J. Spectr. Theory 11 (2021), 229–279
DOI 10.4171/JST/340

Wildly perturbed manifolds: norm resolvent and spectral convergence

Colette Anné and Olaf Post

Abstract. The publication of the important work of Rauch and Taylor [J. Funct. Anal. 18 (1975)] started a hole branch of research on wild perturbations of the Laplace-Beltrami operator. Here, we extend certain results and show *norm* convergence of the resolvent. We consider a (not necessarily compact) manifold with many small balls removed, the number of balls can increase as the radius is shrinking, the number of balls can also be infinite. If the distance of the balls shrinks less fast than the radius, then we show that the Neumann Laplacian converges to the unperturbed Laplacian, i.e., the obstacles vanish. In the Dirichlet case, we consider two cases here: if the balls are too sparse, the limit operator is again the unperturbed one, while if the balls concentrate at a certain region (they become "solid" there), the limit operator is the Dirichlet Laplacian on the complement of the solid region. Norm resolvent convergence in the limit case of homogenisation is treated by Khrabustovskyi and the second author in another article (see also the references therein). Our work is based on a norm convergence result for operators acting in varying Hilbert spaces described in a book from 2012 by the second author.

Mathematics Subject Classification (2010). Primary: 58J50; Secondary: 35P15, 53C23, 58J32.

Keywords. Laplacian on a manifold, perturbation of manifolds, manifolds with many holes, norm resolvent convergence.

Contents

1. Introduction

In this article, we present *norm* convergence of the resolvents of Laplacians on manifolds with wild perturbations. Wild perturbations refers here to increase the complexity of topology. In particular, we show convergence of the Laplace-Beltrami operator on manifolds with an increasing number of small holes.

1.1. Main results. Since the perturbation changes the space on which the operators act, we need to define a *generalised norm resolvent convergence* for operators on varying spaces (see Definition 1.1). This powerful tool and many consequences (like convergence of eigenvalues, eigenfunctions, functions of the operators such as spectral projections, heat operators etc.) is explained in detail in a book by the second author [35]. Let us stress here that we do not need a compactness assumption on the space or the resolvents as in many of the previous works (see Section 1.2). Moreover, the abstract convergence result shows its full strengths especially when the perturbed space is not a subset of the unperturbed one or vice versa: an example is given by adding many small handles to a manifold; we treat this problem in a subsequent publication [3].

We give sufficient conditions on the obstacles in Theorems 4.3 and 5.2 to have (generalised norm resolvent) convergence to the unperturbed situation (obstacles without an effect) where we remove a family of obstacles and consider on the remaining manifold either the Neumann or Dirichlet Laplacian. In the Dirichlet case, there is a regime when the obstacles can become "solid" (Theorem 6.4). These abstract results use as assumptions e.g. non-concentrating of energy-bounded functions on the obstacles and extension properties in the Neumann case.

We make these abstract results concrete in Theorems 4.7, 5.6 and 6.16, where we assume that the obstacles consist of many small balls having a certain minimal distance, and filling up the "solid" region for Theorem 6.16, a terminology introduced in [40] to describe the situation under the name "crushed ice problem" where small obstacles such as holes maintained at zero temperature increase in number while their size converge to 0 in such a way that they *freeze* at the limit. A typical assumption here is that small balls in the manifold look everywhere roughly the same; this is assured if the harmonic radius is uniformly positive; and the latter follows if the manifold has *bounded geometry*, see Definition 3.2 and Proposition 3.5.

Let us first explain the main idea behind the abstract convergence tool: In all our results, we deal with an ε-dependent space X_ε and suitable Laplace operators

Δ_ε acting on X_ε for each $\varepsilon \ge 0$. We define a *generalised norm resolvent convergence* for Δ_ε to a limit Laplacian Δ_0. To do so, we need so-called *identification* or *transplantation* operators $J = J_\varepsilon : \mathcal{H}_0 := \mathsf{L}_2(X_0) \to \mathcal{H}_\varepsilon := \mathsf{L}_2(X_\varepsilon)$, which are asymptotically unitary (cf. (1.1a)) and intertwine the resolvents (cf. (1.1b)) in the following sense:

Definition 1.1. We say that Δ_ε *converges in general norm resolvent sense* to Δ_0 if there exist bounded operators $J = J_\varepsilon$ and $m \ge 0$ such that

$$\|(\mathrm{id}_{\mathcal{H}_0} - J^*J)R_0\| \le \delta_\varepsilon, \quad \|(\mathrm{id}_{\mathcal{H}_\varepsilon} - JJ^*)R_\varepsilon\| \le \delta_\varepsilon, \tag{1.1a}$$

$$\|(JR_0 - R_\varepsilon J)R_0^{m/2}\| \le \delta_\varepsilon, \tag{1.1b}$$

where $R_0 := (\Delta_0 + 1)^{-1}$ and $R_\varepsilon := (\Delta_\varepsilon + 1)^{-1}$ for $\varepsilon > 0$ and where $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$. Moreover, $\|\cdot\|$ denotes the operator norm for operators $\mathcal{H}_0 \to \mathcal{H}_0$, $\mathcal{H}_\varepsilon \to \mathcal{H}_\varepsilon$ and $\mathcal{H}_0 \to \mathcal{H}_\varepsilon$ in (1.1a)–(1.1b), respectively.

The name is justified as follows: if $\mathcal{H}_\varepsilon = \mathcal{H}_0$, then generalised norm resolvent convergence (with $m = 0$) is just the classical norm resolvent convergence if one chooses $J = \mathrm{id}_{\mathcal{H}_0}$. In Section 2, we interpret δ_ε as a sort of "distance" between Δ_0 and Δ_ε, or more, precisely, between their corresponding quadratic forms $\mathfrak{d}_0$ and $\mathfrak{d}_\varepsilon$, and call such forms *δ_ε-quasi-unitarily equivalent*. If this distance converges to 0, then Δ_ε converges to Δ_0 in generalised norm resolvent convergence, see Section 2.

Once we have this generalised norm resolvent convergence, similar conclusions as for the classical norm resolvent convergence are valid. In particular, we have norm convergence (using also J and J^*) of the corresponding functional calculus, i.e., of $\varphi(\Delta_\varepsilon)$ towards $\varphi(\Delta_0)$ for suitable functions φ such as $\varphi = \mathbf{1}_{[a,b]}$ with $a, b \notin \sigma(\Delta_0)$ (spectral projections) or $\varphi(\lambda) = \mathrm{e}^{-t\lambda}$ (heat operator), see Theorem 2.4. Moreover, we conclude the following spectral convergence:

Theorem 1.2 ([35, Theorems 4.3.3–4.3.5], [27, Theorem 2.7]). *Assume that Δ_ε converges to Δ_0 in generalised norm resolvent sense then*

$$\sigma_\bullet(\Delta_\varepsilon) \longrightarrow \sigma_\bullet(\Delta_0)$$

uniformly (i.e., in Hausdorff distance) on any compact interval $[0, \Lambda]$. Here, $\sigma_\bullet(\Delta_\varepsilon)$ stands for the entire spectrum or the essential spectrum of Δ_ε for $\varepsilon \ge 0$.

If $\lambda_0 \in \sigma_{\mathrm{disc}}(\Delta_0)$ *is an eigenvalue of multiplicity* $\mu > 0$, *then there exist* μ *eigenvalues* (*not necessarily all distinct*) $\lambda_{\varepsilon,j}$, $j = 1 \dots \mu$, *such that* $\lambda_{\varepsilon,j} \to \lambda_0$ *as* $\varepsilon \to 0$. *In particular, if* $\mu = 1$ *and if* $\psi_0 \in \mathcal{H}_0$ *is the corresponding normalised eigenvector, then there exists a family of normalised eigenvectors* ψ_ε *of* Δ_ε *such that*

$$\|J\psi_0 - \psi_\varepsilon\|_{\mathcal{H}_\varepsilon} \longrightarrow 0 \quad \textit{and} \quad \|J^*\psi_\varepsilon - \psi_0\|_{\mathcal{H}_0} \longrightarrow 0 \tag{1.2a}$$

as $\varepsilon \to 0$.

If Δ_ε *has purely discrete spectrum* $(\lambda_k(\varepsilon))_{k\in\mathbb{N}}$ *written in increasing order and repeated according to multiplicity for each* $\varepsilon \ge 0$, *then we have*

$$|\lambda_k(\varepsilon) - \lambda_k(0)| \le 4C_\varepsilon(\lambda_k(\varepsilon) + 1)(\lambda_k(0) + 1)\delta_\varepsilon \tag{1.2b}$$

with $\lim_{\varepsilon\to 0} C_\varepsilon = 1$.

Let us also stress that we have a convergence of a (suitably sandwiched) difference of the resolvents R_0 and R_ε as operators

$$\mathsf{L}_2(X) \longrightarrow \mathsf{H}^1, \tag{1.3}$$

where H^1 is a first order Sobolev space, i.e., $\mathsf{H}^1(X_\varepsilon)$ or a closed subspace, see Proposition 2.5 and Remark 2.6 for details. Moreover, one can also show convergence of eigenvectors in *energy* norm, see (2.7).

1.2. Previous works. The results of Rauch and Taylor in [40] inspired a lot of works (cited by 85 papers in MathSciNet in November 2019), and served as a starting point of our analysis here. In particular, we borrowed the names “wild perturbations”, “fading”, “solidification” and “crushed ice” from their article, the latter three appearing already in the earlier lectures of Jeffrey Rauch [39]. It is impossible to give a comprehensive review of all literature on domain perturbations after Rauch and Taylor’s paper (and even before): we will only emphasise on the following aspects here:

Asymptotic behaviour of eigenvalues. A classical topic is how eigenvalues change under small singular domain perturbations: asymptotic expansions on Dirichlet eigenvalues on bounded domains with small obstacles taken out is given e.g. in [9, 32, 31, 10, 18, 14, 7]; the difference of the unperturbed and perturbed Dirichlet eigenvalues is of order as the capacity of the obstacle set; e.g., for balls of radius ε the capacity is of order $1/|\log \varepsilon|$ and ε^{m-2} in dimension $m = 2$ and $m \ge 3$, hence the difference of the unperturbed and perturbed k-th eigenvalue is of order ε if $m = 3$ ([32]). Using the eigenvalue estimate (1.2b), we obtain for a single ball removed in dimension $m = 3$ as error estimate $\delta_\varepsilon = \mathrm{O}(\varepsilon^{(1/6-0)})$ (see Corollary 5.7 with $\alpha = 0$), i.e., for a single obstacle, our analysis is far from being optimal.

Similarly, the asymptotic behaviour of Neumann eigenvalues has been studied for a single hole for bounded domains or compact manifolds e.g. in [33, 31, 21, 28]; again the asymptotic expansion for a single ball as obstacle gives a better estimate on the difference of the unperturbed and perturbed eigenvalues, see Example 4.8.

It seems that our method always gives only the square root of the optimal estimate (or even worse); a similar phenomenon appears for manifolds converging to metric graphs (see e.g. [38, Remark 3.9]). Nevertheless, our analysis shows its full power when considering non-compact domains and manifolds and when one is interested in the entire spectrum; as well as convergence of operator functions of the Laplacians such as the heat operators (see for instance [37, Example 1.11]). Also, we believe that our approach gives rather abstract conditions from which it follows that an obstacle "fades" in the limit, i.e., from which (generalised) norm resolvent convergence of the Neumann resp. Dirichlet Laplacian on the manifold without obstacles towards the original ("free") Laplacian follows.

Domain perturbations and convergence results. Weidmann [43] proved strong resolvent convergence (in a generalised sense) of elliptic differential operators under perturbation of the domain. Moreover, he also developed a general (strong resolvent) convergence theory for sequences of operators acting in different Hilbert spaces (which can be embedded in a larger common Hilbert space).

Daners [15] considers the *norm* convergence of resolvents of Dirichlet Laplacians for perturbations of Euclidean *bounded* domains (or at least those with compact resolvent), the norm convergence follows from the strong one under the assumption of compactness of the limit resolvent, see also [16] for a survey and the references therein. Our approach is more general as we do not assume a priori that the perturbed and unperturbed domains are embedded in a common space as in [15, 16]. Moreover, we obtain explicit error estimates in terms of δ_ε. For an older survey about strong resolvent convergence and perturbations of Euclidean domains, we refer to [22].

Homogenisation theory. Finally, Rauch and Taylor [39, 40] inspired with their *crushed ice problem* also the study of homogenisation problems (see also [29, 13] for some other pioneering works on this topic). If the density of small balls is removed from the domain is too low, then the limit of the corresponding Dirichlet Laplacian is "fading", i.e., converging to the original Laplacian. If it is too high, then in the limit "solidification" takes place, i.e., the limit Laplacian only survives on some subsets, the other became "solid". The critical parameter here is the capacity: In [6] Balzano and Notarantonio consider a compact Riemannian manifold with an increasing finite number of small balls removed. They show that if the balls are placed randomly and if their capacity converges, then the Dirichlet Laplacian on the manifold less the holes converges in *strong* resolvent sense to a Laplacian plus a potential given by the random distribution of ball centres. The proof is based on earlier works of Balzano [5] using Γ-convergence, see [17]. More recent works can be found in [26] and references therein.

For a similar approach using the above mentioned generalised *norm* resolvent convergence in the homogenisation case, we refer to [27] and the references cited therein. For an approach using the already shown strong resolvent convergence to upgrade to norm resolvent convergence (similarly as in [15, 16], but even for general unbounded domains) we refer to [12]. The very recent work [42] also treats norm resolvent convergence as operators $\mathsf{L}_2(X) \to \mathsf{H}^1(X_\varepsilon)$ on *periodic* spaces. We are also able to show estimates like (1.3), see Proposition 2.5 and Remark 2.6.

In [8] the authors show also *norm* resolvent convergence of type $\mathsf{L}_2(X) \to \mathsf{H}^1(X_\varepsilon)$ in a homogenisation problem: this time they place small balls along a curve in an infinite horizontal 2-dimensional strip as obstacles. They have a fading case and also a case of homogenisation: Here, the little holes become a delta interaction supported on the curve in the limit. The proof of norm resolvent convergence is established directly along the problem (see also the formulation of the problem in [27, Section 2]). It is straightforward to see that if we place small balls of radius ε along a curve such that they are η_ε-separated, then the fading results of Theorems 4.7 and 5.6 remain true (provided the conditions on ε and η_ε are true). We strongly believe that it is also possible to apply our concept of quasi-unitary equivalence to the homogenisation problem of [8] using basic estimates from [8] and ideas of [27].

1.3. Structure of the article. In Section 2 we briefly describe the main tool of norm convergence of operators on varying Hilbert spaces. In Section 3 we briefly introduce Laplacians and Sobolev spaces on manifolds, the harmonic radius and manifolds of bounded geometry. Moreover, we introduce the concept of non-concentration in Definition 3.7 and Proposition 3.9.

In Section 4 we present the situation for obstacles with Neumann boundary condition, the main result is Theorem 4.3 for abstract fading obstacles, and Theorem 4.7 deals with the situation where each obstacle is a disjoint union of many small balls of radius ε. Similarly, Section 5 contains results for fading Dirichlet obstacles and many balls in Theorems 5.2 and 5.6. Finally, Section 6 is about

Dirichlet obstacles that become "solid", again an abstract version and one for many balls removed in Theorems 6.4 and 6.16. We conclude with an appendix, where we collect some estimates on manifolds.

Acknowledgements. Olaf Post would like to thank the *Fédération Recherche Mathématiques des Pays de Loire* for the hospitality, through its programmes *Géanpyl* and *DéfiMaths*, at the *Université de Nantes*. Colette Anné would like to thank the *Universität Trier* for the hospitality. Both authors would like to thank the anonymous referee for carefully reading the manuscript and for many useful suggestions.

2. Main tool: norm convergence of operators on varying Hilbert spaces

The second author of the present article proposed in [34] and in more detail in the monograph [35] a general framework which assures a *generalised* norm resolvent convergence for operators Δ_ε converging to Δ_0 as $\varepsilon \to 0$. Here, each operator Δ_ε acts in a Hilbert space $\mathcal{H}_\varepsilon$ for $\varepsilon \geq 0$; and the Hilbert spaces are allowed to depend on ε. In typical applications, the Hilbert spaces $\mathcal{H}_\varepsilon$ are of the form $\mathsf{L}_2(X_\varepsilon)$ for some metric measure space X_ε which is considered as a perturbation of a "limit" metric measure space X_0; and typically, there is a topological transition between $\varepsilon > 0$ and $\varepsilon = 0$.

In order to define the convergence, we define a sort of "distance" δ_ε between $\widetilde{\Delta} := \Delta_\varepsilon$ and $\Delta := \Delta_0$, in the sense that if $\delta_\varepsilon \to 0$ then Δ_ε converges to Δ_0 in the above-mentioned generalised norm resolvent sense.

Let $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ be two separable Hilbert spaces. We say that $(\mathfrak{d}, \mathcal{H}^1)$ is an *energy form in* $\mathcal{H}$ if $\mathfrak{d}$ is a closed, non-negative and densely defined quadratic form in $\mathcal{H}$ with domain $\mathcal{H}^1$, i.e., if $\mathfrak{d}(f) := \mathfrak{d}(f, f) \geq 0$ for some sesquilinear form $\mathfrak{d}\colon \mathcal{H}^1 \times \mathcal{H}^1 \to \mathbb{C}$, denoted by the same symbol, with $\mathcal{H}^1 =: \operatorname{dom} \mathfrak{d}$ endowed with the norm defined by

$$\|f\|_1^2 := \|f\|_{\mathcal{H}^1}^2 := \|f\|_{\mathcal{H}}^2 + \mathfrak{d}(f), \tag{2.1}$$

so $\mathcal{H}^1$ is itself a Hilbert space and a dense set in $\mathcal{H}$. We denote by Δ the corresponding non-negative, self-adjoint operator the *energy operator* associated with $(\mathfrak{d}, \mathcal{H}^1)$ (see e.g. [25, Section VI.2]). Similarly, let $(\tilde{\mathfrak{d}}, \widetilde{\mathcal{H}}^1)$ be an energy form in $\widetilde{\mathcal{H}}$ with energy operator $\widetilde{\Delta}$.

Associated with an energy operator Δ, we can define a natural *scale of Hilbert spaces* $\mathcal{H}^k$ defined via the *abstract Sobolev norms*

$$\|f\|_{\mathcal{H}^k} := \|f\|_k := \|(\Delta + 1)^{k/2} f\|. \tag{2.2}$$

Then $\mathcal{H}^k = \operatorname{dom} \Delta^{k/2}$ if $k \geq 0$ and $\mathcal{H}^k$ is the completion of $\mathcal{H}$ with respect to the norm $\|\cdot\|_k$ for $k < 0$. Obviously, the scale of Hilbert spaces for $k = 1$ and its associated norm agrees with $\mathcal{H}^1$ and $\|\cdot\|_1$ defined above (see [35, Section 3.2] for details). Similarly, we denote by $\widetilde{\mathcal{H}}^k$ the scale of Hilbert spaces associated with $\widetilde{\Delta}$.

We denote by $\sigma(\Delta)$ the spectrum of the energy operator and by $R(z) = (\Delta - z)^{-1}$ its resolvent at $z \in \mathbb{C} \setminus \sigma(\Delta))$ and for short $R = R(-1) = (\Delta + 1)^{-1}$, we use similar notations for $\widetilde{\Delta}$.

We now need pairs of so-called *identification* or *transplantation operators* acting on the Hilbert spaces and later also pairs of identification operators acting on the form domains. Note that our definition is slightly more general than the one in [35, Section 4.4]. The new point here is that we allow the (somehow "smoothing") resolvent power of order $k/2$ on the right hand side in (2.3d) and (2.3d′) also for $k > 0$ (see Remark 2.7 for more details).

Definition 2.1. Let $\delta \geq 0$, and let $J: \mathcal{H} \to \widetilde{\mathcal{H}}$ and $J': \widetilde{\mathcal{H}} \to \mathcal{H}$ be linear bounded operators.[1] Moreover, let $\delta \geq 0$, and let $J^1: \mathcal{H}^1 \to \widetilde{\mathcal{H}}^1$ and $J'^1: \widetilde{\mathcal{H}}^1 \to \mathcal{H}^1$ be linear bounded operator on the energy form domains.

(i) We say that J is δ-*quasi-unitary* with δ-*quasi-adjoint* J' if

$$\|Jf\| \leq (1+\delta)\|f\|, \qquad \left|\langle Jf, u\rangle - \langle f, J'u\rangle\right| \leq \delta \|f\|\|u\| \tag{2.3a}$$

for $f \in \mathcal{H}$, $u \in \widetilde{\mathcal{H}}$, and

$$\|f - J'Jf\| \leq \delta\|f\|_1, \qquad \|u - J'Ju\| \leq \delta\|u\|_1 \tag{2.3b}$$

for $f \in \mathcal{H}^1, u \in \widetilde{\mathcal{H}}^1$.

(ii) We say that J^1 and J'^1 are δ-*compatible* with the identification operators J and J' if

$$\|J^1 f - Jf\| \leq \delta\|f\|_1, \qquad \|J'^1 u - J'u\| \leq \delta\|u\|_1 \tag{2.3c}$$

for $f \in \mathcal{H}^1, u \in \widetilde{\mathcal{H}}^1$.

