathbb{1}$ is the biinfinte identity matrix.

Along with the infinite-volume operators H, one considers their finite-volume versions H_N of dimension $N \times N$, called *random band matrices.* Again, we single out Gaussian band matrices, and among them – those with the covariances

$$\mathbb{E}H_{x,y}\overline{H_{x',y'}} = \frac{1}{W}\delta_{x,x'}\delta_{y,y'}\mathbb{1}_{|x-y|\leq W}. \tag{1.3}$$

and

$$\mathbb{E}H_{x,y}\overline{H_{x',y'}} = \delta_{x,x'}\delta_{y,y'}(-W^2\Delta_N + \mathbb{1}_N)^{-1}_{xy}, \tag{1.4}$$

where

$$\Delta_N = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 \\ 0 & 1 & -2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{pmatrix}$$

is the Neumann Laplacian on $\{1, \cdots, N\}$. We regard (1.3) and (1.4) as finite-volume versions of (1.1) and (1.2), respectively.

Localization length   By the general theory of one-dimensional random operators [31], finite-range band operators (including (1.1)) exhibit localization for any value of W, manifesting itself in pure point spectrum with exponentially decaying eigenfunctions. The rate of exponential decay of the eigenfunctions is known as the localization length and denoted L_{loc}. An essentially equivalent quantity

is the minimal value of N such that 9/10 of the ℓ_2 mass of the eigenvectors is concentrated in a subinterval of length $N/10$. Anderson localization also occurs for long-range random operators with sufficiently fast decay of the off-diagonal elements (such as 1.2); see [39].

A long-standing question is to determine the asymptotic dependence of the localization length on the bandwidth. It is widely believed that L_{loc} scales as W^2 for large W, for eigenvectors corresponding to energies $|E| < 2$. This belief is supported by various convincing albeit not mathematically rigorous arguments [11, 10, 24, 25]; see further below.

On the rigorous side, Schenker proved [33] that $L_{\text{loc}} \leq C W^8$ for a class of band matrices including (1.1). His argument is reminiscent of the Mermin–Wagner theorem in statistical mechanics. A combination of the result of [33] with the recent Wegner estimate from [32] yields a slight improvement $L_{\text{loc}} \leq C W^7$.

As to lower bounds, the results of [20, 19] pertaining to the quantum evolution imply a weak delocalization result for $W \gg N^{\frac{6}{7}}$. A stronger form of delocalization was established in [23] for $W \gg N^{\frac{4}{5}}$, and in [26], the constraint was relaxed to $W \gg N^{\frac{7}{9}}$. A genuine delocalization result was obtained for $W \gg N^{\frac{6}{7}}$ by Bao and Erdős [2]. The argument of [2] combines the methods developed for the proof of universality for Wigner matrices ($W \sim N$, see [22] for a review, as well as the monograph [21]) with a supersymmetric analysis incorporating superbosonization formulas [8] and the asymptotic method of [37]. In the recent series of works [7, 6, 42] delocalization and the convergence of local eigenvalue statistics in the bulk to the sine process was established for $W \gg N^{\frac{3}{4}}$; see [5] for a review.

We mention that band matrices and band operators admit a generalization to higher spatial dimension. If the spatial structure of the band is d dimensional with $d \geq 3$, they are expected to exhibit an Anderson-type spectral phase transition similar to the conjectural metal-insulator transition in realistic solid state models. The dimension $d = 2$ is critical, and localization is expected for all values of the band width. See further the review [38].

Density of states The behaviour of the eigenvectors of H is controlled by the quantity $\langle |G_{xy}(E + i\varepsilon)|^2 \rangle$, where

$$G_{xy}(z) = G_{xy}[H](z) = (z - H)^{-1}_{xy}$$

is the Green's function of the random matrix H, $E \in \mathbb{R}$ is a spectral parameter, and $\langle \cdot \rangle$ denotes averaging over the disorder. In infinite volume, ϵ should be sent to zero, while W is large but fixed. In finite volume, ϵ should be taken of order $1/N$. The quantity $\langle |G_{xy}(E + i\varepsilon)|^2 \rangle$ also controls the properties of the (quenched)

spectral measure μ_H of H, which is defined by

$$G_{00}(z) = \int (z - \lambda)^{-1} d\mu_H(\lambda), \quad z \in \mathbb{C} \setminus \mathbb{H}.$$

In this paper we focus on a simpler quantity, the average Green function $\langle G_{xx}(E + i\varepsilon) \rangle$, and the related quantity $\langle G_{xy}(E + i\varepsilon)G_{yx}(E + i\varepsilon) \rangle$. The former one controls the behaviour of the average density of states $\rho = \rho[H]$, which can be defined as the Radon derivative of the disorder average $\langle \mu_H \rangle$ of μ_H. We also consider the average density of states in finite volume, $\rho_N = \rho[H_N]$.

For Gaussian random band matrices including both (1.1) and (1.2), the density of states exists in both finite and infinite volume by a general argument of Wegner [40]. The average density of states is related to $\langle G_{00} \rangle$:

$$\int (z - E)^{-1} \rho(E) dE = \langle G_{00}[H](z) \rangle,$$

$$\int (z - E)^{-1} \rho_N(E) dE = \frac{1}{N} \sum_{x=1}^{N} \langle G_{xx}[H_N](z) \rangle$$

and (consequently)

$$\rho(E) = -\frac{1}{\pi} \lim_{\varepsilon \searrow 0} \Im \langle G_{00}[H](E + i\varepsilon) \rangle, \tag{1.5a}$$

$$\rho_N(E) = -\frac{1}{\pi N} \lim_{\varepsilon \searrow 0} \Im \langle \operatorname{Tr} G[H_N](E + i\varepsilon) \rangle. \tag{1.5b}$$

In finite volume, ρ_N can be also written as

$$\rho_N(E) = \partial_E \frac{1}{N} \langle \# \{\text{eigenvalues of } H_N \text{ in } (-\infty, E]\} \rangle$$

justifying its name. Also, one has $\lim_{N \to \infty} \rho_N(E) = \rho(E)$ (see the proof of Proposition 4.2). It is believed that the average $\langle G_{xy}(E + i\varepsilon)G_{yx}(E + i\varepsilon) \rangle$ and the average density of states do not reflect the localization properties of the eigenvectors and the spectral type of H, see e.g. [40]. Still, the former quantities are of intrinsic interest.

In the works [4, 30], it was proved that as $W \to \infty$ the densities $\rho(\lambda)$ converge weakly to the semicircle density

$$\rho_{\text{s.c.}}(E) = \frac{1}{2\pi} \sqrt{4 - E^2} \, \mathbb{1}_{|E| \le 2},$$

meaning that

$$\lim_{W \to \infty} \int \phi(E) \rho(E) dE = \int \phi(E) \rho_{\text{s.c.}}(E) dE \tag{1.6}$$

for any bounded continuous test function ϕ. The arguments in these works apply to for a wide class of band matrices including (1.1) and (1.2).

The available pointwise results are much less general, and we list them below following a brief discussion of supersymmetry.

Supersymmetry. One of the powerful methods for the study of random operators is the supersymmetric formalism. First introduced by Wegner and Schäfer and developed in the works of Efetov, it allows to rewrite the disorder averages of various observables as high-dimensional superintegrals. A general introduction may be found in the monographs [41, 18].

In the context of random band matrices, the supersymmetric approach was applied by Fyodorov and Mirlin [24, 25], who confirmed the dependence of the localisation length on the bandwidth and also described the crossover occurring as $W \asymp \sqrt{N}$. For Gaussian random band matrices, the average $\langle |G_{xy}(E + i\varepsilon)|^2 \rangle$ corresponds, through Berezin integration and superbosonization or certain formal versions of the Hubbard–Stratonovich transformation [43], to a high dimensional super-integral dominated by a complicated saddle *manifold*. The average density of states ρ_N and the average $\langle G_{xy}(E + i\varepsilon) G_{yx}(E + i\varepsilon) \rangle$ lead, in the same way, to an integral dominated by saddle *points*.

The four main steps in the works [24, 25] are the derivation of the supersymmetric integral representation; the σ-model approximation, in which the integration domain is restricted to the saddle manifold; the continuum limit; and a semiclassical analysis of the infinitesimal transfer operator. All these steps, and particularly the last three, have so far not been put on firm mathematical ground. (See [36] for recent progress on the second step.)

A version of the SUSY formalism that uses similar algebraic identities (Berezin integration), but simpler supersymmetries than those involved in superbosonization, had been used early on in rigorous investigations, e.g., of localization in $d = 1$ random Schrödinger operators [9, 29].

Pointwise estimates. The models (1.2) and their counterparts in higher dimension are especially convenient for supersymmetric analysis, since the dual supersymmetric model has nearest neighbour coupling (see Proposition 2.1; in classical statistical mechanics, the idea of duality between carefully chosen long-range models and nearest neighbour models goes back to the work of Mark Kac [28, Section 9]).

This is why most of the pointwise results established to date pertain to this class of operators. One exception is the upper bound

$$\rho(E) \leq C, \quad \max_N \rho_N(E) \leq C$$

valid [32] for a reasonably wide class of Gaussian random band matrices including both (1.1) and (1.2), and their counterparts in arbitrary dimension.

Much more is known for models of the form (1.2). In dimension $d = 3$, it was proved in [16] that

$$\partial_E^n \rho(E) \leq C_n \quad \text{for } n, \, W \geq W_0(n), \, |E| \leq 1.8,$$

and that

$$|\rho(E) - \rho_{\text{s.c.}}(E)| \leq C W^{-2} \quad \text{for } |E| \leq 1.8.$$

Corresponding results were also proved in finite volume. The argument of [16] relies on a cluster expansion similar to the one used for the Wegner orbital model in [13]. Recently, a parallel result was also proved for $d = 2$ in [15].

In $d = 1$, the cluster expansion methods of [16, 15] run into difficulties. Instead, the method of transfer matrices can be used. While the method of transfer matrices has been successfully applied in the physical literature for this and more involved problems [24, 25], a mathematical justification is far from straightforward since the transfer matrix is not self-adjoint. One possible strategy to perform semi-classical analysis for non-self-adjoint operators was suggested in [17]; for now, this strategy was only implemented for toy operators much simpler than the one considered here.

Recently, Mariya and Tatyana Shcherbina developed a different and very general method of semi-classical analysis for non-self-adjoint operators [35]. In the work [34], they applied the method to the problem under discussion and proved

Theorem ([34]). *If $N \geq C W \log W$ and $|E| \leq \sqrt{\frac{32}{9}}$, then*

$$|\rho_N(E) - \rho_{sc}(E)| \leq \frac{C'}{W}.$$

The main goal of the current paper is to provide an alternative and arguably simpler proof for the result of [34]. Most of the work in [34] is devoted to proving the spectral gap for the transfer operator and some properties of its top eigenfunction (Theorem 4.2 therein). The argument is based on methods developed in the non-supersymmetric framework in [35]. Our proof, by contrast, exploits the supersymmetry of the problem. Apart from important simplification and reorganization of the algebra, supersymmetry manifestly fixes the top eigenvalue to 1, implies a simple equation for the top eigenfunction, and reduces the proof of the spectral gap to an elementary spectral bound. We also emphasize that, similarly to the classical applications of transfer operators in statistical mechanics, the original

problem with two large parameters, N and W, is reduced to a bunch of asymptotic problems containing only one parameter.

Our method is non-applicable to problems lacking supersymmetry, such as the correlation length of characteristic polynomials considered in [35], but has the advantage of simplicity. It also allows to obtain stronger results, summarized in the theorems below.

Theorem 1. *For $|E| < 2$ and $N \geq C(E)W \log W$*

$$|\rho_N(E) - \rho(E)| \leq \frac{C(E)}{N}. \tag{1.7}$$

The infinite volume density is well approximated by the semi-circle law:

$$|\rho(E) - \rho_{\text{s.c.}}(E)| \leq \frac{C(E)}{W^2}. \tag{1.8}$$

Remark 1.1. The estimate (1.7) together with (1.8) yields

$$|\rho_N(E) - \rho_{\text{s.c.}}(E)| \leq \frac{C(E)}{W \log W}, \quad N \geq C(E)W \log W. \tag{1.9}$$

Note that, for large band width $N \sim W \log W$, the finite volume deviation (1.7) is substantially bigger than the accuracy (1.8) of the semicircle law.

Theorem 2. *Assume that $|E| < 2$. For $N \geq C(E)W \log W$ the following hold.*

(1) *For any $1 \leq y, y' \leq N$,*

$$|\langle G_{yy'}[H_N](E) G_{y'y}[H_N](E) \rangle| \leq \frac{C(E)}{W} \exp\left(-\frac{c(E)|y' - y|}{W}\right). \tag{1.10}$$

(2) *The finite volume density ρ_N is smooth and for all $n \geq 1$*

$$|\partial_E^n \rho_N(E)| \leq C(E)^n n! \, W^{n-1}. \tag{1.11}$$

Remark 1.2. From (1.7) and (1.11) we obtain that the densities $\rho_N(E)$ and $\rho(E)$ admit an analytic extension to a domain of complex energies of the form

$$\{E + i\epsilon \mid |E| < 2, |\epsilon| \leq c(E)/W\}.$$

Remark 1.3. Throughout the paper, expressions of the form $C(E)$, $C'(E)$, $c(E)$ are bounded from above and below on compact subintervals of $(-2, 2)$. The values of $C(E)$, $C'(E)$, $c(E)$ and of the numerical constants C, C', c, et cet. may change from line to line.

Plan of the paper. In Section 2, we recall the representation of the mean density of states $\rho_N(E)$, as well as of the left-hand sides of (1.10)–(1.11), via Berezin integration, and set up a supersymmetric transfer operator $\mathfrak{T}$, (2.16). Then we discuss a certain supersymmetry, that has already been described in [12], and is the supermatrix analogue of rotational symmetry and polar coordinates. This supersymmetry allows to reduce the transfer operator (2.16) to the much simpler one, $\mathcal{T}$ of (2.37), acting on functions of two real variables.

In Section 3 we analyse the transfer operator $\mathcal{T}$. The two main steps are (a) the construction of an approximate top eigenfunction u_0, $\mathcal{T}u_0 \approx u_0$, and (b) a bound on the restriction of $\mathcal{T}$ to a complement of the top eigenfunction. For a weakened version of Theorem 1 (with a worse error term), it would suffice to use the simple ansatz

$$u_0^{(0)}(\lambda) = \exp(-\alpha W(\lambda_1^2 + \lambda_2^2))$$

with an appropriately chosen α. In order to prove the results as stated, we develop, in Section 3.2, a systematic construction ("supersymmetric WKB") which allows to find an approximate solution to an arbitrary order in $W^{-1/2}$ (for the purposes of the current paper, we use this construction up to order 5). As for step (b), the key fact is Proposition 3.1, proved via a simple semiclassical argument. Corollary 3.4 and Lemma 3.5 contain refinements needed to obtain the improved error term in Theorem 1 and for Theorem 2.

In Section 4 we prove Theorems 1 and 2, for $|E| \leq \sqrt{\frac{32}{9}}$. The relatively simple Proposition 3.7 would suffice to prove a version of Theorem 1 with a worse error term; we work a bit more to prove the results as stated.

In Section 5 we extend the argument to all $|E| < 2$, using a contour deformation and applying the general strategy of the Section 4 to a power of the transfer operator (where the power is chosen depending on the energy E).

With the exception of the Berezin integral representation (well explained elsewhere) and undergraduate analysis, our proof is self contained.

2. Berezin algebra for the DOS and SUSY

2.1. Supersymmetric integral representation. The Berezin integral representation of $\rho_N(E)$ involves the use of Grassmann variables. These will be denoted by the symbols $\rho, \bar{\rho}, \eta, \bar{\eta}$ and $\xi, \bar{\xi}$, possibly with an index. All Grassmann variables anticommute with one another. The Grassmann algebra $\mathfrak{G}$ ($\equiv$ free anticommutative) generated (over $\mathbb{C}$) by all Grassmann variables admits a $\mathbb{Z}/2\mathbb{Z}$ grading, in

which a monomial in the generators is even or odd according to the parity of the number of symbols.

On the Grassmann algebra we define complex conjugation by $\rho \to \bar{\rho}, \bar{\rho} \to -\rho$ (etc.) on the generators, and ordinary complex conjugation on the coefficients.[1] An element f in the Grassmann algebra generated by the family $\{\rho_x, \bar{\rho}_x\}_{x=1}^N$ has a unique decomposition as

$$f = f(\{\rho_x, \bar{\rho}_x\}_x) = \sum_{I,J \subset \{1,\ldots,N\}} c_{IJ} \rho_I \bar{\rho}_J \qquad (2.1)$$

where $c_{IJ} \in \mathbb{C}$, $\rho_I = \prod_{i \in I} \rho_i$ and $\bar{\rho}_J := \prod_{j \in J} \bar{\rho}_j$ and the products are understood under some fixed arbitrary order on the set $\{1, \ldots, N\}$. In particular, every element can be written as $f = f_0 + n$ where $f_0 \in \mathbb{C}$ and n is nilpotent, i.e. there exists $k \geq 1$ such that $n^k = 0$. The Grassmann algebra has a natural decomposition into even and odd elements.

Having a Grassmann algebra $\mathfrak{G}$ at hand, we consider the algebra of C^∞-smooth functions $f \colon \mathbb{C}^m \to \mathfrak{G}$. A general element of this larger algebra has the form

$$f(z_1, \ldots, z_m, \{\rho_x, \bar{\rho}_x\}_{x \in \{1,\cdots,N\}}) = \sum_{I,J \subset \{1,\ldots,N\}} c_{IJ}(z_1, \ldots, z_m) \rho_I \bar{\rho}_J. \qquad (2.2)$$

Smooth multivariate functions $(z_1, \ldots, z_m) \mapsto f(z_1, \ldots, z_m)$ can be extended to functions of the form (2.2) taking values in the Grassmann algebra by replacing the variables by some even elements of the algebra $z_i \to z_i + n_i$, where n_i is nilpotent, and taking their Taylor series around $(z_1, \ldots, z_m)$ until it terminates by nilpotency. For example, for $m = 1$ one has

$$f(z + n) = f(z) + \sum_{j=1}^{k-1} \frac{f^{(j)}(z)}{j!} n^j, \quad n^k = 0.$$

The Berezin integral is the linear functional $\int$ that selects the top degree (in the generators) coefficient in (2.1) and multiplies it with $(2\pi)^{-\frac{m}{2}}$, where m is the number of generators. Thus, if $\rho, \bar{\rho}$ are the only generators,

$$\int \mathrm{d}\bar{\rho}\, \mathrm{d}\rho (a + b\rho + c\bar{\rho} + d\rho\bar{\rho}) = \frac{d}{2\pi}.$$

where the differentials $\mathrm{d}\rho$, $\mathrm{d}\bar{\rho}$ anticommute with each other and with all Grasmann generators.

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[1] In the terminology of [41, §6.1], this is the conjugate of the second kind; see further the discussion in [41, §6.2].

A $2m \times 2m$ supermatrix is a matrix of the form

$$\begin{pmatrix} A & \Sigma \\ \Gamma & B \end{pmatrix},$$

where A, B (resp. Σ, Γ) are $m \times m$ matrices whose matrix elements are even (respectively, odd) elements of the Grassmann algebra. The supertrace and superdeterminant of a supermatrix are defined as

$$\mathrm{Str} \begin{pmatrix} A & \Sigma \\ \Gamma & B \end{pmatrix} = \mathrm{tr}\, A - \mathrm{tr}\, B, \quad \mathrm{Sdet} \begin{pmatrix} A & \Sigma \\ \Gamma & B \end{pmatrix} = \frac{1}{\det B} \det(A - \Sigma B^{-1} \Gamma),$$

where tr and det are the ordinary trace and determinant. We shall use the symbol R (possibly with an index) to denote 2×2 supermatrices of the form

$$R = \begin{pmatrix} a & \bar{\rho} \\ \rho & ib \end{pmatrix}, \tag{2.3}$$

where a, b are real numbers, and ρ, $\bar{\rho}$ are generators of the Grassmann algebra. By a superfunction F of one or several variables R we mean an expression of the form (2.2) in which the z variables are replaced by a, b. See [3] for more details on these conventions and definitions.

We will use a supersymmetric representation for the average Green's function, for which we will now introduce some notation.

Let

$$J := -W^2 \Delta_N + \mathbb{1},$$

the inverse of the covariance introduced in (1.4). Abbreviate

$$E_\varepsilon = E + i\varepsilon, \quad \varepsilon > 0.$$

Let Γ_a and Γ_b be two horizontal lines in $\mathbb{C}$ oriented from left to right, such that Γ_a lies below the pole $E + i\varepsilon \in \mathbb{C}$, while no additional constraint is needed for Γ_b. For any suitable[2] superfunction $F = F((R_x)_{x=1}^N)$, introduce the SUSY average

$$\langle F \rangle_{\mathrm{SUSY}} = \int \prod_{x=1}^N \mathrm{d}\, R_x \, e^{-\frac{1}{2} \mathrm{Str}(R, JR)} \prod_{x=1}^N \frac{1}{\mathrm{Sdet}(E_\varepsilon - R_x)} F((R_x)_{x \in \{1,\cdots,N\}}), \tag{2.4}$$

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[2] It will be enough to consider superfunctions whose coefficients $c_{IJ}(\{a_x, b_x\}_{x \in \{1,\cdots,N\}})$ in the expansion (2.2) are smooth functions of polynomial growth

where

$$\operatorname{Str}(R, JR) := \sum_{xy} J_{xy} \operatorname{Str} R_x R_y = \sum_x \operatorname{Str} R_x^2 + W^2 \sum_{x=1}^{N-1} \operatorname{Str}(R_{x+1} - R_x)^2$$

and

$$E_\varepsilon - R_x := E_\varepsilon \mathbb{1}_2 - R_x$$

is a 2×2 supermatrix. Finally

$$\mathrm{d} R_x := \mathrm{d} a_x \, \mathrm{d} b_x \, \mathrm{d} \bar{\rho}_x \, \mathrm{d} \rho_x,$$

the integral in a_x is taken along Γ_a, and the integral in b_x is taken along Γ_b. We will denote $\mathbf{a} = (a_x)_{x=1}^N$ and $\mathbf{b} = (b_x)_{x=1}^N$.

