J. Spectr. Theory 11 (2021), 193–228
DOI 10.4171/JST/339

On open scattering channels for a branched covering of the Euclidean plane

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Abstract. We study the interaction of two scattering channels for a simple geometric model consisting in a double covering of the plane with two branch points, equipped with the Euclidean metric. We show that the scattering channels are open in the sense of [11] and that this property is stable under suitable perturbations of the metric.

Mathematics Subject Classification (2010). Primary: 35P25; Secondary: 35Q40, 81U35

Keywords. Laplacian on Riemannian manifolds, multi-channel scattering, harmonic radius.

Contents

1. Introduction

Let M denote a branched covering of the plane, obtained by glueing two copies of $\mathbb{R}^2$ along a straight-line cut between the points $q_- = (-1, 0)$ and $q_+ = (+1, 0)$, where the northern edge of the upper copy of $\mathbb{R}^2$ is joined to the southern edge of the lower copy, and vice-versa (see Figure 1). The branch points $q_\pm$ do *not* belong to M. The manifold M is a real version of the complex Riemann surface associated with the function $\sqrt{z^2-1}$. With the Euclidean metric g_{E} of $\mathbb{R}^2$, we obtain a smooth, connected Riemannian manifold $\mathcal{M} = (M, g_{\mathrm{E}})$ with curvature zero; note, however, that $\mathcal{M}$ is not complete. In the second part of the paper we will consider Riemannian metrics g on M which are close to g_{E} in a suitable sense so that the perturbational results of [11] can be applied.

We let H denote the Laplacian of $\mathcal{M}$, a self-adjoint operator acting in the Hilbert space $\mathcal{H} = \mathsf{L}_2(\mathcal{M})$. For a metric g on M, different from the Euclidean metric, we denote the associated Laplacian by H_g. It is the aim of this paper to study some asymptotic properties of the unitary groups $(\mathrm{e}^{-\mathrm{i}tH}; t \in \mathbb{R})$ and $(\mathrm{e}^{-\mathrm{i}tH_g}; t \in \mathbb{R})$. In particular, we are interested in the question whether there is transmission from the lower to the upper sheet and vice versa. As noted by Percy Deift (private communication), this amounts to the question

“When I shout on the lower plane, will I be heard on the upper plane?”

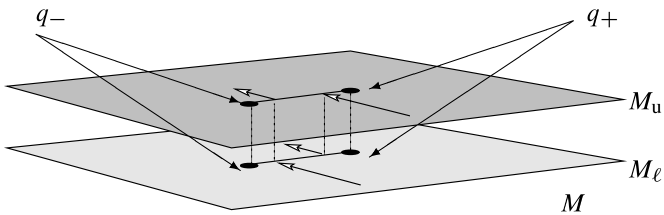

Figure 1. The double covering M with two branch points q_- and q_+, and the straight line cut Γ between q_- and q_+. If one arrives from the lower sheet M_ℓ from below (in the picture from the right), then one continues on the upper sheet M_{u} and vice versa. Points along the dashed lines are identified as explained above.

For the comparison dynamics (with two scattering channels) we take the free Laplacian on two copies of $\mathbb{R}^2$ which we may imagine to lie one atop of the other. In other words, we consider the Hilbert space $\mathcal{H}_0 = \mathsf{L}_2(\mathbb{R}^2) \oplus \mathsf{L}_2(\mathbb{R}^2)$ and we let H_0 denote the direct sum of two copies of the self-adjoint Laplacian in $\mathsf{L}_2(\mathbb{R}^2)$,

$$H_0 = H_{0,\ell} \oplus H_{0,\mathrm{u}},$$

where the indices ℓ and u mean “lower” and “upper,” respectively. H_0 is (purely) absolutely continuous. With a natural (unitary) identification $J\colon \mathcal{H}_0 \to \mathcal{H}$ the wave operators

$$W_\pm(H, H_0, J) = \operatorname*{s-lim}_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH} J \mathrm{e}^{-\mathrm{i}tH_0},$$

exist, are complete, and isometric, as will be seen in Section 2. Since also H is absolutely continuous the wave operators $W_\pm(H, H_0, J)$ are in fact unitary. Writing

$J = J_\ell \oplus J_u$, the *channel wave operators* $W_\pm(H, H_{0,\ell}, J_\ell)$ and $W_\pm(H, H_{0,u}, J_u)$ are given by

$$W_\pm(H, H_{0,k}, J_k) = \operatorname*{s-lim}_{t\to\pm\infty} e^{itH} J_k e^{-itH_{0,k}}, \quad k \in \{\ell, u\}.$$