(iii) We say that the energy forms $\mathfrak{d}$ and $\tilde{\mathfrak{d}}$ are δ-*close* (*of order* $k \geq 1$) if

$$\left|\tilde{\mathfrak{d}}(J^1 f, u) - \mathfrak{d}(f, J'^1 u)\right| \leq \delta\|f\|_k\|u\|_1 \tag{2.3d}$$

for $f \in \mathcal{H}^k, u \in \widetilde{\mathcal{H}}^1$.

[1] In our applications here, we set $J' = J^*$.

(iv) We say that $\mathfrak{d}$ and $\tilde{\mathfrak{d}}$ are *δ-quasi unitarily equivalent* (*of order* $k \geq 1$), if (2.3a)–(2.3d) are fulfilled, i.e.,

- if there exists identification operators J and J' such that J is δ-quasi-unitary with δ-adjoint J' (i.e., (2.3a)–(2.3b) hold);
- if there exists identification operators J^1 and J'^1 which are δ-compatible with J and J' (i.e., (2.3c) holds);
- and if $\mathfrak{d}$ and $\tilde{\mathfrak{d}}$ are δ-close (of order k) (i.e., (2.3d) holds).

We comment on the asymmetry in (2.3d) with respect to the norms $\|f\|_k$ and $\|u\|_1$ in Remark 2.7 at the end of this section.

In operator norm notation, δ-quasi-unitary equivalence means

$$\|J\| \leq 1+\delta, \quad \|J^* - J'\| \leq \delta \tag{2.3a$'$}$$

$$\|(\mathrm{id}_{\mathcal{H}} - J'J)R^{1/2}\| \leq \delta, \quad \|(\mathrm{id}_{\tilde{\mathcal{H}}} - JJ')\tilde{R}^{1/2}\| \leq \delta, \tag{2.3b$'$}$$

$$\|(J^1 - J)R^{1/2}\| \leq \delta, \quad \|(J'^1 - J')\tilde{R}^{1/2}\| \leq \delta, \tag{2.3c$'$}$$

$$\|\tilde{R}^{1/2}(\tilde{\Delta}J^1 - (J'^1)^*\Delta)R^{k/2}\| \leq \delta, \tag{2.3d$'$}$$

where $R := (\Delta+1)^{-1}$ resp. $\tilde{R} := (\tilde{\Delta}+1)^{-1}$ denotes the resolvent of Δ resp. $\tilde{\Delta}$ in -1. Moreover, $(J'^1)^*\colon \mathcal{H}^{-1} \to \tilde{\mathcal{H}}^{-1}$ where $(\cdot)^*$ denotes here the dual map with respect to the dual pairing $\mathcal{H}^1 \times \mathcal{H}^{-1}$ induced by the inner product on $\mathcal{H}$ and similarly on $\tilde{\mathcal{H}}$. Moreover, Δ is interpreted as $\Delta\colon \mathcal{H}^1 \to \mathcal{H}^{-1}$, and similarly for $\tilde{\Delta}$.

To give a flavour of the ideas, we give a short proof of the following result:

Proposition 2.2. *Let $\mathfrak{d}$ and $\tilde{\mathfrak{d}}$ be δ-quasi-unitarily equivalent* (*of order* $k \geq 1$), *then we have*

$$\big\|\big(JR - \tilde{R}J\big)R^{m/2}\big\| \leq 7\delta \quad \textit{for } m = \max\{k-2, 0\}. \tag{2.4}$$

In particular, if the energy forms $\mathfrak{d}_\varepsilon$ and $\mathfrak{d}_0$ are δ_ε-quasi-unitarily equivalent of order $k \geq 1$ then the corresponding operators Δ_ε converge in generalised norm resolvent sense to Δ_0 of order m (*cf. Definition* 1.1) *and the conclusions of Theorem* 1.2 *hold.*

Note that we can ignore the factors $R^{m/2}$ in (2.4) and (2.6a) if $k \in \{1, 2\}$.

Proof. We have the expansion

$$(JR - \tilde{R}J)R^{m/2} = (J - J^1)R^{m/2+1} + \big(J^1R - \tilde{R}(J'^1)^*\big)R^{m/2} + \tilde{R}^{1/2}(\tilde{R}^{1/2}((J'^1)^* - (J')^*))R^{m/2} + \tilde{R}((J')^* - J)R^{m/2},$$

where the second term can be further expanded into

$$\begin{aligned}(J^1 R - \widetilde{R}(J'^1)^*)R^{m/2} \\ &= \widetilde{R}((\widetilde{\Delta}+1)J^1 - (J'^1)^*(\Delta+1))R^{m/2+1} \\ &= \widetilde{R}(\widetilde{\Delta}J^1 - (J'^1)^*\Delta)R^{m/2+1} \\ &\quad + \widetilde{R}((J^1 - J) + (J - (J')^*) + ((J')^* - (J'^1)^*))R^{m/2+1}.\end{aligned} \tag{2.5}$$

Taking the operator norm, and using $\|A^*\| = \|A\|$ for the dual of an operator, we obtain from the last two equations (as $m \geq 0$ and $m+2 \geq k$)

$$\begin{aligned}\|(JR - \widetilde{R}J)R^{m/2}\| &\leq 2\|(J - J^1)R^{1/2}\| + \|\widetilde{R}^{1/2}(\widetilde{\Delta}J^1 - (J'^1)^*\Delta)R^{k/2}\| \\ &\quad + 2\|(J'^1 - J')\widetilde{R}^{1/2}\| + 2\|J' - J^*\| \\ &\leq 7\delta.\end{aligned}$$

□

Remark 2.3. The last proposition explains the notation in two extreme cases.

(i) *"0-quasi-unitary equivalence" is "unitary equivalence."* If $\delta = 0$ then J is 0-quasi-unitary if and only if J is unitary with $J^* = J'$. Moreover, $\mathfrak{d}$ and $\widetilde{\mathfrak{d}}$ are 0-quasi-unitarily equivalent (of order $k \geq 1$) if and only if Δ and $\widetilde{\Delta}$ are unitarily equivalent (in the sense that $JR = \widetilde{R}J$, see (2.4)). In this sense, δ-quasi unitary equivalence is a *quantitative generalisation of unitary equivalence.*

(ii) *"δ_ε-quasi-unitary equivalence" (with $\delta_\varepsilon \to 0$) is a generalisation of "norm resolvent convergence."* If $\mathcal{H} = \widetilde{\mathcal{H}}$ and $\mathcal{H}^1 = \widetilde{\mathcal{H}}^1$ (i.e., $\operatorname{dom}\mathfrak{d} = \operatorname{dom}\widetilde{\mathfrak{d}}$), and if we choose all identification operators to be the respective identity maps, then δ_ε-quasi-unitary equivalence of order $k \in \{1,2\}$ (with $\delta_\varepsilon \to 0$) implies (classical) norm resolvent convergence. In particular, Proposition 2.2 is a generalisation of a result by Kato [25, Theorem VI.3.6]; see also the discussion in [27, Remark 3.4] and the one in great detail in [37, pp. 6–7].

We also have the following functional calculus result.

Theorem 2.4 (see [35, Section 4.2, Theorem 4.2.11, Lemma 4.2.13]). *Let $U \subset (-1,\infty)$ be open and unbounded, and let $\varphi\colon [0,\infty) \to \mathbb{R}$ be analytic on U such that $\lim_{\lambda\to\infty}\varphi(\lambda)$ exists, then there exists a constant C_φ depending only on φ and U such that*

$$\|(J\varphi(\Delta) - \varphi(\widetilde{\Delta})J)R^{m/2}\| \leq C_\varphi\delta \tag{2.6a}$$

for all $\mathfrak{d}$ *and* $\tilde{\mathfrak{d}}$ *being* δ*-quasi-unitary equivalent energy forms (of order* $k \geq 1$*) with* $\sigma(\Delta) \subset U$ *or* $\sigma(\widetilde{\Delta}) \subset U$*. Moreover, if* $k \in \{1, 2\}$ *then we can replace* (2.6a) *by*

$$\|\varphi(\widetilde{\Delta}) - J\varphi(\Delta)J'\| \leq 5C'_\varphi\delta + C_\varphi\delta, \quad \textit{where } C'_\varphi := \sup_{\lambda \in U}(\lambda+1)^{1/2}|\varphi(\lambda)|. \tag{2.6b}$$

In particular, if $\varphi = \mathbf{1}_{[a,b]}$ with $a, b \notin \sigma(\Delta)$ then (2.6a)–(2.6b) are norm estimates of *spectral projections*. Moreover, if $\varphi_t(\lambda) = \mathrm{e}^{-t\lambda}$ for $t > 0$, then we have norm estimates of the *heat operators*. One can also prove similar operator norm estimates on $J'\varphi(\widetilde{\Delta})J - \varphi(\Delta)$. If φ is only continuous on U, then one has to replace $C_\varphi\delta$ by δ_φ with $\delta_\varphi \to 0$ as $\delta \to 0$.

As a conclusion, spectral convergence as in Theorem 1.2 follows. Note that we also have convergence of eigenfunctions in energy norm, namely we can replace (1.2a) by

$$\|J^1\psi_0 - \psi_\varepsilon\|_1 \leq C'_1\delta_\varepsilon \longrightarrow 0 \tag{2.7}$$

as $\varepsilon \to 0$ using a similar argument as in [36, Proposition 2.6].

A slight modification of the proof of Proposition 2.2 gives us a norm estimate of a suitably sandwiched resolvent difference as operator $\mathcal{H} \to \widetilde{\mathcal{H}}^1$; for simplicity we assume $k \in \{1, 2\}$ here:

Proposition 2.5. *Let* $\mathfrak{d}$ *and* $\tilde{\mathfrak{d}}$ *be* δ*-quasi-unitarily equivalent (of order* $k \in \{1, 2\}$*), then we have*

$$\|J^1R - \widetilde{R}J\|_{\mathcal{H}\to\widetilde{\mathcal{H}}^1} = \|(\widetilde{H}+1)^{1/2}(J^1R - \widetilde{R}J)\| \leq 6\delta. \tag{2.8}$$

Proof. The proof is similar to the one of Proposition 2.2 (with $m = 0$). Here, we have the expansion

$$(J^1R - \widetilde{R}J) = (J^1R - \widetilde{R}(J'^1)^*) + \widetilde{R}^{1/2}(\widetilde{R}^{1/2}((J'^1)^* - (J')^*)) + \widetilde{R}((J')^* - J).$$

The first term can again be expanded as in (2.5); note that we can factor out $\widetilde{R}^{1/2}$ from the left, and all remaining terms can be estimated by (2.3a′)–(2.3d′). As we have one term less than in the proof of Proposition 2.2, we end up with 6δ. □

Remark 2.6. In our applications, the space $\mathcal{H}$ is an L_2-space of an unperturbed set X such as $\mathsf{L}_2(X)$ and $\widetilde{\mathcal{H}}$ is a perturbed space $\mathsf{L}_2(X_\varepsilon)$ where $X_\varepsilon = X \setminus B_\varepsilon$ for some obstacle set B_ε shrinking in a suitable manner. Moreover, the operators are Neumann or Dirichlet Laplacians (see the next section for details). The above convergence (2.8) then means convergence of the resolvents as operators $\mathsf{L}_2(X) \to \mathsf{H}^1$ if H^1 denotes the first order Sobolev space associated with the form domain of the perturbed Laplacian Δ_ε, typically $\mathsf{H}^1(X_\varepsilon)$ or a closed subspace.

We can also formulate similar results as in Theorem 2.4 as conclusions of (2.8).

Remark 2.7. The asymmetry of (2.3d) with respect to the norms $\|f\|_k$ and $\|u\|_1$ has the following reason: As explained in the previous remark, $\widetilde{\mathcal{H}} = \mathsf{L}_2(X_\varepsilon)$ will be a parameter dependent space, hence $\|u\|_1$ is just the energy norm with respect to a Laplacian. Dealing here with higher order norms $\|u\|_k$ ($k \geq 2$) would force us to control the estimate in terms of the *graph norm* of the corresponding Laplacians. We normally use the corresponding Sobolev norm of order k, but then we need an elliptic estimate of the form $\|u\|_{\mathsf{H}^k(X_\varepsilon)} \leq C_\varepsilon \|(\Delta_{X_\varepsilon} + 1)^{k/2} u\|_{\mathsf{L}_2(X_\varepsilon)}$ on the parameter-depending manifolds X_ε; and we would then need information about the (complicated) geometry of X_ε in order to have some control over the dependency of C_ε on ε. Instead, we use such arguments only on the parameter-independent manifold $\mathcal{H} = \mathsf{L}_2(X)$ with its parameter-independent Laplacian.

The asymmetry seems to be a key ingredient in order to use the concept of quasi-unitary equivalence for perturbed domains; see also Remark 4.4 why the energy norm is not enough.

3. Laplacians on manifolds

3.1. Energy form, Laplacian and Sobolev spaces associated with a Riemannian manifold. Let (X, g) be a complete[2] Riemannian manifold of dimension $n \geq 2$, for the moment without boundary. Denote by $\mathrm{d}g$ the Riemannian measure induced by the metric g on X (we often omit the measure if it is clear from the context). Then $\mathsf{L}_2(X) = \mathsf{L}_2(X, g)$ is the usual L_2-space with norm given by

$$\|u\|^2_{\mathsf{L}_2(X,g)} := \int_X |u|^2 \, \mathrm{d}g.$$

The *energy form* associated with (X, g) is defined by

$$\mathfrak{d}_{(X,g)}(u) := \int_X |du|^2_g \, \mathrm{d}g$$

for u in the first Sobolev space $\mathsf{H}^1(X) = \mathsf{H}^1(X, g)$, which can be defined as the completion of smooth functions with compact support, under the so-called *energy*

[2] Most of the results are also true for incomplete manifolds, but then we have some more technicalities with fixing different boundary conditions and with elliptic regularity. In order to keep this presentation readable, we simply assume that the manifold is complete.

norm given by

$$\|u\|^2_{\mathsf{H}^1(X,g)} := \int_X (|u|^2 + |du|^2_g)\,\mathrm{d}g.$$

Here, du is a section into the cotangent bundle T^*M and g the corresponding metric on it. Note that by definition, $\mathfrak{d}_{(X,g)}$ is a closed form with $\operatorname{dom}\mathfrak{d}_{(X,g)} = \mathsf{H}^1(X,g)$. The *Laplacian* $\Delta_{(X,g)}$ associated with (X,g) is the energy operator associated with the energy form $\mathfrak{d}_{(X,g)}$. The Laplacian is a self-adjoint non-negative operator and hence introduces a scale of Hilbert spaces

$$\mathcal{H}^k := \mathsf{H}^k(\Delta_{(X,g)}) := \operatorname{dom}((\Delta_{(X,g)} + 1)^{k/2})$$

with norm

$$\|u\|_{\mathsf{H}^k(\Delta_{(X,g)})} := \|(\Delta_{(X,g)} + 1)^{k/2} u\|_{\mathsf{L}_2(X,g)},$$

this definition extends to negative exponents k as already explained in the text after (2.2). We also call $\mathsf{H}^k(\Delta_{(X,g)})$ the *k-th Laplacian-Sobolev space*. Obviously, we have $\mathsf{H}^1(X,g) = \mathsf{H}^1(\Delta_{(X,g)})$ with identical norms.

If X is a manifold with (smooth) boundary, then we define the *Neumann energy form* $\mathfrak{d}^{\mathrm{N}}_{(X,g)}$ as above with domain $\operatorname{dom}\mathfrak{d}^{\mathrm{N}}_{(X,g)} = \mathsf{H}^1(X,g)$, where the latter is the closure of all functions, smooth up to the boundary and with compact support, with respect to the energy norm. The corresponding operator $\Delta^{\mathrm{N}}_{(X,g)}$ is called the *Neumann Laplacian on* (X,g).

Similarly, we define the *Dirichlet energy form* $\mathfrak{d}^{\mathrm{D}}_{(X,g)}$ as above with domain $\operatorname{dom}\mathfrak{d}^{\mathrm{D}}_{(X,g)} = \mathring{\mathsf{H}}^1(X,g)$, where the latter is the closure of all functions with compact support *away from the boundary* with respect to the energy norm. The corresponding operator $\Delta^{\mathrm{D}}_{(X,g)}$ is called the *Dirichlet Laplacian on* (X,g).

We denote by $\mathsf{L}_2(T^*X^{\otimes k}, g)$ the L_2-space of k-tensors with the pointwise norm on the tensors induced by g, i.e., of sections into $T^*X^{\otimes k} = T^*X \otimes \cdots \otimes T^*X$ with norm given by

$$\|u\|^2_{\mathsf{L}_2(T^*X^{\otimes k},g)} := \int_X |u|^2_g \,\mathrm{d}g,$$

where $|\cdot|^2_g$ is the canonical extension of g onto the corresponding tensor bundle. Here and in the sequel, we are often sloppy and just write $\|u\|^2_{\mathsf{L}_2(X,g)}$ for the corresponding norm (assuming that the fibre norm $|\cdot|_g$ is clear from the context).

Denote by ∇ the extension of the Levi-Civita connection on the tensor bundle $T^*X^{\otimes k}$. For $k = 0$, we have $\nabla u = du$. Moreover, we set $\nabla^2 u := \nabla\nabla u$, which is in $T^*X \otimes T^*X$ if u is a function. We have for instance $\nabla^2_{V_1,V_2} := \nabla_{V_1}\nabla_{V_2} - \nabla_{\nabla_{V_1}V_2}$ for vector fields V_1, V_2, and similarly for higher derivatives. We say that u has a *k-th weak derivative* if there exists a measurable section $v \in \mathsf{L}_{1,\mathrm{loc}}(X, (T^*X)^{\otimes k})$ such that

$$\int_X u \cdot (\nabla^*)^k \varphi \, \mathrm{d}g = \int_X \langle v, \varphi \rangle_g \, \mathrm{d}g$$

for all $\varphi \in \mathsf{C}_\mathrm{c}^\infty(X, (T^*X)^{\otimes k})$, where ∇^* denotes the (formal) adjoint of ∇. We set $\nabla^k u := v$ and

$$\mathsf{H}_p^k(X, g) := \{ u \in \mathsf{L}_p(X, g) \mid \nabla^j u \in \mathsf{L}_p(X, g) \text{ for } j \le k \},$$

with norm given by

$$\|u\|_{\mathsf{H}_p^k(X,g)}^p := \sum_{j=0}^k \|\nabla^j u\|_{\mathsf{L}_p(T^*X^{\otimes j}, g)}^p$$

for $p \ge 1$, and $\mathsf{H}^k(X, g) := \mathsf{H}_2^k(X, g)$.

Note that the above defined Sobolev space $\mathsf{H}^1(X, g)$ agrees with the one defined in the beginning of the section, i.e., $\mathsf{H}^1(X, g) = \operatorname{dom} \mathfrak{d}_{(X,g)} = \mathsf{H}^1(\Delta_{(X,g)})$ and the corresponding norms agree.

3.2. Bounded geometry, harmonic radius and Euclidean balls. We also need some estimates of higher order Sobolev spaces in terms of Laplace-graph norms:

Definition 3.1. We say that (X, g) is an *elliptically regular* Riemannian manifold (of order $k \ge 2$) if $\operatorname{dom}(\Delta_{(X,g)} + 1)^{k/2} \subset \mathsf{H}^k(X, g)$ and if there is $C_{\mathrm{ell.reg},k} \ge 1$ such that

$$\|f\|_{\mathsf{H}^k(X,g)} \le C_{\mathrm{ell.reg},k} \|(\Delta_{(X,g)} + 1)^{k/2} f\|_{\mathsf{L}_2(X,g)}$$

for all $f \in \operatorname{dom}(\Delta_{(X,g)} + 1)^{k/2}$. We say that (X, g) is *elliptically regular*, if (X, g) is elliptically regular of order $k = 2$.

An immediate consequence of elliptic regularity (of order k) is that the Sobolev and Laplace-Sobolev spaces agree, i.e.,

$$\mathsf{H}^k(X, g) = \mathsf{H}^k(\Delta_{(X,g)})(= (\operatorname{dom} \Delta_{(X,g)} + 1)^{k/2}).$$

Typically, assumptions assuring elliptic regularity of order k also imply elliptic regularity of lower order, but we will not put this in our definition.

The elliptic regularity of a manifold is not given for higher order without further assumptions:

Definition 3.2. We say that a complete Riemannian manifold (X, g) has *bounded geometry* if the injectivity radius is uniformly bounded from below by some constant $\iota_0 > 0$ and if the Ricci tensor Ric is uniformly bounded from below by some constant $\kappa_0 \in \mathbb{R}$, i.e.,

$$\mathrm{Ric}_x \geq \kappa_0 g_x \quad \text{for all } x \in X \tag{3.1}$$

as symmetric 2-tensors.

We will not need assumptions on *derivatives* of the curvature tensor (i.e., bounded geometry of higher order) in this article.