Proposition 2.1. *In the above notation the following identities hold*:

$$1 = \langle 1 \rangle = \langle 1 \rangle_{\mathrm{SUSY}}, \tag{2.5}$$

$$\begin{aligned}
\langle G_{yy}[H_N](E_\varepsilon) \rangle &= \langle (J\mathbf{a})_y \rangle_{\mathrm{SUSY}} \\
&= \langle (iJ\mathbf{b})_y \rangle_{\mathrm{SUSY}} \\
&= \sum_x J_{yx} \langle a_x \rangle_{\mathrm{SUSY}} \\
&= \sum_x J_{yx} \langle (ib)_x \rangle_{\mathrm{SUSY}},
\end{aligned} \tag{2.6}$$

$$\begin{aligned}
\langle G_{yy'}[H_N](E_\varepsilon) G_{y'y}[H_N](E_\varepsilon) \rangle &= -J_{yy'} + \langle (J\mathbf{a})_{y'}[J(\mathbf{a} - i\mathbf{b})]_y \rangle_{\mathrm{SUSY}} \\
&= -J_{yy'} + \sum_{xx'} J_{yx} J_{y'x'} \langle a_{x'} \operatorname{Str} R_x \rangle_{\mathrm{SUSY}},
\end{aligned} \tag{2.7}$$

and, for $n \geq 1$,

$$\begin{aligned}
\partial_E^n \langle \operatorname{Tr} G[H_N](E_\varepsilon) \rangle &= N\delta_{n,1} + (-1)^n \left\langle \left[\sum_x a_x \right] \left[\sum_y (a_y - ib_y) \right]^n \right\rangle_{\mathrm{SUSY}} \\
&= N\delta_{n,1} + (-1)^n \left\langle \left[\sum_x a_x \right] \left[\sum_y \operatorname{Str} R_y \right]^n \right\rangle_{\mathrm{SUSY}}.
\end{aligned} \tag{2.8}$$

Proof. Let $\mathbf{E} = \operatorname{diag}(E_1, \ldots, E_N), \widetilde{\mathbf{E}} = \operatorname{diag}(\widetilde{E}_1, \ldots \widetilde{E}_N) \in \mathbb{R}^{N \times N}$ and consider the generating function

$$\mathcal{J}(\mathbf{E}, \widetilde{\mathbf{E}}) := \left\langle \frac{\det(\mathbf{E} + i\varepsilon \mathbb{1}_N - H)}{\det(\widetilde{\mathbf{E}} + i\varepsilon \mathbb{1}_N - H)} \right\rangle.$$

Then $\mathcal{J}(E\mathbb{1}_N, E\mathbb{1}_N) = 1$ for all $E \in \mathbb{R}$. Moreover

$$\langle G_{yy}[H_N](E_\varepsilon)\rangle = -\partial_{\widetilde{E}_y}\mathcal{J}(E\mathbb{1}_N, E\mathbb{1}_N) = \partial_{E_y}\mathcal{J}(E\mathbb{1}_N, E\mathbb{1}_N),$$

$$\langle G_{yy'}[H_N](E_\varepsilon)G_{y'y}[H_N](E_\varepsilon)\rangle = \partial_{\widetilde{E}_{y'}}(\partial_{\widetilde{E}_y} + \partial_{E_y})\mathcal{J}(E\mathbb{1}_N, E\mathbb{1}_N),$$

and setting $E, \widetilde{E} \in \mathbb{R}$

$$\partial_E^n\langle \operatorname{Tr} G[H_N](E_\varepsilon)\rangle = -\partial_{\widetilde{E}}(\partial_E + \partial_{\widetilde{E}})^n\mathcal{J}(E\mathbb{1}_N, \widetilde{E}\mathbb{1}_N)_{|E=\widetilde{E}}.$$

By the same arguments as in [16, Section 3] the generating function can be written as follows

$$\mathcal{J}(\mathbf{E}, \widetilde{\mathbf{E}}) = \int \prod_{x=1}^N dR_x\, e^{-\frac{1}{2}\operatorname{Str}(R,JR)} \prod_{x=1}^N \frac{1}{\operatorname{Sdet}(\widehat{E}_x + i\varepsilon\mathbb{1}_2 - R_x)},$$

where $\widehat{E}_x$ is the 2×2 supermatrix

$$\widehat{E}_x := \begin{pmatrix} \widetilde{E}_x & 0 \\ 0 & E_x \end{pmatrix},$$

Setting

$$\mathbf{E} = \widetilde{\mathbf{E}} = E\mathbb{1}_N$$

we obtain the measure in (2.4) whence $1 = \mathcal{J}(E\mathbb{1}_N, E\mathbb{1}_N) = \langle 1\rangle_{\text{SUSY}}$. Moreover

$$
\begin{aligned}
&-\partial_{\widetilde{E}_y}\mathcal{J}(E\mathbb{1}_N, E\mathbb{1}_N) \\
&= \int \prod_{x=1}^N dR_x e^{-\frac{1}{2}\operatorname{Str}(R,JR)}(-\partial_{\widetilde{E}_y})\prod_{x=1}^N \frac{1}{\operatorname{Sdet}(\widehat{E}_x + i\varepsilon\mathbb{1}_2 - R_x)}\bigg|_{\mathbf{E}=\widetilde{\mathbf{E}}=E\mathbb{1}_N} \\
&= \int \prod_{x=1}^N dR_x e^{-\frac{1}{2}\operatorname{Str}(R,JR)}(\partial_{a_y})\prod_{x=1}^N \frac{1}{\operatorname{Sdet}(E_\varepsilon - R_x)} \\
&= \Big\langle \partial_{a_y}\frac{\operatorname{Str}(R, JR)}{2}\Big\rangle_{\text{SUSY}} \\
&= \langle (J\mathbf{a})_y\rangle_{\text{SUSY}},
\end{aligned}
$$

where we used

$$(\partial_{\widetilde{E}_y} + \partial_{a_y})\operatorname{Sdet}(\widehat{E}_y + i\varepsilon\mathbb{1}_2 - R_y) = 0$$

and we performed integration by parts in a_y. The same argument holds for $\partial_{E_y}\mathcal{J}(E\mathbb{1}_N, E\mathbb{1}_N)$ using

$$(\partial_{E_y} - i\partial_{b_y})\operatorname{Sdet}(\widehat{E}_y + i\varepsilon\mathbb{1}_2 - R_y) = 0.$$

This completes the proof of (2.6). Performing integration by parts twice we obtain

$$\partial_{\widetilde{E}_{y'}}(\partial_{\widetilde{E}_y} + \partial_{E_y})\mathcal{J}(E\mathbb{1}, E\mathbb{1})$$

$$= \int \prod_{x=1}^{N} dR_x \prod_{x=1}^{N} \frac{1}{\text{Sdet}(E_\varepsilon - R_x)}(-\partial_{a_{y'}})[J(\mathbf{a} - i\mathbf{b})]_y e^{-\frac{1}{2}\text{Str}(R,JR)}$$

$$= -J_{yy'} + \langle (J\mathbf{a})_{y'}[J(\mathbf{a} - i\mathbf{b})]_y \rangle_{\text{SUSY}}.$$

This completes the proof of (2.7). Similar arguments yield

$$-\partial_{\widetilde{E}}(\partial_E + \partial_{\widetilde{E}})^n \mathcal{J}(E\mathbb{1}_N, \widetilde{E}\mathbb{1}_N)_{|E=\widetilde{E}}$$

$$= (-1)^{n+1} n N \Big\langle \Big[\sum_y \text{Str } R_y \Big]^{n-1} \Big\rangle_{\text{SUSY}} + (-1)^n \Big\langle \Big[\sum_x a_x \Big]\Big[\sum_y \text{Str } R_y \Big]^n \Big\rangle_{\text{SUSY}},$$

where we used that $(\partial_{a_y} - i\partial_{b_y})(a_y - ib_y) = 0$ for any y. The first summand can be written as

$$\Big\langle \Big[\sum_y \text{Str } R_y \Big]^{n-1} \Big\rangle_{\text{SUSY}} = \partial_\gamma^{n-1} \langle e^{\gamma \sum_y \text{Str } R_y} \rangle_{\text{SUSY}}|_{\gamma=0=0} \quad \text{all } n > 1 \qquad (2.9)$$

since by performing the translation $R \to R + \gamma \mathbb{1}_2$ we get

$$\langle e^{\gamma \sum_y \text{Str } R_y} \rangle_{\text{SUSY}} = \mathcal{J}((E - \gamma)\mathbb{1}, (E - \gamma)\mathbb{1}) = 1 \quad \text{for } \gamma \in \mathbb{R}.$$

This completes the proof of (2.8). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Identities of the form (2.5) for supersymmetric averages of supersymmetric functions go back to the work of Parisi and Sourlas, cf. [41, Chapter 15]. Note that the first two items appear also in this form, for example, in [34, 15].

Translation to the saddle   Disregarding mathematical precision in our thinking about Grassmann variables, we have the following heuristics. The factor

$$\exp\Big(-\frac{W^2}{2} \sum \text{Str}(R_{x+1} - R_x)^2 \Big)$$

forces the "integration field" R_x to be almost constant, and the Laplace method then implies that the main contribution is from field R_x in the vicinity of the "saddle points" of $e^{-\frac{1}{2}\text{Str } R_x^2}(\text{Sdet}(E_\varepsilon - R_x))^{-1}$. Now we proceed with the rigorous argument. Setting the Grassmann variables equal to 0 and neglecting the ε contribution it is easy to see that the saddle points (i.e. the critical points of the logarithm) are at

$$a^\pm = \frac{1}{2}(E \pm i\sqrt{4 - E^2}), \quad b^\pm = (-i)\frac{1}{2}(E \pm i\sqrt{4 - E^2}). \qquad (2.10)$$

a^+ is in the same half space as the pole, therefore we choose as contour Γ_a the translate of the real axis with origin at a^-. We will abbreviate

$$\mathcal{E} = a^- = \frac{E}{2} - i\sqrt{1 - \frac{E^2}{4}}.$$

Note that $\mathcal{E}\overline{\mathcal{E}} = 1$. It will turn out later that b^- is the dominant saddle point.[3] We therefore choose as contour Γ_b the translate of the real axis with origin at b^-. After this translation, the limit $\varepsilon \searrow 0$ can be performed by dominated convergence. We introduce the modified SUSY average

$$\langle F \rangle'_{\text{SUSY}} := \int \prod_{x=1}^N dR_x e^{-\frac{1}{2}\text{Str}(R,JR)} e^{-\mathcal{E}\sum_x \text{Str} R_x} \tag{2.11}$$
$$\cdot \prod_{x=1}^N \frac{1}{\text{Sdet}(\overline{\mathcal{E}} - R_x)} F((R_x)_{x \in \{1,\cdots,N\}}),$$

where ϵ is now set equal to 0 and the integrals in a_x, b_x are taken along the real axis. Moreover we used $E - \mathcal{E} = \overline{\mathcal{E}}$,

$$\text{Str}(R + \mathcal{E}\mathbb{1}_2, J(R + \mathcal{E}\mathbb{1}_2)) = \text{Str}(R, JR) + 2\mathcal{E}\,\text{Str}(R, J\mathbb{1}_2) + \mathcal{E}^2\,\text{Str}(\mathbb{1}_2, J\mathbb{1}_2)$$
$$= \text{Str}(R, JR) + 2\mathcal{E}\sum_x \text{Str} R_x,$$

where the last term vanishes since $\text{Str}\,\mathbb{1}_2 = 0$. We then have

Corollary 2.2. *Let* $|E| < 2$. *Then*

$$\lim_{\varepsilon \searrow 0} \langle G_{yy}[H_N](E_\varepsilon) \rangle = \mathcal{E} + \langle (J\mathbf{a})_y \rangle'_{\text{SUSY}} = \mathcal{E} + \langle (J(i\mathbf{b}))_y \rangle'_{\text{SUSY}}, \tag{2.12}$$

$$\lim_{\varepsilon \searrow 0} \langle G_{yy'}[H_N](E_\varepsilon) G_{y'y}[H_N](E_\varepsilon) \rangle = -J_{yy'} + \langle (J\mathbf{a})_{y'}[J(\mathbf{a} - i\mathbf{b})]_y \rangle'_{\text{SUSY}}, \tag{2.13}$$

and, for $n \geq 1$,

$$\lim_{\varepsilon \searrow 0} \partial_E^n \langle \text{Tr}\, G[H_N](E_\varepsilon) \rangle = N\delta_{n,1} + (-1)^n \Big\langle \Big[\sum_x a_x\Big]\Big[\sum_y (a_y - ib_y)\Big]^n \Big\rangle'_{\text{SUSY}}. \tag{2.14}$$

---

[3] If, as is the case for GUE, $\rho_x \equiv \rho$ were a collective variable, i.e. a single Grassmann generator, and similarly for $\bar{\rho}$, this could be seen heuristically by evaluating $I = e^{-\frac{N}{2}\text{Str} R^2}\text{Sdet}(E_\varepsilon - R)^{-N}$ at $R = R^\pm = \begin{pmatrix} a^- & \bar{\rho} \\ \rho & ib^\pm \end{pmatrix}$. Set $I|_{R=R^\pm, \varepsilon=0} = I_0 \pm I^\pm \bar{\rho}\rho$. If there were no observable, the Grassmann integral $d\rho\, d\bar{\rho}$ would select $I^\pm$. Explicit calculation gives $I^- = -N(2 - \frac{E^2}{2} + iE\sqrt{4 - E^2})$, which is large (proportional to N), while $I^+ = -N(1 - |\mathcal{E}|^2) = 0$. A more convincing argument would have to analyze "fluctuations" in $\rho, \bar{\rho}$.

Proof. Eq. (2.12) follows from (2.6) by replacing $\mathbf{a}$ by $\mathbf{a}+\mathcal{E}$ and $i\,\mathbf{b}$ by $i\,\mathbf{b}+\mathcal{E}$. In the same way, (2.13) follows from (2.7) by observing that, in addition, (2.12) implies $\langle (J\mathbf{a})_y \rangle'_{\text{SUSY}} = \langle (J(i\,\mathbf{b}))_y \rangle'_{\text{SUSY}}$. Finally (2.14) follows from (2.8) observing that $\langle [\sum_y \text{Str } R_y]^n \rangle'_{\text{SUSY}} = 0$ for all $n \geq 1$ by the same argument as in (2.9). $\qquad \square$

Note that $-\frac{1}{\pi}\Im\mathcal{E} = \frac{1}{2\pi}\sqrt{4 - E^2}$ is the semicircle density. Thus, for example, eq. (1.9) amounts to showing that the remaining term in (2.12) is small for large W and N.

2.2. Supersymmetric transfer operator.

The representation of Corollary 2.2 can be stated using a supersymmetric transfer operator. Recall that

$$\text{Str}(R + \mathcal{E}, J(R + \mathcal{E})) = \sum_x \text{Str}(R_x + \mathcal{E})^2 + W^2 \sum_{x=1}^{N-1} \text{Str}(R_{x+1} - R_x)^2.$$

We denote

$$e^{-V(R)} := \frac{e^{-\frac{1}{2}\text{Str}(R+\mathcal{E})^2}}{\text{Sdet}(\bar{\mathcal{E}} - R)} \tag{2.15}$$

and define

$$(\mathfrak{T}F)(R) := e^{-V(R)} \int d R' e^{-\frac{W^2}{2}\text{Str}(R-R')^2} F(R'). \tag{2.16}$$

Remark 2.3. Here we consider e^{-V} as a single typographic symbol, therefore the reader need not be concerned, for example, with the choice of a branch of the logarithm. Also, for now we treat $\mathfrak{T}$ as a formal operation rather than as an operator, therefore we do not worry about the domain.

Using (2.15) and (2.16), for all $1 \leq y \leq y' \leq N$,

$$\langle a_y \rangle'_{\text{SUSY}} = \int d R \, [\mathfrak{T}^{y-1}e^{-V}](R)e^{V(R)}a[\mathfrak{T}^{N-y}e^{-V}](R) \tag{2.17a}$$

and

$$\langle a_y(a_{y'} - i b_{y'}) \rangle'_{\text{SUSY}} = \langle a_y \text{ Str } R_{y'} \rangle'_{\text{SUSY}} = I_{yy'}, \tag{2.17b}$$

where

$$I_{yy'} := \int d R[\mathfrak{T}^{y-1}e^{-V}](R)e^{V(R)}a[\mathfrak{T}^{y'-y} \text{ Str}(\cdot)\mathfrak{T}^{N-y'}e^{-V}](R),$$

and $\text{Str}(\cdot)$ is the "multiplication operator" by $(\text{Str } R)$ and we use the convention $\mathfrak{T}^0 = \text{id}$. In the same way we get, for any $n \geq 1$,

$$\left\langle \left[\sum_x a_x \right] \left[\sum_y (a_y - i b_y) \right]^n \right\rangle'_{\text{SUSY}}$$
$$= \sum_{q=1}^{n} \sum_{\substack{n_1,\ldots n_q \geq 1 \\ n_1 + \cdots + n_q = n}} \frac{n!}{n_1! \cdots n_q!} \sum_{m=0}^{n} \sum_{\substack{1 \leq y_1 < y_2 < \cdots \leq y_q \leq N \\ y_m \leq x < y_{m+1}}} I^{m,n_1,\ldots,n_q}_{x,y_1,\ldots,y_q}, \qquad (2.18)$$

where we defined $y_0 = 1$, $y_{q+1} = N + 1$, $n_0 = n_{q+1} = 0$, and

$$I^{m,n_1,\ldots,n_q}_{x,y_1,\ldots,y_q}$$
$$:= \int d R [\mathfrak{T}^{x-y_m} \text{Str}(\cdot)^{n_m} \cdots \mathfrak{T}^{y_2-y_1} \text{Str}(\cdot)^{n_1} \mathfrak{T}^{y_1-1} e^{-V}](R)$$
$$\cdot [e^{V(R)} a][\mathfrak{T}^{y_{m+1}-x} \text{Str}(\cdot)^{n_{m+1}} \mathfrak{T}^{y_{m+2}-y_{m+1}} \cdots \text{Str}(\cdot)^{n_q} \mathfrak{T}^{N-y_q} e^{-V}](R). \qquad (2.19)$$

Moreover, in the special case $m = 0$ the first product is replaced by $\mathfrak{T}^{x-1} e^{-V}$ while for $m = q$ the second product is replaced by $\mathfrak{T}^{N-x} e^{-V}$. For $n = 1$, eq. (2.18) can be simplified as follows

$$\left\langle \left[\sum_x a_x \right] \left[\sum_y (a_y - i b_y) \right] \right\rangle'_{\text{SUSY}} = -\sum_{1 \leq x < y \leq N} I^{0,1}_{x,y} - \sum_{1 \leq y \leq x \leq N} I^{1,1}_{x,y}$$
$$= -2 \sum_{x < x'} I_{xx'} - I_{xx},$$

where for $x \leq x'$ $I_{xx'}$ was defined in (2.17) and we used

$$I^{0,1}_{xy} = I_{xy} \quad \text{and} \quad I^{1,1}_{xy} = I_{N-x+1, N-y+1}.$$

We want to reduce $\mathfrak{T}$ to an ordinary transfer operator, involving no Grassmann integration variables. This could be done by hand, i.e. by expanding in the Grassmann variables and selecting top degree coefficients. In this paper, we will instead exploit a supersymmetry that morally states that $\mathfrak{T}$ commutes with the superrotations of the supermatrix R. This will yield a purely bosonic (Grassmann free) representation of $\mathfrak{T}$ in the appropriately defined polar coordinates.

To make this intuition precise, let ξ, η be two new Grassmann generators (that may or may not depend on $\rho, \bar{\rho}$). Then the matrix

$$U_{\eta,\xi} := \exp \begin{pmatrix} 0 & \eta \\ \xi & 0 \end{pmatrix} = \begin{pmatrix} e^{\frac{1}{2}\eta\xi} & \eta \\ \xi & e^{-\frac{1}{2}\eta\xi} \end{pmatrix} \qquad (2.20)$$

is a "superunitary" rotation in the following sense: $U_{\eta,\xi}^{-1} = U_{-\eta,-\xi} = \exp\begin{pmatrix} 0 & -\eta \\ -\xi & 0 \end{pmatrix}$.
It is easy to see that $\text{Str } R^n = \text{Str}(U_{\eta\xi}^{-1} R U_{\eta\xi})^n$ and $\text{Sdet } R = \text{Sdet}(U_{\eta\xi}^{-1} R U_{\eta\xi})$ for
any $n \geq 1$.

Lemma 2.4. *The measure* dR *is invariant under superunitary rotations*: *for any Schwartz function* $F(R)$ *one has*

$$\int dR \, F(R) = \int dR \, F(U_{\eta\xi}^{-1} R U_{\eta\xi}) \quad \text{for all } \eta, \xi. \tag{2.21}$$

Proof. This is a special case of the general framework of Berezin integration [3] as applied to the conjugation supersymmetry at hand. We will prove it here by direct computation. The function F has a unique decomposition as

$$F(R) = F_0(a, b) + \rho F_1(a, b) + \bar{\rho} F_2(a, b) + \rho \bar{\rho} F_3(a, b).$$

Replacing in this formula

$$R' = U_{\eta\xi}^{-1} R U_{\eta\xi} = \begin{pmatrix} a + \mathfrak{n} & \bar{\rho} + \eta(a - ib) \\ \rho - \xi(a - ib) & ib + \mathfrak{n} \end{pmatrix}, \tag{2.22}$$

where $\mathfrak{n} := \eta\xi(a - ib) + (\bar{\rho}\xi + \rho\eta)$, one gets

$$F(R') = \tilde{F}_0 + (\rho - \xi(a - ib))\tilde{F}_1 + (\bar{\rho} + \eta(a - ib))\tilde{F}_2 + (\rho - \xi(a - ib))(\bar{\rho} + \eta(a - ib))\tilde{F}_3,$$

where we used $\mathfrak{n}^2 = -2\rho\bar{\rho}\eta\xi$ and $\mathfrak{n}^3 = 0$ and we defined

$$\tilde{F}_i := F_i(a + \mathfrak{n}, b - i\mathfrak{n}) = F_i(a, b) + \mathfrak{n}(\partial_a - i\partial_b)F_i(a, b) + \frac{\mathfrak{n}^2}{2}(\partial_a - i\partial_b)^2 F_i(a, b).$$

Note that

$$(\rho - \xi(a - ib))\mathfrak{n} = \rho\bar{\rho}\xi, \qquad (\rho - \xi(a - ib))\mathfrak{n}^2 = 0,$$
$$(\bar{\rho} + \eta(a - ib))\mathfrak{n} = -\rho\bar{\rho}\eta, \quad (\bar{\rho} + \eta(a - ib))\mathfrak{n}^2 = 0.$$

Then

$$\begin{aligned} F(R') - F(R) = \tilde{F}_0 - F_0 &+ (a - ib)[-\xi F_1 + \eta F_2] \\ &+ \rho\bar{\rho}(\partial_a - i\partial_b)[\xi F_1 - \eta F_2] \\ &+ (a - ib)[\bar{\rho}\xi + \rho\eta + \eta\xi(a - ib)]F_3 \\ &- \rho\bar{\rho}\eta\xi(a - ib)(\partial_a - i\partial_b)F_3. \end{aligned}$$

Performing now integration over $\rho, \bar{\rho}$ we obtain a sum of terms of the form $\partial_a F_i, \partial_b F_i, (\partial_a - i\partial_b)^2 F_i$ or $(a - ib)(\partial_a - i\partial_b)F_i$, hence the integral over a and b yields 0. This completes the proof. $\quad\square$

Polar decomposition. Every supermatrix $\begin{pmatrix} a & \bar{\rho} \\ \rho & ib \end{pmatrix}$ with $(a, b) \neq (0, 0)$ has a polar decomposition as follows

$$
\begin{aligned}
\begin{pmatrix} a & \bar{\rho} \\ \rho & ib \end{pmatrix} &= \left(\exp \begin{pmatrix} 0 & -\eta \\ -\xi & 0 \end{pmatrix} \right) \begin{pmatrix} \lambda_1 & 0 \\ 0 & i\lambda_2 \end{pmatrix} \left(\exp \begin{pmatrix} 0 & \eta \\ \xi & 0 \end{pmatrix} \right) \\
&= U_{\eta\xi}^{-1} \begin{pmatrix} \lambda_1 & 0 \\ 0 & i\lambda_2 \end{pmatrix} U_{\eta\xi}
\end{aligned}
\tag{2.23}
$$

with (cf. (2.22))

$$
\begin{aligned}
\lambda_1 &= a + \frac{\bar{\rho}\rho}{a - ib} & \lambda_2 &= b - i\frac{\bar{\rho}\rho}{a - ib}, \\
\eta &= \frac{\bar{\rho}}{a - ib} & \xi &= -\frac{\rho}{a - ib}.
\end{aligned}
\tag{2.24}
$$

We call $\lambda = (\lambda_1, \lambda_2)$ the *eigenvalues of R,* and (λ, η, ξ) the *polar coordinates for R*. Note that $\lambda_1 - i\lambda_2 = a - ib \neq 0$ for $(a, b) \neq (0, 0)$. Any function $F(R)$ invariant under superunitary rotations depends on the eigenvalues λ_1, λ_2 only. In this case we will write

$$
f(\lambda) := F(\mathrm{diag}(\lambda_1, i\lambda_2)) = F(R).
$$

In particular we have $\mathrm{Str}\, R^n = \lambda_1^n - (i\lambda_2)^n$ for all $n \geq 0$ and $\mathrm{Sdet}\, R = \frac{\lambda_1}{i\lambda_2}$. Hence the potential term in $\mathfrak{T}$ (2.15) becomes

$$
\begin{aligned}
e^{-V(R)} &= \frac{\overline{\mathcal{E}} - i\lambda_2}{\overline{\mathcal{E}} - \lambda_1} e^{-\frac{1}{2}(\lambda_1 + \varepsilon)^2 + (\lambda_2 - i\varepsilon)^2} \\
&= \frac{\overline{\mathcal{E}} - i\lambda_2}{\overline{\mathcal{E}} - \lambda_1} e^{-\frac{1}{2}(\lambda_1^2 + \lambda_2^2) - \varepsilon(\lambda_1 - i\lambda_2)} =: e^{-V(\lambda)}.
\end{aligned}
\tag{2.25}
$$

Note that λ_1, λ_2 are *even* elements of the Grassmann algebra. We will see below that inside an integral we can replace them by ordinary real variables.