Note that $f \in \operatorname{Ran} W_+(H, H_{0,u}, J_u)$ means that there exists $h \in \mathscr{H}_u$ such that

$$\|e^{-itH} f - J_u e^{-itH_{0,u}} h\| \longrightarrow 0, \quad t \to +\infty;$$

in particular, $e^{-itH} f$ is asymptotically in the *upper* sheet, as $t \to +\infty$. This leads to the question whether states which come in on the lower sheet will also go out on the lower sheet, or whether there are states which change sheets as t goes from $-\infty$ to $+\infty$. We construct, indeed, states that move from the lower to the upper sheet, up to a small error. It follows that there is non-zero transmission between the upper and the lower sheets of $\mathcal{M}$, or, in the terminology of [11], that the upper and the lower channels are open. By symmetry there is also transmission from the upper to the lower sheet; since it is more or less trivial that there is transmission within the two sheets we find that all scattering channels are open one to another. This is stated as Theorem 2.6.

We next ask whether the scattering channels remain open when the Euclidean metric g_E on M is replaced with a more general metric g on M which is close to g_E at infinity in the sense of [11]. The corresponding assumptions concern, in particular, the harmonic radius [1, 11]) and the injectivity radius of (M, g), and the difference of the Riemannian metrics g_E and g in a suitable distance function. Here we profit in several ways from the fact that the geometry of $\mathcal{M}$ is so simple. We require that the metrics g and g_E are *quasi-isometric* in the usual sense (cf. Definition 3.2), and we assume a global bound on the curvature of (M, g). Under additional assumptions on g, expressed in terms of the distance $\tilde{d}_1(g_E, g)$ in eq. (3.6), Theorem 3.3 states that the wave operators

$$W_\pm(H_g, H_{g_E}, I_g) := \operatorname*{s-lim}_{t\to\pm\infty} e^{itH_g} I_g e^{-itH_{g_E}} \tag{1.1}$$

exist and are complete, where $H_{g_E} = H$ is the Laplacian of (M, g_E), H_g is the Laplacian of (M, g), and I_g is the natural identification between $\mathsf{L}_2(M, g_E)$ and $\mathsf{L}_2(M, g)$; as was mentioned earlier, H_{g_E} is purely absolutely continuous.

In Theorem 3.3, smallness of the perturbation is only required at infinity. In contrast, for the question of openness of the scattering channels the deviation of g from g_E has to satisfy a global, quantitative smallness condition. Then Theorem 3.4 establishes the strong convergence of the scattering operators

$$S(H_{g_\varepsilon}, H_0, I_{g_\varepsilon} J) := (W_+(H_{g_\varepsilon}, H_0, I_{g_\varepsilon} J))^* \circ W_-(H_{g_\varepsilon}, H_0, I_{g_\varepsilon} J)$$

to $S(H, H_0, J)$ for a family of metrics g_ε on M tending to g_E as $\varepsilon \downarrow 0$. In Corollary 3.5 we then obtain the openness of all scattering channels for small ε.

This paper is a companion paper to [11]; its main intention is to enrich the set of examples where the perturbational results of [11] can be applied. The framework of [11] is rather abstract and conditions are formulated in terms of curvature and the harmonic radius. It is not trivial to go from there to the analysis of concrete examples. Indeed, the examples presented in [11] start from unperturbed manifolds with rotational symmetry which makes it somewhat easier to construct wave packets and to obtain lower bounds for the injectivity radius and the harmonic radius. The double covering of the plane is a manifold of classical beauty, and a good theory should be able to handle such an example. In this context, we have been led to provide lower bounds for the injectivity radius for metrics different from the Euclidean metric; these estimates, discussed in Appendix C, may be of independent interest. While the branch points pose no major difficulty in the construction of self-adjoint extensions and in the setup of scattering, they clearly limit the injectivity radius and therefore influence several estimates.

The paper is organized as follows. In Section 2 we introduce most of our notation and we discuss some basic spectral properties of the manifold $\mathcal{M} = (M, g_E)$, deferring the details and proofs to Appendix A. We then turn to scattering for the pair (H, H_0) where we establish existence and completeness of the wave operators. The technically difficult part of Section 2 concerns the construction of a wave packet that comes in from infinity on the lower sheet and moves out to infinity on the upper sheet. Here we use ideas from Enß' theory of scattering and stationary phase estimates to construct states that pass between the branch points $q_\pm$ at time $t = 0$ at high speed, and which are essentially localized to a double cone.

In Section 3 we consider metrics g on M that are close (or, at least, close at infinity) to the Euclidean metric g_E. In essence, we only have to write down what the basic definitions and results of [11] mean in the present context. We then find simple conditions for the existence and completeness of the wave operators (1.1) as well as for a non-trivial interaction between the scattering channels for (M, g).

The main results of Section 3 are illustrated in Section 4 by a simple class of metrics on M, namely metrics $g = g_f$ that come from the graph of smooth functions f on M. It turns out that it is fairly easy to indicate conditions on f so that the metric g_f satisfies the requirements of Theorem 3.3. We finally discuss branched coverings with more than two sheets and corresponding generalizations of the present results.