Proposition 3.3 ([19, Proposition 2.10]). *Suppose that (X, g) is a complete manifold with bounded geometry, then the set of smooth functions with compact support $\mathcal{D}(X)$ is dense in the Sobolev space $\mathsf{H}^2(X, g)$. Moreover, (X, g) is elliptically regular (of order 2), and the constant $C_{\mathrm{ell.reg}}$ depends only on the lower bound κ_0 on the Ricci curvature.*

Proof. For the proof of the first claim, we refer to the proof of Prp. 2.10 in [19]. For sufficiently smooth metrics, there is a constant $c_{\mathrm{ell.reg}} > 0$ depending on g and its first derivatives such that

$$c_{\mathrm{ell.reg}} \|(\Delta_{(X,g)} + 1) f\|_{\mathsf{L}_2(X,g)} \leq \|f\|_{\mathsf{H}^2(X,g)}$$

for all $f \in \mathcal{D}(X)$. For the estimate of the Sobolev norm in terms of the (Laplace) graph norm, we use the following consequence of the Bochner-Lichnerowicz-Weitzenböck formula, namely,

$$\|\nabla^2 u\|^2_{\mathsf{L}_2(T^*X^{\otimes 2})} = \|\Delta_{(X,g)} u\|^2_{\mathsf{L}_2(X,g)} - \langle \mathrm{Ric}\, du, du \rangle_{\mathsf{L}_2(T^*X,g)} \tag{3.2}$$

for all $u \in \mathcal{D}(X)$, where we understand Ric as endomorphism on T^*X (an idea appearing already in [4]). From this equality and the spectral calculus for the self-adjoint operator $\Delta_{(X,g)}$ we obtain the desired result, namely that $C_{\mathrm{ell.reg}}$ of Definition 3.1 depends only on κ_0. □

We now give some estimates on the Riemannian metric in order to compare small balls with Euclidean balls. To this purpose, we recall the useful notion of a harmonic chart:

Definition 3.4 ([19, Definition 1.1]). Let U be an open subset of a Riemannian manifold (X, g). A chart $\varphi = (y^1, ..., y^n): U \to \mathbb{R}^n$ on (X, g) is called *harmonic* if $\Delta_{(X,g)} y^k = 0$ for all $k = 1, \dots, n$.

Since $\Delta_{(X,g)} y^k = \sum_{i,j=1}^n g^{ij} \Gamma_{ij}^k$, a chart $\varphi = (y^1, \dots, y^n)$ is harmonic if and only if $\sum_{i,j=1}^n g^{ij} \Gamma_{ij}^k = 0$ for all $k = 1, \dots, n$. Here, g^{ij} and Γ_{ij}^k are as usual the components of the inverse metric tensor and the Christoffel symbols with respect to the chart φ, respectively.

We now give some estimates on the Riemannian metric in order to compare small balls with Euclidean balls:

Proposition 3.5 ([19, Theorem 1.3]). *Assume that* (X, g) *is complete and has bounded geometry* (*with constants* $\kappa_0 \in \mathbb{R}$ *and* $\iota_0 > 0$). *Then for all* $a \in (0, 1)$ *there exist* $r_0 > 0$, $K \geq 1$ *and* $k > 0$ *depending only on* κ_0, ι_0 *and* a, *such that around any point* $x \in X$ *there exists a harmonic chart* $\varphi_x = (y^1, \dots, y^n)$ *defined on* $\overline{B_{r_0}(x)}$, *and in these charts we have*

$$K^{-1}(\delta_{ij}) \leq (g_{ij}) \leq K\,(\delta_{ij}) \quad (\textit{as bilinear forms}) \tag{3.3a}$$

and

$$|g_{ij}(x') - g_{ij}(x'')| \leq k\, d_g(x', x'')^a. \tag{3.3b}$$

for all $x', x'' \in B_{r_0}(x)$.

The radius r_0 will be called *harmonic radius* in the following. We refer to [24, 19, 20] and the references therein for more details. We assume $r_0 \leq 1$ here, as it simplifies some estimates later on, when using estimates of cut-off functions on small balls, see e.g. Lemma 3.10.

Denote by $g_{\mathrm{eucl},x}$ the Euclidean metric in the harmonic chart φ_x defined in the ball $B_{r_0}(x)$ by

$$g_{\mathrm{eucl},x}(\partial_{y_i}, \partial_{y_j}) = \delta_{ij}. \tag{3.4}$$

We immediately conclude from (3.3a):

Corollary 3.6. *Let* $p \in X$ *and let* $B := B_r(p)$ *with*

$$B_r(p) := \{\, x \in X \mid d_g(x, p) < r \,\} \tag{3.5}$$

be a ball around p *with geodesic radius* $r \in (0, r_0)$ *in* (M, g). *Then*

(i) *the volume measures and the cotangent norm satisfy the estimates*

$$K^{-n/2}\, \mathrm{d}g_{\mathrm{eucl},x} \leq \mathrm{d}g_x \leq K^{n/2}\, \mathrm{d}g_{\mathrm{eucl},x}$$

and

$$K^{-1}|\xi|^2_{g_{\mathrm{eucl},x}} \leq |\xi|^2_{g_x} \leq K|\xi|^2_{g_{\mathrm{eucl},x}}$$

for all $x \in B$ *and* $\xi \in T_x^* X$;

(ii) *we have the following norm estimates*

$$K^{-n/4}\|u\|_{\mathsf{L}_2(B,g_{\mathrm{eucl}})} \le \|u\|_{\mathsf{L}_2(B,g)} \le K^{n/4}\|u\|_{\mathsf{L}_2(B,g_{\mathrm{eucl}})},$$
$$K^{-(n+2)/4}\|du\|_{\mathsf{L}_2(T^*B,g_{\mathrm{eucl}})} \le \|du\|_{\mathsf{L}_2(T^*B,g)} \le K^{(n+2)/4}\|du\|_{\mathsf{L}_2(T^*B,g_{\mathrm{eucl}})},$$
$$K^{-(n+2)/4}\|u\|_{\mathsf{H}^1(B,g_{\mathrm{eucl}})} \le \|u\|_{\mathsf{H}^1(B,g)} \le K^{(n+2)/4}\|u\|_{\mathsf{H}^1(B,g_{\mathrm{eucl}})}$$

for all $u \in \mathsf{L}_2(B,g)$ *resp.* $u \in \mathsf{H}^1(B,g)$.

3.3. The non-concentrating property. We now formulate a property which will be used in all our examples. Typically, $A = A_\varepsilon \subset B$ and $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$. The definition of "δ-non-concentrating" allows us to quantify how much a function $f \in \mathsf{H}^1(B)$ is not concentrated in A.

Definition 3.7. Let (X, g) be a Riemannian manifold, $A \subset B \subset X$ and $\delta > 0$. We say that (A, B) is δ-*non-concentrating* (of order 1) if

$$\|f\|_{\mathsf{L}_2(A,g)} \le \delta\|f\|_{\mathsf{H}^1(B,g)} \tag{3.6}$$

for all $f \in \mathsf{H}^1(B, g)$.

Note that if $\widetilde{B} \supset B$ and if (A, B) is δ-non-concentrating, then $(A, \widetilde{B})$ is also δ-non-concentrating.

Remark 3.8. In the Euclidean setting (i.e., if $X \subset \mathbb{R}^n$ and $g = g_{\mathrm{eucl}}$), we could use a result by Marchenko and Khruslov [30, Lemma 4.9], namely

$$\int_A |u|^2 \le \frac{2\,\mathrm{vol}\,A}{\mathrm{vol}\,G}\int_G |u|^2 + C(n)\frac{(\mathrm{diam}\,B)^{n+1}(\mathrm{vol}\,A)^{1/n}}{\mathrm{vol}\,G}\int_B |du|^2 \tag{3.7}$$

for measurable sets A, G with $A, G \subsetneq B \subset \mathbb{R}^n$, where B is a parallelepiped and where $\mathrm{diam}\,B$ denotes the diameter of B. Moreover, $C(n)$ depends only on the dimension. In this situation, (A, B) is δ-non-concentrating with

$$\delta^2 := \max\Big\{\frac{2\,\mathrm{vol}\,A}{\mathrm{vol}\,G}, C(n)\frac{(\mathrm{diam}\,B)^{n+1}(\mathrm{vol}\,A)^{1/n}}{\mathrm{vol}\,G}\Big\}.$$

In particular, if $(A_\varepsilon)_\varepsilon$ is a family of subsets such that $\mathrm{vol}\,A_\varepsilon \to 0$, then (A_ε, B) is δ_ε-non-concentrating for some $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$ (for G choose some fixed smaller parallelepiped included in B). If A_ε are balls of radius ε, then δ_ε is of order $\varepsilon^{1/2}$. Note that this error is worse than the one we obtain in Lemma 3.10, which is of order ε (resp. $\varepsilon|\log\varepsilon|$ in dimension 2) in the situation here.

Once we have the non-concentrating property, we can immediately conclude a similar estimate for the derivatives:

Proposition 3.9. *Assume that* (A, B) *is* δ*-non-concentrating, then* (A, B) *is* δ*-non-concentrating of order* 2*, i.e.,*

$$\|df\|_{\mathsf{L}_2(A,g)} \le \delta \|f\|_{\mathsf{H}^2(B,g)} \tag{3.8}$$

for all $f \in \mathsf{H}^2(B, g)$.

Proof. Let $f \in \mathsf{H}^2(X, g)$. We apply (3.6) to the function $\varphi = |df|_g$ and calculate for any $x \in X$ with $df(x) \ne 0$ and any $V \in T_x X$:

$$d_V \varphi = d_V \sqrt{\langle df, df \rangle_g} = \frac{1}{\sqrt{\langle df, df \rangle_g}} \langle \nabla_V df, df \rangle_g. \tag{3.9}$$

We conclude $|d_V \varphi| \le |\nabla df|_g |V|_g$ by the Cauchy–Schwarz inequality. In particular, $|d\varphi|_g \le |\nabla df|_g = |\nabla^2 f|_g$, and this inequality (also called *Kato's inequality*) is also true if $df(x) = 0$. Inequality (3.6) now yields

$$\begin{aligned}
\|df\|_{\mathsf{L}_2(A,g)} &= \|\varphi\|_{\mathsf{L}_2(A,g)} \\
&\le \delta \|\varphi\|_{\mathsf{H}^1(B,g)} \\
&= \delta \big(\|df\|^2_{\mathsf{L}_2(B,g)} + \|d\varphi\|^2_{\mathsf{L}_2(B,g)}\big)^{1/2} \\
&\le \delta \big(\|df\|^2_{\mathsf{L}_2(B,g)} + \|\nabla^2 f\|^2_{\mathsf{L}_2(B,g)}\big)^{1/2} \\
&\le \delta \|f\|_{\mathsf{H}^2(B,g)}.
\end{aligned}$$

□

Let us now check the non-concentrating property for balls of different radii.

Lemma 3.10. *Assume that* (X, g) *has bounded geometry with harmonic radius* $r_0 \in (0, 1]$. *Let* $\eta \in (0, r_0)$ *and* $\varepsilon \in (0, \eta/2)$ *then* $(B_\varepsilon(p), B_\eta(p))$ *are* $\tau_n(\varepsilon/\eta)$*-non-concentrating for all* $p \in X$*, i.e.,*

$$\|f\|_{\mathsf{L}_2(B_\varepsilon(p),g)} \le \tau_n\Big(\frac{\varepsilon}{\eta}\Big) \|f\|_{\mathsf{H}^1(B_\eta(p),g)}$$

for all $f \in \mathsf{H}^1(B_\eta(p), g)$. *Here,*

$$\tau_n(\omega) := \sqrt{8} K^{(n+1)/2} \omega \quad \textit{resp.} \quad \tau_2(\omega) := \sqrt{8} K^{3/2} \omega \sqrt{|\log \omega|} \tag{3.10}$$

if $n \ge 3$ *resp.* $n = 2$.

Proof. We apply the results of [35, Section A.2]. We first consider Euclidean balls: note that in polar coordinates the Euclidean metric is a warped product $g_{\mathrm{eucl}} = \mathrm{d}s^2 + s^2 h$ with density function $\varrho(s) = s^{n-1}$, where h is the standard metric on the $(n-1)$-dimensional sphere. We then apply [35, Corollary A.2.7 (A.9b)] with $s_0 = 0$, $s_1 = \varepsilon$, $s_2 = \eta$, $a = \eta - \varepsilon$. We conclude

$$\|f\|^2_{\mathsf{L}_2(B_\varepsilon, g_{\mathrm{eucl}})} \le 2\eta_2(0,\varepsilon,\eta)\Big(\|f'\|^2_{\mathsf{L}_2(B_\eta, g_{\mathrm{eucl}})} + \frac{1}{(\eta-\varepsilon)^2}\|f\|^2_{\mathsf{L}_2(B_\eta, g_{\mathrm{eucl}})}\Big),$$

where f' denotes the radial derivative and where

$$\eta_2(0,\varepsilon,\eta) := \int_0^\varepsilon \Big(\int_t^\eta \frac{1}{\varrho(s)}\,\mathrm{d}s\Big)\varrho(t)\,\mathrm{d}t \le \begin{cases} \varepsilon^2 \log(\eta/\varepsilon) & \text{if } n = 2, \\ \varepsilon^2 & \text{if } n \ge 3, \end{cases}$$

provided $\varepsilon \le \eta/2 < \mathrm{e}^{-1/2}\eta$. In particular,

$$\frac{\varepsilon^2}{(\eta-\varepsilon)^2} = \frac{\omega^2}{(1-\omega)^2} \le 4\omega^2$$

with $\omega = \varepsilon/\eta \le 1/2$. We then use Corollary 3.6 (ii) to carry over the estimates to the original metric g, namely

$$\begin{aligned} \|f\|^2_{\mathsf{L}_2(B_\varepsilon(p), g)} \le K^{n/2}\|f\|^2_{\mathsf{L}_2(B_\varepsilon, g_{\mathrm{eucl}})} &\le 8K^{n/2}[|\log\omega|]\omega^2\|f\|^2_{\mathsf{H}^1(B_\eta, g_{\mathrm{eucl}})} \\ &\le 8K^{n+1}[|\log\omega|]\omega^2\|f\|^2_{\mathsf{H}^1(B_\eta, g)}, \end{aligned}$$

where $[|\log\omega|]$ appears only if $n = 2$. □

Let us now consider a disjoint union of small balls as obstacle; in our setting, I is a discrete subset of X:

Definition 3.11. We denote by

$$B_r(I) := \big\{ x \in X \,\big|\, d_g(x, I) := \inf_{p\in I} d_g(x,p) \le r \big\} \tag{3.11}$$

the *r-neighbourhood* of a subset $I \subset X$. We say that $I \subset X$ is an *r-separated set* if for all $p_1, p_2 \in I$, $p_1 \ne p_2$, we have $d(p_1, p_2) \ge 2r$.

Let I be an η-separated set in X, then $B_\varepsilon(I)$ consists of $|I|$-many *disjoint* balls of radius $\varepsilon \in (0, \eta)$ around each point in I.

Let us now check the non-concentrating property for the union of balls:

Proposition 3.12. *Let (X,g) be a complete Riemannian manifold with bounded geometry and harmonic radius $r_0 > 0$. Let $\eta \in (0, r_0)$ and $\varepsilon \in (0, \eta/2)$. Assume that I is η-separated, then $(B_\varepsilon(I), B_\eta(I))$ are $\tau_n(\varepsilon/\eta)$-separated, i.e.,*

$$\|f\|_{\mathsf{L}_2(B_\varepsilon(I),g)} \le \tau_n\Big(\frac{\varepsilon}{\eta}\Big)\|f\|_{\mathsf{H}^1(B_\eta(I),g)}$$

for all $f \in \mathsf{H}^1(B_\eta(I),g)$.

Proof. The estimate follows from

$$\begin{aligned}\|f\|^2_{\mathsf{L}_2(B_\varepsilon(I),g)} &= \sum_{p\in I}\|f\|^2_{\mathsf{L}_2(B_\varepsilon(p),g)}\\ &\le \sum_{p\in I}\tau_n\Big(\frac{\varepsilon}{\eta}\Big)^2\|f\|^2_{\mathsf{H}^1(B_\eta(p),g)}\\ &= \tau_n\Big(\frac{\varepsilon}{\eta}\Big)^2\|f\|^2_{\mathsf{H}^1(B_\eta(I),g)}\end{aligned}$$

using Lemma 3.10 and the disjointness of the balls in B_η. □

4. Neumann obstacles without an effect

4.1. Abstract Neumann obstacles without effect. Let (X,g) be a Riemannian manifold of dimension $n \ge 2$ and let $B_\varepsilon \subset X$ be a closed subset for each $\varepsilon \in (0,\varepsilon_0]$. We will impose conditions on the family $(B_\varepsilon)_\varepsilon$ such that the Neumann Laplacian on $X_\varepsilon := X \setminus B_\varepsilon$ converges to the Laplacian on X. Later in Subsection 4.2, B_ε will be the disjoint union of many balls, and we show there that the abstract properties of the following definition can actually be realised:

Definition 4.1. We say that a family $(B_\varepsilon)_\varepsilon$ of closed subsets of a Riemannian manifold (X,g) is *Neumann-asymptotically fading* if the following conditions are fulfilled.

(i) *Non-concentrating property.* We assume that (B_ε, X) is δ'_ε-non-concentrating with $\delta'_\varepsilon \to 0$.

(ii) *Uniform extension property.* We assume that there is a constant $C_{\mathrm{ext}} \ge 1$ such that $\|E_\varepsilon\| \le C_{\mathrm{ext}}$ for all $\varepsilon \in (0,\varepsilon_0]$, where

$$E_\varepsilon \colon \mathsf{H}^1(X_\varepsilon,g) \longrightarrow \mathsf{H}^1(X,g)$$

is an extension operator, i.e., $(E_\varepsilon u)\restriction_{X_\varepsilon} = u$ for all $u \in \mathsf{H}^1(X_\varepsilon,g)$.

Remark 4.2. (i) Note that if $\operatorname{vol} B_\varepsilon \to 0$ as $\varepsilon \to 0$ in the Euclidean case (and if B_ε is included in a bounded set for ε small enough), then (3.7) implies that (B_ε, X) is non-concentrating with δ'_ε of order $(\operatorname{vol} B_\varepsilon)^{1/(2n)}$. A possible counterexample for B_ε not fulfilling the non-concentrating property but still fulfilling $\operatorname{vol} B_\varepsilon \to 0$ would be a rectangle in dimension 2 of length $r_\varepsilon > 0$ and width $\varepsilon > 0$ such that $r_\varepsilon \to \infty$, $\operatorname{vol} B_\varepsilon = r_\varepsilon \varepsilon \to 0$ and $r_\varepsilon \varepsilon^\alpha \to \infty$ for some $\alpha \in (0,1)$.

But the non-concentrating property does not imply that $\operatorname{vol} B_\varepsilon \to 0$: in Subsection 4.2 we allow that B_ε consists of an infinite number of small (disjoint) balls (in a non-compact manifold), hence $\operatorname{vol} B_\varepsilon = \infty$.

(ii) The *uniform extension property* of Definition 4.1 (ii) is closely related to a property of a (bounded) domain X in $\mathbb{R}^n$, called *strongly connected* in [30], we refer to the discussion in Chapter 4, especially of Section 4.2 of this book, for further details; a counterexample is given in [30, Example 4.6].

We now show our first main result:

Theorem 4.3. *Let (X, g) be an elliptically regular Riemannian manifold and $(B_\varepsilon)_\varepsilon$ be a family of closed subsets of X. If $(B_\varepsilon)_\varepsilon$ is Neumann-asymptotically fading, then the energy form $\mathfrak{d}_{(X,g)}$ of (X, g) and the (Neumann) energy form $\mathfrak{d}^{\mathrm{N}}_{(X_\varepsilon,g)}$ of (X_ε, g) with $X_\varepsilon = X \setminus B_\varepsilon$ are δ_ε-quasi-unitarily equivalent of order $k = 2$ with $\delta_\varepsilon = C_{\mathrm{ext}} C_{\mathrm{ell.reg}} \delta'_\varepsilon$.*

Proof. We show that the hypotheses of Definition 2.1 are fulfilled. To do so, we first need to specify the spaces and transplantation operators. Namely, we set

$$\begin{aligned}
J\colon \mathcal{H} &:= \mathsf{L}_2(X,g) \longrightarrow \widetilde{\mathcal{H}} := \mathsf{L}_2(X_\varepsilon, g), & f &\longmapsto f\restriction_{X_\varepsilon},\\
J^1\colon \mathcal{H}^1 &:= \mathsf{H}^1(X,g) \longrightarrow \widetilde{\mathcal{H}}^1 := \mathsf{H}^1(X_\varepsilon, g), & f &\longmapsto f\restriction_{X_\varepsilon},\\
J'\colon \widetilde{\mathcal{H}} &= \mathsf{L}_2(X_\varepsilon,g) \longrightarrow \mathcal{H} = \mathsf{L}_2(X, g), & u &\longmapsto \bar{u},\\
J^{1\prime}\colon \widetilde{\mathcal{H}}^1 &= \mathsf{H}^1(X_\varepsilon,g) \longrightarrow \mathcal{H}^1 = \mathsf{H}^1(X, g), & u &\longmapsto E_\varepsilon u,
\end{aligned}$$

where $\bar{u}$ denotes the extension of $u\colon X_\varepsilon \to \mathbb{C}$ by 0 on B_ε.

We check the hypotheses of Definition 2.1: We easily see that

$$J' = J^*, \quad JJ' = \mathrm{id}_{\widetilde{\mathcal{H}}}, \quad J^1 = J\restriction_{\mathcal{H}^1}.$$

Moreover, we have

$$\|Jf\|^2_{\mathsf{L}_2(X_\varepsilon,g)} = \int_{X_\varepsilon} |f|^2 \,\mathrm{d}g \le \int_X |f|^2 \,\mathrm{d}g = \|f\|^2_{\mathsf{L}_2(X,g)},$$

and if $\operatorname{supp} f \subset X_\varepsilon$, then $\|Jf\| = \|f\|$, hence we have $\|J\| = 1$; in particular, (2.3a) is fulfilled with $\delta = 0$.