Lemma 2.5. *The transfer operator $\mathfrak{T}$ preserves superunitary rotation invariance. Precisely, let $F(R)$ be such that all coefficients $F_I(a, b)$ are smooth and $|F_I(a, b)| \leq e^{K(a^2 + b^2)}$ with $K < \frac{W^2}{2}$. Assume F is invariant under superunitary rotations $F(R) = F(U_{\eta\xi}^{-1} R U_{\eta\xi})$ for all η, ξ. Then $\mathfrak{T}F$ is also invariant: for all η, ξ, $[\mathfrak{T}F](R) = [\mathfrak{T}F](U_{\eta\xi}^{-1} R U_{\eta\xi})$.*

Proof. We abbreviate $U_{\eta\xi}$ by U. We have

$$(\mathfrak{T}F)(U^{-1}RU) = e^{-V(U^{-1}RU)} \int d R' \, e^{-\frac{W^2}{2} \operatorname{Str}(U^{-1}RU - R')^2} F(R')$$

$$= e^{-V(R)} \int d R' \, e^{-\frac{W^2}{2} \operatorname{Str}(R - UR'U^{-1})^2} F(UR'U^{-1})$$

$$= e^{-V(R)} \int d R' \, e^{-\frac{W^2}{2} \operatorname{Str}(R - R')^2} F(R')$$

$$= (\mathfrak{T}F)(R).$$

where in the second line we used $V(U^{-1}RU) = V(R)$ (since both Sdet and Str are invariant) and $F(R') = F(UR'U^{-1})$ (since we assumed F is invariant). Finally in the last line we used (2.21) with U replaced by U^{-1}. □

Remark 2.6. This lemma implies that we can always replace R by the diagonal matrix $\operatorname{diag}(\lambda_1, i\lambda_2)$ when evaluating $(\mathfrak{T}F)(R)$, as long as F is invariant under superunitary rotations, and $\operatorname{Str} R = a - ib \neq 0$.

2.3. Some useful identities. For certain functions the transformed $\mathfrak{T}F$ can be explicitly computed. Let $z \in \mathbb{C}$, with $\Re(z) > 0$. We abbreviate

$$d\mu_z(R) := d R \, e^{-\frac{z}{2} \operatorname{Str} R^2} \tag{2.26}$$

the supergaussian measure with variance z^{-1}. By direct computation

$$\int d\mu_z(R) = \int d R e^{-\frac{z}{2} \operatorname{Str} R^2} = \int d a \, d b \, d \bar{\rho} \, d \rho e^{-\frac{z}{2}(a^2 + b^2 + 2\bar{\rho}\rho)} = 1. \tag{2.27}$$

Lemma 2.7. *Let $\mathbf{1}(R) := 1$ be the constant function, and $\Omega_\alpha(R) := e^{-W\alpha \operatorname{Str} R^2}$, the gaussian with $\alpha \in \mathbb{C}$, $\Re\alpha > 0$. Then*

$$(\mathfrak{T}\mathbf{1})(R) = e^{-V(R)} \tag{2.28}$$

and

$$(\mathfrak{T}\Omega_\alpha)(R) = e^{-V(R)}\Omega_{\alpha\mu}(R), \tag{2.29}$$

where

$$\mu := \frac{W}{W + 2\alpha}.$$

Moreover, for any function $F(R)$ such that F is invariant under superunitary rotations, $F(R) = f(\lambda)$, and f is a polynomial in λ one has

$$(\mathfrak{T}(\Omega_\alpha F))(R) = e^{-V(R)} \Omega_{\alpha\mu}(R) \int d\mu_{\frac{W^2}{\mu}}(R') F(R' + \mu R) \tag{2.30}$$

where we used the measure defined in (2.26) with $z = W^2/\mu$.

Remark 2.8. Since $\Re \alpha > 0$ we have $\Re \mu > 0$ and $\Re \alpha \mu > 0$.

Proof. The first identity above follows by translating all variables $R' \mapsto R' + R$ and applying (2.27). For the second identity we first complete the square, then perform the translation $R' \mapsto R' + \mu R$:

$$e^{V(R)} (\mathfrak{T}\Omega_\alpha)(R) = e^{-W\alpha\mu \operatorname{Str} R^2} \int d R' \, e^{-\frac{W_2}{2\mu} \operatorname{Str}(R' - \mu R)^2} = e^{-W\alpha\mu \operatorname{Str} R^2},$$

The second equality follows from (2.27), since $\frac{W}{\mu_\alpha} = W + 2\alpha$ has positive real part. Repeating the same arguments we get (2.30). □

The following identities will be useful later.

Lemma 2.9. *For any $z \in \mathbb{C}$, with $Re(z) > 0$ and supermatrix R the following identities hold:*

(a)
$$\int d\mu_z(R') \, [\operatorname{Str}(R' + R)]^n = (\operatorname{Str} R)^n, \quad \text{for all } n \geq 1,$$

(b)
$$\int d\mu_z(R') \, \operatorname{Str}(R' + R)^n = \operatorname{Str} R^n, \quad n = 2, 3,$$

(c)
$$\int d\mu_z(R') \, \operatorname{Str}(R' + R)^4 = \operatorname{Str} R^4 + \frac{2}{z}(\operatorname{Str} R)^2,$$

(d)
$$\int d\mu_z(R') \, [\operatorname{Str}(R' + R)^3]^2 = [\operatorname{Str}(R)^3]^2 + \frac{9}{z}\operatorname{Str} R^4 + \frac{9}{z^2}(\operatorname{Str} R)^2.$$

Proof. Without loss of generality we can assume $R = \operatorname{diag}(\lambda_1, i\lambda_2)$. Indeed by superunitary rotations any $R = \begin{pmatrix} a & \bar{\rho} \\ \rho & ib \end{pmatrix}$ with $(a, b) \neq (0, 0)$ can be reduced to a diagonal supermatrix. Finally the case $(a, b) = (0, 0)$ can be recovered from $(a, b) \neq (0, 0)$ by continuity remarking that both sides of the identities above are polynomials in the variables a and b.

(a) We have

$$[\operatorname{Str}(R' + R)]^n = [\operatorname{Str} R' + \operatorname{Str} R)]^n = \sum_{j=0}^{n} C_j^n (\operatorname{Str} R')^j (\operatorname{Str} R)^{n-j}.$$

By identity (2.38),

$$\int d\mu_z(R')(\operatorname{Str} R')^j = (\operatorname{Str} 0)^j = 0, \quad \text{for all } j > 0.$$

(b) For $n = 2$,

$$\int d\mu_z(R') \operatorname{Str}(R' + R)^2$$

$$= \operatorname{Str} R^2 + 2 \int d\mu_z(R') \operatorname{Str}(R'R) + \int d\mu_z(R') \operatorname{Str}(R')^2.$$

By (2.38)

$$\int d\mu_z(R') \operatorname{Str}(R')^2 = 0$$

and the remaining integral vanishes by parity under $R' \to -R'$. For $n = 3$,

$$\operatorname{Str}(R' + R)^3 = \operatorname{Str} R^3 + 3 \operatorname{Str}(R' R^2) + 3 \operatorname{Str} R(R')^2 + \operatorname{Str}(R')^3.$$

The second and fourth terms are odd under $R' \to -R'$, hence the corresponding integrals vanish. Note that

$$(R')^2 = \begin{pmatrix} a'^2 + \bar{\rho}'\rho' & (a' + ib')\bar{\rho}' \\ (a' + ib')\rho' & -b'^2 - \bar{\rho}'\rho' \end{pmatrix}.$$

Direct computation shows that the integral of each matrix component equals zero.

(c) Expanding,

$$\operatorname{Str}(R' + R)^4 = \operatorname{Str} R^4 + 4 \operatorname{Str} R'^3 R + 4 \operatorname{Str} R^3 R'$$
$$+ 2 \operatorname{Str}(RR')^2 + 4 \operatorname{Str} R^2 R'^2 + \operatorname{Str} R'^4.$$

The second and third term are odd, i.e. change sign under $R' \to -R'$, hence the corresponding integrals vanish. The integral of the first term vanish by (2.38). Since the integral of each matrix component in R'^2 equals zero also the fifth term disappears. It remains to consider $\operatorname{Str}(RR')^2$. Since R is assumed to be diagonal we have

$$\operatorname{Str}(RR')^2 = (a'\lambda_1)^2 + 2\bar{\rho}'\rho'\lambda_1 i\lambda_2 + (b'i\lambda_2)^2.$$

By direct computation,

$$\int d\mu_z(R') \operatorname{Str}(RR')^2 = \frac{1}{z}[\lambda_1^2 + (i\lambda_2)^2 - 2\lambda_1 i\lambda_2]$$
$$= \frac{[\lambda_1 - (i\lambda_2)]^2}{z}$$
$$= \frac{(\operatorname{Str} R)^2}{z}.$$

(d) Using (b),

$$\int d\mu_z(R')[\operatorname{Str}(R' + R)^3]^2 = [\operatorname{Str} R^3]^2 + \int d\mu_z(R')X,$$

where

$$X = [\mathrm{Str}(R' + R)^3 - \mathrm{Str}\, R^3]^2$$
$$= [3\,\mathrm{Str}(R' R^2) + 3\,\mathrm{Str}\, R(R')^2 + \mathrm{Str}(R')^3]^2$$
$$= [\mathrm{Str}(R')^3]^2 + 9[\mathrm{Str}(R' R^2)]^2 + 9[\mathrm{Str}\, R(R')^2]^2 + 6\,\mathrm{Str}(R')^3\,\mathrm{Str}(R' R^2)$$
$$+ 6\,\mathrm{Str}(R')^3\,\mathrm{Str}\, R(R')^2 + 18\,\mathrm{Str}(R' R^2)\,\mathrm{Str}\, R(R')^2.$$

The integral of the last two term vanish by parity while the integral of the first term vanishes by (2.38). Since R is diagonal,

$$\mathrm{Str}(R' R^2) = a'(\lambda_1)^2 - i b'(i\lambda_2)^2.$$

Hence

$$\int d\mu_z(R')[\mathrm{Str}\, R'(R)^2]^2 = \frac{1}{z}[(\lambda_1)^4 - (i\lambda_2)^4] = \frac{\mathrm{Str}\, R^4}{z}.$$

Similarly

$$\mathrm{Str}\, R(R')^2 = a'^2\lambda_1 + b'^2(i\lambda_2) + \bar\rho'\rho'(\lambda_1 + i\lambda_2).$$

Hence

$$\int d\mu_z(R')\,[\mathrm{Str}\, R(R')^2]^2 = \frac{(\lambda_1 - i\lambda_2)^2}{z^2} = \frac{(\mathrm{Str}\, R)^2}{z^2}.$$

Finally

$$\mathrm{Str}\, R'^3 = a'^3 - (i b')^3 + \bar\rho'\rho'3(a' + i b').$$

Direct computation yields

$$\int d\mu_z(R')\,\mathrm{Str}(R')^3\,\mathrm{Str}(R' R^2) = 0.$$

This concludes the proof. □

As a direct consequence of this lemma, we can compute exactly the action of $\mathfrak{T}F$ for a certain functions. This is done in the following corollary.

Corollary 2.10. *Let $\Re\alpha > 0$, and let that $\Omega_\alpha(R) = e^{-W\alpha\,\mathrm{Str}\, R^2}$, $\mu = W/(W + 2\alpha)$ as before. For any supermatrix R,*

$$\mathfrak{T}(\Omega_\alpha(R')[\mathrm{Str}\, R']^n)(R) = e^{-V(R)}\Omega_{\alpha\mu}(R)\,\mu^n[\mathrm{Str}\, R]^n \quad \text{for all } n \geq 1, \quad (2.31)$$

$$\mathfrak{T}(\Omega_\alpha(R')\,\mathrm{Str}\, R'^n)(R) = e^{-V(R)}\Omega_{\alpha\mu}(R)\mu^n\,\mathrm{Str}\, R^n \quad n = 2, 3, \quad (2.32)$$

$$\mathfrak{T}(\Omega_\alpha(R')[\mathrm{Str}\, R']^4)(R) = e^{-V(R)}\Omega_{\alpha\mu}(R)\Big[\mu^4\,\mathrm{Str}\, R^4 + \frac{2\mu^3}{W^2}(\mathrm{Str}\, R)^2\Big], \quad (2.33)$$

and

$$\mathfrak{T}(\Omega_\alpha(R')[\mathrm{Str}\, R'^3]^3)(R)$$
$$= e^{-V(R)}\Omega_{\alpha\mu}(R)\Big[\mu^6[\mathrm{Str}(R)^3]^2 + \frac{9\mu^5}{W^2}\,\mathrm{Str}\, R^4 + \frac{9\mu^4}{W^4}(\mathrm{Str}\, R)^2\Big]. \tag{2.34}$$

Proof. Combine identity (2.30) with Lemma 2.9. □

2.4. Transfer operator in polar coordinates.

In the following we will consider only functions $R \mapsto F(R)$ taking values in even elements of the Grassmann algebra and such that $F(R)$ contains no Grassmann generators except ξ, η (equivalently $\bar{\rho}$, ρ). If such F is invariant under superunitary rotations then the corresponding function f maps $\mathbb{R}^2$ to $\mathbb{R}$. Our goal is to write $\mathfrak{T}F$ as an operator $\mathcal{T}$ acting on f instead. This requires to change coordinates in the integral $(a, b, \bar{\rho}, \rho) \mapsto (\lambda_1, \lambda_2, \eta, \xi)$. The operation is well defined only for functions that vanish at the origin: if $F(R)$ is a "nice" function (we will make this precise below) with $F(0) = 0$, then by Berezin integration formula (cf. [3])

$$\int \mathrm{d}R\, F(R) = \int \mathrm{d}a\, \mathrm{d}b\, \mathrm{d}\bar{\rho}\, \mathrm{d}\rho F(R) = \int \mathrm{d}\lambda_1\, \mathrm{d}\lambda_2\, \mathrm{d}\xi\, \mathrm{d}\eta \frac{F(R(\lambda, \eta, \xi))}{\mathrm{Ber}},$$

where $\mathrm{Ber} = (\lambda_1 - i\lambda_2)^2$ is the so-called *Berezinian* (the super-analog of the Jacobi determinant), and $a, b, \lambda_1, \lambda_2$ are real integration variables. The function appearing in the integral (2.16) defining $\mathfrak{T}F$ is $e^{-\frac{W^2}{2}\mathrm{Str}(R-R')^2}F(R')$ which does not necessary vanish in 0. To solve the problem one can decompose the integral as follows:

$$e^{-\frac{W^2}{2}\mathrm{Str}(R-R')^2}F(R') = \widetilde{F}_1(R') + \widetilde{F}_2(R'),$$

where the functions $\widetilde{F}_1(R')$, $\widetilde{F}_2(R')$ must satisfy: (*a*) both functions are 'nice', in particular integrable, (*b*) $\widetilde{F}_1(0) = 0$ so that we can apply the change of coordinates and (*c*) the integral $\int \mathrm{d}R'\widetilde{F}_2(R')$ is easy to compute without any coordinate change. When F is invariant under superunitary rotations, the following two choices are especially convenient:

$$e^{-\frac{W^2}{2}\mathrm{Str}(R-R')^2}F(R') = e^{-\frac{W^2}{2}\mathrm{Str}(R-R')^2}[F(R') - F(0)] + e^{-\frac{W^2}{2}\mathrm{Str}(R-R')^2}F(0),$$

$$e^{-\frac{W^2}{2}\mathrm{Str}(R-R')^2}F(R')$$
$$= e^{-\frac{W^2}{2}\mathrm{Str}(R-R')^2}[1 - e^{-W^2\,\mathrm{Str}\, RR'}]F(R') + e^{-\frac{W^2}{2}\mathrm{Str}(R^2+R'^2)}F(R').$$

In both cases the function $\widetilde{F}_1$ vanishes at 0. Moreover the function $\widetilde{F}_2$ can be evaluated exactly

$$\int \mathrm{d}R'\, e^{-\frac{W^2}{2}\mathrm{Str}(R-R')^2}F(0) = \int \mathrm{d}R'\, e^{-\frac{W^2}{2}\mathrm{Str}\, R'^2}F(R') = F(0), \tag{2.35}$$

where the first integral can be performed directly as the one in (2.28), while the second integral is a consequence of the rotation symmetry (cf. Remark 2.13 below). We will see below these choices translate in two equivalent representations of the operator $\mathfrak{T}$ acting on f. We first need some notation. For $f \quad \mathbb{R}^2 \to \mathbb{R}$ we define

$$(\delta_0 f)(\lambda) := f(0), \tag{2.36a}$$

$$(\delta_{W^2}^\star f)(\lambda) := \frac{W^2}{2\pi} \int\limits_{\mathbb{R}^2} e^{-\frac{1}{2}W^2(\lambda'-\lambda)^2} f(\lambda') \, d\lambda'_1 \, d\lambda'_2, \tag{2.36b}$$

where

$$(\lambda' - \lambda)^2 = (\lambda_1 - \lambda'_1)^2 + (\lambda_2 - \lambda'_2)^2$$

coincides with $\mathrm{Str}(R'-R)^2$ when both matrices are diagonal. Moreover we denote by Λ the multiplication by

$$\Lambda = \lambda_1 - i\lambda_2 = \mathrm{Str}\, R$$

(from the Ber term in (2.4)) and by $\bar{\Lambda}$ the multiplication by $\bar{\Lambda} = \lambda_1 + i\lambda_2$ so that

$$(e^{-\bar{\Lambda}\Lambda} f)(\lambda) = e^{-\lambda^2} f(\lambda).$$

Finally we denote by e^{-V} — the multiplication by $e^{-V(\lambda)}$ (cf. (2.25))

Proposition 2.11. *Assume $R \mapsto F(R)$ takes values in even elements of the Grassmann algebra and $F(R)$ contains no Grassmann generators except ξ, η (equivalently $\bar{\rho}, \rho$, since $\eta = \bar{\rho}(a - ib)^{-1}$ and $\xi = -\rho(a - ib)^{-1}$, cf. (2.24)). Assume further that F is invariant under superunitary rotations $F(R) = f(\lambda)$ so that*

$$f \in C^1(\mathbb{R}^2), \quad |f(\lambda)| \leq \exp(K|\lambda^2|),$$

where $K < W^2/2$. Then $\mathfrak{T}F$ is also rotation invariant. Moreover, $\mathfrak{T}$ can be represented as an operator acting on $C^1(\mathbb{R}^2)$ $(\mathfrak{T}F)(R) = (\mathfrak{T}f)(\lambda)$, with

$$\mathfrak{T} := e^{-V} e^{-\frac{W^2}{2}\bar{\Lambda}\Lambda} \delta_0 + T, \tag{2.37a}$$

$$T := e^{-V} \Lambda \delta_{W^2}^\star \Lambda^{-1} = e^{-V} \delta_0 + e^{-V} T(\mathrm{id} - \delta_0). \tag{2.37b}$$

Remark 2.12. Note that $(\mathfrak{T}f)(0) = f(0)$, thus if f vanishes at the origin, so does $\mathfrak{T}f$. For f vanishing at the origin we have then $\mathfrak{T}f = Tf$.

Remark 2.13. A direct consequence of this result is the following localization identity:

$$\int d R' e^{-\frac{W^2}{2}\mathrm{Str}\, R'^2} F(R') = (\mathfrak{T}F)(0) = (\mathfrak{T}f)(0) = f(0) = F(0). \tag{2.38}$$

The above decomposition appears already in [12] and is a special case of the general framework of Berezin integration [3] applied to the conjugation supersymmetry at hand. Generalization to more complicated supersymmetries has been an important theoretical tool in condensed matter physics. All such results are proven by an inspired sequence of elementary applications of Stokes' theorem. We give a proof of the present simple case for the convenience of the reader.

Proof. Let F be as above. Then setting

$$\hat{\lambda} = \text{diag}(\lambda_1, i\lambda_2)$$

we get from Lemma 2.5

$$
\begin{aligned}
(\mathfrak{T}F)(R) = (\mathfrak{T}F)(\hat{\lambda}) \\
&= e^{-V(\lambda)} \int \mathrm{d}R' \, e^{-\frac{W^2}{2} \text{Str}(\hat{\lambda}-R')^2} F(R') \\
&= e^{-V(\lambda)} \int \mathrm{d}R' \, e^{-\frac{W^2}{2} \text{Str}(\hat{\lambda}-R')^2} f(\lambda_1', \lambda_2') \\
&= e^{-V(\lambda)} \int \mathrm{d}R' \, e^{-\frac{W^2}{2} \text{Str}(\hat{\lambda}-R')^2} f\left(a' + \frac{\bar{\rho}'\rho'}{a'-ib'}, b' - i\frac{\bar{\rho}'\rho'}{a'-ib'}\right) \\
&= e^{-V(\lambda)} \int \mathrm{d}R' \, e^{-\frac{W^2}{2} \text{Str}(\hat{\lambda}-R')^2} f\left(a' + \frac{\bar{\rho}'\rho'}{a'-ib'}, b' - i\frac{\bar{\rho}'\rho'}{a'-ib'}\right).
\end{aligned}
$$

The arguments of f are even elements of the Grassmann algebra, hence

$$f\left(a' + \frac{\bar{\rho}'\rho'}{a'-ib'}, b' - i\frac{\bar{\rho}'\rho'}{a'-ib'}\right) = f(a', b') + \frac{\bar{\rho}'\rho'}{a'-ib'}(\partial_{a'} - i\partial_{b'})f(a', b').$$

Moreover

$$
\begin{aligned}
e^{-\frac{W^2}{2}\text{Str}(\hat{\lambda}-R')^2} &= e^{-\frac{W^2}{2}((a'-\lambda_1)^2+(b'-\lambda_2)^2+2\bar{\rho}'\rho')} \\
&= e^{-\frac{W^2}{2}((a'-\lambda_1)^2+(b'-\lambda_2)^2)}(1 - W^2\bar{\rho}'\rho').
\end{aligned}
$$

Inserting all this in the integral we obtain

$$
\begin{aligned}
e^{V(\lambda)}(\mathfrak{T}F)(\hat{\lambda}) \\
&= W^2 \int \frac{\mathrm{d}a' \, \mathrm{d}b'}{2\pi} e^{-\frac{W^2}{2}((a'-\lambda_1)^2+(b'-\lambda_2)^2)} f(a', b') \\
&\quad - \int \frac{\mathrm{d}a' \, \mathrm{d}b'}{2\pi} e^{-\frac{W^2}{2}((a'-\lambda_1)^2+(b'-\lambda_2)^2)} \frac{1}{a'-ib'}(\partial_{a'} - i\partial_{b'})f(a', b') \\
&= I_1 - I_2
\end{aligned}
$$

The second integral can be reorganized as follows:

$$
\begin{aligned}
I_2 &= \int \frac{\mathrm{d}a'\,\mathrm{d}b'}{2\pi} \frac{1}{a'-ib'}(\partial_{a'}-i\partial_{b'})[e^{-\frac{W^2}{2}((a'-\lambda_1)^2+(b'-\lambda_2)^2)}f(a',b')] \\
&\quad + W^2 \int \frac{\mathrm{d}a'\,\mathrm{d}b'}{2\pi} \frac{1}{a'-ib'}[(a'-ib')-(\lambda_1-i\lambda_2)] \\
&\qquad\qquad\qquad\qquad \cdot [e^{-\frac{W^2}{2}((a'-\lambda_1)^2+(b'-\lambda_2)^2)}f(a',b')] \\
&= -f(0,0)e^{-\frac{W^2}{2}(\lambda_1^2+\lambda_2^2)} + I_1 - (\Lambda\delta^\star_{W^2}\Lambda^{-1}f)(\lambda).
\end{aligned}
$$

In the first line of the previous chain of equalities we used Stokes theorem in the form of the Cauchy–Pompeiu [= Cauchy–Green] formula

$$
\int_{|\xi|\leq r} \frac{da\,db}{a-ib}2(\bar\partial g)(a,b) = -2\pi g(0) + \oint_{|\xi|=r} \frac{d\zeta}{\zeta}g(\zeta,\bar\zeta), \tag{2.39}
$$

where

$$
\zeta = a+ib, \quad \bar\partial = \frac{\partial_a - i\partial_b}{2}.
$$

The first term on the right hand side vanishes in the limit $r \to \infty$ as long as our test function $f(\lambda)$ does not increase too fast. This concludes the proof of the first representation for $\mathfrak{T}$. The second one follows from

$$
\begin{aligned}
I_1 &= f(0,0) + W^2 \int \frac{\mathrm{d}a'\,\mathrm{d}b'}{2\pi} e^{-\frac{W^2}{2}((a'-\lambda_1)^2+(b'-\lambda_2)^2)}[f(a',b')-f(0,0)] \\
&= f(0,0) + \tilde{I}_1
\end{aligned}
$$

and

$$
(\partial_{a'}-i\partial_{b'})f(a',b') = (\partial_{a'}-i\partial_{b'})[f(a',b')-f(0,0)]. \qquad\square
$$

Recall that $G[H_N](E_\varepsilon) := (E+i\varepsilon-H)^{-1}$ and that J_{xy} are the matrix elements of $J = -W^2\Delta_N + \mathbb{1}$, and set

$$
\begin{aligned}
&I^{m,n_1,\ldots,n_q}_{x,y_1,\ldots,y_q} \\
&\quad := \int \frac{\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2}{2\pi}[\mathfrak{T}^{x-y_m}\Lambda^{n_m}\cdots\mathfrak{T}^{y_2-y_1}\Lambda^{n_1}\mathfrak{T}^{y_1-1}e^{-V}](\lambda) \\
&\qquad\quad \cdot [e^{V(\lambda)}\Lambda^{-1}][\mathfrak{T}^{y_{m+1}-x}\Lambda^{n_{m+1}}\mathfrak{T}^{y_{m+2}-y_{m+1}}\cdots\Lambda^{n_q}\mathfrak{T}^{N-y_q}e^{-V}](\lambda)
\end{aligned}
\tag{2.40}
$$

where q, m, n_i, x, y_i are defined as in (2.19). Also set, for $x \leq x'$,

$$
I_{xx'} := \frac{1}{2\pi} \int \frac{\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2}{\lambda_1-i\lambda_2} e^{V(\lambda)}[\mathfrak{T}^{x-1}e^{-V}](\lambda)[\mathfrak{T}^{x'-x}\Lambda\mathfrak{T}^{N-x'}e^{-V}](\lambda).
$$

and for $x' < x$ set $I_{xx'} = I_{N-x+1,N-x'+1}$. As we see from the proof below, these definitions are consistent with (2.19) and (2.17).