The paper comes with three appendices; the first two of them are mainly included for the convenience of the reader. Appendix A is devoted to self-adjoint extensions, compactness and spectral properties of the Laplacian with metrics g_{E} and g. As for the absolute continuity of $H_{g_{\mathrm{E}}}$ and H_g, we mainly refer to some work of Donnelly [5] and Kumura [17, 18].

In Appendix B we recall a basic estimate from stationary phase theory to establish an estimate on the localization error for the Schrödinger evolution. More precisely, for suitably chosen initial data u_0 in the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ we multiply $u(t) := \mathrm{e}^{it\Delta}u_0$ by a cut-off function χ and obtain estimates for $\nabla\chi\cdot\nabla u(t)$ and $(\Delta\chi)u(t)$ in the L_2-norm.

Appendix C is devoted to lower bounds for the injectivity radius of (M, g) where the metric g on $\mathbb{R}^2$ or on M is close to the Euclidean metric. Starting from a comparison result of Müller and Salomonsen [19] we obtain "local" versions by means of cut-offs and extension theorems, proceeding from $\mathbb{R}^2$ via $\mathbb{R}^2 \setminus \{(0,0)\}$ to M.

We conclude the introduction with a few remarks concerning the literature. The paper [11] and the literature quoted there give a partial overview of Riemannian scattering on manifolds with ends. Recent progress in this direction can be found in Güneysu and Thalmaier [9]. The specific case of manifolds with branch points has been studied in recent years under various aspects and our results have some overlap with the work of Hillairet and others; cf. [12] and [7]. There is a connection between the analysis of the Aharonov-Bohm effect in Quantum Mechanics and branched coverings of Euclidean space; cf. [3]. Scattering for magnetic Schrödinger operators with two magnetic point charges has been studied in a number of papers; as an example, we mention Ito and Tamura [14] which has some connection with our investigations.

Acknowledgements. The authors thank Luc Hillairet (Univ. d'Orléans) for an interesting discussion and comments. Rainer Hempel would like to express his gratitude to Brian Davies (King's College, London), Percy Deift (Courant Institute, New York), Ira Herbst (Univ. of Virginia, Charlottesville), Barry Simon (Caltech, Pasadena), and Larry Thomas (Univ. of Virginia) for valuable discussions and suggestions concerning the matter of the present paper.

Note added in proof: After the completion of the present paper the following relevant work came to our attention: Ito and Skibsted [13] show that scattering channels are open within the framework of their theory, answering hence partially a conjecture of [11, Remark 5.7].

2. Wave operators for the Euclidean metric

Let us begin with some notation. As far as general notation for self-adjoint operators T in a Hilbert space $\mathscr{H}$ is concerned we mostly follow [15] and [23]. In particular, we let $\mathscr{H}_{\mathrm{ac}}(T)$ denote the absolutely continuous subspace of $\mathscr{H}$ associated with T, and $P_{\mathrm{ac}}(T)$ the orthogonal projection onto $\mathscr{H}_{\mathrm{ac}}(T)$. For the general formal setup of multi-channel scattering we refer to Section 4 of [11] and the literature quoted there. Since the model studied in the present paper is so simple, we develop most notions in multi-channel scattering directly as we go along.

Let M be defined as in the Introduction. We then denote the points p of M by $((x, y), \ell)$ or $((x, y), \mathrm{u})$ where $(x, y) \in \mathbb{R}^2$ and "ℓ" means "lower," "u" means "upper". This works for all points of M with the exception of the points with $-1 < x < 1$ and $y = 0$; note that these exceptional points form a set of measure zero. With g_{E} denoting the metric tensor $g_{\mathrm{E}} = (\delta_{ij})$ we obtain the Riemannian manifold $\mathcal{M} := (M, g_{\mathrm{E}})$. For the remainder of this section we will be cavalier about the distinction between M and $\mathcal{M} = (M, g_{\mathrm{E}})$ and we will mostly write M. For two points $p_1, p_2 \in M$ the (geodesic) distance is then given by

$$\operatorname{dist}(p_1, p_2) := \inf\{\, |\gamma| \,;\, \gamma(0) = p_1, \gamma(1) = p_2 \,\} \tag{2.1}$$

where $\gamma\colon [0, 1] \to M$ is a rectifiable curve and $|\gamma|$ denotes the length of γ. It will be useful to extend the definition of distance to the branch points q_- and q_+. The infimum in (2.1) is attained either for a straight line segment connecting p_1 and p_2 or for (the union of) two straight line segments that meet at one of the branch points. E.g., if $p_1 = ((0, y), \ell)$, $p_2 = ((0, -y), \ell)$ with $y > 0$, then $\operatorname{dist}(p_1, p_2) = \operatorname{dist}(p_1, q_-) + \operatorname{dist}(q_-, p_2) = 2\sqrt{1 + y^2}$ (see Figure 2 left).