The first estimate in (2.3b) follows since (B_ε, X) is δ'_ε-non-concentrating (see (3.6)), namely we have

$$\|f - J'Jf\|_{\mathsf{L}_2(X,g)} = \|f\|_{\mathsf{L}_2(B_\varepsilon,g)} \le \delta'_\varepsilon \|f\|_{\mathsf{H}^1(X,g)}.$$

Moreover, $J^{1\prime}u - J'u = \mathbf{1}_{B_\varepsilon} E_\varepsilon u$ (the uniform extension onto B_ε), hence

$$\|J^{1\prime}u - J'u\|_{\mathsf{L}_2(X,g)} = \|E_\varepsilon u\|_{\mathsf{L}_2(B_\varepsilon,g)} \le \delta'_\varepsilon \|E_\varepsilon u\|_{\mathsf{H}^1(X,g)} \le \delta'_\varepsilon C_{\mathrm{ext}} \|u\|_{\mathsf{H}^1(X_\varepsilon,g)}$$

by the non-concentrating property (3.6) and the uniform extension property Definition 4.1 (ii). Finally,

$$\begin{aligned}
\left|\mathfrak{d}_\varepsilon(J^1 f, u) - \mathfrak{d}(f, J^{1\prime}u)\right| &= \left|\langle df, d(E_\varepsilon u)\rangle_{\mathsf{L}_2(B_\varepsilon,g)}\right| \\
&\le \|df\|_{\mathsf{L}_2(B_\varepsilon,g)} \|d(E_\varepsilon u)\|_{\mathsf{L}_2(B_\varepsilon,g)} \\
&\le \delta'_\varepsilon \|f\|_{\mathsf{H}^2(X,g)} C_{\mathrm{ext}} \|u\|_{\mathsf{H}^1(X_\varepsilon,g)} \\
&\le C_{\mathrm{ext}} C_{\mathrm{ell.reg}} \delta'_\varepsilon \|(\Delta_{(X,g)} + 1) f\|_{\mathsf{L}_2(X,g)} \|u\|_{\mathsf{H}^1(X_\varepsilon,g)}
\end{aligned} \tag{4.1}$$

by the non-concentrating property (3.8), the elliptic regularity assumption and again the uniform extension property in Definition 4.1 (ii). □

Remark 4.4. Note that we have to use the estimate against the graph norm on (X, g) (i.e., the unitary equivalence of order $k = 2$ and *not* of order $k = 1$), as the following example shows: Let $X = B_1(0)$ be the Euclidean ball of radius 1, $B_\varepsilon = B_\varepsilon(0)$ and $X_\varepsilon = X \setminus B_\varepsilon$ the annulus with inner radius ε and outer radius 1. We will show that estimate (4.1) cannot hold if we replace the graph norm $\|f\|_2 = \|(\Delta_{(X,g)} + 1)f\|$ by the quadratic form norm $\|f\|_1 = \|f\|_{\mathsf{H}^1(X,g)}$:

Namely, let $u \in \mathsf{H}^1(X_\varepsilon)$ be given in polar coordinates $(r, \theta) \in (\varepsilon, 1) \times (0, 2\pi)$ by $u(r, \theta) = r^\beta \cos\theta$ for some $\beta \in \mathbb{R}$. Then the harmonic extension $\widetilde{u}_\varepsilon = E_\varepsilon u$ (used also in the next Subsection 4.2) is given by $\widetilde{u}_\varepsilon(r, \theta) = \varepsilon^{\beta-1} r \cos\theta$. Moreover, we have

$$\|d\widetilde{u}_\varepsilon\|^2_{\mathsf{L}_2(B_\varepsilon,g)} = \pi \varepsilon^{2\beta}, \qquad \|\widetilde{u}_\varepsilon\|^2_{\mathsf{L}_2(B_\varepsilon,g)} = \frac{\pi}{4}\varepsilon^{2(\beta+1)},$$

$$\|d\widetilde{u}_\varepsilon\|^2_{\mathsf{L}_2(X_\varepsilon,g)} = \frac{\pi(\beta^2+1)}{2\beta}(1 - \varepsilon^{2\beta}), \quad \|\widetilde{u}_\varepsilon\|^2_{\mathsf{L}_2(X_\varepsilon,g)} = \frac{\pi}{2(\beta+1)}(1 - \varepsilon^{2(\beta+1)}),$$

hence we have (with $f = \widetilde{u}_\varepsilon$, the optimal case for the Cauchy–Schwarz estimate in (4.1))

$$\frac{\left|\mathfrak{d}_\varepsilon(J^1 f, u) - \mathfrak{d}(f, J^{1\prime}u)\right|}{\|f\|_{\mathsf{H}^1(X,g)} \|u\|_{\mathsf{H}^1(X_\varepsilon,g)}} = \frac{\|d\widetilde{u}_\varepsilon\|^2_{\mathsf{L}_2(B_\varepsilon,g)}}{\|\widetilde{u}_\varepsilon\|_{\mathsf{H}^1(X,g)} \|u\|_{\mathsf{H}^1(X_\varepsilon,g)}} \longrightarrow -\frac{2\beta}{\beta^2+1} > 0 \tag{4.2}$$

as $\varepsilon \to 0$ provided $\beta < 0$. In particular, Estimate (4.1) cannot hold with the quadratic form norm instead of the graph norm of $f = \widetilde{u}_\varepsilon$. Note also, that we have chosen the harmonic extension, which minimises $\|d\widetilde{u}_\varepsilon\|^2_{\mathsf{L}_2(B_\varepsilon,g)}$ among all extensions with given boundary values $\widetilde{u}_\varepsilon(\varepsilon,\theta) = \varepsilon^\beta \cos\theta$, hence Estimate (4.1) cannot hold either for *any* extension operator E_ε having the uniform extension property Definition 4.1 (ii), as (setting $f = E_\varepsilon u$)

$$\frac{\left|\mathfrak{d}_\varepsilon(J^1 f, u) - \mathfrak{d}(f, J^{1\prime} u)\right|}{\|f\|_{\mathsf{H}^1(X,g)}\|u\|_{\mathsf{H}^1(X_\varepsilon,g)}} = \frac{\|d(E_\varepsilon u)\|^2_{\mathsf{L}_2(B_\varepsilon,g)}}{\|E_\varepsilon u\|_{\mathsf{H}^1(X,g)}\|u\|_{\mathsf{H}^1(X_\varepsilon,g)}}$$
$$\geq \frac{\|d\widetilde{u}_\varepsilon\|^2_{\mathsf{L}_2(B_\varepsilon,g)}}{C_{\mathrm{ext}}\|u\|^2_{\mathsf{H}^1(X_\varepsilon,g)}} \longrightarrow \frac{-2\beta}{C_{\mathrm{ext}}(\beta^2+1)} > 0.$$

4.2. Application: many small balls as Neumann obstacles. We now let B_ε be the disjoint union of many balls: Assume that for each $\varepsilon > 0$ there is η_ε such that $\varepsilon/\eta_\varepsilon \to 0$ (e.g., $\eta_\varepsilon = \varepsilon^\alpha$ for some $0 < \alpha < 1$). Assume additionally, that $(I_\varepsilon)_\varepsilon$ is a family of η_ε-separated subsets $I_\varepsilon \subset X$ (i.e., different points in I_ε have distance at least $2\eta_\varepsilon$, see Definition 3.11). We denote by

$$B_\varepsilon := B_\varepsilon(I_\varepsilon) \quad \text{and} \quad X_\varepsilon = X \setminus B_\varepsilon$$

the ε-neighbourhood of all points in I_ε resp. its complement in X. Note that, by the η_ε-separation, B_ε consists of $|I_\varepsilon|$-many *disjoint* balls around each point in I_ε.

Let us first show the uniform extension property of Definition 4.1 (ii): We define

$$E_\varepsilon \colon \mathsf{H}^1(X_\varepsilon, g) \longrightarrow \mathsf{H}^1(X, g), \quad u \longmapsto \widetilde{u},$$

where $\widetilde{u}$ denotes the *harmonic extension* on B_ε with respect to the *Euclidean* metric g_{eucl} on B_ε (the metric $g_{\mathrm{eucl},p}$ is defined in (3.4) on each small ball $B_\varepsilon(p)$ in B_ε, $p \in I_\varepsilon$, $\varepsilon \le r_0$).

We first need an estimate of the harmonic extension from an annulus to the inside ball:

Lemma 4.5. *For $0 < \varepsilon \le 1$, let B_ε and $B_{2\varepsilon}$ be Euclidean balls in $\mathbb{R}^n$ of radius ε and 2ε around 0. For $u \in \mathsf{H}^1(B_{2\varepsilon} \setminus B_\varepsilon)$, denote by $\widetilde{u}$ the* ***harmonic extension*** *of u into B_ε. Then $\widetilde{u} \in \mathsf{H}^1(B_\varepsilon)$ and there exist constants $C_0, C_1 > 0$ depending only on the dimension n such that*

$$\int_{B_\varepsilon} |\widetilde{u}|^2 \le C_0 \int_{B_{2\varepsilon}\setminus B_\varepsilon} (|u|^2 + \varepsilon^2 |du|^2) \quad \text{and} \quad \int_{B_\varepsilon} |\,d\widetilde{u}|^2 \le C_1 \int_{B_{2\varepsilon}\setminus B_\varepsilon} |du|^2$$

for all $u \in \mathsf{H}^1(B_{2\varepsilon} \setminus B_\varepsilon)$.

Proof. This result is proven in [40]. For the convenience of the reader, we repeat the proof using a scaling argument.

For $u \in \mathsf{H}^1(B_{2\varepsilon} \setminus B_\varepsilon)$ let $f(x) = u(\varepsilon x)$. Then $f \in \mathsf{H}^1(B_2 \setminus B_1)$ and we have the scaling behaviour

$$\int_{B_2 \setminus B_1} |f|^2 = \varepsilon^{-n} \int_{B_{2\varepsilon} \setminus B_\varepsilon} |u|^2 \quad \text{and} \quad \int_{B_2 \setminus B_1} |df|^2 = \varepsilon^{2-n} \int_{B_{2\varepsilon} \setminus B_\varepsilon} |du|^2$$

We know that $\widetilde{\cdot} \colon \mathsf{H}^1(B_2 \setminus B_1) \to \mathsf{H}^1(B_1)$, $f \mapsto \widetilde{f}$, is a continuous operator. In particular, there exists a constant $C_0 > 0$ depending only on n such that

$$\int_{B_1} (|\widetilde{f}|^2 + |d\widetilde{f}|^2) \le C_0 \int_{B_2 \setminus B_1} (|f|^2 + |df|^2)$$

holds. After scaling, we obtain

$$\int_{B_\varepsilon} |\tilde{u}|^2 \le C_0 \int_{B_{2\varepsilon} \setminus B_\varepsilon} (|u|^2 + \varepsilon^2 |du|^2) \le C_0 \int_{B_{2\varepsilon} \setminus B_\varepsilon} (|u|^2 + |du|^2)$$

as $\varepsilon \le 1$. For the control of the derivative, we remark that the harmonic extension of the constant function $\mathbf{1}$ on $B_2 \setminus B_1$ is the constant function $\mathbf{1}$ on B_1. Therefore, we can assume that u (and after rescaling also f) is orthogonal to $\mathbf{1}$. If λ_1 denote the first positive eigenvalue of the Neumann problem of the standard annulus $B_2 \setminus \overline{B}_1$, we can conclude with the min-max principle and obtain

$$\int_{B_2 \setminus B_1} |f|^2 \le \frac{1}{\lambda_1} \int_{B_2 \setminus B_1} |df|^2,$$

so that

$$\int_{B_1} |d\widetilde{f}|^2 \le C_0 \Big(1 + \frac{1}{\lambda_1}\Big) \int_{B_2 \setminus B_1} |df|^2.$$

Since both sides scale with the same order, rescaling gives

$$\int_{B_\varepsilon} |d\widetilde{u}|^2 \le \underbrace{C_0 \Big(1 + \frac{1}{\lambda_1}\Big)}_{=:C_1} \int_{B_{2\varepsilon} \setminus B_\varepsilon} |du|^2.$$ □

Proposition 4.6. *Assume that (X, g) is a Riemannian manifold with harmonic radius $r_0 > 0$. Assume additionally that I_ε is 2ε-separated for each $\varepsilon \in (0, r_0/2)$. Then there is a constant $C_{\mathrm{ext}} > 0$ such that*

$$\|\widetilde{u}\|_{\mathsf{H}^1(B_{2\varepsilon}, g)} \le C_{\mathrm{ext}} \|u\|_{\mathsf{H}^1(B_{2\varepsilon} \setminus B_\varepsilon, g)}$$

for all $u \in \mathsf{H}^1(X_\varepsilon, g)$ *and all* ε. *In particular, there exists* $C_{\mathrm{ext}} \geq 1$ *such that* $\|E_\varepsilon\| \leq C_{\mathrm{ext}}$ *for all* $\varepsilon \in (0, r_0/2)$, *i.e., the extension operator given by* $E_\varepsilon u = \widetilde{u}$ *has the uniform extension property* (*Definition* 4.1 (ii)).

Proof. We have

$$\begin{aligned}
\|\widetilde{u}\|^2_{\mathsf{H}^1(B_\varepsilon,g)} &= \sum_{p\in I_\varepsilon} \|\widetilde{u}\|^2_{\mathsf{H}^1(B_\varepsilon(p),g)} \\
&\leq K^{n/2+1} \sum_{p\in I_\varepsilon} \|\widetilde{u}\|^2_{\mathsf{H}^1(B_\varepsilon(p),g_{\mathrm{eucl}})} \\
&\leq K^{n/2+1}(C_0 + C_1) \sum_{p\in I_\varepsilon} \|u\|^2_{\mathsf{H}^1(B_{2\varepsilon}(p)\setminus B_\varepsilon(p),g_{\mathrm{eucl}})} \\
&\leq K^{(n+2)}(C_0 + C_1) \sum_{p\in I_\varepsilon} \|u\|^2_{\mathsf{H}^1(B_{2\varepsilon}(p)\setminus B_\varepsilon(p),g)} \\
&=: C^2_{\mathrm{ext}} \|u\|^2_{\mathsf{H}^1(B_{2\varepsilon}\setminus B_\varepsilon,g)}
\end{aligned}$$

using Corollary 3.6 (ii) and Lemma 4.5. □

The proof of the following theorem follows now directly from Theorem 4.3 together with Proposition 3.12 ($(B_\varepsilon, B_\eta(I_\varepsilon))$ and hence (B_ε, X) are $\tau_n(\varepsilon/\eta_\varepsilon)$-non-concentrating, see Definition 4.1 (i)), Proposition 3.3 (for the elliptic regularity assumption) and Proposition 4.6 (Recall that, by Proposition 3.5, bounded geometry implies that the harmonic radius r_0 is strictly positive; we always assume that the separation distance η_ε fulfils $0 < 2\varepsilon < \eta_\varepsilon < r_0$ for all ε small enough):

Theorem 4.7. *Let* (X, g) *be a complete Riemannian manifold with bounded geometry, and let* $B_\varepsilon = \dot{\bigcup}_{p\in I_\varepsilon} B_\varepsilon(p)$ *be the union of* η_ε*-separated balls of radius* ε. *If* $\varepsilon/\eta_\varepsilon \to 0$, *then* $(B_\varepsilon)_\varepsilon$ *is Neumann-asymptotically fading, i.e., the energy form* $\mathfrak{d}_{(X,g)}$ *and the* (*Neumann*) *energy form* $\mathfrak{d}^{\mathrm{N}}_{(X_\varepsilon,g)}$ *are* δ_ε*-quasi-unitarily equivalent of order* $k = 2$ *with*

$$\delta_\varepsilon = \mathrm{O}(\varepsilon/\eta_\varepsilon) \qquad \text{if } n \geq 3,$$

resp.

$$\delta_\varepsilon = \mathrm{O}(\sqrt{\log(\eta_\varepsilon/\varepsilon)}\varepsilon/\eta_\varepsilon) \quad \text{if } n = 2.$$

The error depends only on m, K *and* κ_0, *see* (3.3a) *and* (3.1). *In particular, if* $\eta_\varepsilon = \varepsilon^\alpha$ *with* $\alpha \in (0, 1)$, *then* $\delta_\varepsilon = \mathrm{O}(\varepsilon^{1-\alpha})$ *if* $n \geq 3$ *resp.* $\delta_\varepsilon = \mathrm{O}(\varepsilon^{1-\alpha}\sqrt{|\log \varepsilon|})$ *if* $n = 2$.

Example 4.8. For a single ball of radius ε removed from a bounded subdomain X of $\mathbb{R}^2$ with Neumann boundary conditions on the ball and Dirichlet ones on ∂X, Ozawa [33] proved that, for simple eigenvalues, the difference of the perturbed and unperturbed k-th eigenvalue is of order ε^2 (he even gave a precise asymptotic expression in terms of the k-th eigenfunction and its gradient). Hempel [21] generalised the result for the first eigenvalue to irregular obstacles and higher dimensions $n \geq 2$ (obtaining the convergence rate ε^n in some cases). Our results (together with the eigenvalue convergence (1.2b) of Theorem 1.2) only give the weaker estimate $\mathrm{O}(\delta_\varepsilon) = \mathrm{O}(\varepsilon^{1-0})$ for the eigenvalue difference. Here, notation $\delta_\varepsilon = \mathrm{O}(\varepsilon^{\gamma-0})$ means that there is $\tau_0 > 0$ such that $\delta_\varepsilon/\varepsilon^{\gamma-\tau} \to 0$ as $\varepsilon \to 0$ for all $\tau \in (0, \tau_0)$.

Remark 4.9. If $\alpha = 1$, or, more generally, $\eta_\varepsilon/\varepsilon$ converges to a constant, then we do not expect that the Neumann Laplacian converges to the free Laplacian on X in general. If the balls are placed on a periodic lattice of order ε, and if their radius is ε, then we are in the setting of *homogenisation* (with Neumann boundary conditions), and we expect that the limit operator is no longer the free Laplacian, see e.g. [2] and also [30, Chapter 5] and very recently [42]. Suslina proved operator norm estimates for the resolvents on a periodic problem. Using a scaling argument, she works on an ε-independent space.

5. Dirichlet obstacles without an effect

5.1. Abstract Dirichlet obstacles without effect. Let us now consider the same problem, but with Dirichlet boundary conditions on the obstacles:

Definition 5.1. We say that a family $(B_\varepsilon)_\varepsilon$ of closed subsets of a Riemannian manifold (X, g) is *Dirichlet-asymptotically fading* (of order $k \geq 2$) if there exists a sequence $(\chi_\varepsilon)_\varepsilon$ of Lipschitz-continuous cut-off functions $\chi_\varepsilon\colon X \to [0, 1]$ with $\operatorname{supp} \chi_\varepsilon \subset X_\varepsilon$ such that the following conditions are fulfilled.

(i) *Non-concentrating property.* We assume that (B_ε^+, X) is δ'_ε-non-concentrating with $\delta'_\varepsilon \to 0$, where $B_\varepsilon^+ := \operatorname{supp}(1 - \chi_\varepsilon)$. (It follows that $B_\varepsilon \subset B_\varepsilon^+$.)

(ii) The cut-off function has *moderate decay* of order $k \geq 2$, i.e.,

$$T_\varepsilon^+\colon \mathsf{H}^k(X, g) \longrightarrow \mathsf{L}_2(T^* B_\varepsilon^+, g), \qquad f \longmapsto f \restriction_{B_\varepsilon^+} d\chi_\varepsilon$$

has norm $\|T_\varepsilon^+\| = \delta_\varepsilon^+ \to 0$ as $\varepsilon \to 0$.

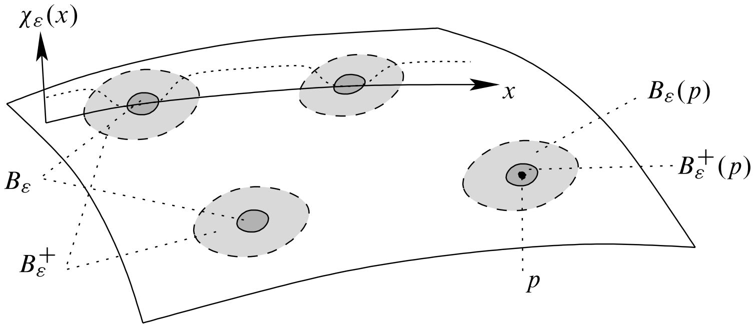

Figure 1. Dark grey: the obstacle set B_ε (consisting here of the disjoint union of balls $B_\varepsilon(p)$ as in Subsection 5.2); dark and light grey: the set B_ε^+ (again consisting of the disjoint union of balls $B_\varepsilon^+(p)$), and a profile of the cut-off function χ_ε (dotted line, 0 on B_ε, 1 outside B_ε^+).

If B_ε is a union of small balls, then this problem is the famous *crushed ice problem* of [40], see below in Subsection 5.2.