Corollary 2.14. *For* $y, y' = 1, \cdots, N$, *we have the following representation*:

$$\lim_{\varepsilon \searrow 0} \langle G_{yy}[H_N](E_\varepsilon) \rangle$$
$$= \mathcal{E} + \sum_{x=1}^{N} J_{yx} \frac{1}{2\pi} \int \frac{d\lambda_1 \, d\lambda_2}{\lambda_1 - i\lambda_2} e^{V(\lambda)} [\mathcal{J}^{x-1} e^{-V}](\lambda) [\mathcal{J}^{N-x} e^{-V}](\lambda), \tag{2.41}$$

$$\lim_{\varepsilon \searrow 0} \langle G_{yy'}[H_N](E_\varepsilon) G_{y'y}[H_N](E_\varepsilon) \rangle = -J_{yy'} + \sum_{xx'=1}^{N} J_{yx} J_{y'x'} I_{xx'}. \tag{2.42}$$

For $n > 1$,

$$\lim_{\varepsilon \searrow 0} \partial_E^n \langle \operatorname{Tr} G[H_N](E_\varepsilon) \rangle$$
$$= (-1)^n \sum_{q=1}^{n} \sum_{\substack{n_1,\ldots n_q \geq 1 \\ n_1 + \cdots + n_q = n}} \frac{n!}{n_1! \cdots n_q!} \sum_{m=0}^{n} \sum_{\substack{1 \leq y_1 < y_2 < \cdots < y_q \leq N \\ y_m \leq x < y_{m+1}}} I_{x, y_1, \ldots, y_q}^{m, n_1, \ldots, n_q}. \tag{2.43}$$

Proof. From Corollary 2.2 and relations (2.17) the proof of the first two identities is reduced to study

$$I_i := \int da \, db \, d\bar{\rho} \, d\rho \, af_i(\lambda), \quad i = 1, 2,$$

with

$$f_1(\lambda) := e^{V(\lambda)} [\mathcal{J}^{x-1} e^{-V}](\lambda) [\mathcal{J}^{N-x} e^{-V}](\lambda),$$

$$f_2(\lambda) := e^{V(\lambda)} [\mathcal{J}^{x-1} e^{-V}](\lambda) [\mathcal{J}^{x'-x} \Lambda \mathcal{J}^{N-x'} e^{-V}](\lambda).$$

Then

$$I_i := \int da \, db \, d\bar{\rho} \, d\rho \, af_i \left(a + \frac{\bar{\rho}\rho}{a - ib}, b - i\frac{\bar{\rho}\rho}{a - ib} \right)$$
$$= \int da \, db \, d\bar{\rho} \, d\rho \, a \left[f_i(a, b) + \frac{\bar{\rho}\rho}{a - ib} (\partial_a - i\partial_b) f_i(a, b) \right]$$
$$= -\int \frac{da \, db}{2\pi} a \frac{1}{a - ib} (\partial_a - i\partial_b) f_i(a, b)$$
$$= \int \frac{da \, db}{2\pi} \frac{1}{a - ib} f_i(a, b)$$

where in the last line we applied again Stokes theorem (2.39) and we used $af_i(a, b) = 0$ for $a = b = 0$. The proof for the third identity is done in the same way. □

3. Analysis of the transfer operator

So far we treated $\mathfrak{T}$ and $\mathcal{T}$ as formal operations. To apply spectral-theoretic methods, recall that for f vanishing at the origin, $\mathcal{T}f = Tf$, and hence

$$\mathcal{T}^n f = T^n f = \Lambda (e^{-V} \delta^\star_{W^2})^n \Lambda^{-1} f = \Lambda e^{-\frac{1}{2}V} (e^{-\frac{1}{2}V} \delta^\star_{W^2} e^{-\frac{1}{2}V})^n e^{\frac{1}{2}V} \Lambda^{-1} f. \tag{3.1}$$

It is safe to use expressions such as $e^{-\frac{V}{2}}$ and V, since e^{-V} does not vanish except if $E = 0$ at $\lambda_2 = 1$ (cf.(2.15)). We will show in Sect. 4.4 below this problem is easy to deal with. For f sufficiently regular (but not necessarily vanishing at zero) the computation of $\mathcal{T}^n f$ will be reduced to the following two ingredients.

i. We show in Section 3.1 that the operator $e^{-\frac{1}{2}V} \delta^\star_{W^2} e^{-\frac{1}{2}V}$ is bounded in $L_p(\mathbb{R}^2)$ for any $p \in [1, \infty]$, and spectral theory yields a good bound on its norm for $p \in (1, \infty)$.

ii. In Section 3.2, we construct a solution u of the eigenvalue equation $\mathcal{T}u = u$ satifying in addition $u(0) = 1$. Unlike Section 3.1, the argument relies on explicit elementary computations rather than on methods of operator theory, and the term "eigenfunction" is used without specifying the construction of the operator.[4]

We recover then $\mathcal{T}^n f$ from (i) and (ii) via the decomposition

$$f = f(0)u + [f - f(0)u]$$

as follows:

$$\mathcal{T}^n f = f(0)u + \Lambda e^{-\frac{1}{2}V} (e^{-\frac{1}{2}V} \delta^\star_{W^2} e^{-\frac{1}{2}V})^n e^{\frac{1}{2}V} \Lambda^{-1} (f - f(0)u).$$

Note that $u(0) = 1$ ensures $(f - f(0)u)(0) = 0$. The results of this section mostly pertain to the case $|E| \le \sqrt{\frac{32}{9}}$. The extension to all $|E| < 2$ is discussed in Section 5.

3.1. Operator norm bound. Denote by $\| \cdot \|_p$ the operator norm of an operator from $L_p(\mathbb{R}^2) = \{ f : \mathbb{R}^2 \to \mathbb{C} \mid \|f\|_p < \infty \}$ to itself.

We recall for further use that, for an integral operator with kernel K,

$$\|K\|_\infty = \sup_\lambda \int d\lambda'_1 d\lambda'_2 |K(\lambda, \lambda')|, \quad \|K\|_1 = \sup_{\lambda'} \int d\lambda_1 d\lambda_2 |K(\lambda, \lambda')|, \tag{3.2}$$

---

[4] A proper operator-theoretic meaning can be given by constructing the operator in an appropriate weighted Hilbert space; this, however, is not required for our argument.

and that for all $1 < p < \infty$

$$\|K\|_p \leq \|K\|_1^{\frac{1}{p}} \|K\|_\infty^{1-\frac{1}{p}}. \tag{3.3}$$

The combination of the relations (3.2) and (3.3) is known as Schur's bound. It immediately follows from (2.36) and (3.3) that $\|\delta_{W^2}^\star\|_p \leq 1$ for any $p \in [1, \infty]$. It is also easy to check that, for [5] $|E| \leq \sqrt{\frac{32}{9}}$, $\Re V(\lambda)$ has non-degenerate global minima at $\lambda = 0$, and $\lambda = \left(0, 2\sqrt{1 - \frac{E^2}{4}}\right)$, with value 0. In particular, $\Re V \geq 0$, so that

$$\|e^{-V}\|_p \leq 1 \quad \text{for } |E| \leq \sqrt{\frac{32}{9}}, \; p \in [1, \infty], \tag{3.4}$$

where $\|e^{-V}\|_p$ means the operator norm. In particular,

$$\|\delta_{W^2}^\star e^{-V}\|_p, \|e^{-V}\delta_{W^2}^\star\|_p, \|e^{-V/2}\delta_{W^2}^\star e^{-V/2}\|_p \leq 1 \quad \text{for } |E| \leq \sqrt{\frac{32}{9}}, \; p \in [1, \infty]. \tag{3.5}$$

The following standard semiclassical argument improves these bounds for $1 < p < \infty$.

Proposition 3.1. *For any $p \in (1, \infty)$ there exists $c > 0$ such that*

$$\|e^{-\frac{1}{2}V}\delta_{W^2}^\star e^{-\frac{1}{2}V}\|_p \leq 1 - \frac{c}{W} \quad \text{for } |E| \leq \sqrt{\frac{32}{9}}.$$

Remark 3.2. The estimate fails for $p = 1, \infty$: indeed,

$$\|e^{-\frac{1}{2}V}\delta_{W^2}^\star e^{-\frac{1}{2}V}\|_1 = \|e^{-\frac{1}{2}V}\delta_{W^2}^\star e^{-\frac{1}{2}V}\|_\infty$$
$$= \sup_\lambda \int d\lambda_1' d\lambda_2' e^{-\frac{1}{2}\Re V(\lambda)} \frac{W^2}{2\pi} e^{-\frac{1}{2}W^2(\lambda-\lambda')^2} e^{-\frac{1}{2}\Re V(\lambda')}$$
$$= 1 - \mathcal{O}(W^{-2}).$$

Remark 3.3. The proposition implies that

$$\|(e^{-V}\delta_{W^2}^\star)^k\|_p, \|(\delta_{W^2}^\star e^{-V})^k\|_p \leq (1 - c/W)^{k-1} \quad \text{for } p \in (1, \infty).$$

Proof. We prove the estimate for $p = 2$, in which case this is a standard semiclassical argument (see [27]), which we repeat for the convenience of the reader. The general case follows from the case $p = 2$ and (3.5) by Riesz–Thorin interpolation.

---

[5] At $E = \sqrt{\frac{32}{9}}$, $\Re V$ develops two new local minima. The value at these minima is positive until $|E| = 1.893 \cdots > \sqrt{\frac{32}{9}} = 1.885 \ldots$ (no analytical expression available), and the results of [34] as well as the argument of this section actually hold until this threshold.

It suffices to show that $\|e^{-\frac{1}{2}\Re V}\delta^{\star}_{W^2}e^{-\frac{1}{2}\Re V}\|_2 \leq 1 - \frac{c}{W}$. Let χ_1, χ_2 be smooth bump functions in a sufficiently small (but W-independent) neighbourhood of the minima 0, $\left(0, 2\sqrt{1 - \frac{E^2}{4}}\right)$ of $\Re V$, and let χ_3 be so that $\chi_1^2 + \chi_2^2 + \chi_3^2 = 1$. We have

$$e^{-\frac{1}{2}\Re V}\delta^{\star}_{W^2}e^{-\frac{1}{2}\Re V} = \sum_{i=1,2,3}\chi_i e^{-\frac{1}{2}\Re V}\delta^{\star}_{W^2}e^{-\frac{1}{2}\Re V}\chi_i + e^{-\frac{1}{2}\Re V}Ae^{-\frac{1}{2}\Re V},$$

where A has the kernel $\frac{1}{2}\delta^{\star}_{W^2}(\lambda, \lambda')\sum_i[\chi_i(\lambda) - \chi_i(\lambda')]^2$ and we used the relation

$$\chi_i(\lambda)^2 + \chi_i(\lambda')^2 = [\chi_i(\lambda) - \chi_i(\lambda')]^2 + 2\chi_i(\lambda)\chi_i(\lambda').$$

Therefore, using $\|e^{-\frac{1}{2}\Re V}\|_2 \leq 1$,

$$\|e^{-\frac{1}{2}\Re V}\delta^{\star}_{W^2}e^{-\frac{1}{2}\Re V}\|_2 \Bigg\|\sum_{i=1,2,3}\chi_i e^{-\frac{1}{2}\Re V}\delta^{\star}_{W^2}e^{-\frac{1}{2}\Re V}\chi_i\Bigg\|_2 + \|A\|_2.$$

From

$$\sum_i[\chi_i(\lambda) - \chi_i(\lambda')]^2 = (\lambda - \lambda')^2\tilde{\chi}(\lambda, \lambda')$$

with bounded $\tilde{\chi}$, it is easy to show with the Schur bound (3.2) (3.3) that $\|A\|_2 = \mathcal{O}(W^{-2})$. Further,

$$\Bigg\|\sum_{i=1,2,3}\chi_i e^{-\frac{1}{2}\Re V}\delta^{\star}_{W^2}e^{-\frac{1}{2}\Re V}\chi_i\Bigg\|_2$$

$$= \sup_{\|\phi\|_2=1}\sum_{i=1,2,3}(\chi_i\phi, \mathbb{1}_{\operatorname{supp}\chi_i}e^{-\frac{1}{2}\Re V}\delta^{\star}_{W^2}e^{-\frac{1}{2}\Re V}\mathbb{1}_{\operatorname{supp}\chi_i}\chi_i\phi)$$

$$\leq \max_i\|\mathbb{1}_{\operatorname{supp}\chi_i}e^{-\frac{1}{2}\Re V}\delta^{\star}_{W^2}e^{-\frac{1}{2}\Re V}\mathbb{1}_{\operatorname{supp}\chi_i}\|_2 \sup_{\|\phi\|_2=1}\sum_{i=1,2,3}\|\chi_i\phi\|_2^2$$

$$= \max_i\|\mathbb{1}_{\operatorname{supp}\chi_i}e^{-\frac{1}{2}\Re V}\delta^{\star}_{W^2}e^{-\frac{1}{2}\Re V}\mathbb{1}_{\operatorname{supp}\chi_i}\|_2.$$

For $i = 3$, since $\Re V|_{\operatorname{supp}\chi_3} > 0$,

$$\|\mathbb{1}_{\operatorname{supp}\chi_3}e^{-\frac{1}{2}\Re V}\delta^{\star}_{W^2}e^{-\frac{1}{2}\Re V}\mathbb{1}_{\operatorname{supp}\chi_3}\|_2 \leq c < 1,$$

uniformly in W. For $i = 1, 2$, since the minima of $\Re V$ are non-degenerate,

$$\|\mathbb{1}_{\operatorname{supp}\chi_i}e^{-\frac{1}{2}\Re V}\delta^{\star}_{W^2}e^{-\frac{1}{2}\Re V}\mathbb{1}_{\operatorname{supp}\chi_i}\|_2 \leq \|e^{-c'\lambda^2}\delta^{\star}_{W^2}e^{-c'\lambda^2}\|_2$$

for some $c' > 0$. For self adjoint operators, the L^2 operator norm is equal to the spectral radius, and the spectrum of the harmonic Kac operator $e^{-c'\lambda^2}\delta^{\star}_{W^2}e^{-c'\lambda^2}$ can be computed explicitly, giving $\|e^{-c'\lambda^2}\delta^{\star}_{W^2}e^{-c'\lambda^2}\|_2 \leq 1 - \frac{c''}{W}$. The claim follows for $p = 2$. $\qquad\square$

The following corollary will be a key ingredient in the proof of our theorems.

Corollary 3.4. *Let $f \in L_p$ with $1 < p < \infty$ be such that $\Lambda^{-1} f \in L_p$. Then we have for all $n \geq 1$, $m \geq 0$*

$$\|\Lambda^m T^n f\|_p \leq C^{m+1} \sqrt{(m+1)!} \left(1 - \frac{c}{W}\right)^{n-1} \|\Lambda^{-1} f\|_p, \tag{3.6a}$$

$$\|\Lambda^{-1} T^n f\|_p \leq \left(1 - \frac{c}{W}\right)^{n-1} \|\Lambda^{-1} f\|_p. \tag{3.6b}$$

Proof. For the first bound we use

$$\begin{aligned}
\|\Lambda^m T^n f\|_p &= \|\Lambda^{m+1} e^{-\frac{1}{2}V} (e^{-\frac{1}{2}V} \delta_{W^2}^\star e^{-\frac{1}{2}V})^{n-1} e^{-\frac{1}{2}V} \delta_{W^2}^\star \Lambda^{-1} f\|_p \\
&\leq \|\Lambda^{m+1} e^{-\frac{1}{2}V}\|_\infty \|(e^{-\frac{1}{2}V} \delta_{W^2}^\star e^{-\frac{1}{2}V})^{n-1} e^{-\frac{1}{2}V} \delta_{W^2}^\star \Lambda^{-1} f\|_p \\
&\leq \|\Lambda^{m+1} e^{-\frac{1}{2}V}\|_\infty \left(1 - \frac{c}{W}\right)^{n-1} \|\Lambda^{-1} f\|_p,
\end{aligned} \tag{3.7}$$

where in the first line we applied equation (3.1) and in the second line of (3.5) and Proposition 3.1. For the second bound we use

$$\begin{aligned}
\|\Lambda^{-1} T^n f\|_p &= \|e^{-\frac{1}{2}V} (e^{-\frac{1}{2}V} \delta_{W^2}^\star e^{-\frac{1}{2}V})^{n-1} e^{-\frac{1}{2}V} \delta_{W^2}^\star \Lambda^{-1} f\|_p \\
&\leq \|e^{-\frac{1}{2}V}\|_\infty \|(e^{-\frac{1}{2}V} \delta_{W^2}^\star e^{-\frac{1}{2}V})^{n-1} e^{-\frac{1}{2}V} \delta_{W^2}^\star \Lambda^{-1} f\|_p \\
&\leq \left(1 - \frac{c}{W}\right)^{n-1} \|\Lambda^{-1} f\|_p.
\end{aligned} \tag{3.8}$$

The bound $\|\Lambda^{m+1} e^{-V}\|_\infty \leq C^{m+1} \sqrt{(m+1)!}$ follows from the quadratic growth of $\Re V$ at infinity. $\square$

We conclude this subsection with a few additional properties of the T operator that will prove important later.

Lemma 3.5. *The following holds.*

i. *Let $p \in (1, \infty)$, $n \in \mathbb{N}$ and $1 \leq q \leq p$ such that f and $\Lambda^n f \in L_p$ and $\Lambda^{-1} f \in L_q$. Then $Tf \in C^\infty(\mathbb{R}^2)$ and there exists a constant $C = C_{p,q} > 0$ such that*

$$\|e^V \Lambda^n T f\|_p \leq C^{n+1} [\|\Lambda^n f\|_p + \sqrt{(n+1)!} W^{-1-n} W^{\frac{2(p-q)}{pq}} \|\Lambda^{-1} f\|_q]. \tag{3.9}$$

ii. *Let $n \in \mathbb{N}$ and $f \in L_1 \cap L_\infty$ be a C^1 function such that $\Lambda^n f \in L_1 \cap L_\infty$. Then $\mathfrak{T} f$ is smooth, and for all $p \in (1, \infty)$ $e^V \mathfrak{T} f \in L_p$, and satisfies the bound*

$$\begin{aligned}
&\|e^V \Lambda^n \mathfrak{T} f\|_p \\
&\leq C^{n+1} [\|\Lambda^n f\|_p + \sqrt{(n+1)!} \{W^{1-\frac{2}{p}-n} \|\Lambda^{-1} f\|_1 + W^{-\frac{2}{p}-n} |f(0)|\}].
\end{aligned}$$

Proof. Recall that $Tf = e^{-V}\Lambda\delta^{\star}_{W^2}\Lambda^{-1}f$ where $\delta^{\star}_{W^2}$ is the convolution with the heat kernel in dimension $d = 2$ at time $t = \frac{1}{2W^2}$, and hence $e^V\Lambda^n Tf = \Lambda^{n+1}\delta^{\star}_{W^2}\frac{1}{\Lambda}f$. Smoothness follows from the regularizing effect of the heat kernel.

(i) We drag the multiplier Λ^{n+1} through $\delta^{\star}_{W^2}$, as follows. The decomposition $|\Lambda^{n+1}| \le 2^n(|\Lambda - \Lambda'|^{n+1} + |\Lambda'|^{n+1})$ yields

$$
\begin{aligned}
2^{-n}&|e^V\Lambda^n Tf(\lambda)| \\
&= 2^{-n}|\Lambda^{n+1}\delta^{\star}_{W^2}\Lambda^{-1}f(\lambda)| \\
&\le (\delta^{\star}_{W^2}|\Lambda^n f|)(\lambda) + \frac{W^2}{2\pi}\int d\lambda'_1 d\lambda'_2 e^{-\frac{W^2}{2}|\Lambda-\Lambda'|^2}|\Lambda - \Lambda'|^{n+1}|\Lambda'|^{-1}|f(\lambda')| \\
&\le (\delta^{\star}_{W^2}|\Lambda^n f|)(\lambda) + C^{n+1}\sqrt{(n+1)!}\,W^{-1-n}\delta^{\star}_{W^2/2}(|\Lambda^{-1}f|)(\lambda),
\end{aligned}
$$

where in the last term we extracted a fraction of the exponential decay to bound the factor $|\Lambda - \Lambda'|$. The result (i) now follows from

$$
\|\delta^{\star}_{W^2}\phi\|_p \le C^{p-q}\Big(\frac{W^2}{2\pi}\Big)^{\frac{p-q}{pq}}\|\phi\|_q, \quad \text{for all } p \in (1,\infty),\ 1 \le q \le p. \tag{3.10}
$$

(ii) From Definition (2.37),

$$
\|e^V\Lambda^n \mathcal{T}f\|_p \le \|\Lambda^n e^{-\frac{W^2}{2}\lambda^2}\|_p |f(0)| + \|e^V\Lambda^n Tf\|_p.
$$

The first term is bounded by

$$
\|\Lambda^n e^{-\frac{W^2}{2}\lambda^2}\|_p |f(0)| \le C^{n+1}\sqrt{(n+1)!}W^{-n-\frac{2}{p}}|f(0)|.
$$

The result now follows from (3.9) setting $q = 1$. The condition $\Lambda^{-1}f \in L_1$ is ensured by $f \in C_1 \cap L_1$ and $f(0) = 1$. Finally $f, \Lambda^n f \in L_1 \cap L_\infty$ ensures f and $\Lambda^n f \in L_p$ for all $p \in (1,\infty)$. $\qquad\square$

Remark 3.6. Note that while the function e^{-V} is bounded, the function e^V develops a singularity of the form $1/(\lambda_2 - 1)$ at $\lambda_2 = 1$ for $E = 0$. The lemma above shows this causes no problem, as long as the function appears together with the operator T.

3.2. The top eigenfunction. Using the bound of the previous paragraph, we will construct a solution u of the equation $\mathcal{T}u = u$ normalized to $u(0) = 1$. We call this solution an *eigenfunction* (with eigenvalue 1), without specifying the operator-theoretic construction.

The strategy is as follows. First, we guess an approximate eigenfunction u_0 such that $\mathfrak{T}u_0 \approx u_0$. This is done below in Proposition 3.8. Then we upgrade it to the true eigenfunction

$$u = u_0 + \sum_{n=0}^{\infty} \mathfrak{T}^n(\mathfrak{T}u_0 - u_0). \tag{3.11}$$

The next proposition ensures that this procedure is justified, and yields an eigenfunction that is close to u_0.