Our next main result is the following:

Theorem 5.2. *Let (X, g) be an elliptically regular Riemannian manifold of order 2 and k and $(B_\varepsilon)_\varepsilon$ be a family of closed subsets of X. If $(B_\varepsilon)_\varepsilon$ is Dirichlet-asymptotically fading (of order k), then the energy form $\mathfrak{d}_{(X,g)}$ of (X, g) and the (Dirichlet) energy form $\mathfrak{d}^{\mathrm{D}}_{(X_\varepsilon,g)}$ of (X_ε, g) with $X_\varepsilon = X \setminus B_\varepsilon$ are δ_ε-quasi-unitarily equivalent of order k with $\delta_\varepsilon = \max\{\delta'_\varepsilon, C_{\mathrm{ell.reg},2}\delta'_\varepsilon + C_{\mathrm{ell.reg},k}\delta_\varepsilon^+\}$.*

Proof. We show again that the hypotheses[3] of Definition 2.1 are fulfilled, and specify the spaces and transplantation operators by

$$\begin{aligned}
J\colon \mathcal{H} &:= \mathsf{L}_2(X,g) \longrightarrow \widetilde{\mathcal{H}} := \mathsf{L}_2(X_\varepsilon,g), & f &\longmapsto f\restriction_{X_\varepsilon},\\
J^1\colon \mathcal{H}^1 &:= \mathsf{H}^1(X,g) \longrightarrow \widetilde{\mathcal{H}}^1 := \mathring{\mathsf{H}}^1(X_\varepsilon,g), & f &\longmapsto \chi_\varepsilon f,\\
J'\colon \widetilde{\mathcal{H}} &= \mathsf{L}_2(X_\varepsilon,g) \longrightarrow \mathcal{H} = \mathsf{L}_2(X,g), & u &\longmapsto \bar{u},\\
J^{1\prime}\colon \widetilde{\mathcal{H}}^1 &= \mathring{\mathsf{H}}^1(X_\varepsilon,g) \longrightarrow \mathcal{H}^1 = \mathsf{H}^1(X,g), & u &\longmapsto \bar{u},
\end{aligned}$$

where $\bar{u}$ denotes the extension of $u\colon X_\varepsilon \to \mathbb{C}$ by 0 on B_ε.

We check the hypotheses of Definition 2.1: We easily see that

$$J' = J^*, \qquad JJ' = \mathrm{id}_{\widetilde{\mathcal{H}}}, \qquad J^{1\prime} = J'\restriction_{\widetilde{\mathcal{H}}^1}.$$

As in the Neumann case, we have $\|J\| = 1$ and (2.3a) is fulfilled with $\delta = 0$.

[3] Note that the Dirichlet fading case is in some sense dual to the Neumann case, as here, J^1 needs a (more complicated) cut-off function and $J^{1\prime}$ is simply the extension by $\mathbf{0}$.

The first estimate in (2.3b) follows from the non-concentrating property Definition 5.1 (i), namely we have

$$\|f - J'Jf\|_{\mathsf{L}_2(X,g)} = \|f\|_{\mathsf{L}_2(B_\varepsilon,g)} \le \|f\|_{\mathsf{L}_2(B_\varepsilon^+,g)} \le \delta'_\varepsilon \|f\|_{\mathsf{H}^1(X,g)}.$$

Moreover, $Jf - J^1 f = (1-\chi_\varepsilon)f$, hence

$$\begin{aligned}\|Jf - J^1 f\|_{\mathsf{L}_2(X_\varepsilon,g)} = \|(1-\chi_\varepsilon)f\|_{\mathsf{L}_2(X_\varepsilon,g)} &\le \|f\|_{\mathsf{L}_2(B_\varepsilon^+\cap X_\varepsilon,g)}\\ &\le \|f\|_{\mathsf{L}_2(B_\varepsilon^+,g)} \le \delta'_\varepsilon \|f\|_{\mathsf{H}^1(X,g)}\end{aligned}$$

by the same argument. Finally,

$$\begin{aligned}&|\mathfrak{d}(f, J^{1\prime}u) - \mathfrak{d}_\varepsilon(J^1 f, u)|\\ &\quad= |\langle df - d(\chi_\varepsilon f), du\rangle_{\mathsf{L}_2(T^*B_\varepsilon^+,g)}|\\ &\quad\le |\langle (1-\chi_\varepsilon)df, du\rangle_{\mathsf{L}_2(T^*B_\varepsilon^+,g)}| + |\langle f d\chi_\varepsilon, du\rangle_{\mathsf{L}_2(T^*B_\varepsilon^+,g)}|\\ &\quad\le (\|df\|_{\mathsf{L}_2(T^*B_\varepsilon^+,g)} + \|f d\chi_\varepsilon\|_{\mathsf{L}_2(T^*B_\varepsilon^+,g)})\|du\|_{\mathsf{L}_2(T^*B_\varepsilon^+,g)}\\ &\quad\le (\delta'_\varepsilon \|f\|_{\mathsf{H}^2(X,g)} + \delta_\varepsilon^+ \|f\|_{\mathsf{H}^k(X,g)})\|u\|_{\mathsf{H}^1(X_\varepsilon,g)}\\ &\quad\le (C_{\mathrm{ell.reg},2}\delta'_\varepsilon \|(\Delta_{(X,g)}+1)f\| + \delta_\varepsilon^+ C_{\mathrm{ell.reg},k}\|(\Delta_{(X,g)}+1)^{k/2} f\|)\|u\|_{\mathsf{H}^1(X_\varepsilon,g)}\\ &\quad= (C_{\mathrm{ell.reg},2}\delta'_\varepsilon + C_{\mathrm{ell.reg},k}\delta_\varepsilon^+)\|f\|_k \|u\|_1\end{aligned}$$

by the non-concentrating property together with Proposition 3.9 and the elliptic regularity assumption and the moderate decay property Definition 5.1 (ii). ☐

5.2. Application: many small balls as Dirichlet obstacles. The obstacles are of the same kind as in Subsection 4.2. Let I_ε be η_ε-separated as before with $0 < \eta_\varepsilon < r_0$ for $\varepsilon \in (0, \varepsilon_0)$ and some $\varepsilon_0 > 0$, where r_0 denotes the harmonic radius of (X, g). Let $(\cdot)^+ \colon (0, \varepsilon_0) \to (0, r_0)$ be a function such that $\varepsilon < \varepsilon^+ \le \eta_\varepsilon/2$ for all $\varepsilon \in (0, \varepsilon_0)$.

Let

$$B_\varepsilon^+ := B_{\varepsilon^+}(I_\varepsilon) = \bigcup_{p\in I_\varepsilon} B_{\varepsilon^+}(p).$$

We now check the conditions of Definition 5.1 and need good cut-off functions. Define by $h = h_n$ the radially symmetric, harmonic function in dimension n given by

$$h(s) := \begin{cases} -\dfrac{1}{(n-2)s^{n-2}}, & n > 2,\\ \ln s, & n = 2.\end{cases} \tag{5.1}$$

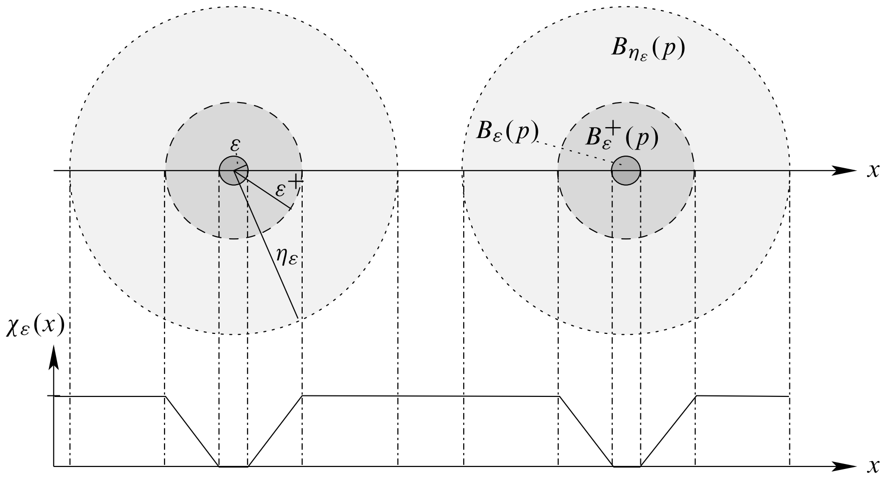

Figure 2. The obstacle (union of balls) of radius ε (dark grey); the separation balls (very light grey and dotted balls) of radius η_ε and the intermediate balls (light grey and dashed) of radius ε^+.

Note that $h'(s) = 1/s^{n-1}$. Furthermore, let $\widetilde{\chi}_\varepsilon\colon X \to [0,1]$ be the radial cut-off function given by

$$\widetilde{\chi}_\varepsilon(r) = \begin{cases} 0, & 0 \le r \le \varepsilon, \\ \dfrac{h(r) - h(\varepsilon)}{h(\varepsilon^+) - h(\varepsilon)}, & \varepsilon \le r \le \varepsilon^+, \\ 1, & \varepsilon^+ \le r. \end{cases}$$

This function is Lipschitz-continuous. We define the cut-off function of Definition 5.1 by

$$\chi_\varepsilon(x) := \widetilde{\chi}_\varepsilon(d(x,p)) \quad \text{for } x \in B_{\eta_\varepsilon}(p) \tag{5.2}$$

for each $p \in I_\varepsilon$ and extend it by 1 on $X \setminus B_{\eta_\varepsilon}$; again χ_ε is Lipschitz-continuous. Clearly, $\operatorname{supp}(1-\chi_\varepsilon) = B_\varepsilon^+$ and $\chi_\varepsilon\restriction_{B_\varepsilon} = 0$ by definition.

Remark 5.3. For the moderate decay property of Definition 5.1 (ii), we need to control $\|f d\chi_\varepsilon\|_{\mathsf{L}_2(B_\varepsilon^+,g)}$ and will use Sobolev embedding theorems. If we stay in the L_2-world, the order k must satisfy $k > \dim X/2$ to have control of the L_∞-norm of f by its H^k-norm, and we only need cut-off functions satisfying $\|d\chi_\varepsilon\|_{\mathsf{L}_2(T^*B_\varepsilon^+,g)} \to 0$ as $\varepsilon \to 0$. The counterpart are stronger assumptions concerning the sectional curvature to control the norm of H^k with the graph norm in $\mathsf{H}^k(\Delta_{(X,g)})$ in Definition 3.1: typically, one needs uniform bounds on the derivatives of the sectional curvature up to order $(k-2)$. We explain another approach in Remark 5.8.

In the sequel, we prefer to use only a lower bound on the Ricci curvature, using Hölder inequalities and the Sobolev embeddings given in Proposition A.1. For this argument, we need the estimate $\|d\chi_\varepsilon\|_{\mathsf{L}_q(T^*B_\varepsilon^+,g)} \to 0$ as $\varepsilon \to 0$ for some q, see Proposition 5.5.

As proposed, we now want to use the Hölder estimate

$$\|f d\chi_\varepsilon\|_{\mathsf{L}_2(T^*B_\varepsilon^+,g)} \le \|f\|_{\mathsf{L}_{2p_n}(B_\varepsilon^+)} \|d\chi_\varepsilon\|_{\mathsf{L}_{2q_n}(T^*B_\varepsilon^+)} \tag{5.3}$$

with $1 \le p_n, q_n \le \infty$ such that $1/p_n + 1/q_n = 1$. For this estimates it is good that q_n is as small as possible, but the Sobolev embedding forces that p_n is not too large, at least for higher dimensions. This restriction leads us to introduce the following definition of p_n and q_n, namely

$$p_m = \frac{n}{n-4} \quad \text{if } n \ge 5, \qquad p_4 = \frac{2}{\beta}, \qquad p_3 = p_2 = \infty, \tag{5.4a}$$

$$q_n = \frac{n}{4} \quad \text{if } n \ge 5, \qquad q_4 = \frac{2}{2-\beta}, \qquad q_3 = q_2 = 1, \tag{5.4b}$$

with $\beta \in (0,1]$ if $n = 4$, similarly as in [27].

Lemma 5.4. *The cut-off function χ_ε at a ball $B_{\varepsilon^+}(p)$ satisfies*

$$\|d\chi_\varepsilon\|_{\mathsf{L}_{2q_n}(T^*B_{\varepsilon^+}(p),g)} = \hat{\delta}_\varepsilon$$

for all $p \in I_\varepsilon$, where

$$\hat{\delta}_\varepsilon = \mathrm{O}(\varepsilon^{1-\beta}) \quad \text{if } n \ge 3 \text{ with } \beta = \beta_n \begin{cases} = 0, & n \ge 5, \\ \in (0,1), & n = 4, \\ = 1/2, & n = 3, \end{cases}$$

resp.

$$\hat{\delta}_\varepsilon = \mathrm{O}(1/\sqrt{\log(\varepsilon^+/\varepsilon)}) \quad \text{if } n = 2.$$

Proof. We calculate

$$\begin{aligned}\|d\chi_\varepsilon\|^{2q_n}_{\mathsf{L}_{2q_n}(T^*B_{\varepsilon^+}(x),g)} &\le K^{q_n+n/2}\operatorname{vol}_{n-1}(\mathbb{S}^{n-1})\int_\varepsilon^{\varepsilon^+}|\chi'_\varepsilon(r)|^{2q_n}r^{n-1}\,\mathrm{d}r\\ &= K^{q_n+n/2}\frac{\operatorname{vol}_{n-1}(\mathbb{S}^{n-1})}{(h(\varepsilon^+)-h(\varepsilon))^{2q_n}}\int_\varepsilon^{\varepsilon^+}r^{(1-2q_n)(n-1)}\,\mathrm{d}r\\ &=: (\hat\delta_\varepsilon)^{2q_n}\end{aligned}$$

using Corollary 3.6 (ii). If $n \neq 2$ the exponent of r in the integral is different to -1, thus

$$\hat\delta_\varepsilon^{2q_n} = \begin{cases} K^{q_n+n/2}\dfrac{\operatorname{vol}_{n-1}(\mathbb{S}^{n-1})(\varepsilon^{n-2q_n(n-1)}-(\varepsilon^+)^{(n-2q_n(n-1))})}{(h(\varepsilon^+)-h(\varepsilon))^{2q_n}(2q_n(n-1)-n)} & \text{if } n\ge 3,\\ K^2\dfrac{2\pi}{(\log\varepsilon^+-\log\varepsilon)} & \text{if } n=2,\end{cases}$$

by the definition of h in (5.1). The result follows. □

We can now show the moderate decay property of Definition 5.1 (ii):

Proposition 5.5. *Assume that (X, g) is a complete manifold with bounded geometry and let I_ε be η_ε-separated, then there exists δ_ε^+ such that*

$$\|f\,d\chi_\varepsilon\|_{\mathsf{L}_2(T^*B_\varepsilon^+,g)} \le \delta_\varepsilon^+\|f\|_{\mathsf{H}^2(\Delta_{(X,g)})}$$

for all $\varepsilon > 0$ with $\varepsilon^+ \le \eta_\varepsilon/4$ and $f \in \operatorname{dom}\Delta_{(X,g)}$, where

$$\delta_\varepsilon^+ = \begin{cases}\mathrm{O}\Big(\Big(\dfrac{\varepsilon}{\varepsilon^+}\Big)^{1-\beta}\dfrac{1}{\varepsilon^+}\Big) & \text{if } n\ge 3,\\ \mathrm{O}(1/(\varepsilon^+\sqrt{\log(\varepsilon^+/\varepsilon)})) & \text{if } n=2,\end{cases}$$

with $\beta = \beta_n$ as in Lemma 5.4. In particular, if $\delta_\varepsilon^+ \to 0$ as $\varepsilon \to 0$, then the cut-off function has moderate decay of order $k = 2$, i.e., Definition 5.1 (ii) is fulfilled.

Proof. We have

$$\begin{aligned}\|f d\chi_\varepsilon\|^2_{\mathsf{L}_2(T^*B^+_\varepsilon,g)} &= \sum_{p\in I_\varepsilon} \|f d\chi_\varepsilon\|^2_{\mathsf{L}_2(T^*B_{\varepsilon^+}(p),g)}\\ &\le \sum_{p\in I_\varepsilon} \|f\|^2_{\mathsf{L}_{2p_n}(B_{\varepsilon^+}(p),g)} \|d\chi_\varepsilon\|^2_{\mathsf{L}_{2q_n}(T^*B_{\varepsilon^+}(p),g)}\\ &\le C^2_{\mathrm{Sob}}(\varepsilon^+)^{-2a_n}\hat\delta^2_\varepsilon \sum_{p\in I_\varepsilon} \|f\|^2_{\mathsf{H}^2(B_{4\varepsilon^+}(p),g)}\\ &\le \underbrace{C^2_{\mathrm{ell.reg},2} C^2_{\mathrm{Sob}}(\varepsilon^+)^{-2a_n}\hat\delta^2_\varepsilon}_{=:(\delta^+_\varepsilon)^2} \|f\|^2_{\mathsf{H}^2(\Delta_{(X,g)})}\end{aligned}$$

where we used Hölder's inequality for the first inequality, Proposition A.1 and Lemma 5.4 for the second inequality and Proposition 3.3 for the last one. □

Note that we have the integral estimate in Lemma 5.4 only for single balls, and used the supremum when considering all balls in the previous proof.

Recall that, by Proposition 3.5, bounded geometry implies that the harmonic radius r_0 is strictly positive; we always assume that the separation distance η_ε fulfils $0 < 2\varepsilon < \eta_\varepsilon < r_0$ for all ε small enough. Recall that the exponent of ε in the following theorem has the form

$$\frac{1-\beta}{2-\beta}\begin{cases} = \dfrac{1}{2} & \text{if } n \ge 5,\\ \in \left(0, \tfrac{1}{2}\right) \text{ for } \beta \in (0,1) & \text{if } n = 4,\\ = \dfrac{1}{3} & \text{if } n = 3,\end{cases}$$

where $\beta = \beta_n$ is defined in Lemma 5.4.

Theorem 5.6. *Let (X, g) be a complete Riemannian manifold of bounded geometry. Moreover, let $B_\varepsilon = \dot\bigcup_{p\in I_\varepsilon} B_\varepsilon(p)$ be the union of balls of radius ε centred at the points of the η_ε-separated set I_ε. If $n \ge 3$ assume that*

$$\omega_\varepsilon := \frac{\varepsilon^{(1-\beta)/(2-\beta)}}{\eta_\varepsilon} \longrightarrow 0 \quad \text{as } \varepsilon \to 0.$$

If $n = 2$ assume that

$$\omega_\varepsilon := \frac{1}{\eta_\varepsilon\sqrt{|\log\varepsilon|}} \longrightarrow 0 \quad \text{as } \varepsilon \to 0.$$

Then $(B_\varepsilon)_\varepsilon$ is Dirichlet-asymptotically fading, i.e., the energy form $\mathfrak{d}_{(X,g)}$ and the (Dirichlet) energy form $\mathfrak{d}^{\mathrm{D}}_{(X_\varepsilon,g)}$ are δ_ε-quasi-unitarily equivalent (of order $k = 2$) with $\delta_\varepsilon = \mathrm{O}(\sqrt{\omega_\varepsilon})$ if $n \ge 3$ and $\delta_\varepsilon = \mathrm{O}(\sqrt{|\log\omega_\varepsilon|\omega_\varepsilon})$ if $n = 2$.

Proof. According to Definition 5.1, Theorem 5.2 and Proposition 5.5, we have to find ε^+ such that $\delta'_\varepsilon = \mathrm{O}(\tau_n(\varepsilon^+/\eta_\varepsilon)) \to 0$ and $\delta^+_\varepsilon = \mathrm{O}((\varepsilon/\varepsilon^+)^{1-\beta}/\varepsilon_+) \to 0$. We set

$$\varepsilon^+ = \eta_\varepsilon \sqrt{\omega_\varepsilon}.$$

First, we have $\varepsilon^+/\eta_\varepsilon = \sqrt{\omega_\varepsilon} \to 0$ and $\varepsilon^+ \to 0$ by our assumption, hence $\delta'_\varepsilon = \mathrm{O}(\tau_n(\sqrt{\omega_\varepsilon}))$ by Proposition 3.12.

If $n \geq 3$, then δ^+_ε is of order

$$\Big(\frac{\varepsilon}{\varepsilon^+}\Big)^{1-\beta}\frac{1}{\varepsilon^+} = \Big(\frac{\varepsilon}{\eta_\varepsilon}\Big)^{1-\beta}\frac{1}{\eta_\varepsilon}\cdot\omega_\varepsilon^{-(2-\beta)/2} = \omega_\varepsilon^{2-\beta}\cdot\omega_\varepsilon^{-(2-\beta)/2} = \omega_\varepsilon^{(2-\beta)/2}.$$

Since $\beta < 1$ we have $(2-\beta)/2 \geq 1/2$, and the error term from $\delta'_\varepsilon = \mathrm{O}(\sqrt{\omega_\varepsilon})$ wins, hence $\delta_\varepsilon = \mathrm{O}(\sqrt{\omega_\varepsilon})$ as error in the quasi-unitary equivalence.