Proposition 3.7. *Let $u_0 \in L_1 \cap L_\infty$ be a C^1 function such that $u_0(0) = 1$. We define*

$$v := \mathfrak{T}u_0 - u_0. \tag{3.12}$$

Then for all $p \in (1, \infty)$ it holds $v \in L_p$ and $\Lambda^{-1}v \in L_p$. Moreover, the series

$$u = u_0 + \sum_{n=0}^{\infty} \mathfrak{T}^n v \tag{3.13}$$

converges in L_p to a solution u of the equation $\mathfrak{T}u = u$ which satisfies

$$\|u - u_0\|_p \leq \|v\|_p + \mathcal{O}(W)\|\Lambda^{-1}v\|_p, \tag{3.14a}$$

$$\|\Lambda^{-1}(u - u_0)\|_p \leq \mathcal{O}(W)\|\Lambda^{-1}v\|_p, \tag{3.14b}$$

and

$$\|e^V \Lambda^n (u - \mathfrak{T}u_0)\|_p \leq C^{n+1}\Big(\|\Lambda^n(u - u_0)\|_p + \frac{\sqrt{(n+1)!}}{W^{1+n}}\|\Lambda^{-1}(u - u_0)\|_p\Big), \tag{3.15a}$$

$$\|\Lambda^n(u - \mathfrak{T}u_0)\|_p \leq C^{n+1}\sqrt{(n+1)!}\|\Lambda^{-1}(u - u_0)\|_p. \tag{3.15b}$$

The limit function u is independent of the initial choice for u_0.

Proof. Inserting the definition (2.37) of $\mathfrak{T}$ and using $u_0(0) = 1$ we have

$$v = \mathfrak{T}u_0 - u_0 = e^{-V - \frac{W^2}{2}\lambda^2} - u_0 + Tu_0.$$

From $u_0 \in C^1 \cap L_1$ and $u_0(0) = 1$ follows $\Lambda^{-1}u_0 \in L_1$ and

$$\|\delta^{\star}_{W^2}\Lambda^{-1}u_0\|_p \leq \Big(\frac{W^2}{2\pi}\Big)^{\frac{p-1}{p}}\|\Lambda^{-1}u_0\|_1, \quad \text{for all } p \in (1, \infty), \tag{3.16}$$

whence $Tu_0 \in L_p$ and $\Lambda^{-1}Tu_0 \in L_p$. Moreover $Tu_0 \in C^\infty$, hence v is continuous and $v \in L_p$. Finally $u_0 \in C^1 \cap L_1$ and $e^{-V(0)} = 1 = u_0(0)$ yields $\Lambda^{-1}[e^{-V-\frac{W^2}{2}\lambda^2} - u_0] \in L_p$, hence $\|\Lambda^{-1}v\|_p < \infty$. Since $(\mathfrak{T}u_0)(0) = u_0(0)$ we have $v(0) = 0$ hence $\mathfrak{T}v = Tv$ and for all $n \geq 1$ it holds

$$\|\mathfrak{T}^n v\|_p = \|T^n v\|_p \leq \|\Lambda e^{-\frac{1}{2}V}\|_\infty \Big(1 - \frac{c}{W}\Big)^{n-1} \|\Lambda^{-1}v\|_p, \tag{3.17}$$

where we applied equation (3.1) and estimate (3.6). This inequality implies convergence of the geometric series and the relations (3.14). Since $(\mathfrak{T}^n v)(0) = v(0) = 0$ we have $u(0) = 1$ and $u(\lambda) = \lim_{N \to \infty} \mathfrak{T}^{N+1}u_0$, hence $\mathfrak{T}u = u$. Finally, for any two initial functions $u_0, \tilde{u}_0$, $(u_0 - \tilde{u}_0)(0) = 0$ implies $\mathfrak{T}^n(u_0 - \tilde{u}_0) = T^n(u_0 - \tilde{u}_0)$.

The estimates 3.15) follow from (3.9) together with the relation

$$u = \mathfrak{T}u_0 + T(u - u_0), \tag{3.18}$$

which is obtained from (3.11).

The independence of the limit of u_0 follows from (3.17) again. $\qquad\square$

3.2.1. Approximate eigenfunction. We need to make a good choice for u_0 in order for $u - u_0$ to be relatively small. Since we expect u to be also eigenfunction of the supersymmetric operator $\mathfrak{T}$, we take as initial ansatz some function $U_0(R)$ invariant under superunitary rotations hence $U_0(R) = u_0(\lambda)$. Guided from equation (2.29) and Corollary 2.10 and in analogy with semiclassical analysis we take as ansatzs

$$U_0^{(0)}(R) := e^{-\alpha W \operatorname{Str} R^2},$$

$$U_0^{(M)}(R) := e^{-\alpha W \operatorname{Str} R^2}\Big(1 + \sum_{j=1}^M W Q^{(j)}(R)\Big), \quad M \geq 1,$$

where $Q^{(j)}$ is a polynomial in R of degree j, consisting of sums of products of supertraces, which we assume invariant under superunitary rotations $Q^{(j)}(R) = q^{(j)}(\lambda)$. Finally $\alpha \in \mathbb{C}$ is some constant. Then from (2.30)

$$(\mathfrak{T}(U_0^{(M)}))(R) = e^{-V(R)}e^{-\alpha\mu W \operatorname{Str} R^2}\Big(1 + \sum_{j=1}^M W \tilde{Q}^{(j)}(R)\Big)$$

$$= e^{-\alpha W \operatorname{Str} R^2}\Big[e^{\alpha(1-\mu)W \operatorname{Str} R^2 - V(R)}\Big(1 + \sum_{j=1}^M W \tilde{Q}^{(j)}(R)\Big)\Big],$$

where

$$\tilde{Q}^{(j)}(R) = \int d\mu_{\frac{W^2}{\mu}}(R')Q^{(j)}(R' + \mu R).$$

We introduce the rotation invariant error function

$$\begin{aligned}
\mathrm{Err}^{(M)}(R) &:= e^{\alpha W \,\mathrm{Str}\, R^2}[\mathfrak{T}(U_0^{(M)}) - U_0^{(M)}](R) \\
&= \sum_{j=1}^{M} W(\tilde{Q}^{(j)}(R) - Q^{(j)}(R)) \\
&\quad + [e^{\alpha(1-\mu)W \,\mathrm{Str}\, R^2 - V(R)} - 1]\Big(1 + \sum_{j=1}^{M} W\tilde{Q}^{(j)}(R)\Big).
\end{aligned} \tag{3.19}$$

By abuse of notation we will use the same letter $\mathrm{Err}^{(M)}(\cdot)$ to denote the function of R and the function of λ. Then

$$v^{(M)}(\lambda) = \mathfrak{T}u_0^{(M)} - u_0^{(M)}(\lambda) = e^{-\alpha W \lambda^2} \,\mathrm{Err}^{(M)}(\lambda). \tag{3.20}$$

Hence

$$\|v^{(M)}\|_p^p = \int d\lambda |v^{(M)}(\lambda)|^p = \frac{1}{W} \int d\lambda |e^{-\alpha\lambda^2} \,\mathrm{Err}^{(M)}(\lambda/\sqrt{W})|^p.$$

The precision of our approximation is therefore determined by the leading order in $W^{-\frac{1}{2}}$ from $\mathrm{Err}(R/\sqrt{W})$. The following proposition shows we can make this error function at least as small as $O(W^{-3})$.

Proposition 3.8. *Let α be the solution of the equation $\alpha^2 = \frac{1}{4}(1 - \mathcal{E}^2)$ with $\Re\alpha > 0$. Set $Q^{(1)} = Q^{(2)} = 0$ and*

$$Q^{(3)}(R) := c_3 \,\mathrm{Str}\, R^3,$$

$$Q^{(4)}(R) := c_4^0 \,\mathrm{Str}\, R^4 + c_4^1 W^{-1} \,\mathrm{Str}\, R^2 + c_4^2 W^{-1}(\mathrm{Str}\, R)^2 + c_4^3 W(\mathrm{Str}\, R^3)^2,$$

$$\begin{aligned}
Q^{(5)}(R) := {}& c_5^0 \,\mathrm{Str}\, R^5 + c_5^1 \,\mathrm{Str}\, R^3 \,\mathrm{Str}\, R^2 + c_5^2 \,\mathrm{Str}\, R^3(\mathrm{Str}\, R)^2 \\
& + c_5^3 W^{-1} \,\mathrm{Str}\, R \,\mathrm{Str}\, R^2 + c_5^4 W^{-1} \,\mathrm{Str}\, R^3 + c_5^5 W^{-2} \,\mathrm{Str}\, R \\
& + c_5^6 W \,\mathrm{Str}\, R^3 \,\mathrm{Str}\, R^4 + c_5^7 W^2(\mathrm{Str}\, R^3)^3,
\end{aligned}$$

$$\tag{3.21}$$

where $c_3, c_4^j, j = 0, \ldots, 3$, and $c_5^j, j = 0, \ldots, 7$, form the solution of the following

system of equations:

$$3(2\alpha)c_3 = \frac{\mathcal{E}^3}{3}, \qquad 4(2\alpha)c_4^0 - 9c_4^3 = \frac{\mathcal{E}^4}{4}, \qquad c_4^1 = -\alpha^2,$$

$$(2\alpha)c_4^2 - c_4^0 = 0, \qquad 6(2\alpha)c_4^3 = \frac{\mathcal{E}^3}{3}c_3 \qquad 5(2\alpha)c_5^0 = \frac{\mathcal{E}^5}{5} + 12c_5^6,$$

$$5(2\alpha)c_5^1 = \frac{\mathcal{E}^3}{3}c_4^1, \qquad 5(2\alpha)c_5^2 - 2c_5^6 = \frac{\mathcal{E}^3}{3}c_4^2, \quad 3(2\alpha)c_5^3 - 5c_5^0 - 6c_5^1 = 0,$$

$$(2\alpha)c_5^4 - 2c_5^2 = 2(2\alpha)^2 c_3, \qquad (2\alpha)c_5^5 - 2c_5^3 = 0, \qquad 7(2\alpha)c_5^6 - 3^3 c_5^7 = \frac{\mathcal{E}^3}{3}c_4^0,$$

$$9(2\alpha)c_5^7 = \frac{\mathcal{E}^3}{3}c_4^3.$$

$$(3.22)$$

Then for all $p \in [1, \infty]$ it holds

$$\|v^{(0)}\|_p = \mathcal{O}(W^{-\frac{1}{p}}W^{-\frac{3}{2}}), \qquad \|\Lambda^{-1}v^{(0)}\|_p = \mathcal{O}(W^{-\frac{1}{p}}W^{-1}),$$

$$\|v^{(M)}\|_p = \mathcal{O}(W^{-\frac{1}{p}}W^{-\frac{M+1}{2}}), \quad \|\Lambda^{-1}v^{(M)}\|_p = \mathcal{O}(W^{-\frac{1}{p}}W^{-\frac{M}{2}}), \quad M = 3, 4, 5.$$

$$(3.23)$$

Proof. It is more convenient to study the error term in R coordinates. After rescaling $R \to W^{-\frac{1}{2}}R$ the polynomials Q^3, Q^4 can be written as

$$Q^{(3)}(W^{-\frac{1}{2}}R) = \frac{1}{\sqrt{W}^3}P_3(R),$$

$$Q^{(4)}(W^{-\frac{1}{2}}R) = \frac{1}{\sqrt{W}^4}P_4(R),$$

$$Q^{(5)}(W^{-\frac{1}{2}}R) = \frac{1}{\sqrt{W}^5}P_5(R),$$

where we defined

$$P_3(R) := c_3 \operatorname{Str} R^3,$$

$$P_4(R) := c_4^1 \operatorname{Str} R^4 + c_4^2 \operatorname{Str} R^2 + c_4^3(\operatorname{Str} R)^2 + c_4^4(\operatorname{Str} R^3)^2,$$

$$P_5(R) := c_5^0 \operatorname{Str} R^5 + c_5^1 \operatorname{Str} R^3 \operatorname{Str} R^2 + c_5^2 \operatorname{Str} R^3(\operatorname{Str} R)^2 + c_5^3 \operatorname{Str} R \operatorname{Str} R^2$$
$$+ c_5^4 \operatorname{Str} R^3 + c_5^5 \operatorname{Str} R + c_5^6 \operatorname{Str} R^3 \operatorname{Str} R^4 + c_5^7(\operatorname{Str} R^3)^3.$$

The W prefactors ensure that all terms in P_4 contribute to the same order $\sqrt{W}^{-4}$ and all terms in P_5 contribute to the same order $\sqrt{W}^{-5}$. It follows from Corollary 2.10 and the additional formulas in the appendix

$$\tilde{Q}^{(M)}(W^{-\frac{1}{2}}R) = \frac{1}{\sqrt{W}^M}\tilde{P}_M(R), \quad M = 3, 4, 5,$$

where

$$\widetilde{P}_3(R) = P_3(\mu R) = \mu^3 c_3 \operatorname{Str} R^3,$$

$$\widetilde{P}_4(R) = P_4(\mu R) + \frac{1}{W}[2c_4^1 \mu^3 (\operatorname{Str} R)^2 + 9c_4^4 \mu^5 \operatorname{Str} R^4] + \mathcal{O}(W^{-3}),$$

$$\begin{aligned}
\widetilde{P}_5(R) = P_5(\mu R) + \frac{1}{W}[&5c_5^0 \operatorname{Str} R \operatorname{Str} R^2 + 6c_5^1 \operatorname{Str} R \operatorname{Str} R^2 + 6c_5^2 \operatorname{Str} R^3 \\
&+ 2c_5^3 \operatorname{Str} R + c_5^6[2(\operatorname{Str} R)^2 \operatorname{Str} R^3 + 12 \operatorname{Str} R^5] \\
&+ c_5^7 3^3 \operatorname{Str} R^3 \operatorname{Str} R^4] \\
&+ \mathcal{O}(W^{-2}).
\end{aligned}$$

On the other hand

$$\begin{aligned}
V(R) &= \frac{1 - \mathcal{E}^2}{2} \operatorname{Str} R^2 - \sum_{q \ge 3} \frac{\mathcal{E}^q}{q!} \operatorname{Str} R^q \\
&= \frac{1 - \mathcal{E}^2}{2}(\lambda_1^2 + \lambda_2^2) - \sum_{q \ge 3} \frac{\mathcal{E}^q}{q!}(\lambda_1^q - (i\lambda_2)^q),
\end{aligned} \tag{3.24}$$

where the sum is absolutely convergent for $|\lambda|$ small. Moreover

$$\mu = \left(1 + \frac{2\alpha}{W}\right)^{-1} = \sum_{n \ge 0} (-2\alpha)^n W^{-n},$$

$$W(1 - \mu) = 2\alpha\mu = (2\alpha) - (2\alpha)^2 W^{-1} + \mathcal{O}(W^{-2}).$$

Rescaling $R \to W^{-\frac{1}{2}}R$ and setting $2\alpha^2 = \frac{1}{2}(1 - \mathcal{E}^2)$ we obtain

$$\begin{aligned}
[\alpha W(1 - \mu)&W^{-1} \operatorname{Str} R^2 - V(RW^{-\frac{1}{2}})] \\
&= \frac{1}{W^{\frac{3}{2}}} V_3(R) + \frac{1}{W^{\frac{4}{2}}} V_4(R) + \frac{1}{W^{\frac{5}{2}}} V_5(R) + \mathcal{O}(W^{-3}),
\end{aligned}$$

where

$$V_3(R) := \frac{\mathcal{E}^3}{3} \operatorname{Str} R^3,$$

$$V_4(R) := \frac{\mathcal{E}^4}{4} \operatorname{Str} R^4 - 4\alpha^3 \operatorname{Str} R^2,$$

$$V_5(R) := \frac{\mathcal{E}^5}{5} \operatorname{Str} R^5.$$

Hence,

$$\begin{aligned}
e^{\alpha(1-\mu) \operatorname{Str} R^2 - V(RW^{-\frac{1}{2}})} \\
= 1 + W^{-\frac{3}{2}} V_3(R) + W^{-\frac{4}{2}} V_4(R) + W^{-\frac{5}{2}} V_5(R) + \mathcal{O}(W^{-3}).
\end{aligned}$$

Inserting all this in the rescaled error function yields for $M = 0$

$$\text{Err}^{(0)}(RW^{-\frac{1}{2}}) = W^{-\frac{3}{2}}V_3(R) + \mathcal{O}(W^{-2}) = \mathcal{O}(W^{-\frac{3}{2}}). \qquad (3.25)$$

Setting $6\alpha c_3 = \frac{\mathcal{E}^3}{3}$ we obtain for $M = 3$

$$\begin{aligned}
\text{Err}^{(3)}(RW^{-\frac{1}{2}}) &= W[\tilde{Q}^{(3)}(RW^{-\frac{1}{2}}) - Q^{(3)}(RW^{-\frac{1}{2}})] \\
&\quad + [1 + W\tilde{Q}^{(3)}(RW^{-\frac{1}{2}})][W^{-\frac{3}{2}}V_3(R) + \mathcal{O}(W^{-2})] \\
&= \frac{1}{\sqrt{W}}(\mu^3 - 1)P_3(R) + \frac{1}{\sqrt{W}^3}V_3(R) + \mathcal{O}(W^{-2}) \qquad (3.26) \\
&= W^{-\frac{3}{2}}(-6\alpha P_3(R) + V_3(R)) + \mathcal{O}(W^{-2}) \\
&= \mathcal{O}(W^{-2}).
\end{aligned}$$

For $M = 4$, inserting the values (3.22) we get

$$\begin{aligned}
&\text{Err}^{(4)}(RW^{-\frac{1}{2}}) \\
&= W[\tilde{Q}^{(3)}(RW^{-\frac{1}{2}})Q^{(3)}(RW^{-\frac{1}{2}}) + \tilde{Q}^{(4)}(RW^{-\frac{1}{2}}) - Q^{(4)}(RW^{-\frac{1}{2}})] \\
&\quad + [1 + W(\tilde{Q}^{(3)}(RW^{-\frac{1}{2}}) + \tilde{Q}^{(4)}(RW^{-\frac{1}{2}}))] \\
&\qquad \cdot [W^{-\frac{3}{2}}V_3(R) + W^{-2}V_4(R) + \mathcal{O}(W^{-\frac{5}{2}})] \\
&= \frac{1}{W^2}\Big[\Big(-8c_4^1\alpha + 9c_4^4 + \frac{\mathcal{E}^4}{4}\Big)\,\text{Str}\,R^4 + (-4c_4^2\alpha - 4\alpha^3)\,\text{Str}\,R^2 \\
&\qquad + (-4c_4^3\alpha + 2c_4^1)(\text{Str}\,R)^2 + \Big(\frac{\mathcal{E}^3}{3}c_3 - 12\alpha c_4^4\Big)(\text{Str}\,R^3)^2\Big] \\
&\quad + \mathcal{O}(W^{-\frac{5}{2}}) \\
&= \mathcal{O}(W^{-\frac{5}{2}}).
\end{aligned}$$

Finally, the same arguments yield $\text{Err}^{(5)}(RW^{-\frac{1}{2}}) = \mathcal{O}(W^{-3})$. The result follows. $\qquad \square$

Remark 3.9. Note that Proposition 3.8 remains valid for all $|E| < 2$, since it only relies on the properties of the kernel in the vicinity of the origin. For the same reason, the conclusion remains valid if the kernel (or the contour) is deformed outside a vicinity of the origin. We shall use this in Section 5.

A first consequence of these results is the following Corollary.

Corollary 3.10. *Let* $u_0^{(0)}, u_0^{(3)}, u_0^{(4)},$ *and* $u_0^{(5)}$ *as above. Then for all* $p \in (1, \infty)$,

$$\|u - u_0^{(0)}\|_p \le \mathcal{O}(W^{-\frac{1}{p}-\frac{1}{2}}), \qquad \|\Lambda^{-1}(u - u_0^{(0)})\|_p \le \mathcal{O}(W^{-\frac{1}{p}})$$

for $M = 3, 4$,

$$\|u - u_0^{(M)}\|_p \leq \mathcal{O}(W^{-\frac{1}{p} - \frac{(M-1)}{2}}), \quad \|\Lambda^{-1}(u - u_0^{(M)})\|_p \leq \mathcal{O}(W^{-\frac{1}{p} - \frac{(M-2)}{2}}),$$

and

$$\|u - u_0^{(5)}\|_p \leq \mathcal{O}(W^{-\frac{1}{p} - \frac{3}{2}}), \qquad \|\Lambda^{-1}(u - u_0^{(5)})\|_p \leq \mathcal{O}(W^{-\frac{1}{p} - \frac{3}{2}}).$$

Moreover

$$\|u\|_p \leq \mathcal{O}(W^{-\frac{1}{p}}),$$

$$\|\Lambda^n u\|_p* \leq C^n \sqrt{n!} \mathcal{O}(W^{-\frac{1}{p} - \frac{1}{2}}) \qquad \text{for all } n \geq 1,$$

$$\|e^V \Lambda^n u\|_p \leq C^{n+1} \sqrt{n!} (\mathcal{O}(W^{-\frac{1}{p} - \frac{1}{2}}) + \mathcal{O}(W^{-\frac{1}{p} - \frac{n}{2}})) \quad \text{for all } n \geq 0.$$

Note that combining Proposition 3.8 with equations (3.14), we have

$$\|u - u_0^{(0)}\|_p \leq \mathcal{O}(W^{-\frac{1}{p}}), \qquad \|\Lambda^{-1}(u - u_0^{(0)})\|_p \leq \mathcal{O}(W^{-\frac{1}{p}}),$$

and, for $M = 3, 4, 5$,

$$\|u - u_0^{(M)}\|_p \leq \mathcal{O}(W^{-\frac{1}{p} - \frac{(M-2)}{2}}), \quad \|\Lambda^{-1}(u - u_0^{(M)})\|_p \leq \mathcal{O}(W^{-\frac{1}{p} - \frac{(M-2)}{2}}).$$

To improve this bounds, we need a longer argument.

Proof. First, note that

$$u_0^{(3)} - u_0^{(0)} = e^{-\alpha W \lambda^2} W q^{(3)}(\lambda),$$

and

$$u_0^{(M+1)} - u_0^{(M)} = e^{-\alpha W \lambda^2} W q^{(M+1)}(\lambda) \quad \text{for } M = 3, 4.$$

Then $\|u - u_0^{(0)}\|_p \leq \|u - u_0^{(3)}\|_p + \|u_0^{(3)} - u_0^{(0)}\|_p \leq \mathcal{O}(W^{-\frac{1}{p} - \frac{1}{2}})$. Similar arguments show that the norms for the cases $M = 3, 4$ are also improved by a factor $\frac{1}{2}$. This argument yields no improvement on $\|\Lambda^{-1}(u - u_0^{(M)})\|_p$, because the Λ^{-1} term generates an additional factor $W^{\frac{1}{2}}$. To prove the last inequalities note that $\|u_0^{(M)}\|_p = \mathcal{O}(W^{-\frac{1}{p}})$ (for, say, $M = 0$) yields $\|u\|_p = \mathcal{O}(W^{-\frac{1}{p}})$. Moreover inserting (3.18),

$$\begin{aligned}
\|\Lambda^n u\|_p &\leq \|\Lambda^n \mathcal{T} u_0^{(3)}\|_p + \|\Lambda^n (u - \mathcal{T} u_0^{(3)})\|_p \\
&\leq \|\Lambda^n \mathcal{T} u_0^{(3)}\|_p + C^{n+1} \sqrt{n!} \|\Lambda^{-1}(u - u_0)\|_p \\
&\leq C^{n+1} \sqrt{n!} (\mathcal{O}(W^{-\frac{1}{p} - \frac{n}{2}}) + \mathcal{O}(W^{-\frac{1}{p} - \frac{1}{2}})) \\
&= C^{n+1} \sqrt{n!} \mathcal{O}(W^{-\frac{1}{p} - \frac{1}{2}}),
\end{aligned}$$

where in the first line we used (3.15) and in the last line we used the explicit expression $\mathfrak{T}u_0^{(3)} = e^{-V(\lambda)}e^{-\alpha W\mu\lambda^2}(1+W\mu^3 q^{(3)}(\lambda))$, together with the constraint $n \geq 1$. The same argument yields the bound on $\|e^V \Lambda^n u\|_p$. $\qquad\qquad \square$

Remark 3.11. In the rest of the paper we will mostly use $u_0^{(0)}$, $u_0^{(3)}$ and $u_0^{(5)}$. While the latter gives a better approximation of u, the first two are easier to deal with. This last feature is particularly useful in some parts of the proof.

A key ingredient of our proofs will be the comparison of the function e^{-V}, where the operator $\mathfrak{T}^n$ applies, with the exact eigenfunction u. This is done in the following lemma.