If $n = 2$, then δ^+_ε is of order

$$\frac{1}{\varepsilon^+\sqrt{|\log\varepsilon|}} = \frac{1}{\eta_\varepsilon\sqrt{\omega_\varepsilon}\sqrt{|\log\varepsilon|}} = \sqrt{\omega_\varepsilon}.$$

As a consequence,

$$(\varepsilon^+)^2\log\Big(\frac{\varepsilon^+}{\varepsilon}\Big) = (\varepsilon^+)^2(|\log\varepsilon| + \log\varepsilon^+) \geq \frac{1}{2}(\varepsilon^+)^2|\log\varepsilon|,$$

for $\varepsilon \in (0,1)$ as $(\varepsilon^+)^2\log(\varepsilon^+) \to 0$ and $(\varepsilon^+)^2|\log\varepsilon| \to \infty$. Finally, we have

$$\frac{1}{\varepsilon^+\sqrt{|\log(\varepsilon^+/\varepsilon)|}} \leq \frac{\sqrt{2}}{\varepsilon^+\sqrt{|\log\varepsilon|}} = \sqrt{2\omega_\varepsilon}.$$

In particular, the error term from δ'_ε wins again, hence we obtain

$$\delta_\varepsilon = \mathrm{O}(\sqrt{|\log\omega_\varepsilon|\omega_\varepsilon}). \qquad \square$$

We now make the previous theorem more explicit by assuming that $\eta_\varepsilon = \varepsilon^\alpha$ for some $\alpha \in (0,1)$:

Corollary 5.7. *Let (X,g) be a complete Riemannian manifold of bounded geometry. Moreover, let $B_\varepsilon = \bigcup_{p\in I_\varepsilon} B_\varepsilon(p)$ be the union of balls of radius ε centred at the points of the η_ε-separated set I_ε. Assume that $\eta_\varepsilon = \varepsilon^\alpha$ for $\alpha \in (0,1/2)$ if $n \geq 4$ and $\alpha \in (0,1/3)$ if $n = 3$ and $\eta_\varepsilon = |\log\varepsilon|^{-\alpha}$ if $n = 2$ for $\alpha \in (0,1/2)$.*

Then $(B_\varepsilon)_\varepsilon$ is Dirichlet-asymptotically fading, i.e., the energy form $\mathfrak{d}_{(X,g)}$ and the (Dirichlet) energy form $\mathfrak{d}^{\mathrm{D}}_{(X_\varepsilon,g)}$ are δ_ε-quasi-unitarily equivalent (of order $k = 2$) with

$$\delta_\varepsilon = \begin{cases} \mathrm{O}(\varepsilon^{(1/2-\alpha)/2}), & n \geq 5, \\ \mathrm{O}(\varepsilon^{(1/2-\alpha)/2-0}), & n = 4, \\ \mathrm{O}(\varepsilon^{(1/3-\alpha)/2}), & n = 3, \\ \mathrm{O}(|\log\varepsilon|^{(\alpha-1/2)/2}\log|\log\varepsilon|), & n = 2. \end{cases}$$

For the notation $\delta_\varepsilon = \mathrm{O}(\varepsilon^{\gamma-0})$ see the end of Example 4.8.

Proof. If $n \geq 3$ we just have to assume that

$$\omega_\varepsilon = \varepsilon^{\frac{1-\beta}{2-\beta}-\alpha} \longrightarrow 0$$

as $\varepsilon \to 0$, and this is equivalent with $\frac{1-\beta}{2-\beta} > \alpha$. If $n \geq 5$ this means $1/2 > \alpha$, if $n = 3$ it is $1/3 > \alpha$. If $n = 4$, we can choose $\beta \in (0,1)$ for given $\alpha \in (0, \frac{1}{2})$ such that $\frac{1}{2} > \frac{1-\beta}{2-\beta} > \alpha$; the smaller we choose β, the better and closer the error δ_ε comes to $\mathrm{O}(\varepsilon^{(1/2-\alpha)/2})$. If $n = 2$, we have

$$\omega_\varepsilon = |\log\varepsilon|^{\alpha-1/2} \longrightarrow 0$$

as $\varepsilon \to 0$, since $\alpha \in (0, 1/2)$. The error term is then as given above. □

Remark 5.8. Note that the critical parameter for the balls to fade is the *capacity* (see the discussion in [40] or [27]). In our notation, the capacity of the balls of radius ε with η_ε-separated balls ($\eta_\varepsilon = \varepsilon^\alpha$) is vanishing if $\varepsilon^{n-2} \ll \eta_\varepsilon^n$, i.e., if $(n-2)/n > \alpha$ for $n \geq 3$, or $|\log\varepsilon|^{-1/2} \ll \eta_\varepsilon$ for $n = 2$. In particular, our result is optimal in small dimensions $n \in \{2, 3, 4\}$, as we can come arbitrarily close to the critical separation parameter. If $n \geq 5$, our result is no longer optimal (as we have to assume $\alpha < 1/2$ instead of the optimal bound $\alpha < (n-2)/n$). This is the price to pay for only staying at second order Sobolev spaces (see also Remark 5.3): If we actually use a result by [11, Proposition 1.3] stating that

$$|f(x_0)| \leq c(n) \sum_{j=0}^{N} r^{-n/2+2j} \|(\Delta_{(X,g)})^j f\|_{\mathsf{L}_2(B_r(x_0))},$$

provided $0 < r \leq \min\{|K|^{-1/2}, \iota_0\}$, where $|K|$ is the maximal absolute value of the sectional curvature, ι_0 is the injectivity radius and $N = [n/4]+1$. In particular, we can use a similar argument as in Proposition 5.5 to show that

$$\begin{aligned}\|f d\chi_\varepsilon\|^2_{\mathsf{L}_2(T^*B_\varepsilon^+,g)} &\le \sum_{p\in I_\varepsilon} \|f\|^2_{\mathsf{L}_\infty(B_{\varepsilon^+}(p),g)} \|d\chi_\varepsilon\|^2_{\mathsf{L}_2(T^*B_{\varepsilon^+}(p),g)} \\ &\le c'(n)\varepsilon^{n-2}(\varepsilon^+)^{-n}\|(\Delta_{(X,g)}+1)^N f\|^2_{\mathsf{L}_2(X,g)}\end{aligned}$$

as $\|d\chi_\varepsilon\|^2_{\mathsf{L}_2(T^*B_{\varepsilon^+}(p),g)} = \mathrm{O}(\varepsilon^{n-2})$ uniformly in $p \in I_\varepsilon$ (for $n \ge 3$). In particular, if we choose again $\varepsilon^+ = \eta_\varepsilon\sqrt{\omega_\varepsilon}$ and $\eta_\varepsilon = \varepsilon^\alpha$, we can find for any $\alpha \in (0,(n-2)/n)$ a sequence $\omega_\varepsilon \to 0$ such that $\delta_\varepsilon^+ = \mathrm{O}((\varepsilon^+)^{-n/2}\varepsilon^{(n-2)/2}) = \mathrm{O}(\varepsilon^{(n-2-n\alpha)/2}\omega_\varepsilon^{-n})$ and $\delta'_\varepsilon = \mathrm{O}(\sqrt{\omega_\varepsilon})$. As a consequence, the energy form $\mathfrak{d}_{(X,g)}$ and the (Dirichlet) energy form $\mathfrak{d}^{\mathrm{D}}_{(X_\varepsilon,g)}$ are δ_ε-quasi-unitarily equivalent with $\delta_\varepsilon = \mathrm{O}(\delta_\varepsilon^+ + \delta'_\varepsilon)$, but now of order $k = 2N = 2+2[n/4]$. Hence we obtain also the optimal ball density for dimensions $n \ge 5$, but the price is a higher resolvent power (namely the power $m = \max\{k-2,0\} = 2[n/4]$, see Definition 1.1 and Section 2) entering in the resolvent convergence.

The opposite effect of solidifying happens if $\alpha > (n-2)/n$, see (6.4).

6. Solidifying obstacles for Dirichlet boundary conditions

6.1. Abstract solidifying Dirichlet obstacles. Let us now consider the case, when the obstacles fill out some closed subset S, on which the limit operator has a Dirichlet boundary condition (it "solidifies" on S). We assume that the obstacles B_ε in some sense "converge" to S in the following sense:

Definition 6.1. We say that a family $(B_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]}$ of closed subsets of a Riemannian manifold (X,g) is *Dirichlet-asymptotically solidifying* towards a closed subset S if there is a sequence $(\chi_\varepsilon)_\varepsilon$ of Lipschitz-continuous cut-off functions $\chi_\varepsilon: X \to [0,1]$ with $\mathrm{supp}(\chi_\varepsilon) \subset X_0 := X \setminus S$ such that the following conditions are fulfilled (we let $X_\varepsilon := X \setminus B_\varepsilon$).

(i) *Non-concentrating property.* We assume that $(A_\varepsilon, X_\varepsilon)$ is δ'_ε-non-concentrating of order 1 with $\delta'_\varepsilon \to 0$, and (A_ε, X_0) is δ''_ε-non-concentrating of order 2 with $\delta''_\varepsilon \to 0$, where $A_\varepsilon := \mathrm{supp}(d\chi_\varepsilon)$ is an annulus region around the boundary of S.

(ii) *Spectrally solidifying.* We assume $B_\varepsilon \subset S$ and that there is $\bar\delta_\varepsilon \to 0$ as $\varepsilon \to 0$ such that

$$\|u\|_{\mathsf{L}_2(\mathring{S}\setminus\bar B_\varepsilon,g)} \le \bar\delta_\varepsilon\|u\|_{\mathsf{H}^1(X_\varepsilon,g)}$$

for all $u \in \mathring{\mathsf{H}}^1(X_\varepsilon,g)$ and $\varepsilon \in (0,\varepsilon_0]$.

(iii) The cut-off functions χ_ε have *moderate decay* in the sense that

$$\delta_\varepsilon^+ := \delta'_\varepsilon \delta''_\varepsilon \|d\chi_\varepsilon\|_\infty \longrightarrow 0$$

as $\varepsilon \to 0$, where δ'_ε and δ''_ε are given in (i).

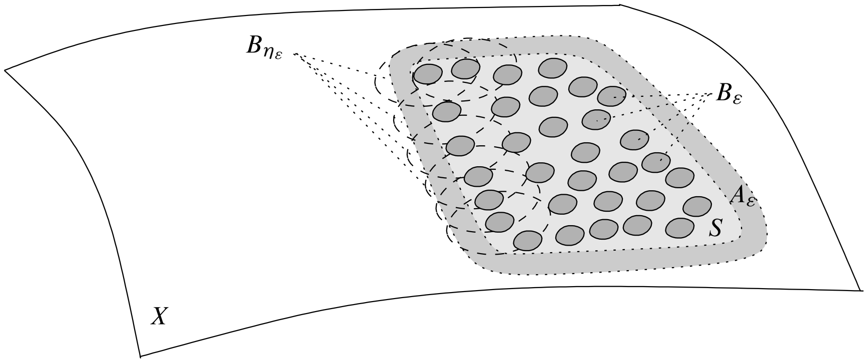

Figure 3. The solidifying part S (light grey and dotted) with the annulus region A_ε (middle dark grey and dotted) and the obstacles B_ε inside (dark grey balls); the larger balls B_{η_ε} (dashed lines) for a uniformly locally finite cover of the annulus region and the solidifying part S.

A sufficient condition for the spectral non-concentration property of Definition 6.1 (ii) is as follows (explaining also the terminology) (Rauch and Taylor [40] say that such obstacles "become solid" in S).

Proposition 6.2. *Assume that λ_ε is the bottom of the spectrum of the Laplacian on $\mathring{S} \setminus \overline{B}_\varepsilon$ with Dirichlet boundary conditions on $\partial B_\varepsilon \setminus \partial S$ and Neumann boundary condition on ∂S. If $\lim_{\varepsilon\to 0} \lambda_\varepsilon = \infty$, then $(B_\varepsilon)_\varepsilon$ is spectrally solidifying.*

Proof. Note that the mentioned Laplacian is the operator associated with the quadratic form given by $\|du\|^2_{\mathsf{L}_2(T^*(\mathring{S}\setminus\overline{B}_\varepsilon),g)}$ with domain

$$u \in \{\, f\restriction_{\mathring{S}\setminus\overline{B}_\varepsilon} \mid f \in \mathring{\mathsf{H}}^1(X_\varepsilon)\,\}.$$

By the variational characterisation of the first eigenvalue, we have

$$\lambda_\varepsilon = \inf\left\{ \frac{\int_{\mathring{S}\setminus\overline{B}_\varepsilon} |du|^2 \,\mathrm{d}g}{\int_{\mathring{S}\setminus\overline{B}_\varepsilon} |u|^2 \,\mathrm{d}g} \,\middle|\, u \in \mathring{\mathsf{H}}^1(X_\varepsilon)\setminus\{0\} \right\}.$$

From this characterisation via an infimum, we conclude

$$\|u\|_{\mathsf{L}_2(\mathring{S}\setminus\overline{B}_\varepsilon,g)} \le \frac{1}{\sqrt{\lambda_\varepsilon}}\|du\|_{\mathsf{L}_2(T^*(\mathring{S}\setminus\overline{B}_\varepsilon),g)} \le \frac{1}{\sqrt{\lambda_\varepsilon}}\|u\|_{\mathsf{H}^1(X_\varepsilon,g)}.$$

As $\lambda_\varepsilon \to \infty$, we can choose $\bar{\delta}_\varepsilon = 1/\sqrt{\lambda_\varepsilon} \to 0$ as $\varepsilon \to 0$. □

Remark 6.3. There is a subtle point in Definition 6.1 (i) and (iii): if we would assume that (A_ε, X_0) is δ_ε-non-concentrating for the *same* $\delta_\varepsilon = \delta'_\varepsilon = \delta''_\varepsilon$, then δ^+_ε will most likely not converge to 0 as it contains the cut-off function, see also Remark 6.15 for details. This is why we have two different assumptions of non-concentration in Definition 6.1 (i).

In the applications below in Subsection 6.2 we show similarly as in Proposition 6.2 that $(A_\varepsilon, X_\varepsilon)$ is non-concentrating of order 1, see Proposition 6.7.

We extend our notion of elliptic regularity (Definition 3.1) of a manifold (X_0, g) with boundary and Dirichlet boundary conditions as follows: We say that (X_0, g) is *elliptically regular,* i.e., we have $\operatorname{dom} \Delta^{\mathrm{D}}_{(X_0,g)} \subset \mathsf{H}^2(X_0, g)$ and there is $C_{\mathrm{ell.reg}} \geq 1$ such that

$$\|f\|_{\mathsf{H}^2(X_0,g)} \leq C_{\mathrm{ell.reg}} \|(\Delta^{\mathrm{D}}_{(X_0,g)} + 1)f\|_{\mathsf{L}_2(X_0,g)}$$

for all $f \in \mathsf{H}^2(\Delta^{\mathrm{D}}_{(X_0,g)}) = \operatorname{dom} \Delta^{\mathrm{D}}_{(X_0,g)}$, where $\Delta^{\mathrm{D}}_{(X_0,g)}$ denotes the Dirichlet Laplacian on (X_0, g).

Our next main result is as follows:

Theorem 6.4. *Let (X, g) be a Riemannian manifold and $(B_\varepsilon)_\varepsilon$ be a family of closed subsets of X that is Dirichlet-asymptotically solidifying towards S. Assume in addition that (X_0, g) is elliptically regular where $X_0 = X \setminus S$.*

Then the Dirichlet energy form $\mathfrak{d}^{\mathrm{D}}_{(X_0,g)}$ of (X_0, g) and the Dirichlet energy form $\mathfrak{d}^{\mathrm{D}}_{(X_\varepsilon,g)}$ of (X_ε, g) with $X_\varepsilon = X \setminus B_\varepsilon$ are δ_ε-quasi-unitarily equivalent of order 2 with $\delta_\varepsilon = \max\{\bar{\delta}_\varepsilon, C_{\mathrm{ell.reg}}(\delta''_\varepsilon + \delta^+_\varepsilon)\}$.

Proof. We show again that the hypotheses[4] of Definition 2.1 are fulfilled. Here, $X_0 \subset X_\varepsilon$, so extension by 0 and restriction are swapped. We set

$$\begin{aligned}
J\colon \mathcal{H} &:= \mathsf{L}_2(X_0, g) \longrightarrow \widetilde{\mathcal{H}} := \mathsf{L}_2(X_\varepsilon, g), & f &\longmapsto \bar{f},\\
J^1\colon \mathcal{H}^1 &:= \mathring{\mathsf{H}}^1(X_0, g) \longrightarrow \widetilde{\mathcal{H}}^1 := \mathring{\mathsf{H}}^1(X_\varepsilon, g), & f &\longmapsto \bar{f},\\
J'\colon \widetilde{\mathcal{H}} &= \mathsf{L}_2(X_\varepsilon, g) \longrightarrow \mathcal{H} = \mathsf{L}_2(X_0, g), & u &\longmapsto u\restriction_{X_0},\\
J^{1\prime}\colon \widetilde{\mathcal{H}}^1 &= \mathring{\mathsf{H}}^1(X_\varepsilon, g) \longrightarrow \mathcal{H}^1 = \mathring{\mathsf{H}}^1(X_0, g), & u &\longmapsto \chi_\varepsilon u,
\end{aligned}$$

where $\bar{f}$ denotes the extension of $f\colon X_0 \to \mathbb{C}$ by 0 onto X_ε, as $X_0 \subset X_\varepsilon$.

[4] Note that the Dirichlet solidifying case is in some sense dual to the Dirichlet fading case: Here, we have again $X_0 \subset X_\varepsilon$, hence $J^{1\prime}$ is more complicated (as in the Neumann fading case).

We check the hypotheses of Definition 2.1: We easily see that

$$J' = J^*, \quad J'J = \mathrm{id}_{\mathcal{H}}, \quad J^1 = J \restriction_{\mathcal{H}^1}.$$

As in the Neumann case, we have $\|J\| = 1$ and (2.3a) is fulfilled with $\delta = 0$.

The second estimate in (2.3b) follows from the spectral non-concentrating property Definition 6.1 (ii), namely we have

$$\|u - JJ'u\|_{\mathsf{L}_2(X,g)} = \|u\|_{\mathsf{L}_2(\mathring{S} \setminus \bar{B}_\varepsilon, g)} \le \bar{\delta}_\varepsilon \|u\|_{\mathsf{H}^1(X_\varepsilon, g)}.$$

Moreover, $J'u - J^{1\prime}u = ((1-\chi_\varepsilon)u)\restriction_{X_0}$, hence

$$\begin{aligned}\|J'u - J^{1\prime}u\|_{\mathsf{L}_2(X_0,g)} &= \|(1-\chi_\varepsilon)u\|_{\mathsf{L}_2(X_0,g)} \\ &\le \|u\|_{\mathsf{L}_2(A_\varepsilon,g)} \le \delta'_\varepsilon \|u\|_{\mathsf{H}^1(X_\varepsilon,g)}\end{aligned}$$

by the non-concentration property of (A_ε, X_0) in Definition 6.1 (i) (implying the same property for $(A_\varepsilon, X_\varepsilon)$ as $X_0 \subset X_\varepsilon$). Finally,

$$\begin{aligned}&|\mathfrak{d}_\varepsilon(J^1 f, u) - \mathfrak{d}(f, J^{1\prime}u)| \\ &\quad = |\langle df, d((1-\chi_\varepsilon)u)\rangle_{\mathsf{L}_2(T^*A_\varepsilon,g)}| \\ &\quad \le |\langle df, (1-\chi_\varepsilon)du\rangle_{\mathsf{L}_2(T^*A_\varepsilon,g)}| + |\langle df, u\, d\chi_\varepsilon\rangle_{\mathsf{L}_2(T^*A_\varepsilon,g)}| \\ &\quad \le \|df\|_{\mathsf{L}_2(T^*A_\varepsilon,g)}(\|du\|_{\mathsf{L}_2(T^*A_\varepsilon,g)} + \|u\|_{\mathsf{L}_2(A_\varepsilon,g)}\|d\chi_\varepsilon\|_\infty) \\ &\quad \le \delta''_\varepsilon \|f\|_{\mathsf{H}^2(X_0,g)}(1 + \delta'_\varepsilon\|d\chi_\varepsilon\|_\infty)\|u\|_{\mathsf{H}^1(X_\varepsilon,g)} \\ &\quad \le C_{\mathrm{ell.reg}}(\delta''_\varepsilon + \delta^+_\varepsilon)\|(\Delta^{\mathrm{D}}_{(X_0,g)} + 1)f\|\,\|u\|_1\end{aligned}$$

by the non-concentrating property of order 2 in Definition 6.1 (i) for the second last estimate and the elliptic regularity assumption and the moderate decay property (Definition 6.1 (iii)) for the last estimate. □

6.2. Application: many solidifying small balls as Dirichlet obstacles. The obstacles are of the same kind as in Subsection 4.2 but denser: let now I_ε be ε-separated and let $B_\varepsilon = \bigcup_{p \in I_\varepsilon} B_\varepsilon(p)$ be the disjoint union of balls of radius ε. Before checking the conditions of Definition 6.1, we first need the following result:

Lemma 6.5 (Rauch and Taylor [40]). *Assume that $\eta > \varepsilon$ and that*

$$A_{\varepsilon,\eta}(0) := B_\eta(0) \setminus \overline{B_\varepsilon(0)}$$

is an annulus with inner radius ε and outer radius η in Euclidean space $\mathbb{R}^n$.

Denote by $\lambda_\varepsilon^{\mathrm{eucl}}$ *the first eigenvalue of the Laplacian with Dirichlet boundary condition on the inner sphere, and Neumann on the outer sphere. Then there exists a constant* $C_{\mathrm{eucl}} > 0$ (*depending only on the dimension*) *such that*

$$\lambda_\varepsilon^{\mathrm{eucl}} \geq \frac{C_{\mathrm{eucl}}\varepsilon^{n-2}}{\eta^n} \quad \text{for } n \geq 3,$$

resp.

$$\lambda_\varepsilon^{\mathrm{eucl}} \geq \frac{C_{\mathrm{eucl}}}{\eta^2|\log\varepsilon|} \quad \text{for } n = 2.$$

for all $0 < \varepsilon < \eta < r_0$.