Lemma 3.12. *Let $u_0^{(M)}$ as above and u be the solution of $\mathfrak{T}u = u$ constructed from $u_0^{(M)}$ via (3.11). For $p \in (1, \infty)$, it holds*

$$\|(e^{-V} - u)\|_p \leq C, \tag{3.27a}$$

$$\|\Lambda^{-1}(e^{-V} - u)\|_p \leq \begin{cases} CW^{\frac{1}{2}-\frac{1}{p}} & \text{for } p > 2, \\ C \ln W & \text{for } p = 2, \\ C & \text{for } p < 2. \end{cases} \tag{3.27b}$$

Proof. The first bound follows directly from $\|e^{-V}\|_p \leq \mathcal{O}(1)$ and Remark 3.11. It suffices to prove the second bound replacing u by $u_0^{(0)}$. Indeed by Corollary 3.10 $\|\Lambda^{-1}(u - u_0^{(0)})\|_p \leq \mathcal{O}(W^{\frac{1}{2}-\frac{1}{p}})$ where $W^{\frac{1}{2}-\frac{1}{p}} \leq \ln W$ for $p = 2$ and $W^{\frac{1}{2}-\frac{1}{p}} \leq 1$ for $p > 2$. By similar arguments we can replace e^{-V} by $e^{-\frac{1}{2}\lambda^2}$ in the estimate. We have

$$\|\Lambda^{-1}(e^{-\frac{1}{2}\lambda^2} - e^{-W\alpha\lambda^2})\|_p^p \leq C \int_0^\infty t^{1-p}|e^{-\frac{t^2}{2}} - e^{-W\alpha t^2}|^p dt.$$

We use different bounds in the various integration regions. For $t \leq W^{-\frac{1}{2}}$ we have

$$\int_0^{\frac{1}{\sqrt{W}}} t^{1-p}|e^{-\frac{t^2}{2}} - e^{-W\alpha t^2}|^p dt \leq CW^p \int_0^{\frac{1}{\sqrt{W}}} t^{p+1} = \mathcal{O}(W^{\frac{p}{2}-1}).$$

In all other regions we estimate $e^{-\frac{t^2}{2}}$ and $e^{-W\alpha t^2}$ separately. Direct computation give

$$\int_{\frac{1}{\sqrt{W}}}^\infty t^{1-p}e^{-Wp\Re\alpha\, t^2} dt = CW^{\frac{p}{2}-1} \quad \text{and} \quad \int_1^\infty t^{1-p}e^{-\frac{1}{2}pt^2} dt = \mathcal{O}(1).$$

Finally

$$
\int_{\frac{1}{\sqrt{W}}}^{1} t^{1-p} e^{-\frac{1}{2}pt^2} dt \le
\begin{cases}
C & \text{for } p < 2, \\
C \ln W & \text{for } p = 2, \\
C W^{\frac{p}{2}-1} & \text{for } p > 2.
\end{cases}
$$

This concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4. Proof of Theorems 1 and 2 away from the edges

Throughout this section we assume that $|E| \le \sqrt{\frac{32}{9}}$. This assumption will be relaxed in Section 5. For technical reasons we will initially assume also $E \ne 0$. We will explain at the end of this section how to deal with $E = 0$.

4.1. Preliminary results.
Recalling the definition of $\rho(E)$ and $\rho_N(E)$ (1.5), the relevant quantities to study are

$$
\lim_{\varepsilon \searrow 0} \frac{1}{N} \langle \operatorname{Tr} G[H_N](E_\varepsilon) \rangle \quad \text{and} \quad \lim_{\varepsilon \searrow 0} \langle G_{00}[H](E_\varepsilon) \rangle,
$$

where H_N is the finite-volume operator, whereas H is the operator in infinite volume. From (2.41) we have

$$
\lim_{\varepsilon \searrow 0} \frac{1}{N} \langle \operatorname{Tr} G[H_N](E_\varepsilon) \rangle - \mathcal{E} = \frac{1}{N} \sum_{y=1}^{N} I_N(y), \tag{4.1}
$$

where

$$
I_N(y) = \frac{1}{2\pi} \int \frac{d\lambda_1 \, d\lambda_2}{\lambda_1 - i\lambda_2} e^{V(\lambda)} [\mathcal{T}^{y-1} e^{-V}](\lambda) [\mathcal{T}^{N-y} e^{-V}](\lambda).
$$

For the individual diagonal matrix elements of the finite volume resolvent, we have a similar expression

$$
\begin{aligned}
&\lim_{\varepsilon \searrow 0} \langle G_{yy}[H_N](E_\varepsilon) \rangle - \mathcal{E} \\
&= -W^2 (I_N(y-1) + I_N(y+1)) + (2W^2 + 1) I_N(y), \quad y \ne 1, N.
\end{aligned} \tag{4.2}
$$

Recall that u is the top eigenfunction constructed in Sect. 3.2 above, i.e. $\mathcal{T}u = u$ and $u(0) = 1$. Inserting the decomposition $e^{-V} = u + (e^{-V} - u)$ we get for $k \ge 1$ (cf. (3.1))

$$
\mathcal{T}^k e^{-V} = u + \mathcal{T}^k[e^{-V} - u] = u + T^k(e^{-V} - u)
$$

where we used $(e^{-V} - u)(0) = 0$. Therefore,

$$I_N(y) = I_1 + I_2(y-1) + I_2(N-y) + I_3(y-1),$$

where

$$I_1 := \frac{1}{2\pi} \int d\lambda_1 \, d\lambda_2 [\Lambda^{-1} e^V] u^2, \tag{4.3a}$$

$$I_2(k) := \frac{1}{2\pi} \int d\lambda_1 \, d\lambda_2 [\Lambda^{-1} e^V] u [T^{k-1}(e^{-V} - u)], \quad 1 \le k \le N, \tag{4.3b}$$

$$I_3(k) := \frac{1}{2\pi} \int d\lambda_1 \, d\lambda_2 [\Lambda^{-1} e^V][T^{k-1}(e^{-V} - u)][T^{N-k}(e^{-V} - u)]. \tag{4.3c}$$

Note that I_1 is independent of N and y. The following proposition estimates the decay of I_2 and I_3 and is a key ingredient for the proof of our results. The estimate we obtain here for I_2 is not optimal. We will prove a stronger bound in Proposition 4.3 below.

Proposition 4.1. *Let I_2, I_3 as above. Then for all $k \ge 1$*

$$|I_2(k)| \le C W^{-\frac{1}{2}} (1 - c/W)^{k-1}, \tag{4.4a}$$

$$|I_3(k)| \le C(\ln W)^2 (1 - c/W)^{N-4}. \tag{4.4b}$$

Proof. For $k \ge 2$ we write

$$I_2(k) = \int \frac{d\lambda_1 \, d\lambda_2}{2\pi} [\delta_{W^2}^\star u](\lambda) [\Lambda^{-1} T^{k-2}(e^{-V} - u)(\lambda)].$$

Inserting absolute values and setting $\frac{1}{q} + \frac{1}{p} = 1$, $p > 2$ yields

$$\begin{aligned}
2\pi |I_2(k)| &\le \|\delta_{W^2}^\star u\|_q \|\Lambda^{-1} T^{k-2}(e^{-V} - u)\|_p \\
&\le \|u\|_q \|\Lambda^{-1}(e^{-V} - u)\|_p \left(1 - \frac{c}{W}\right)^{\max\{0,k-3\}} \\
&\le C W^{-\frac{1}{q}} W^{\frac{1}{2} - \frac{1}{p}} \left(1 - \frac{c}{W}\right)^{k-2} \\
&= C W^{-\frac{1}{2}} \left(1 - \frac{c}{W}\right)^{k-2},
\end{aligned}$$

where in the first line we applied eq. 3.6, while in the second line we used (3.27) for $p > 2$ and Corollary 3.10. For $k = 1$, using again (3.27) for $p > 2$ and

Corollary 3.10, the bound reduces to

$$2\pi |I_2(1)| = \left| \int d\lambda_1 \, d\lambda_2 u e^V \Lambda^{-1}(e^{-V} - u) \right|$$
$$\leq \|u e^V\|_q \|\Lambda^{-1}(e^{-V} - u)\|_p$$
$$\leq \mathcal{O}(W^{-\frac{1}{2}}).$$

It remains to prove the decay of $I_3(k)$. Since $N \gg 1$, it holds $k - 1 > 2$ or $N - k > 2$. We assume without loss of generality $k - 1 > 2$. Then applying (3.6) and (3.27) for $p = 2$ we get

$$2\pi |I_3(k)| = \left| \int d\lambda_1 \, d\lambda_2 \delta^\star_{W^2} [\Lambda^{-1} T^{k-2}(e^{-V} - u)][T^{N-k}(e^{-V} - u)] \right|$$
$$\leq C \|\Lambda^{-1}(e^{-V} - u)\|_2 \|\Lambda^{-1}(e^{-V} - u)\|_2 \left(1 - \frac{c}{W}\right)^{N-4}$$
$$= \mathcal{O}((\ln W)^2)\left(1 - \frac{c}{W}\right)^{N-4}. \qquad \square$$

A first consequence of this estimates is the following representation for the infinite volume Green's function.

Proposition 4.2. *Let I_1, I_2, I_3 be as in* (4.3). *Then*

$$\lim_{\varepsilon \searrow 0} \langle G_{00}[H](E_\varepsilon) \rangle - \mathcal{E} = I_1, \tag{4.5a}$$

$$\lim_{\varepsilon \searrow 0} \left\{ \langle G_{00}[H](E_\varepsilon) \rangle - \frac{1}{N} \langle \mathrm{Tr}\, G[H_N](E_\varepsilon) \rangle \right\} = \frac{1}{N} \sum_{y=0}^{N-1} \{2 I_2(y) + I_3(y)\}. \tag{4.5b}$$

Proof. Let $y = y(N)$ so that $\min(y, N - y) \geq N^{0.01}$, and rewrite

$$\langle G_{00}[H](E_\varepsilon) \rangle = \langle G_{yy}[H](E_\varepsilon) \rangle$$
$$= \langle G_{yy}[H_N](E_\varepsilon) \rangle + \langle (G_{yy}[H](E_\varepsilon) - G_{yy}[H_N](E_\varepsilon)) \rangle.$$

For fixed ϵ, the second term vanishes in the limit $N \to \infty$. Indeed, it is equal to a sum of several boundary terms such as $\langle G_{y1}[H_N](E_\varepsilon) G_{1y}[H](E_\varepsilon) \rangle$. Each of these terms tends to zero: indeed, $|G_{1y}[H](E_\varepsilon)| \leq \epsilon^{-1}$, whereas $\langle |G_{y1}[H_N](E_\varepsilon)| \rangle$ tends to zero by (an appropriate version of) the Combes–Thomas bound (see e.g. [1]). Precisely let X_N denote the event $|(H_N)_{jk}| \leq K \, N^\alpha J_{jk}^{1/2}$, where $J_{jk} = (-W^2 \Delta_N + \mathrm{id})^{-1}$ is the covariance of the random matrix H_N and decays exponentially $J_{jk} \leq c_W e^{-|j-k|/W}$, $0 < \alpha < 1$ and $K \gg 1$ are some fixed

parameters. Then $\sup_x \sum_y |(H_n)_{xy}(e^{\delta(|x-y|)}-1)| \leq K'N^\alpha$ as long as $J_{jk}^{1/2}e^{\delta|j-k|}$ retains some exponential decay. By Combes–Thomas,

$$\langle |G_{y1}[H_N](E_\varepsilon)|\mathbf{1}_{X_N}\rangle \leq \epsilon^{-1}e^{-c|y-1|\epsilon/N^\alpha} \xrightarrow[N\to\infty]{} 0,$$

where $c > 0$ is some constant and we used $y/N^\alpha \to \infty$ as $N \to \infty$. To conclude we show that X_N^c has vanishing probability:

$$\begin{aligned}
\langle |G_{y1}[H_N](E_\varepsilon)|\mathbf{1}_{X_N^c}\rangle &\leq \epsilon^{-1}\mathbb{P}(X_N^c)\\
&\leq K\epsilon^{-1}\sum_{ij} e^{-cN^{2\alpha}}\\
&\leq K\epsilon^{-1}N^2 e^{-cN^{2\alpha}} \xrightarrow[N\to\infty]{} 0.
\end{aligned}$$

We obtain (recall $y = y(N)$)

$$\lim_{\varepsilon\searrow 0}\langle G_{00}[H](E_\varepsilon)\rangle = \lim_{\varepsilon\searrow 0}\lim_{N\to\infty}\langle G_{yy}[H_N](E_\varepsilon)\rangle = \lim_{N\to\infty}\lim_{\varepsilon\searrow 0}\langle G_{yy}[H_N](E_\varepsilon)\rangle$$

where in the last equality we can exchange limits since after translating to the saddle in the integral representation for $\langle |G_{y1}[H_N](E_\varepsilon)|\rangle$, all integrals are bounded uniformly in ϵ. The result now follows from representation (4.2) and estimates (4.4). □

Using this Proposition, (1.8) and (1.8) of Theorem 1 reduce to a study of I_1 and I_2, I_3, respectively. However, to obtain the error estimate (1.8) we will need an improved version of (4.4) for $I_2(k)$ that requires substantial more work. This will be done in Proposition 4.3 below. The bound on I_1 follows from the properties of u, in particular, its approximate symmetry under $\lambda \to -\lambda$. The argument is given in section 4.3 below.

4.2. Proof of Theorem 1

4.2.1. Convergence to ρ. Our goal is to prove $|\rho_N(E) - \rho(E)| \leq \frac{C'(E)}{N}$ for $|E| < \sqrt{32/9}$ (away from the edge) and $N \geq C(E)W\log W$. Recalling the definition of $\rho(E)$ and $\rho_N(E)$ (1.5), Proposition 4.2 implies

$$\rho_N(E) - \rho(E) = -\frac{1}{\pi}\Im\left[\frac{1}{N}\sum_{y=0}^{N-1}\{2I_2(y) + I_3(y)\}\right].$$

A direct application of (4.4) yields $|\rho_N(E) - \rho(E)| \leq \frac{C'(E)}{N}W^{\frac{1}{2}}$. To extract the correct W prefactor we need the following improved version of (4.4).

Proposition 4.3 (improved estimate on $I_2(k)$). *Let I_2 be as as above. Then*

$$|I_2(k)| \leq \mathcal{O}(W^{-1})(1 - c/W)^{k-4}, \quad k \geq 4. \tag{4.6}$$

Proof. Note that for any (regular enough) functions f, g it holds

$$\int d\lambda_1 \, d\lambda_2 g[T^n f] = \int d\lambda_1 \, d\lambda_2[T^n(\Lambda^2 e^{-V} g)][e^V \Lambda^{-2} f]. \tag{4.7}$$

Replacing $f = (e^{-V} - u)$ and $g = \Lambda^{-1} e^V u$ the integral $I_2(k)$ can be written as

$$\begin{aligned} 2\pi I_2(k) &= \int d\lambda_1 \, d\lambda_2[\Lambda^{-1} e^V u][T^{k-1}(e^{-V} - u)] \\ &= \int d\lambda_1 \, d\lambda_2[\Lambda^{-2} e^V (e^{-V} - u)][T^{k-1}(\Lambda u)]. \end{aligned} \tag{4.8}$$

The proof of the Proposition relies on two main ingredients:(*a*) Λu is an approximate eigenfunction of T with eigenvalue μ, i.e. $T(\Lambda u) \simeq \mu(\Lambda u)$ and (*b*) the contribution $I_2(0)$ from $k = 0$ is smaller than expected. More precisely

Proposition 4.4. *Let $u_0^{(0)}(\lambda) = e^{-\alpha W \lambda^2}$ and $u_0^{(3)}(\lambda) = e^{-\alpha W \lambda^2}(1 + W q^{(3)}(\lambda))$, as in Proposition* 3.8, *with $q^{(3)}(\lambda) = c_3[\lambda_1^3 - (i\lambda_2^3)]$. Recall that $\mu = \left(1 + \frac{2\alpha}{W}\right)^{-1}$ and the definition of $v^{(M)}$ in* (3.20). *Then*

$$T(\Lambda u_0^{(3)}) = \mu(\Lambda u_0^{(3)}) + \mu \Lambda \, \text{Rem}(\lambda) \tag{4.9}$$

where

$$\text{Rem} := 3c_3 \frac{\mu^3}{W} \bar{\Lambda}(u_0^{(0)} + v^{(0)}) + v^{(3)}.$$

Moreover,

$$\|\Lambda^{-1} e^V[T^n(\Lambda u_0^{(3)}) - \mu^n(\Lambda u_0^{(3)})]\|_p \leq C W^{-\frac{1}{2} - \frac{1}{p}} \left(1 - \frac{c}{W}\right)^{n-3} \quad \text{for } p \in (1, \infty). \tag{4.10}$$

Proposition 4.5. *Let*

$$I_{2,0}(k) := \frac{1}{2\pi} \mu^{k-1} \int d\lambda_1 \, d\lambda_2[\Lambda^{-1}(e^{-V} - u)][e^V u_0^{(3)}].$$

It holds

$$|I_{2,0}(k)| \leq C|\mu|^{k-1} W^{-\frac{3}{2}}.$$

The proofs are given below. We decompose now the integral in (4.8) as

$$I_2(k) = I_{2,0}(k) + I_{2,1}(k) + I_{2,3}(k),$$

where $I_{2,0}$ was defined above and

$$I_{2,1}(k) := \frac{1}{2\pi} \int d\lambda_1 \, d\lambda_2 [\Lambda^{-2} e^V (e^{-V} - u)][T^{k-1}(\Lambda u_0^{(3)}) - \mu^{k-1} \Lambda u_0^{(3)}],$$

$$I_{2,2}(k) := \frac{1}{2\pi} \int d\lambda_1 \, d\lambda_2 [\Lambda^{-2} e^V (e^{-V} - u)][T^{k-1}(\Lambda(u - u_0^{(3)}))].$$

For the first integral we obtain

$$\begin{aligned}
|2\pi \, I_{2,1}(k)| &\leq \|\Lambda^{-1}(e^{-V} - u)\|_p \|\Lambda^{-1} e^V [T^{k-1}(\Lambda u_0^{(3)}) - \mu^{k-1} \Lambda u_0^{(3)}]\|_q \\
&\leq C W^{\frac{1}{2} - \frac{1}{p}} W^{-\frac{1}{2} - \frac{1}{q}} \left(1 - \frac{c}{W}\right)^{k-4} \\
&= C W^{-1} \left(1 - \frac{c}{W}\right)^{k-4}.
\end{aligned}$$

where we applied (3.27) for $p > 2$. Finally, applying Corollary 3.10

$$\begin{aligned}
|2\pi \, I_{2,2}(k)| &\leq \|\Lambda^{-1}(e^{-V} - u)\|_p \|\Lambda^{-1} e^V [T^{k-1}(\Lambda(u - u_0^{(3)}))]\|_q \\
&= \|\Lambda^{-1}(e^{-V} - u)\|_p \|\delta_{W^2}^\star \Lambda^{-1} [T^{k-2}(\Lambda(u - u_0^{(3)}))]\|_q \\
&\leq C \left(1 - \frac{c}{W}\right)^{k-3} W^{\frac{1}{2} - \frac{1}{p}} \|u - u_0^{(3)}\|_q \\
&= \mathcal{O}(W^{-\frac{3}{2}}) \left(1 - \frac{c}{W}\right)^{k-3}.
\end{aligned}$$

This concludes the proof of Proposition 4.3. $\square$

Proof of Proposition 4.4. Note that both V and $u_0^{(M)}$ have a representation as functions of the supermatrix R. Moreover $\Lambda u_0^{(3)}$ vanishes at 0. Then we can rewrite

$$[T(\Lambda u_0^{(3)})](\lambda) = [\mathcal{T}(\Lambda u_0^{(3)})](\lambda) = [\mathfrak{T}[\mathrm{Str}(\cdot) U_0^{(3)}](R)_{|R=\mathrm{diag}(\lambda_1, i\lambda_2)}$$

where we used $\Lambda = \mathrm{Str}\, R$. From equation (2.30) and Lemma A.1(d) we find

$$\begin{aligned}
\mathfrak{T}&[\mathrm{Str}(\cdot) U_0^{(3)}](R) \\
&= \mu \, \mathrm{Str}\, R \Big[\mathfrak{T}[U_0^{(3)}](R) + \frac{3\mu^3 c_3}{W} \frac{\mathrm{Str}\, R^2}{\mathrm{Str}\, R} \mathfrak{T}[U_0^{(0)}](R) \Big] \\
&= \mu U_0^{(3)}(R) \, \mathrm{Str}\, R \\
&\quad + \mu U_0^{(0)}(R) \, \mathrm{Str}\, R \Big[\mathrm{Err}^{(3)}(R) + \frac{3\mu^3 c_2}{W} \frac{\mathrm{Str}\, R^2}{\mathrm{Str}\, R} (\mathrm{Err}^{(0)}(R) + 1) \Big] \\
&= \mu \, \mathrm{id}[\mathrm{Str}(\cdot) U_0^{(3)}](R) + \mu \, \mathrm{Str}(R) \, \mathrm{Rem}(R).
\end{aligned}$$

where

$$\mathrm{Rem}(R) := U_0^{(0)}(R) \, \mathrm{Err}^{(3)}(R) + \frac{3\mu^3 c_3}{W} \frac{\mathrm{Str}\, R^2}{\mathrm{Str}\, R} U_0^{(0)}(R)(1 + \mathrm{Err}^{(0)}(R)),$$

and $\mathrm{Err}^{(0)}, \mathrm{Err}^{(3)}$ were introduced in (3.19). Then (3.20) yields (4.9). In order to prove (4.10) we insert the decomposition

$$\mathfrak{T}^n = \mu^n \operatorname{id} + \sum_{l=0}^{n-1} \mu^{n-1-l} \mathfrak{T}^l (\mathfrak{T} - \mu \operatorname{id})$$

and we obtain

$$T^n(\Lambda u_0^{(3)}) = \mathfrak{T}^n [\mathrm{Str}(\cdot) U_0^{(3)}](R)_{|R = \mathrm{diag}(\lambda_1, i\lambda_2)}$$

$$= \mu^n \Lambda u_0^{(3)}(\lambda) + \sum_{l=0}^{n-1} \mu^{n-1-l} T^l [\Lambda \operatorname{Rem}(\lambda)].$$

Using $|\mu| \leq \left(1 - \frac{c}{W}\right)$ we get

$$\|\Lambda^{-1} e^V [T^n(\Lambda u_0^{(3)}) - \mu^n (\Lambda u_0^{(3)})]\|_p$$

$$\leq \sum_{l=0}^{n-1} \left(1 - \frac{c}{W}\right)^{n-1-l} \|\Lambda^{-1} e^V [T^l (\Lambda \operatorname{Rem})]\|_p$$

$$\leq \left(1 - \frac{c}{W}\right)^{n-3} [\|e^V \operatorname{Rem}\|_p + n\| \operatorname{Rem}\|_p]$$

$$\leq \left(1 - \frac{c}{W}\right)^{n-3} [\|e^V \operatorname{Rem}\|_p + W\| \operatorname{Rem}\|_p].$$

The estimates (3.23) now yield

$$\| \operatorname{Rem}\|_p \leq \|v^{(3)}\|_p + W^{-1}\|\bar{\Lambda} u_0^{(0)}\|_p + W^{-1}\|\bar{\Lambda} v^{(0)}\|_p$$

$$= \mathcal{O}(W^{-\frac{1}{p}-2}) + W^{-1}\mathcal{O}(W^{-\frac{1}{2}-\frac{1}{p}}) + W^{-1}\mathcal{O}(W^{-\frac{1}{p}-2})$$

$$= \mathcal{O}(W^{-\frac{3}{2}-\frac{1}{p}}).$$

The same bound holds for $\|e^V \operatorname{Rem}\|_p$. This completes the proof. $\qquad\square$

Proof of Proposition 4.5. Inserting the decomposition

$$(e^{-V} - u) = (e^{-V} - u_0^{(3)}) + (u_0^{(3)} - u),$$

we get

$$I_{2,0}(k) = \frac{1}{2\pi} \mu^{k-1} [I_{2,0,a} + I_{2,0,b}],$$

where

$$I_{2,0,a} = \int \mathrm{d}\lambda_1 \, \mathrm{d}\lambda_2 [\Lambda^{-1}(e^{-V} - u_0^{(3)})][e^V u_0^{(3)}],$$

$$I_{2,0,b} = \int \mathrm{d}\lambda_1 \, \mathrm{d}\lambda_2 [\Lambda^{-1}(u_0^{(3)} - u)][e^V u_0^{(3)}].$$