Definition 6.6. We say that $\{B_{\eta_\varepsilon}(p)\}_{p\in I_\varepsilon}$ is a *uniformly locally finite cover* of S if there is $\varepsilon_0 > 0$ and $N \in \mathbb{N}$ such that

$$|\{\, p \in I_\varepsilon \mid B_{\eta_\varepsilon}(p) \cap B_{\eta_\varepsilon}(q) \neq \emptyset \,\}| \leq N \tag{6.1a}$$

and

$$S \subset B_{\eta_\varepsilon} = \bigcup_{p\in I_\varepsilon} B_{\eta_\varepsilon}(p) \tag{6.1b}$$

for all $q \in I_\varepsilon$ and all $\varepsilon \in (0, \varepsilon_0]$.

Proposition 6.7. *Assume that* (X, g) *is a Riemannian manifold with bounded geometry with harmonic radius* $r_0 > 0$*. Let* $\varepsilon, \eta_\varepsilon \in (0, r_0)$ *such that* $0 < \varepsilon < \eta_\varepsilon < r_0$*. Assume that* I_ε *is* ε*-separated and that* $(B_{\eta_\varepsilon}(p))_{p\in I_\varepsilon}$ *is a uniformly locally finite cover of* S*.*

Then we have

$$\|u\|_{\mathsf{L}_2(\mathring{S}\setminus \bar{B}_\varepsilon, g)} \leq \|u\|_{\mathsf{L}_2(A_{\varepsilon,\eta_\varepsilon}, g)} \leq \bar{\delta}_\varepsilon \|u\|_{\mathsf{H}^1(X_\varepsilon, g)} \tag{6.2}$$

for all $u \in \mathsf{H}^1(X_\varepsilon, g)$*, where* $A_{\varepsilon,\eta_\varepsilon} = B_{\eta_\varepsilon} \setminus \bar{B}_\varepsilon$ *and* $\bar{\delta}_\varepsilon = C\omega_\varepsilon$ *for some constant* $C > 0$ *depending only on* N*,* K *and* n *and where*

$$\omega_\varepsilon = \sqrt{\eta_\varepsilon^n/\varepsilon^{n-2}} \quad (n \geq 3), \tag{6.3a}$$

resp.

$$\omega_\varepsilon = \eta_\varepsilon\sqrt{|\log\varepsilon|} \quad (n = 2). \tag{6.3b}$$

In particular, if $\omega_\varepsilon \to 0$ *as* $\varepsilon \to 0$ *then* $(B_\varepsilon)_\varepsilon$ *is spectrally solidifying* (*see Definition* 6.1 (ii)).

Proof. Note first that $\mathring{S} \setminus \overline{B}_\varepsilon \subset A_{\varepsilon,\eta_\varepsilon}$, hence we have

$$\begin{aligned}
\|u\|^2_{\mathsf{L}_2(\mathring{S}\setminus\overline{B}_\varepsilon,g)} \le \|u\|_{\mathsf{L}_2(A_{\varepsilon,\eta_\varepsilon},g)} &\le \sum_{p\in I_\varepsilon} \|u\|^2_{\mathsf{L}_2(A_{\varepsilon,\eta_\varepsilon}(p),g)} \\
&\le \frac{K^{n+1}}{C_{\mathrm{eucl}}} \cdot \frac{\eta_\varepsilon^n}{\varepsilon^{n-2}} \sum_{p\in I_\varepsilon} \|du\|^2_{\mathsf{L}_2(T^*A_{\varepsilon,\eta_\varepsilon}(p),g)} \\
&\le \underbrace{\frac{NK^{n+1}}{C_{\mathrm{eucl}}}}_{=:C^2} \cdot \frac{\eta_\varepsilon^n}{\varepsilon^{n-2}} \|du\|^2_{\mathsf{L}_2(T^*A_{\varepsilon,\eta_\varepsilon},g)}
\end{aligned}$$

using Corollary 3.6 (ii) and Lemma 6.5, where $A_{\varepsilon,\eta}(p) := \mathring{B}_\eta(p) \setminus \overline{B}_\varepsilon(p)$ is the annulus with inner radius ε and outer radius η around p and $A_{\varepsilon,\eta} := \bigcup_{p\in I_\varepsilon} A_{\varepsilon,\eta}(p)$. □

Remark 6.8. If $\eta_\varepsilon = \varepsilon^\alpha$ with $\alpha \in (0,1)$, then B_ε is spectrally solidifying, i.e., $\omega_\varepsilon = \varepsilon^{(n\alpha-(n-2))/2} \to 0$ if and only if

$$\frac{n-2}{n} < \alpha. \tag{6.4}$$

The value $\alpha_0 = (n-2)/n$ is actually the critical parameter for the ε^α-separation of balls when the behaviour changes from *fading* ($\alpha \in (0,\alpha_0)$ as in Section 5 to solidifying ($\alpha \in (\alpha_0, 1)$ as in this section. If $\alpha = (n-2)/2$ and under suitable additional assumptions on the spacing of the obstacles, one obtains a different limit, due to a *homogenisation* effect, see e.g. [27]) and the references cited therein and in Section 1.2.

To check the remaining properties of Definition 6.1 we need some regularity on $Y = \partial S$.

Assumption 6.9 (geometric assumption on the boundary of the solidifying set). We assume that $Y = \partial S$ is a smooth manifold with embedding $\iota\colon Y \hookrightarrow X$ and induced metric $h := \iota^* g$, we assume also that Y *admits a uniform tubular neighbourhood*, i.e., that Y has a global normal unit vector field $\vec{N}$ (so Y is orientable) and that there is $r_0 > 0$ such that

$$\exp\colon Y \times [0, r_0) \longrightarrow X, \quad (y,t) \longmapsto \exp_y(t\vec{N}(y)) \tag{6.5}$$

is a diffeomorphism.

Remark 6.10. This assumption includes the fact that the principal curvatures of the hypersurface Y are bounded by a constant depending on $1/r_0$ and κ_0, see e.g. [23, Corollory 3.3.2]. But our assumption is stronger: we need also that Y does not admit infinitely close points which are far away for the inner distance.

Let $\varepsilon \mapsto \tilde{\varepsilon} \in (0, r_0)$ be a function of ε such that $\tilde{\varepsilon} \to 0$ as $\varepsilon \to 0$ (to be specified later). Moreover set

$$A_\varepsilon := \{ x \in X_0 = X \setminus S \mid d(x, S) < \tilde{\varepsilon} \}.$$

Then A_ε has tubular coordinates $(r, y) \in (0, \tilde{\varepsilon}) \times Y$ by Assumption 6.9.

Let $\widetilde{\chi}: \mathbb{R} \to [0, 1]$ be a smooth function with $\widetilde{\chi}(r) = 0$ for $r \leq 0$, χ strictly monotone on $(0, 1)$ and $\widetilde{\chi}(r) = 1$ for $r \geq 1$ and $\|\widetilde{\chi}'\|_\infty \leq 2$. We then define

$$\chi_{\tilde{\varepsilon}}(x) := \widetilde{\chi}\Big(\frac{d(x, S)}{\tilde{\varepsilon}}\Big) \tag{6.6}$$

as cut-off function. We clearly have $\|d\chi_{\tilde{\varepsilon}}\|_\infty \leq 2/\tilde{\varepsilon}$ and $A_\varepsilon = \operatorname{supp}(d\chi_{\tilde{\varepsilon}}) \cap X_0$

Before using the cut-off function $\chi_{\tilde{\varepsilon}}$, we use Proposition 6.7 to show the following result:

Proposition 6.11. *Assume that (X, g) has bounded geometry with harmonic radius $r_0 > 0$. Assume additionally that*

$$A_\varepsilon \subset B_{\eta_\varepsilon} \tag{6.7}$$

(*it then follows that $A_\varepsilon \subset B_{\eta_\varepsilon} \setminus \overline{B}_\varepsilon$*) *and that* (6.2) *holds. Then*

$$\|u\|_{\mathsf{L}_2(A_\varepsilon, g)} \leq \bar{\delta}_\varepsilon \|u\|_{\mathsf{H}^1(X_\varepsilon, g)}$$

for all $u \in \mathsf{H}^1(X_\varepsilon, g)$ and $\tilde{\varepsilon} \in (0, r_0)$ ($\bar{\delta}_\varepsilon = \mathrm{O}(\omega_\varepsilon)$ and ω_ε are given in Proposition 6.7) *and* (6.3)*, respectively. In particular, $(A_\varepsilon, X_\varepsilon)$ is $\bar{\delta}_\varepsilon$-non-concentrating of order* 1.

Proof. As $A_\varepsilon \subset A_{\varepsilon,\eta_\varepsilon} = B_{\eta_\varepsilon} \setminus \overline{B}_\varepsilon$, we have

$$\|u\|_{\mathsf{L}_2(A_\varepsilon, g)} \leq \|u\|_{\mathsf{L}_2(A_{\varepsilon,\eta_\varepsilon}, g)} \leq \bar{\delta}_\varepsilon \|u\|_{\mathsf{H}^1(X_\varepsilon, g)}$$

using (6.2). □

Remark 6.12. Note that there is a hidden assumption on $\tilde{\varepsilon}$ and η_ε in $A_\varepsilon \subset B_{\eta_\varepsilon}$: namely, as A_ε is the $\tilde{\varepsilon}$-neighbourhood of S and $B_\varepsilon \subset S$, such an inclusion can only be true if $\tilde{\varepsilon}/\eta_\varepsilon$ tends to 0 or at least is bounded. This assumption is the reason why we will not come arbitrarily close to the critical parameter for the spacing of the balls, where the behaviour changes from fading to solidifying, see Remark 6.19.

Proposition 6.13. *Assume that (X, g) has bounded curvature with harmonic radius $r_0 > 0$. Assume additionally that (Y, h) is a complete smooth orientable hypersurface admitting a uniform tubular neighbourhood also with radius $r_0 > 0$. Then there is $\delta''_\varepsilon = \mathrm{O}(\sqrt{\tilde{\varepsilon}})$ depending only on Y and r_0 such that*

$$\|df\|_{\mathsf{L}_2(A_\varepsilon, g)} \le \delta''_\varepsilon \|f\|_{\mathsf{H}^2(X_0, g)}$$

for all $f \in \mathsf{H}^2(X_0, g)$ and $\tilde{\varepsilon} \in (0, r_0)$. In particular, (A_ε, X_0) is δ''_ε-non-concentrating of order 2.

Proof. From Lemma A.3 (with $\tilde{\varepsilon}$ and r_0 instead of ε and ε^+) we conclude that (A_ε, X_0) is δ''_ε-non-concentrating with $\delta''_\varepsilon = C_{r_0,Y}\sqrt{\tilde{\varepsilon}/r_0}$, and Proposition 3.9 then yields

$$\|df\|_{\mathsf{L}_2(A_\varepsilon, g)} \le \delta''_\varepsilon \|f\|_{\mathsf{H}^2(A_{r_0}, g)} \le \delta''_\varepsilon \|f\|_{\mathsf{H}^2(X_0, g)}$$

for all $f \in \mathsf{H}^2(X_0, g)$. □

Recall that the parameter ω_ε is defined in (6.3).

Corollary 6.14. *Let $\varepsilon \mapsto \tilde{\varepsilon} \in (0, r_0)$ be a function with $\tilde{\varepsilon} \to 0$ as $\varepsilon \to 0$. Assume that $\omega_\varepsilon^2/\tilde{\varepsilon} \to 0$ as $\varepsilon \to 0$. Then the cut-off function $\chi_{\tilde{\varepsilon}}$ has moderate decay, i.e., Definition 6.1 (iii) is fulfilled with $\delta^+_\varepsilon = \mathrm{O}(\omega_\varepsilon/\sqrt{\tilde{\varepsilon}})$.*

Proof. We have

$$\delta^+_\varepsilon = \bar{\delta}_\varepsilon \delta''_\varepsilon \|d\chi_\varepsilon\|_\infty \le 2C C_{r_0,Y}\omega_\varepsilon \sqrt{\frac{\tilde{\varepsilon}}{r_0}} \cdot \frac{2}{\tilde{\varepsilon}}$$

as $\|d\chi_\varepsilon\|_\infty \le 2/\tilde{\varepsilon}$, and hence $\delta^+_\varepsilon \to 0$ as $\varepsilon \to 0$ by the assumption $\omega_\varepsilon^2/\tilde{\varepsilon} \to 0$. □

Remark 6.15. There is a subtle point in the combination of arguments for the non-concentrating property: If we used for Proposition 6.13 an analogue result as for Proposition 6.11 (with δ'_ε instead of $\bar{\delta}_\varepsilon$ also of order $\sqrt{\tilde{\varepsilon}}$), then δ^+_ε would not tend to 0, as $\delta'_\varepsilon \delta''_\varepsilon$ is of order $\tilde{\varepsilon}$, but $\|d\chi_\varepsilon\|_\infty$ is of order $\tilde{\varepsilon}^{-1}$. So we need somehow also $\mathring{S} \setminus B_\varepsilon$ for the convergence. In particular, we need that A_ε is covered by B_{η_ε}, which assures that the balls in B_ε are not too far separated, see Remark 6.12. This is also the reason why we need the additional regularity on ∂S in Assumption 6.9.

We can now state our main result of solidifying of a union of many balls.

Theorem 6.16. *Let* (X, g) *be a complete Riemannian manifold of bounded geometry with harmonic radius* $r_0 > 0$ *and let* $B_\varepsilon = \dot{\bigcup}_{p \in I_\varepsilon} B_\varepsilon(p)$ *be the union of* ε*-separated balls of radius* ε*. Assume that there are* $\eta_\varepsilon \in (0, r_0)$ *and* $\tilde{\varepsilon} \in (0, r_0)$ *such that*

$$\eta_\varepsilon \longrightarrow 0 \quad \textit{and} \quad \tilde{\varepsilon} \longrightarrow 0 \quad \textit{as } \varepsilon \to 0,$$

and such that the following holds.

(i) *We have* $\omega_\varepsilon \to 0$*, where*

$$\omega_\varepsilon := \begin{cases} \sqrt{\eta_\varepsilon^n / \varepsilon^{n-2}} & \text{if } n \geq 3 \text{ and} \\ \eta_\varepsilon \sqrt{|\log \varepsilon|} & \text{if } n = 2. \end{cases}$$

(ii) *There is a closed subset* $S \subset X$ *with smooth boundary* $Y = \partial S$ *admitting a uniform tubular neighbourhood of radius* $r_0 > 0$*; denote by* A_ε *the (outer)* $\tilde{\varepsilon}$*-neighbourhood. Moreover,*

$$\frac{\omega_\varepsilon}{\sqrt{\tilde{\varepsilon}}} \longrightarrow 0 \quad \textit{as } \varepsilon \to 0.$$

(iii) *We have* $B_\varepsilon \subset S$ *and* $A_\varepsilon \subset B_{\eta_\varepsilon}$*, and the latter cover* $(B_{\eta_\varepsilon}(p))_{p \in I_\varepsilon}$ *is uniformly locally bounded (see (6.1)). Moreover, assume that*

$$\frac{\tilde{\varepsilon}}{\eta_\varepsilon} \textit{ is bounded as } \varepsilon \to 0.$$

Then $(B_\varepsilon)_\varepsilon$ *is Dirichlet-asymptotically solidifying towards* S*, i.e., the Dirichlet energy form* $\mathfrak{d}^{\mathrm{D}}_{(X_0, g)}$ *and the Dirichlet energy form* $\mathfrak{d}^{\mathrm{D}}_{(X_\varepsilon, g)}$ *are* δ_ε*-quasi-unitarily equivalent with*

$$\delta_\varepsilon = \mathrm{O}\Big(\max\Big\{\omega_\varepsilon, \sqrt{\tilde{\varepsilon}}, \frac{\omega_\varepsilon}{\sqrt{\tilde{\varepsilon}}}\Big\}\Big).$$

(Recall that $X_\varepsilon = X \setminus B_\varepsilon$ *and* $X_0 = X \setminus S$*.)*

Proof. By Proposition 6.11, $(A_\varepsilon, X_\varepsilon)$ is $\bar{\delta}_\varepsilon$-non-concentrating of order 1 with $\delta'_\varepsilon = \bar{\delta}_\varepsilon = \mathrm{O}(\omega_\varepsilon)$. Moreover, by Proposition 6.13, (A_ε, X_0) is δ''_ε-non-concentrating of order 1 with $\delta''_\varepsilon = \mathrm{O}(\sqrt{\tilde{\varepsilon}})$. In particular, Definition 6.1 (i) is fulfilled. For the elliptic regularity assumption we remark that the proof of Proposition 3.3 based on (3.2) works as well for the Dirichlet Laplacian. Definition 6.1 (ii) is fulfilled by Proposition 6.7 with $\bar{\delta}_\varepsilon = \mathrm{O}(\omega_\varepsilon)$, and finally, Definition 6.1 (iii) is fulfilled by Corollary 6.14 with $\delta^+_\varepsilon = 2\bar{\delta}_\varepsilon \delta''_\varepsilon / \tilde{\varepsilon} = \mathrm{O}(\omega_\varepsilon / \sqrt{\tilde{\varepsilon}})$. The total error δ_ε is now of order as the maximum of $\bar{\delta}_\varepsilon$, δ''_ε and δ^+_ε. □

Remark 6.17. There is a competition between $\tilde{\varepsilon}/\eta_\varepsilon$ to be bounded and $\omega_\varepsilon/\sqrt{\tilde{\varepsilon}} \to 0$. Choosing simply $\tilde{\varepsilon} = \varepsilon^\tau$ and $\eta_\varepsilon = \varepsilon^\alpha$ implies that $\tau \geq \alpha$ (by the boundedness of $\tilde{\varepsilon}/\eta_\varepsilon = \varepsilon^{\tau-\alpha}$) and that $(n\alpha - (n-2))/2 > \tau$ (as $\omega_\varepsilon/\sqrt{\tilde{\varepsilon}} = \varepsilon^{(n\alpha-(n-2))/2-\tau} \to 0$). Together, these two requirements imply $n\alpha - (n-2) > 2\alpha$, i.e., $\alpha > 1$. This is in contradiction with $\varepsilon < \eta_\varepsilon = \varepsilon^\alpha$.

We therefore use the more advanced setting $\tilde{\varepsilon} := \omega_\varepsilon^{2\gamma}$ for $\gamma \in (0,1)$ in the next corollary. This setting and the requirement that $\omega_\varepsilon \to 0$ imply that $\tilde{\varepsilon} \to 0$, $\delta''_\varepsilon = \mathrm{O}(\omega_\varepsilon^\gamma)$ and $\delta^+_\varepsilon = \mathrm{O}(\omega_\varepsilon^{1-\gamma})$ as $\varepsilon \to 0$. Only the requirements $\eta_\varepsilon \to 0$ and $\tilde{\varepsilon}/\eta_\varepsilon$ bounded remain to be checked.

Let us now specify η_ε and $\tilde{\varepsilon}$ and show that the assumptions of Theorem 6.4 can actually be fulfilled:

Corollary 6.18. *Let (X,g) be a complete Riemannian manifold of bounded geometry with harmonic radius $r_0 > 0$ and let $B_\varepsilon = \dot{\bigcup}_{p\in I_\varepsilon} B_\varepsilon(p)$ be the union of ε-separated balls of radius ε. Assume that $\eta_\varepsilon = \varepsilon^\alpha$ with $\alpha \in (0,1)$ and that the following holds*:

(i) *there is a closed subset $S \subset X$ with smooth boundary $Y = \partial S$ admitting a uniform tubular neighbourhood of radius $r_0 > 0$;*

(ii) *we have $B_\varepsilon \subset S$ and $A_\varepsilon \subset B_{\eta_\varepsilon}$, and the latter cover $(B_{\eta_\varepsilon})_{p\in I_\varepsilon}$ is uniformly locally bounded (see (6.1)). Moreover, assume that*

$$\frac{n-2}{n-1} < \alpha < 1 \quad \text{if } n \geq 3$$

and

$$0 < \alpha < 1 \qquad \text{if } n = 2.$$

Then $(B_\varepsilon)_\varepsilon$ is Dirichlet-asymptotically solidifying towards S, i.e., the Dirichlet energy form $\mathfrak{d}^{\mathrm{D}}_{(X_0,g)}$ and the Dirichlet energy form $\mathfrak{d}^{\mathrm{D}}_{(X_\varepsilon,g)}$ are δ_ε-quasi-unitarily equivalent with $\delta_\varepsilon \to 0$ given in (6.8).

Proof. We check the conditions of Theorem 6.16. Let $n \geq 3$. From

$$\alpha > \frac{n-2}{n-1} > \frac{n-2}{n},$$

we conclude that $\alpha > (n-2)/n$ and hence $\omega_\varepsilon = \varepsilon^{(n\alpha-(n-2))/2} \to 0$ as $\varepsilon \to 0$. If $n = 2$, then $\omega_\varepsilon = \varepsilon^\alpha |\log \varepsilon| \to 0$ for any $\alpha > 0$. In particular, Theorem 6.16 (i) is fulfilled.