Decomposing further

$$1 - e^V u_0^{(3)} = 1 - u_0^{(3)} - (e^V - 1)u_0^{(3)},$$

we get

$$
\begin{aligned}
I_{2,0,a} = \int d\lambda_1 \, d\lambda_2 \Lambda^{-1} u_0^{(3)} - \int d\lambda_1 \, d\lambda_2 \Lambda^{-1} (u_0^{(3)})^2, \\
- \int d\lambda_1 \, d\lambda_2 \Lambda^{-1} [u_0^{(3)}(1 - e^{-V})][u_0^{(3)} e^V].
\end{aligned}
\tag{4.11}
$$

The first two integrals vanish. This can be more easily seen by going back to R coordinates

$$
\begin{aligned}
\frac{1}{2\pi} \int d\lambda_1 \, d\lambda_2 \Lambda^{-1} u_0^{(3)} \\
= \int d\,Ra u_0^{(3)}(R) \\
= \int d\,Ra e^{-\alpha W \operatorname{Str} R^2} + Wc_3 \int d\,Ra e^{-\alpha W \operatorname{Str} R^2} \operatorname{Str} R^3.
\end{aligned}
\tag{4.12}
$$

The first integral equals 0 by parity, the second by explicit computation. In the same way

$$
\begin{aligned}
\frac{1}{2\pi} \int d\lambda_1 \, d\lambda_2 \Lambda^{-1} (u_0^{(3)})^2 &= \int d\,Ra (u_0^{(3)}(R))^2, \\
&= W^2 c_1^2 \int d\,Ra e^{-2\alpha W \operatorname{Str} R^2} (\operatorname{Str} R^3)^3 \\
&= 0,
\end{aligned}
$$

where the last integral vanishes by parity and the other contributions vanish using eq. (4.12). Hence, using $1 - e^{-V} = \mathcal{O}(\lambda^2)$,

$$
\begin{aligned}
|I_{2,0,a}| &= \left| \int d\lambda_1 \, d\lambda_2 \, \Lambda^{-1} [u_0^{(3)}(1 - e^{-V})][u_0^{(3)} e^V] \right| \\
&\le \|\Lambda^{-1} [u_0^{(3)}(1 - e^{-V})]\|_p \|u_0^{(3)} e^V\|_q \\
&= \mathcal{O}(W^{\frac{1}{2}-1-\frac{1}{p}}) \mathcal{O}(W^{-\frac{1}{q}}) = \mathcal{O}(W^{-\frac{3}{2}}).
\end{aligned}
$$

Finally, using Corollary 3.10 again

$$
|I_{2,0,b}| \le \|\Lambda^{-1} (u_0^{(3)} - u)\|_p \|e^V u_0^{(3)}\|_q \le \mathcal{O}(W^{-\frac{1}{p}-\frac{1}{2}}) \mathcal{O}(W^{-\frac{1}{q}}) = \mathcal{O}(W^{-\frac{3}{2}}).
$$

This concludes the proof. □

4.2.2. Semi-circle law. Our goal is to prove that the infinite volume density of states $\rho(E)$ satisfies $|\rho(E) - \rho_{\text{s.c.}}(E)| \leq \mathcal{O}(W^{-2})$ According to Proposition 4.2,

$$\lim_{\varepsilon \searrow 0} \langle \operatorname{Tr} G_{00}[H](E_\varepsilon) \rangle - \mathcal{E} = I_1[E] = \frac{1}{2\pi} \int \mathrm{d}\lambda_1 \, \mathrm{d}\lambda_2 \Lambda^{-1} e^V u^2,$$

hence

$$\rho(E) - \rho_{\text{s.c.}}(E) = -\frac{1}{\pi} \Im I_1.$$

Inserting the decomposition

$$u = u_0^{(5)} + (u - u_0^{(5)}),$$

we obtain

$$I_1 = I_{1,0} + 2I_{1,1} + I_{1,3},$$

where

$$I_{1,0} = \frac{1}{2\pi} \int \mathrm{d}\lambda_1 \, \mathrm{d}\lambda_2 \Lambda^{-1} e^{V(\lambda)} (u_0^{(5)})^2,$$

$$I_{1,1} = \frac{1}{2\pi} \int \mathrm{d}\lambda_1 \, \mathrm{d}\lambda_2 \Lambda^{-1} e^{V(\lambda)} u_0^{(5)} (u - u_0^{(5)}),$$

$$I_{1,2} = \frac{1}{2\pi} \int \mathrm{d}\lambda_1 \, \mathrm{d}\lambda_2 \Lambda^{-1} e^{V(\lambda)} (u - u_0^{(5)})^2.$$

The second integral is easily bounded by

$$|I_{1,1}| \leq \|e^{V(\lambda)} u_0^{(5)}\|_p \|\Lambda^{-1}(u - u_0^{(5)})\|_q \leq \mathcal{O}(W^{-\frac{1}{p}}) \mathcal{O}(W^{-\frac{1}{q}-\frac{3}{2}}) = \mathcal{O}(W^{-2-\frac{1}{2}}).$$

To control the e^V factor in $I_{1,2}$ we insert the identity

$$u - u_0^{(5)} = v^{(5)} + T(u - u_0^5)$$

and we obtain

$$\begin{aligned}
|I_{1,2}| &\leq \|\Lambda^{-1}(u - u_0^5)\|_p \|e^{V(\lambda)} v^{(5)}\|_q + \|(u - u_0^5)\|_q \|\Lambda^{-1} e^V T(u - u_0^5)\|_p \\
&\leq \|\Lambda^{-1}(u - u_0^5)\|_p (\|e^{V(\lambda)} v^{(5)}\|_q + \|(u - u_0^5)\|_q) \\
&\leq \mathcal{O}(W^{-\frac{1}{p}} W^{-\frac{3}{2}})(\mathcal{O}(W^{-\frac{1}{q}}) + \mathcal{O}(W^{-\frac{1}{q}-\frac{3}{2}})) \\
&= \mathcal{O}(W^{-2-\frac{1}{2}}),
\end{aligned}$$

where we used the fact the

$$v^{(5)} = \mathcal{T} u_0^{(5)} - u_0^{(5)}$$

has always an exponential prefactor $e^{-\alpha W \lambda^2}$. Finally, to compute $I_{1,0}$, we remark that (3.24) yields

$$V(\lambda) = \frac{1-\mathcal{E}^2}{2}\lambda^2 + \mathcal{O}(\lambda^3).$$

We decompose

$$(2\pi)I_{1,0} = I_{1,0,a} + I_{1,0,b}$$

with

$$I_{1,0,a} = \int d\lambda_1 \, d\lambda_2 \Lambda^{-1} e^{V(\lambda)} (1 - e^{\frac{1-\mathcal{E}^2}{2}\lambda^2 - V})(u_0^{(5)})^2,$$

$$I_{1,0,b} = \int d\lambda_1 \, d\lambda_2 \Lambda^{-1} e^{\frac{1-\mathcal{E}^2}{2}\lambda^2} (u_0^{(5)})^2.$$

The first integral is bounded as follows

$$\begin{aligned}
|I_{1,0,a}| &\leq \|\Lambda^{-1}(1 - e^{\frac{1-\mathcal{E}^2}{2}\lambda^2 - V})u_0^{(5)}\|_p \|e^V u_0^{(5)}\|_q \\
&\leq \mathcal{O}(W^{\frac{1}{2}-\frac{3}{2}-\frac{1}{p}})\mathcal{O}(W^{-\frac{1}{q}}) \\
&= \mathcal{O}(W^{-2}).
\end{aligned}$$

The remaining term $I_{1,0,b}$ is estimated using

$$u_0^{(5)} = e^{-\alpha W \lambda^2}(1 + Wq^{(3)} + Wq^{(4)} + Wq^{(5)}),$$

where $q^{(3)}, q^{(5)}$ are odd polynomials while $q^{(4)}$ is even (cf. Proposition 3.8). Replacing α by

$$\tilde{\alpha} := \alpha - \frac{1-\mathcal{E}^2}{4W},$$

we get

$$\begin{aligned}
I_{1,0,b} &= \int d\lambda_1 \, d\lambda_2 \, \Lambda^{-1} e^{-2\tilde{\alpha}W\lambda^2} \\
&\qquad \cdot ((1 + Wq^{(3)})^2 + 2W(q^{(4)} + q^{(5)}) \\
&\qquad\quad + 2W^2 q^{(3)}(q^{(4)} + q^{(5)}) + (Wq^{(5)})^2) \\
&= \int d\lambda_1 \, d\lambda_2 \, \Lambda^{-1} e^{-2\tilde{\alpha}W\lambda^2} (2Wq^{(5)} + 2W^2 q^{(3)}q^{(4)}),
\end{aligned}$$

where the first term vanishes by the same arguments used in 4.11 and the other terms cancel by parity. Finally

$$|I_{1,0,b}| \leq \mathcal{O}(W^{-1+\frac{1}{2}}W^{1-\frac{5}{2}}) + \mathcal{O}(W^{-1+\frac{1}{2}}W^{2-\frac{3}{2}-\frac{4}{2}}) = \mathcal{O}(W^{-2}).$$

This completes the proof of Theorem 1 away from the edges. $\qquad\qquad\square$

4.3. Proof of Theorem 2. Our goal is to prove the estimate

$$|\partial_E^n \rho_N(E)| \leq W^{n-1} C(E)^n n!, \quad n \geq 1,$$

uniformly in N. Using the supersymmetric representation we have seen that

$$-\pi \partial_E^n \rho_N(E) = \frac{1}{N} \partial_E^n \lim_{\varepsilon \searrow 0} \Im \langle \operatorname{Tr} G[H_N](E_\varepsilon) \rangle$$

$$= \frac{1}{N} \Im \lim_{\varepsilon \searrow 0} \partial_E^n \operatorname{Tr} \langle G[H_N](E_\varepsilon) \rangle$$

$$= \frac{(-1)^n}{N} \sum_{q=1}^{n} \sum_{\substack{n_1, \ldots, n_q \geq 1 \\ n_1 + \cdots + n_q = n}} \frac{n!}{n_1! \cdots n_q!} \sum_{m=0}^{q} \sum_{\substack{1 \leq y_1 < y_2 < \cdots < y_q \leq N \\ y_m \leq x < y_{m+1}}} \Im I_{x, y_1, \ldots, y_q}^{m, n_1, \ldots, n_q},$$

where $I_{x, y_1, \ldots, y_q}^{m, n_1, \ldots, n_q}$ was defined in Corollary 2.14 and we use the convention $y_0 = 1$, $y_{q+1} = N+1$. Using $\mathfrak{T}(\Lambda f) = T(\Lambda f)$, we can reorganize the integral as follows:

$$I_{x, y_1, \ldots, y_q}^{m, n_1, \ldots, n_q}$$
$$= \int \frac{\mathrm{d}\lambda_1 \, \mathrm{d}\lambda_2}{2\pi} \Big[T^{x - y_m} \Lambda^{n_m} \prod_{j=m-1}^{1} (T^{y_{j+1} - y_j} \Lambda^{n_j}) \mathfrak{T}^{y_1 - 1} e^{-V} \Big] (\lambda)$$
$$\cdot [e^{V(\lambda)} \Lambda^{-1}] \Big[T^{y_{m+1} - x} \Lambda^{n_{m+1}} \prod_{j=m+1}^{q-1} (T^{y_{j+1} - y_j} \Lambda^{n_{j+1}}) \mathfrak{T}^{N - y_q} e^{-V} \Big] (\lambda),$$

$$(4.13)$$

where we use the convention $n_0 = 0 = n_{q+1}$. Note that by (3.6) for all $m, n \geq 1$ it holds

$$\|\Lambda^{-1} (\Lambda^n T^m) f\|_p \leq C^n \sqrt{n!} (1 - c/W)^{m-1} \|\Lambda^{-1} f\|_p.$$

When $f = \Lambda u$ this estimate gives a factor $\|u\|_p = \mathcal{O}(W^{-\frac{1}{p}})$. The following lemma shows the bound can be improved.

Lemma 4.6. *For all $n, m \geq 1$*

$$\|\Lambda^{-1} (\Lambda^n T^m) \Lambda u\|_p \leq C^n \sqrt{n!} \, (1 - c/W)^{m-1} W^{-\frac{1}{p} - \frac{1}{2}}.$$

Proof. We decompose

$$T^m \Lambda u = T^m \Lambda (u - u_0^{(3)}) + \mu^m \Lambda u_0^{(3)} + [T^m - \mu^m \operatorname{id}] \Lambda u_0^{(3)},$$

where we recall that $\mu = \left(1 + \frac{2\alpha}{W}\right)^{-1}$ and $|\mu| \le \left(1 - \frac{c}{W}\right)^{-1}$. Then

$$
\begin{aligned}
\|\Lambda^{n-1} & T^m \Lambda u\|_p \\
&\le \|\Lambda^{n-1} T^m \Lambda (u - u_0^{(3)})\|_p + |\mu|^m \|\Lambda^n u_0^{(3)}\|_p + \|\Lambda^{n-1}[T^m - \mu^m \, \mathrm{id}]\Lambda u_0^{(3)}\|_p \\
&\le C^n \sqrt{n!}\left(1 - \frac{c}{W}\right)^{m-1} \|(u - u_0^{(3)})\|_p + |\mu|^m \|\Lambda^n u_0^{(3)}\|_p \\
&\quad + \|\Lambda^n e^{-V}\|_\infty \|\Lambda^{-1} e^V [T^m - \mu^m \, \mathrm{id}]\Lambda u_0^{(3)}\|_p \\
&\le C^n \sqrt{n!}\left(1 - \frac{c}{W}\right)^{m-1} [W^{-\frac{1}{p}-1} + W^{-\frac{1}{p}-\frac{n}{2}} + W^{-\frac{1}{p}-\frac{1}{2}}] \\
&= C'^n \sqrt{n!}\left(1 - \frac{c}{W}\right)^{m-1} W^{-\frac{1}{p}-\frac{1}{2}},
\end{aligned}
$$

where in the first line we used (3.6) and in the last line Corollary 3.10, Proposition 4.4 and the explicit form of $u_0^{(3)}$, together with the constraint $n \ge 1$. □

4.3.1. Case $n = 1$. For $n = 1$,

$$
-\pi \partial_E \rho_N(E) = \frac{1}{N} \Im \lim_{\varepsilon \searrow 0} \sum_{yy'} \langle G_{yy'}[H_N](E_\varepsilon) G_{y'y}[H_N](E_\varepsilon)\rangle.
$$

From Corollary 2.14 we have the representation

$$
\lim_{\varepsilon \searrow 0} \langle G_{yy'}[H_N](E_\varepsilon) G_{y'y}[H_N](E_\varepsilon)\rangle = -J_{yy'} + \sum_{xx'=1}^N J_{yx} J_{y'x'} I_{xx'},
$$

where, for $x' \ge x$,

$$
I_{xx'} = \frac{1}{2\pi} \int d\lambda_1 \, d\lambda_2 \Lambda^{-1} e^{V(\lambda)} [\mathfrak{J}^{x-1} e^{-V}](\lambda) [T^{x'-x} \Lambda \mathfrak{J}^{N-x'} e^{-V}](\lambda)
$$

and for $x' < x$ we set

$$
I_{xx'} = I_{N-x+1, N-x'+1}.
$$

We want to prove now

$$
|I_{xx'}| \le C W^{-1}\left(1 - \frac{c}{W}\right)^{x'-x}.
$$

We insert again the decomposition $e^{-V} = u + (e^{-V} - u)$, which yields

$$
\begin{aligned}
I_{xx'} = \; & I_1(x' - x) + I_2(x - 1, x' - x) \\
& + I_2(N - x', x' - x) + I_3(x - 1, x' - x, N - x'),
\end{aligned}
$$

where

$$I_1(k) := \int \frac{\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2}{2\pi} \Lambda^{-1} e^V u[T^k(\Lambda u)], \quad k \geq 0, \tag{4.14a}$$

$$I_2(k,k') := \int \frac{\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2}{2\pi} \Lambda^{-1} e^V [T^k(e^{-V}-u)][T^{k'}(\Lambda u)], \quad k,k' \geq 0, \tag{4.14b}$$

$$I_3(k,k',k'') := \int \frac{\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2}{2\pi} \Lambda^{-1} e^V [T^k(e^{-V}-u)][T^{k'}\Lambda T^{k''}(e^{-V}-u)], \tag{4.14c}$$

and in I_3 we have $k + k' + k'' = N - 1 \gg 1$. To obtain $I_2(N - x', x' - x)$ we used in addition (4.7). The first integral is bounded by

$$2\pi|I_1(k)| \leq \|e^V u\|_p \|\Lambda^{-1} T^k \Lambda u\|_q$$
$$\leq C\Big(1 - \frac{c}{W}\Big)^k \|e^V u\|_p \|u\|_q$$
$$\leq \mathcal{O}(W^{-1})\Big(1 - \frac{c}{W}\Big)^k.$$

where we used (3.6) and Corollary 3.10. The second integral is bounded by

$$2\pi|I_2(k)| \leq \|\Lambda^{-1} T^k (e^{-V}-u)\|_p \|e^V T^{k'} \Lambda u\|_q$$
$$\leq C\Big(1 - \frac{c}{W}\Big)^{k+k'} W^{+\frac{1}{2}-\frac{1}{p}} W^{-\frac{1}{q}-\frac{1}{2}}$$
$$= \mathcal{O}(W^{-1})\Big(1 - \frac{c}{W}\Big)^{k+k'}.$$

where in the first term we used again (3.6) together with (3.27) for $p > 2$. In the second term we used $\|e^V \Lambda u\|_q = \mathcal{O}(W^{-\frac{1}{q}-\frac{1}{2}})$ (cf. Corollary 3.10) for the case $k' = 0$. When $k' \geq 1$ we apply (3.9) to get

$$\|e^V T^{k'} \Lambda u\|_q \leq \|T^{k'-1} \Lambda u\|_q + \frac{C}{W} \|\Lambda^{-1} T^{k'-1} \Lambda u\|_q.$$

The estimate now follows from Lemma 4.6 and (3.6). Note that we are forced to estimate the factor e^V together with $T^{k'}\Lambda u$ since for $k = 0$ the term $e^V(e^{-V}-u)$ is not integrable. Finally the constraint $k + k' + k'' = N - 1$ guarantee that $k \leq 1$ or $k' + k'' \geq 1$. We can assume without loss of generality $k \geq 1$. Then using (3.27)

for $p = 2$,

$$
\begin{aligned}
2\pi |I_3(k)| &\leq \|\Lambda^{-1} e^V T^k (e^{-V} - u)\|_2 \|T^{k'} \Lambda T^{k''} (e^{-V} - u)\|_2 \\
&\leq C (1 - \frac{c}{W})^{N-1} (\ln W)^2 \\
&= \mathcal{O}(W^{-1}) \left(1 - \frac{c}{W}\right)^{k+k'+k''}.
\end{aligned}
$$

if $N \geq C(E) W \ln W$, for $C(E) > 0$ some large constant. This completes the proof of (1.10). Performing the sum over y, y' we obtain $|\partial_E \rho_N(E)| \leq C$.

4.3.2. Case $n > 1$. As in the case $n = 1$ we insert the decomposition $e^{-V} = u + (e^{-V} - u)$, and reorganize the integral (eventually applying also (4.7)) as the sum of three terms of the following form:

$$
I_1 := \int \frac{d\lambda_1 \, d\lambda_2}{2\pi} e^V \Lambda^{-1} \Big[\prod_{j=0}^{l} (T^{m_j} \Lambda^{n_j}) u \Big] \Big[\prod_{k=0}^{l'} (T^{m'_k} \Lambda^{n'_k}) u \Big] \tag{4.15a}
$$

$$
\begin{aligned}
I_2 := \int \frac{d\lambda_1 \, d\lambda_2}{2\pi} e^V \Lambda^{-1} \Big[\prod_{j=1}^{l} (T^{m_j} \Lambda^{n_j}) T^{m_0} (\Lambda^{n_0} u) \Big] \\
\cdot \Big[T^{m'_0} \Lambda^{n'_0} \prod_{k=1}^{l'} (T^{m'_k} \Lambda^{n'_k}) T^{\bar{m}'} (e^{-V} - u) \Big]
\end{aligned} \tag{4.15b}
$$

$$
\begin{aligned}
I_3 := \int \frac{d\lambda_1 \, d\lambda_2}{2\pi} e^V \Lambda^{-1} \Big[\prod_{j=0}^{l} (T^{m_j} \Lambda^{n_j}) T^{\bar{m}} (e^{-V} - u) \Big] \\
\cdot \Big[\prod_{k=0}^{l'} (T^{m'_k} \Lambda^{n'_k}) T^{\bar{m}'} (e^{-V} - u) \Big],
\end{aligned} \tag{4.15c}
$$

where $l \geq 1$, $l' \geq 0$, $n_j, n'_k \geq 1$ and $m_j, m'_k \geq 0$ for all j, k, with the constraint

$$
\sum_{j=0}^{l} n_j + \sum_{k=0}^{l'} n'_k = n.
$$

Finally $\bar{m}, \bar{m}' \geq 0$ but must satisfy the constraints

$$
\sum_{j=0}^{l} m_j + \sum_{k=0}^{l'} m'_k + \bar{m} + \bar{m}' = N - 1.
$$

The proof now works as in the case $n = 1$ and yields

$$|I_{x,y_1,\ldots,y_q}^{m,n_1,\ldots,n_q}| \leq \frac{1}{W} C^n \prod_{j=1}^{q} (\sqrt{n_j!}) \Big[\Big(1 - \frac{c}{W}\Big)^{y_{\max} - y_{\min}} + \Big(1 - \frac{c}{W}\Big)^{y_{\max} - 1}$$
$$+ \Big(1 - \frac{c}{W}\Big)^{N - y_{\min}} + \Big(1 - \frac{c}{W}\Big)^{N} \Big],$$

where

$$y_{\max} := \max[y_q, x] \quad \text{and} \quad y_{\min} := \min[y_1, x].$$

Hence

$$\pi |\partial_E^n \rho_N(E)| \leq \frac{C^n n!}{W} \frac{1}{N} \sum_{q=1}^{n} (N W^q + W^{q+1}) \prod_j \Big(\sum_{n_j} \frac{1}{\sqrt{n_j!}} \Big) \leq C'^n n! W^{n-1}.$$

4.4. The case $E = 0$. At $E = 0$ the factor e^V may develop a pole. To solve the problem we use (3.18) to replace u by $u = \mathfrak{T}u_0 + T(u - u_0)$ before doing any other manipulation. Formulas become slightly more cumbersome, but each factor e^V now comes with a prefactor e^{-V}.

5. Contour deformation

To extend the proof to the entire range $E \in (-2, 2)$, the contour of integration has to be deformed. One possible strategy (followed in [14]) is to rotate the contour. The rotation angle must ensure that $\Re V$ has only one non-degenerate global minimum at the saddle point. This can only be achieved for a rotation angle close to $\pi/6$ (cf. [14, 5.1.2]). However the corresponding transfer operator $e^{-\Re V/2} e^{-W^2(\lambda - \lambda')} e^{-\Re V/2}$ is no longer longer self-adjoint. Another strategy developed in [17] consists in performing a complex rotation that makes the operator approximately normal. The results in [17] require the resulting function $e^{-\Re V}$ to have only one non degenerate global minimum. In the present case we would need to rotate by approximately $\pi/8$, but then $e^{-\,\mathrm{Re}V}$ has two minima.

Here we therefore use the following strategy: first (Section 5.1), we find a good contour Γ for the bosonic variable. After the contour deformation, the operator $\mathfrak{T}$ is transformed to a new operator $\mathfrak{T}_\Gamma$. The main technical difficulty is to find a replacement for the operator norm bound of Proposition 3.1. We show (Proposition 5.2) that a similar bound holds when the operator is replaced with its k-th power, where k is a sufficiently large number, independent of W. Having this bound at hand, the proof follows the lines of its counterparts for $|E| < \sqrt{\frac{32}{9}}$.

We fix an energy $|E| < 2$, the dependence on which is omitted from the notation. An inspection of the argument shows that all the estimates are uniform on compact subintervals of $(-2, 2)$.

5.1. Choice of the contour. Decompose

$$e^{-V(\lambda)} = e^{-V_1(\lambda_1) - V_2(\lambda_2)},$$

$$e^{-V_1(\lambda_1)} = e^{-\frac{1}{2}\lambda_1^2 - \mathcal{E}\lambda_1} \frac{1}{\overline{\mathcal{E}} - \lambda_1},$$

$$e^{-V_2(\lambda_2)} = e^{-\frac{1}{2}\lambda_2^2 + i\mathcal{E}\lambda_2}(\overline{\mathcal{E}} - i\lambda_2).$$

Lemma 5.1. *For any $|E| < 2$ there exists a contour Γ and numbers $C_\Gamma, c_\Gamma > 0$ such that*

1. *Γ contains the segments*

$$(-\infty, -C_\Gamma], \quad [-c_\Gamma, c_\Gamma], \quad [C_\Gamma, \infty);$$

2. *the angle between Γ and the real axis stays in the range*

$$\left(-\frac{\pi}{4}(1 - c_\Gamma), \frac{\pi}{4}(1 - c_\Gamma)\right);$$

3. *e^{-V_1} is analytic in*

$$\Gamma^+ = \bigcup_{\substack{a,a' \in \Gamma \\ |a-a'| < c_\Gamma}} [a, a'];$$

4. *Γ is homologous to $(-\infty, \infty)$ in the domain of analyticity of e^{-V_1};*

5. *$\min_{a \in \Gamma^+} \Re V_1(a)$ is uniquely attained at $a = 0$.*

Note that when $|E| \to 2$ we need to take $c_\Gamma \to 0$ too. The proof is an elementary verification. We reduce it to a similar verification already performed in [14].

Proof. In [14, Section 5.1.2] it is proved that for any $|E| < 2$ there exists ζ with $|\zeta| = 1$, $|\arg \zeta| < \pi/4$ such that $\min_{a \in \zeta \mathbb{R}} \Re V_1(a)$ is uniquely attained at $a = 0$ and the singularity of $\Re V_1$ does not lie between $\zeta \mathbb{R}$ and $\mathbb{R}$. For $C > c > 0$, denote by $\Gamma(c, C)$ the piecewise linear contour going from $-\infty$ to ∞ via the points

$$-c - 2\Re \zeta C, \quad -c - \zeta C, \quad -c, \quad c, \quad c + \zeta C, \quad c + 2\Re \zeta C.$$

We get

$$
\begin{aligned}
\Gamma(c, C) = {} & (-\infty, -c - 2\Re\zeta C] \\
& + [-c - 2\Re\zeta C, -c - \zeta C] \\
& + [-c - \zeta C, -c] \\
& + [-c, c] \\
& + [c, c + \zeta C] \\
& + [c + \zeta C, c + 2\Re\zeta C] \\
& + [c + 2\Re\zeta C, \infty).
\end{aligned}
$$

We first choose a large $C > 0$ and then a small $c > 0$. For sufficiently large C one has $\Re V_1 > \text{const} > 0$ in the entire domain

$$
\{|z| > C, z \text{ lies between } \mathbb{R} \text{ and } \zeta\mathbb{R}\}.
$$

In particular, one has $\Re V_1 > \text{const} > 0$ on the four segments $(1), (2), (6), (7)$. For this value of C, one can choose $c > 0$ sufficiently small so that, by a continuity argument, the minimum of $\Re V_1$ on the union of the remaining segments $(3), (4), (5)$ is uniquely attained at the origin. For these values of C and c, let $\Gamma = \Gamma(c, C)$.

Then Γ satisfies the conditions (1)–(4) of Lemma 5.1, and a weakened form of (5) with Γ in place of Γ^+. By an additional continuity argument, (5) also holds as stated provided that c_Γ is chosen sufficiently small. $\qquad\square$

For a contour Γ, denote by K_Γ the integral operator with kernel

$$
K_\Gamma(\lambda, \lambda') = \frac{W^2}{2\pi} \exp\left\{ -\frac{V(\lambda)}{2} - \frac{W^2}{2}(\lambda - \lambda')^2 - \frac{V(\lambda')}{2} \right\}
$$

acting on $L_p(\Gamma \times \mathbb{R})$. Here we use the convention $\lambda^2 = \lambda_1^2 + \lambda_2^2$. The main technical step is the following proposition, the proof of which will be the subject of the next Section 5.2.

Proposition 5.2. *Let Γ be a contour satisfying (1)–(5) of Lemma 5.1. Then there exists $k \geq 1$ such that for any $p \in (1, \infty)$*

$$
\|K_\Gamma^k\|_p \leq 1 - \frac{c_p}{W}.
$$

5.2. Proof of the operator norm bound. To prove Proposition 5.2, we study the kernel of the operator K_Γ^k.

Lemma 5.3. *Let Γ be a contour satisfying* (2)–(3) *of Lemma* 5.1. *For any k,*

$$K_\Gamma^k(\lambda, \lambda') = \exp\Big\{ -\frac{W^2}{2k}(\lambda - \lambda')^2 - \frac{V(\lambda)}{2} - V(\hat\lambda^1)$$

$$- V(\hat\lambda^2) - \cdots - V(\hat\lambda^{k-1}) - \frac{V(\lambda')}{2}\Big\}$$

$$\cdot \frac{W^2}{2\pi k}(1 + \mathcal{O}(W^{-2})), \quad \lambda, \lambda' \in \Gamma \times \mathbb{R}, \quad |\lambda - \lambda'| < c_\Gamma/2,$$

where $\hat\lambda^j = \lambda + \frac{i}{k}(\lambda' - \lambda)$, and the asymptotics is uniform on compact sets.

Proof. The proof proceeds by a saddle point analysis. Consider the integral

$$K_\Gamma^k(\lambda, \lambda') = \int_{(\Gamma \times \mathbb{R})^{k-1}} K_\Gamma(\lambda, \lambda^1) \cdots K_\Gamma(\lambda^{k-1}, \lambda') \prod_{j=1}^{k-1} d\lambda_1^j \, d\lambda_2^j.$$

The saddle point equations

$$\lambda^j = \frac{\lambda^{j-1} + \lambda^{j+1}}{2} \quad \text{with } \lambda^0 = \lambda, \, \lambda^k = \lambda'$$

have a unique solution given by $\hat{\boldsymbol\lambda} = (\hat\lambda^1, \cdots, \hat\lambda^{k-1})$. Extracting the saddle contribution we get

$$K_\Gamma^k(\lambda, \lambda') = \frac{W^2}{2\pi k} e^{-\frac{W^2}{2k}(\lambda - \lambda')^2} e^{-\frac{V(\lambda)}{2} - V(\hat\lambda^1) - V(\hat\lambda^2) - \cdots - V(\hat\lambda^{k-1}) - \frac{V(\lambda')}{2}} \widetilde{K}_\Gamma^k(\lambda, \lambda')$$

where

$$\widetilde{K}_\Gamma^k(\lambda, \lambda') = k \int_{(\Gamma \times \mathbb{R})^{k-1}} e^{-W^2(\phi(\boldsymbol\lambda_1) + \phi(\boldsymbol\lambda_2))} e^{-\sum_{j=1}^{k-1}[V_\iota(\lambda_\iota^j) - V_\iota(\hat\lambda_\iota^j)]} \prod_{j=1}^{k-1} \frac{W^2}{2\pi} \, d\lambda_1^j \, d\lambda_2^j,$$

$$\phi(\boldsymbol\lambda_\iota) = \frac{1}{2} \sum_{j=0}^{k-1} \Big[(\lambda_\iota^j - \lambda_\iota^{j+1})^2 - \frac{(\lambda_\iota' - \lambda_\iota)^2}{k^2} \Big]$$

$$= \frac{1}{2} \sum_{j=0}^{k-1} \Big(\lambda_\iota^j - \lambda_\iota^{j+1} + \frac{\lambda' - \lambda}{k} \Big)^2.$$

To conclude it is enough to prove that $\widetilde{K}_\Gamma^k(\lambda, \lambda') = (1 + \mathcal{O}(W^{-2}))$ for $\lambda, \lambda' \in \Gamma \times \mathbb{R}$ and $|\lambda - \lambda'| < c_\Gamma/2$.

Perform a contour deformation in each of the variables λ_1^j, so that Γ is replaced with a homologous contour $\Gamma'(\lambda_1, \lambda_1')$ which contains the straight segment

$$L(\lambda_1, \lambda_1') = [\lambda_1 - 3kW^{-2/5}e^{i\arg(\lambda_1' - \lambda_1)}, \lambda_1' + 3kW^{-2/5}e^{i\arg(\lambda_1' - \lambda_1)}]$$

and still satisfies the conclusions (2)–(3) of Lemma 5.1. We claim that for $\iota = 1, 2$ one has

$$\Re\phi(\lambda_\iota) \geq c_k |\lambda_\iota - \hat{\lambda}_\iota|^2 \tag{5.1}$$

on the integration contour. For $\iota = 2$ $\Re\phi(\lambda_2) = \phi(\lambda_2)$ and the result from the positive definiteness of the quadratic form. Also, for $\iota = 1$ we have

$$\Re\phi(\lambda_1) \geq \tilde{c} \sum_{j=0}^{k-1} |(\lambda_1^j - \hat{\lambda}_1^j) - (\lambda_1^{j+1} - \hat{\lambda}_1^{j+1})|^2 \tag{5.2}$$

in each of the following regions:

$$\mathrm{reg}_a = \{\lambda_1 : \text{for all } j \text{ such that } \lambda_1^j \in L(\lambda_1, \lambda_1')\}, \tag{5.3a}$$

$$\mathrm{reg}_b = \{\lambda_1 : \max_j |\lambda_1^j| \geq C_0\}, \tag{5.3b}$$

when $C_0 = C_0(k)$ is chosen to be sufficiently large. For these regions, (5.2) follows from the condition (2) on the slope. To prove (5.2) for the remaining values of λ_1, let $(x(t), y(t))_{t \in \mathbb{R}}$ be a parametrisation of $\Gamma'(\lambda_1, \lambda_1')$. Then for $j = 1, \ldots k - 1$

$$\frac{\partial}{\partial t_j} \Re\phi(x(t_1) + iy(t_1), x(t_2) + iy(t_2), \cdots)$$

$$= \left(x(t_j) - \frac{x(t_{j+1}) + x(t_{j-1})}{2}\right) x'(t_j) - \left(y(t_j) - \frac{y(t_{j+1}) + y(t_{j-1})}{2}\right) y'(t_j). \tag{5.4}$$

Taking into account condition (2), we obtain that (5.4) has a definite sign when λ_1^j lies outside the curvilinear interval containing λ_1^{j-1} to λ_1^{j+1}: indeed, if, for example, λ_1^j lies to the right of this interval, then

$$\arg\left[(x(t_j) - \frac{x(t_{j+1}) + x(t_{j-1})}{2}) + i(y(t_j) - \frac{y(t_{j+1}) + y(t_{j-1})}{2})\right] \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right),$$
$$\arg(x'(t_j) - iy'(t_j)) \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right),$$

whence (5.4) > 0. Therefore the minimum of $\phi(\lambda_1)$ is attained when λ_1^j lies between λ_1^{j-1} and λ_1^{j+1} on the contour. This is true for any j, hence the minimum of ϕ in the part of the contour defined by $\max_j |\lambda_1^j| \leq C_0$ is attained when the coordinates λ_1^j are ordered, and in particular all lie in $L(\lambda_1, \lambda_1')$. Hence $\min_{\Gamma'(\lambda_1, \lambda_1') \backslash \mathrm{reg}_a} \Re\phi(\lambda_1) > c > 0$. This completes the proof of (5.2) and hence also of (5.1).

Now split the integral into two pieces,

$$J_1 = \int\limits_{|\hat{\lambda}-\lambda|<W^{-2/5}} \quad\text{and}\quad J_2 = \int\limits_{|\hat{\lambda}-\lambda|\geq W^{-2/5}}.$$

In J_1 we approximate $e^{-\sum(V(\lambda^j)-V(\hat{\lambda}_j))}$ by a linear function:

$$e^{-\sum_{j=1}^{k-1} V(\lambda^j)} = e^{-\sum_{j=1}^{k-1} V(\hat{\lambda}^j)}\Big\{1 + \sum_{j=1}^{k-1}\sum_{\iota=1,2}(\partial_\iota V)(\hat{\lambda}^j)(\lambda_\iota^j - \hat{\lambda}_\iota^j) + \mathcal{O}(|\hat{\lambda}-\lambda|^2)\Big\}.$$

After replacing the left-hand side with the right-hand side, we may extend the integral to the full straight line containing $L(\lambda_1,\lambda_1')$ (at the expense of adding a negligible term); then the constant term gives the asymptotics, the integral of the linear term vanishes by symmetry, and the error term is $\mathcal{O}(W^{-2})$. For J_2, we insert absolute values and use (5.1). $\qquad\square$

Proof of Proposition 5.2. By Riesz–Thorin interpolation, it suffices to show that, for sufficiently large k,

$$\|K_\Gamma^k\|_1, \|K_\Gamma^k\|_\infty \leq 1 + (W^{-1}), \tag{5.5}$$

$$\|K_\Gamma^k\|_2 \leq 1 - \frac{c}{W}. \tag{5.6}$$

To prove (5.5) we recall (3.2), which implies that

$$\max(\|K_\Gamma^k\|_1, \|K_\Gamma^k\|_\infty) = \sup_\lambda \int\limits_{\Gamma\times\mathbb{R}} |d\lambda_1'||d\lambda_2'||K_\Gamma^k(\lambda,\lambda')|.$$

Consider two cases. If $|\lambda| \leq 2C_\Gamma$, we split

$$\int |d\lambda_1'||d\lambda_2'||K_\Gamma^k(\lambda,\lambda')| = \int\limits_{|\lambda'-\lambda|<c_\Gamma/3} + \int\limits_{|\lambda'-\lambda|\geq c_\Gamma/3} = J_1 + J_2.$$

According to Lemma 5.3,

$$\begin{aligned}
|J_1| &\leq \max_{|\lambda'-\lambda|<c_\Gamma/3} \sqrt{\frac{|\lambda_1-\lambda_1'|^2}{\Re(\lambda_1-\lambda_1')^2}} + (W^{-1})\\
&\leq \begin{cases} 1 + (W^{-1}), & |\lambda_1| < 2c_\Gamma/3,\\ \cos^{-1/2}\left(\frac{\pi}{2}(1-c_\Gamma)\right)\exp(-k\Re V_1(c_\Gamma/3)) + (W^{-1}), & |\lambda_1| \geq 2c_\Gamma/3. \end{cases}
\end{aligned}$$

Choosing k sufficiently large we can ensure that

$$J_1 \leq 1 + (W^{-1}), \quad |\lambda| \leq 2C_\Gamma.$$

The bound obtained by taking absolute values and bounding $\Re V \geq 0$ in the definition of K_Γ suffices to see that J_2 is exponentially small in W^2.

In the case $|\lambda| \geq 2C_\Gamma$ we split

$$\int |d\lambda_1'||d\lambda_2'||K_\Gamma^k(\lambda, \lambda')| = \int\limits_{|\lambda'-\lambda|<C_\Gamma/2} + \int\limits_{|\lambda'-\lambda|\geq C_\Gamma/2}. \tag{5.7}$$

Arguing as before, with the bounds for the non-deformed case in place of Lemma 5.3 for the first integral, we obtain that the left-hand side of (5.7) is less than one and, in, fact, decays exponentially with k. This concludes the proof of (5.5).

To prove (5.6) we apply semiclassical reasoning as in the proof of Proposition 3.1. Let χ_1, χ_2 be smooth bump functions in a small neighbourhood of the minima 0, $\left(0, \sqrt{1 - \frac{E^2}{4}}\right)$ of $\Re V$ (we make sure that the radius of the neighbourhood is $\leq c_\Gamma/10$). Let χ_3 be such that $\chi_1^2 + \chi_2^2 + \chi_3^2 = 1$. Then

$$|K_\Gamma^k(\lambda, \lambda')| = \sum_{i=1,2,3} \chi_i(\lambda) |K_\Gamma^k(\lambda, \lambda')| \chi_i(\lambda')$$
$$+ \frac{1}{2} \sum_{i=1,2,3} |K_\Gamma^k(\lambda, \lambda')| [\chi_i(\lambda) - \chi_i(\lambda')]^2.$$

The norm of the second term is $\mathcal{O}(W^{-2})$. Next,

$$\left\| \sum_{i=1,2,3} \chi_i |K_\Gamma^k| \chi_i \right\|_2 = \sup_{\|\phi\|_2=1} \sum_{i=1,2,3} (\chi_i \phi, |K_\Gamma^k| \chi_i \phi)_\Gamma$$
$$\leq \max_i \|\mathbb{1}_{\operatorname{supp} \chi_i} |K_\Gamma^k| \mathbb{1}_{\operatorname{supp} \chi_i}\|_2 \sup_{\|\phi\|_2=1} \sum_{i=1,2,3} \|\chi_i \phi\|_2^2$$
$$= \max_i \|\mathbb{1}_{\operatorname{supp} \chi_i} |K_\Gamma^k| \mathbb{1}_{\operatorname{supp} \chi_i}\|_2.$$

For $i = 1, 2$ we use Lemma 5.3 and bound

$$\|\mathbb{1}_{\operatorname{supp} \chi_i} |K_\Gamma^k| \mathbb{1}_{\operatorname{supp} \chi_i}\|_2 \leq 1 - c/W$$

as in the proof of Proposition 3.1. For $i = 3$ we argue as in the proof of (5.5) above and show that

$$\|\mathbb{1}_{\operatorname{supp} \chi_3} |K_\Gamma^k| \mathbb{1}_{\operatorname{supp} \chi_3}\|_2 \leq c < 1$$

for sufficiently large k. $\qquad\square$

5.3. Proofs of the theorems close to the edge. Let Γ be a contour as in Lemma 5.1. Define an operation $\mathcal{T}_\Gamma$ on functions $\Gamma \times \mathbb{R} \to \mathbb{C}$ via the same formula

$$\mathcal{T}_\Gamma = e^{-V - W^2 \lambda} \delta_0 + T_\Gamma, \quad T_\Gamma = e^{-V} \Lambda \delta^\star_{W^2, \Gamma} \Lambda^{-1},$$

where for example

$$(\delta^\star_{W^2, \Gamma} f)(\lambda) = \frac{W^2}{2\pi} \int_{\Gamma \times \mathbb{R}} e^{-\frac{1}{2} W^2 (\lambda' - \lambda)^2} f(\lambda') \, d\lambda'_1 \, d\lambda'_2.$$

By analyticity, the formulæ of Corollary 2.14 remain valid with $\mathcal{T}_\Gamma$ in place of $\mathcal{T}$. For the same reason, in R coordinates all formulas still hold replacing the integration contour for the variable a by Γ.

Let us show (cf. Remark 3.9) that the conclusion of Proposition 3.8 remains valid for the deformed operator. Indeed, take $u_0^{(M)}$ from Proposition 3.8. and denote by $u_{0,\Gamma}^{(M)}$ its analytic continuation on Γ. By condition (1) of Lemma 5.1 the contour Γ goes along the real axis in the vicinity of the origin, whereas, by condition (2) $u_{0,\Gamma}^{(M)}$ decays away from the origin. Therefore $u_{0,\Gamma}^{(M)}$ boasts the same properties as the approximate eigenfunction from Proposition 3.8. In particular, setting

$$v_\Gamma^{(M)} := \mathcal{T}_\Gamma u_{0,\Gamma}^{(M)} - u_{0,\Gamma}^{(M)},$$

$\|v_\Gamma^{(M)}\|_p$ and $\|\Lambda^{-1} v_\Gamma^{(M)}\|_p$ enjoy the same bounds as $\|v^{(M)}\|_p$ and $\|\Lambda^{-1} v^{(M)}\|_p$. By Proposition 5.2, one can choose k so that

$$\|(e^{-V/2} \delta^\star_{W^2, \Gamma} e^{-V/2})^k\|_p \leq 1 - c_p W^{-1},$$

hence

$$\|(e^{-V} \delta^\star_{W^2, \Gamma})^n\|_p \leq C^k (1 - c_p W^{-1})^{\lfloor \frac{n-1}{k} \rfloor}.$$

Therefore Proposition 3.7 also remains valid for $\mathcal{T}_\Gamma$, that is, $u_{0,\Gamma}^{(M)}$ can be upgraded to an exact solution of $\mathcal{T}_\Gamma u_\Gamma = u_\Gamma$ with the usual bound on the error $u_\Gamma - u_{0,\Gamma}^{(M)}$. From this point the argument proceeds as in Section 4.

Appendix A. Some additional useful identities

Recall the definition of $d\mu_z(R) := d R \, e^{-\frac{z}{2} \operatorname{Str} R^2}$ from Section 2.3, where $z \in \mathbb{C}$, with $Re(z) > 0$. The following two lemmas extend Lemma 2.9 to a larger set of polynomials. Their proof follows the same strategy as the one of Lemma 2.9.

Lemma A.1. *For any $z \in \mathbb{C}$, with $\mathrm{Re}(z) > 0$ and supermatrix R the following identities hold*:

(a) $\displaystyle\int d\mu_z(R') \, \mathrm{Str}(R' + R)^2 \, \mathrm{Str}(R' + R) = \mathrm{Str} \, R^2 \, \mathrm{Str} \, R + \frac{2}{z} \, \mathrm{Str} \, R,$

(b) $\displaystyle\int d\mu_z(R') \, \mathrm{Str}(R' + R)^2 [\mathrm{Str}(R' + R)]^n$
$\qquad = \mathrm{Str} \, R^2 (\mathrm{Str} \, R)^n + \frac{2n}{z} (\mathrm{Str} \, R)^n \quad \text{for } n \geq 1,$

(c) $\displaystyle\int d\mu_z(R') [\mathrm{Str}(R' + R)^2]^2 = [\mathrm{Str} \, R^2]^2 + \frac{4}{z} \, \mathrm{Str} \, R^2,$

(d) $\displaystyle\int d\mu_z(R') \, \mathrm{Str}(R' + R)^3 \, \mathrm{Str}(R' + R) = \mathrm{Str} \, R^3 \, \mathrm{Str} \, R + \frac{3}{z} \, \mathrm{Str} \, R^2,$

(e) $\displaystyle\int d\mu_z(R') \, \mathrm{Str}(R' + R)^5 = \mathrm{Str} \, R^5 + \frac{5}{z} \, \mathrm{Str} \, R \, \mathrm{Str} \, R^2 + \frac{5}{z} \, \mathrm{Str} \, R,$

(f) $\displaystyle\int d\mu_z(R') \, \mathrm{Str}(R' + R)^2 [\mathrm{Str}(R' + R)]^3 = \mathrm{Str} \, R^2 (\mathrm{Str} \, R)^3 + \frac{6}{z} (\mathrm{Str} \, R)^3,$

(g) $\displaystyle\int d\mu_z(R') \, \mathrm{Str}(R' + R)^2 [\mathrm{Str}(R' + R)]^4 = \mathrm{Str} \, R^2 (\mathrm{Str} \, R)^4 + \frac{8}{z} (\mathrm{Str} \, R)^4.$

Lemma A.2. *Let $\mu = W/(W + 2\alpha)$ and set $W \gg 1$. For any supermatrix R the following identities hold*:

(a) $\displaystyle\int d\mu_{\frac{W^2}{\mu}}(R') \, \mathrm{Str}(R' + R)^2 \, \mathrm{Str}(R' + R)^3$
$\qquad = \mathrm{Str} \, R^2 \, \mathrm{Str} \, R^3 + \frac{6\mu}{W^2} \, \mathrm{Str} \, R \, \mathrm{Str} \, R^2 + \mathcal{O}(W^{-4} R),$

(b) $\displaystyle\int d\mu_{\frac{W^2}{\mu}}(R') \, \mathrm{Str}(R' + R)^3 [\mathrm{Str}(R' + R)]^2$
$\qquad = \mathrm{Str} \, R^3 (\mathrm{Str} \, R)^2 + \frac{6\mu}{W^2} \, \mathrm{Str} \, R^3 + \mathcal{O}(W^{-4} R),$

(c) $\displaystyle\int d\mu_{\frac{W^2}{\mu}}(R') W \, \mathrm{Str}(R' + R)^3 \, \mathrm{Str}(R' + R)^4$
$\qquad = W \, \mathrm{Str} \, R^3 \, \mathrm{Str} \, R^4 + \frac{2\mu}{W} \, \mathrm{Str} \, R^3 (\mathrm{Str} \, R)^2$
$\qquad\quad + \frac{12\mu}{W} \, \mathrm{Str} \, R^5 + \mathcal{O}(W^{-5} R) + \mathcal{O}(W^{-3} R^3),$

(d) $\displaystyle\int d\mu_{\frac{W^2}{\mu}}(R') W^2 \, \mathrm{Str}(R' + W^{-\frac{1}{2}} R)^3]^3$
$\qquad = W^2 [\mathrm{Str}(W^{-\frac{1}{2}} R)^3]^3 + 3^3 \mu \, \mathrm{Str}(W^{-\frac{1}{2}} R)^3 \, \mathrm{Str}(W^{-\frac{1}{2}} R)^4$
$\qquad\quad + \mathcal{O}(W^{-2} \sqrt{W}^{-5}).$

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