For Theorem 6.16 (ii), we set $\tilde{\varepsilon} := \omega_\varepsilon^{2\gamma}$ for $\gamma > 0$, then $\tilde{\varepsilon} \to 0$ as before. Moreover,

$$\frac{\omega_\varepsilon}{\sqrt{\tilde{\varepsilon}}} = \omega_\varepsilon^{1-\gamma} \longrightarrow 0 \quad \text{as } \varepsilon \to 0$$

provided $\gamma < 1$. For the final requirement Theorem 6.16 (iii), we specify $\gamma \in (0, 1)$: if $n \geq 3$ and $\alpha > (n-2)/(n-1)$, then

$$\frac{\tilde{\varepsilon}}{\eta_\varepsilon} = \varepsilon^{(n\alpha-(n-2))\gamma-\alpha} = \mathrm{O}(1) \iff (n\alpha-(n-2))\gamma \geq \alpha \iff \gamma \geq \frac{\alpha}{n\alpha-(n-2)}.$$

The latter can only be true for some $\gamma < 1$ if and only if

$$1 > \frac{\alpha}{n\alpha-(n-2)} \iff \alpha > \frac{n-2}{n-1}.$$

If $n = 2$, then

$$\frac{\tilde{\varepsilon}}{\eta_\varepsilon} = \varepsilon^{\alpha(2\gamma-1)}|\log\varepsilon|^\gamma = \mathrm{O}(1) \iff \gamma > \frac{1}{2}$$

for any $\alpha > 0$. From Theorem 6.16 we conclude the result with error of order

$$\delta_\varepsilon = \mathrm{O}(\omega_\varepsilon^{\max\{\gamma,1-\gamma\}}). \tag{6.8}$$

□

Remark 6.19. Unfortunately, the condition

$$\alpha > \frac{n-2}{n-1}$$

is not the optimal one, namely $\alpha > (n-2)/n$. Note that the condition comes from the boundedness of $\tilde{\varepsilon}/\eta_\varepsilon$ in Theorem 6.16 (iii).

Appendix A. Sobolev estimates on balls on manifolds

Proposition A.1. *Assume that (X, g) is complete and has bounded geometry with harmonic radius $r_0 > 0$. Then there is a constant $C_{\mathrm{Sob}} > 0$ such that*

$$\|f\|_{\mathsf{L}_{2p_n}(B_r(x),g)} \leq C_{\mathrm{Sob}}\, r^{-a_n}\, \|f\|_{\mathsf{H}^2(B_{4r}(x),g)}$$

for all $x \in X$, $r \leq r_0/4$ and $f \in \mathsf{H}^2(B_{4r}(x), g)$, where

$$p_n = \begin{cases} \frac{n}{n-4}, & n \geq 5, \\ \frac{2}{\beta},\ \beta \in (0,1], & n = 4, \\ \infty, & n \in \{2,3\}, \end{cases} \qquad a_n = \begin{cases} 2, & n \geq 5, \\ 2-\beta,\ \beta \in (0,1], & n = 4, \\ 3/2, & n = 3, \\ 1, & n = 2. \end{cases} \tag{A.1}$$

Proof. The Sobolev embedding theorem in $\mathbb{R}^n$ states that $\mathsf{H}^1_q(\mathbb{R}^n) \subset \mathsf{L}_p(\mathbb{R}^n)$ is a continuous embedding provided $1/p = 1/q - 1/n$ (see e.g. [1, Theorem 5.4] or [20, Theorem 2.5]). Thus, using a cut-off function we conclude that there exists a constant $C_{p,q} > 0$ such that

$$\|f\|_{\mathsf{L}_p(B_1(0),g_{\mathrm{eucl}})} \le C_{p,q}\, \|f\|_{\mathsf{H}^1_q(B_2(0),g_{\mathrm{eucl}})}$$

for all $f \in \mathsf{H}^1_q(\mathbb{R}^n)$. By a scaling argument we conclude that

$$\begin{aligned}\|f\|_{\mathsf{L}_p(B_r(0),g_{\mathrm{eucl}})} &\le \frac{C_{p,q}}{2^{n/q}} r^{n(\frac{1}{p}-\frac{1}{q})} \|f\|_{\mathsf{H}^1_q(B_{2r}(0),g_{\mathrm{eucl}})} \\ &= \frac{C_{p,q}}{2^{n/q}} r^{-1} \|f\|_{\mathsf{H}^1_q(B_{2r}(0),g_{\mathrm{eucl}})}\end{aligned}$$

for all $f \in \mathsf{H}^1_{q,\mathrm{loc}}(\mathbb{R}^n)$. Finally, by the hypothesis of bounded geometry, we obtain

$$\|f\|_{\mathsf{L}_p(B_r(x),g)} \le C(p,q,K) r^{-1} \|f\|_{\mathsf{H}^1_q(B_{2r}(x),g)} \tag{A.2}$$

for all $f \in \mathsf{H}^1_{q,\mathrm{loc}}(X,g)$ and $x \in X$ as soon as $2r \le r_0$. To obtain the desired estimate we have to apply this argument twice.

If $n \ge 5$, let p and p' be such that

$$\frac{1}{p'} = \frac{1}{2} - \frac{1}{n} \quad \text{and} \quad \frac{1}{p} = \frac{1}{p'} - \frac{1}{n},$$

thus

$$\frac{1}{p} = \frac{1}{2} - \frac{2}{n} = \frac{n-4}{2n}.$$

Let $f \in \mathsf{H}^2_2(X,g)$, and $r \le r_0/4$. We know already that

$$\|f\|_{\mathsf{L}_p(B_r(x),g)} \le C(p,q,K)\, r^{-1} \|f\|_{\mathsf{H}^1_{p'}(B_{2r}(x),g)}.$$

Moreover, applying (A.2) to the function $\varphi = |df|$ we obtain

$$\|df\|_{\mathsf{L}_{p'}(B_{2r}(x),g)} \le C(p',2,K)\, r^{-1} \|\,|df|\,\|_{\mathsf{H}^1(B_{4r}(x),g)}$$

We now argue as in (3.9) and estimate $|d\varphi|_g \le |\nabla^2 f|_g$, hence we have

$$\|f\|_{\mathsf{L}_p(B_r(x),g)} \le C(p,K)\, r^{-2} \|f\|_{\mathsf{H}^2(B_{4r}(x),g)}$$

for all $f \in \mathsf{H}^2_2(X,g)$ and $x \in X$ with $C(p,K) = C(p',2,K)C(p,p',K)$ and $p_n = p/2 = (n-4)/n$. For small dimensions, we can use the following special

Sobolev embeddings results: there exists a constant $C > 0$ such that

$$\|f\|_{\mathsf{L}_\infty(B_1(0))} \le C\|f\|_{\mathsf{H}^1_q(B_2(0))}, \quad \|f\|_{\mathsf{L}_\infty(B_r(0))} \le r^{-n/q}C\|f\|_{\mathsf{H}^1_q(B_{2r}(0))} \tag{A.3}$$

$$\|f\|_{\mathsf{L}_p(B_1(0))} \le C\|f\|_{\mathsf{H}^1_n(B_2(0))}, \quad \|f\|_{\mathsf{L}_p(B_r(0))} \le r^{n/p-1}C\|f\|_{\mathsf{H}^1_n(B_{2r}(0))} \tag{A.4}$$

for all $f \in \mathsf{H}^1_q(B_2(0), g_{\mathrm{eucl}})$ and $q > n$ resp. $q = n$ and $p \in [n, \infty)$, see [1, Theorem 5.4].

For $n = 4$, choose $p' = 4$ and $p \ge 4$, then we have, applying (A.4) and using the assumption of bounded geometry,

$$\|f\|_{\mathsf{L}_p(B_r(x),g)} \le C(p, K)\, r^{4/p-2}\|f\|_{\mathsf{H}^2(B_{4r}(x),g)}$$

for all $f \in \mathsf{H}^2_2(X, g)$ and $x \in X$. We hence choose $p_4 = p/2 = 2/\beta$ with $\beta \in (0, 1]$.

For $n = 3$, choose $p' = 6$ and $p = \infty$, then we have, applying (A.3) using the assumption of bounded geometry,

$$\|f\|_{\mathsf{L}_\infty(B_r(x),g)} \le C(\infty, K)\, r^{-3/2}\|f\|_{\mathsf{H}^2(B_{4r}(x),g)}.$$

for all $f \in \mathsf{H}^2_2(X, g)$ and $x \in X$.

Finally, for $n = 2$, choose $p' = 4$ and $p = \infty$, then

$$\|f\|_{\mathsf{L}_\infty(B_r(x),g)} \le C(\infty, K)\, r^{-1}\|f\|_{\mathsf{H}^2(B_{4r}(x),g)}$$

for all $f \in \mathsf{H}^2_2(X, g)$ and $x \in X$. □

Remark A.2. If we apply directly the Sobolev embedding theorem [1, Theorem 5.4] for Euclidean balls, then we would obtain an estimate

$$\|f\|_{\mathsf{L}_{2p}(B_1(0),g_{\mathrm{eucl}})} \le C_{p,n}\|f\|_{\mathsf{H}^2(B_1(0),g_{\mathrm{eucl}})}$$

for some $C_{p,n} > 0$ and after a scaling argument we obtain

$$\|f\|_{\mathsf{L}_{2p}(B_r(0),g_{\mathrm{eucl}})} \le C_{p,n} r^{n/(2p)-n/2}\|f\|_{\mathsf{H}^2(B_r(0),g_{\mathrm{eucl}})}$$

for all $r \in (0, 1]$ and $f \in \mathsf{H}^2(B_r(0), g_{\mathrm{eucl}})$. But then, we need an estimate of the Euclidean derivative $|\nabla^2 f|^2$ in terms of $|\nabla^2_g f|^2_g$, but

$$(\nabla^2_g f)_{ij} = \partial_{ij} f - \sum_k \Gamma^k_{ij}\partial_k f,$$

hence we would need additional assumptions on the *derivative* of the metric (entering in the Christoffel symbols Γ^k_{ij}).

Lemma A.3. *Assume that* (X, h) *has bounded geometry with harmonic radius* $r_0 > 0$ *and that* (Y, h) *is a complete orientable submanifold of codimension* 1 *in* X (*a* ***hypersurface***). *We assume that* Y *admits a uniform tubular neighbourhood* (*as defined in Assumption* 6.9) *also with radius* $r_0 > 0$

Let ε *and* ε^+ *such that* $0 < \varepsilon < \varepsilon^+ < r_0 \leq 1$. *Then there is* $C_{r_0,Y} > 0$ *depending only on* Y *and* r_0 *such that*

$$\|f\|_{\mathsf{L}_2(B_\varepsilon(Y),g)} \leq C_{r_0,Y}\Big(\frac{\varepsilon}{\varepsilon^+}\Big)^{1/2}\|f\|_{\mathsf{H}^1(B_{\varepsilon^+}(Y),g)}$$

for all $f \in \mathsf{H}^1(X, g)$.

Proof. In the coordinates defined by exp in (6.5) the metric is of the form $dt^2 + h(t)$ where $h(t)$ is metric on Y equal to h at $t = 0$. We then apply [35, Lemma A.2.16] with $a = \varepsilon$ and $b = \varepsilon^+$ and obtain that $([0, \varepsilon] \times Y, [0, \varepsilon^+] \times Y)$ is $2(\varepsilon/\varepsilon^+)$-non-concentrating (provided $\varepsilon^+ < 1$). Moreover, $(B_\varepsilon(Y), g)$ is an almost product in the sense of App. A.2 in [35], and the relative distortion factor is $\sqrt{C}_{r_0,Y}$. □

References

[1] R. A. Adams, *Sobolev spaces.* Pure and Applied Mathematics, 65. Academic Press, New York and London, 1975. MR 0450957 Zbl 0314.46030

[2] G. Allaire and F. Murat, Homogenization of the Neumann problem with nonisolated holes. With an appendix written jointly with A. K. Nandakumar. *Asymptotic Anal.* **7** (1993), no. 2, 81–95. MR 1225439 Zbl 0823.35014

[3] C. Anné and O. Post, Manifolds with many small handles: norm resolvent and spectral convergence. Work in progress.

[4] T. Aubin, Espaces de Sobolev sur les variétés riemanniennes. *Bull. Sci. Math.* (2) **100** (1976), no. 2, 149–173. MR 0488125 Zbl 0328.46030

[5] M. Balzano, Random relaxed Dirichlet problems. *Ann. Mat. Pura Appl.* (4) **153** (1988), 133–174 MR 1008342 Zbl 0673.60067

[6] M. Balzano and L. Notarantonio, On the asymptotic behavior of Dirichlet problems in a Riemannian manifold less small random holes. *Rend. Sem. Mat. Univ. Padova* **100**(1998), 249–282. MR 1675287 Zbl 0922.31008

[7] J. Bertrand and B. Colbois, Capacité et inégalité de Faber–Krahn dans $\mathbb{R}^n$. *J. Funct. Anal.* **232** (2006), no. 1, 1–28. MR 2200165 Zbl 1100.31005

[8] D. Borisov, G. Cardone, and T. Durante, Homogenization and norm-resolvent convergence for elliptic operators in a strip perforated along a curve. *Proc. Roy. Soc. Edinburgh Sect.* A **146** (2016), no. 6, 1115–1158. MR 3573727 Zbl 1361.35021

[9] I. Chavel and E. A. Feldman, Spectra of domains in compact manifolds. *J. Functional Analysis* **30** (1978), no. 2, 198–222. MR 0515225 Zbl 0392.58016

[10] I. Chavel and E. A. Feldman, Spectra of manifolds less a small domain. *Duke Math. J.* **56** (1988), no. 2, 399–414. MR 0932853 Zbl 0645.58042

[11] J. Cheeger, M. Gromov, and M. Taylor, Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds. *J. Differential Geometry* **17** (1982), no. 1, 15–53. MR 0658471 Zbl 0493.53035

[12] K. Cherednichenko, P. Dondl, and F. Rösler, Norm-resolvent convergence in perforated domains. *Asymptot. Anal.* **110** (2018), no. 3–4, 163–184. MR 3885494 Zbl 1415.35021

[13] D. Cioranescu and F. Murat, Un terme étrange venu d'ailleurs. In H. Brézis and J.-L. Lions (eds.), *Nonlinear partial differential equations and their applications.* Collège de France Seminar. Vol. III. Lectures presented at the Seminar on Applied Mathematics held at the Collège de France, Paris, 1980/1981. Pitman, Boston, MA, and London, 1982, 98–138, 389–390. MR 0652509 Zbl 0496.35030

[14] G. Courtois, Spectrum of manifolds with holes. *J. Funct. Anal.* **134** (1995), no. 1, 194–221. MR 1359926 Zbl 0847.58076

[15] D. Daners, Dirichlet problems on varying domains. *J. Differential Equations* **188** (2003), no. 2, 591–624. MR 1955096 Zbl 1090.35069

[16] D. Daners, Domain perturbation for linear and semi-linear boundary value problems. In M. Chipot (ed.), *Handbook of differential equations*: *stationary partial differential equations.* Vol. VI. Handbook of Differential Equations. Elsevier/North-Holland, Amsterdam, 2008, 1–81. MR 2569323 Zbl 1197.35090

[17] G. Dal Maso, *An introduction to* $\Gamma-$*convergence.* Progress in Nonlinear Differential Equations and their Applications, 8. Birkhäuser Boston, Boston, MA, 1993. MR 1201152 Zbl 0816.49001

[18] M. Flucher, Approximation of Dirichlet eigenvalues on domains with small holes. *J. Math. Anal. Appl.* **193** (1995), no. 1, 169–199. MR 1338506 Zbl 0836.35105

[19] E. Hebey, *Sobolev spaces on Riemannian manifolds.* Lecture Notes in Mathematics, 1635. Springer-Verlag, Berlin etc., 1996. MR 1481970 Zbl 0866.58068

[20] E. Hebey, *Nonlinear analysis on manifolds*: *Sobolev spaces and inequalities.* Courant Lecture Notes in Mathematics, 5. New York University, Courant Institute of Mathematical Sciences, New York, and American Mathematical Society, Providence, R.I., 1999. MR 1688256

[21] R. Hempel, On the lowest eigenvalue of the Laplacian with Neumann boundary condition at a small obstacle. *J. Comput. Appl. Math.* **194** (2006), no. 1, 54–74. MR 2230970 Zbl 1091.35047

[22] A. Henrot, Continuity with respect to the domain for the Laplacian: a survey. *Control Cybernet.* **23** (1994), no. 3, 427–443. (Shape design and optimization.) MR 1303362 Zbl 0822.35029

[23] E. Heintze and H. Karcher, A general comparison theorem with applications to volume estimates for submanifolds. *Ann. Sci. École Norm. Sup.* (4) **11** (1978), no. 4, 451–470. MR R0533065 Zbl 0416.53027

[24] R. Hempel, O. Post, and R. Weder, On open scattering channels for manifolds with ends. *J. Funct. Anal.* **266** (2014), no. 9, 5526–5583. MR 3182951 Zbl 1298.35129

[25] T. Kato, *Perturbation theory for linear operators.* Die Grundlehren der mathematischen Wissenschaften, 132. Springer-Verlag New York, New York, 1966. MR 0203473 Zbl 0148.12601

[26] A. Khrabustovskyi, Homogenization of the spectral problem on the Riemannian manifold consisting of two domains connected by many tubes. *Proc. Roy. Soc. Edinburgh Sect. A* **143** (2013), no. 6, 1255–1289. MR 3134195 Zbl 1293.58012

[27] A. Khrabustovskyi and O. Post, Operator estimates for the crushed ice problem. *Asymptot. Anal.* **110** (2018), no. 3–4, 137–161. MR 3885493 Zbl 1415.35024

[28] M. Lanza de Cristoforis, Simple Neumann eigenvalues for the Laplace operator in a domain with a small hole. A functional analytic approach. *Rev. Mat. Complut.* **25** (2012), no. 2, 369–412. MR 2931418 Zbl 1302.35272

[29] V. A. Marchenko and E. Y. Khruslov, Boundary-value problems with fine-grained boundary. *Mat. Sb.* (*N.S.*) **65**(107) (1964), 458–472. In Russian. MR 0178156 Zbl 0171.08701

[30] V. A. Marchenko and E. Y. Khruslov, *Homogenization of partial differential equations.* Translated from the 2005 Russian original by M. Goncharenko and D. Shepelsky. Progress in Mathematical Physics, 46. Birkhäuser Boston, Boston, MA, 2006. MR 2182441 Zbl 1113.35003

[31] V. G. Maz′ya, S. A. Nazarov, and B. A. Plamenevskiĭ, Asymptotic expansions of the eigenvalues of boundary value problems for the Laplace operator in domains with small holes. *Izv. Akad. Nauk SSSR Ser. Mat.* **48** (1984), no. 2, 347–371. In Russian. English translation, *Math. USSR-Izv.* **24** (1985), no. 2, 321–346. MR 0740795 Zbl 0566.35031

[32] S. Ozawa, Singular variation of domains and eigenvalues of the Laplacian. *Duke Math. J.* **48** (1981), no. 4, 767–778. MR 0782576 Zbl 0483.35064

[33] S. Ozawa, Spectra of domains with small spherical Neumann boundary. *J. Fac. Sci. Univ. Tokyo Sect.* IA **Math. 30** (1983), no. 2, 259–277. MR 0722497 Zbl 0541.35061

[34] O. Post, Spectral convergence of quasi-one-dimensional spaces. *Ann. Henri Poincaré* **7** (2006), no. 5, 933–973. MR 2254756 Zbl 1187.81124

[35] O. Post, *Spectral analysis on graph-like spaces.* Lecture Notes in Mathematics, 2039. Springer, Berlin etc., 2012. MR 2934267 Zbl 1247.58001

[36] O. Post and J. Simmer, Approximation of fractals by discrete graphs: norm resolvent and spectral convergence. *Integral Equations Operator Theory* **90** (2018), no. 6, Paper No. 68, 30 pp. MR 3861286 Zbl 06965995

[37] O. Post and J. Simmer, Quasi-unitary equivalence and generalized norm resolvent convergence. *Rev. Roumaine Math. Pures Appl.* **64** (2019), no. 2–3, 373–391. MR 4012610 Zbl 1438.47028

[38] O. Post and J. Simmer, A distance of operators acting in different Hilbert spaces and operator convergence. In J. Kerner, H. Laasri, and D. Mugnolo (eds), *Control theory of infinite-dimensional systems.* Operator Theory: Advances and Applications, 277. Birkhäuser, Cham, 2020, 53–72.

[39] J. Rauch, Five problems: an introduction to the qualitative theory of partial differential equations. In J. A. Goldstein (ed.), Partial differential equations and related topics. Program held at Tulane University, New Orleans, LA, January to May, 1974. Lecture Notes in Mathematics. 446. Springer-Verlag, Berlin etc., 1975, 355–369. MR 0509045 Zbl 0312.35001

[40] J. Rauch and M. Taylor, Potential and scattering theory on wildly perturbed domains. *J. Functional Analysis* **18** (1975), 27–59. MR 0377303 Zbl 0293.35056

[41] L. Saloff-Coste, *Aspects of Sobolev-type inequalities.* London Mathematical Society Lecture Note Series, 289. Cambridge University Press, Cambridge, 2002. MR 1872526 Zbl 0991.35002

[42] T. A. Suslina, *Spectral approach to homogenization of elliptic operators in a perforated space. Rev. Math. Phys.* **30** (2018), no. 8, 1840016, 57 pp. MR 3846431 Zbl 1411.35029

[43] J. Weidmann, Stetige Abhängigkeit der Eigenwerte und Eigenfunktionen elliptischer Differentialoperatoren vom Gebiet. *Math. Scand.* **54** (1984), no. 1, 51–69. MR 0753063 Zbl 0526.35061

Received October 1, 2018

Colette Anné, Laboratoire de Mathématiques Jean Leray, CNRS – Université de Nantes, Faculté des Sciences, BP 92208, 44322 Nantes, France

e-mail: colette.anne@univ-nantes.fr

Olaf Post, Fachbereich 4 – Mathematik, Universität Trier, 54286 Trier, Germany

e-mail: olaf.post@uni-trier.de