For a point $p_0 \in M$, we denote the (geodesic) disk of radius $r > 0$ and center (x_0, y_0) by $B_r(p_0)$, i.e.,

$$B_r(p_0) = \{\, p \in M \,;\, \operatorname{dist}(p, p_0) < r \,\}; \tag{2.2}$$

such disks may or may not contain points in both sheets (see Figure 2 right), and they may even contain pairs of points (p, p') with the same (x, y)-coordinates and p in the lower, p' in the upper sheet. A disk $B_r(p_0)$ will be "single-valued" if and only if $r \le \min\{\operatorname{dist}(p_0, q_+), \operatorname{dist}(p_0, q_-)\}$. In the extreme case of $p_0 \in \{q_+, q_-\}$ and $0 < r \le 2$ the disk $B_r(p_0)$ will just be a double covering of the punctured disk $\{\, (x, y) \in \mathbb{R}^2 \,;\, 0 < x^2 + y^2 < r^2 \,\}$. The Riemannian manifold $\mathcal{M}$ is not (geodesically) complete.

In order to define the Laplacian H of $\mathcal{M}$, we consider the Hilbert space $\mathscr{H} := \mathsf{L}_2(\mathcal{M})$ with scalar product denoted by $\langle \cdot, \cdot \rangle$, and the Sobolev space $\mathring{\mathsf{H}}^1(\mathcal{M})$,

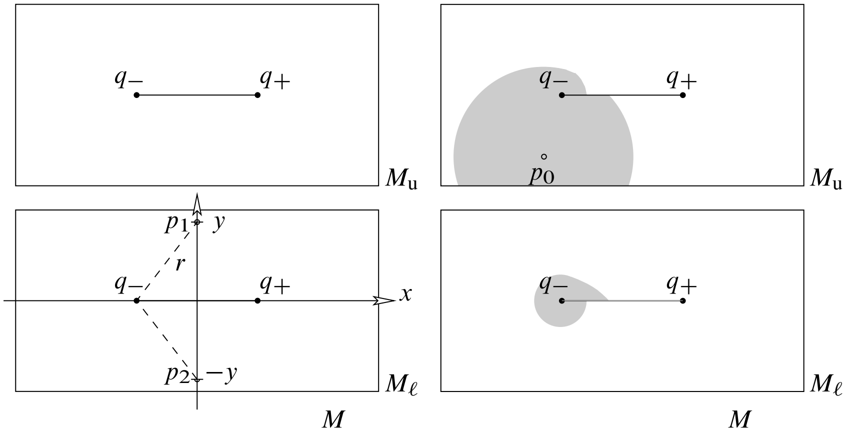

Figure 2. Left: The distance between p_1 and p_2 is $2r = 2\sqrt{1+y^2}$. Right: The shaded area is a disk $B_r(p_0)$ with points in both sheets.

given as the completion of $\mathsf{C}_\mathrm{c}^\infty(M)$ with respect to the norm $\|\cdot\|_1$ defined by

$$\|\psi\|_1^2 := \int_M |\psi(x)|^2 + |\nabla\psi(x)|^2 \,\mathrm{d}x, \qquad \psi \in \mathsf{C}_\mathrm{c}^\infty(M). \tag{2.3}$$

Then H is defined as the unique self-adjoint operator satisfying $\mathrm{Dom}(H) \subset \mathring{\mathsf{H}}^1(\mathcal{M})$ and

$$\langle Hu, v\rangle = \int_M \nabla u \cdot \nabla\bar{v}\,\mathrm{d}x, \qquad u \in \mathrm{Dom}(H),\ v \in \mathring{\mathsf{H}}^1(\mathcal{M}). \tag{2.4}$$

It is easy to see (cf. Appendix A) that $\mathring{\mathsf{H}}^1(\mathcal{M})$ coincides with the Sobolev space $\mathsf{H}^1(\mathcal{M}) = \mathsf{W}_2^1(\mathcal{M})$, consisting of all functions in $\mathsf{L}_2(\mathcal{M})$ that have first order distributional derivatives in $\mathsf{L}_2(\mathcal{M})$. Hence the Laplacian on $\mathsf{C}_\mathrm{c}^\infty(M)$ has only one self-adjoint extension with form domain contained in $\mathsf{H}^1(\mathcal{M})$. However, the Laplacian is *not* essentially self-adjoint on $\mathsf{C}_\mathrm{c}^\infty(M)$. Basic spectral properties of H are also discussed in Appendix A; in particular, H is purely absolutely continuous with $\sigma(H) = \sigma_\mathrm{ac}(H) = [0,\infty)$.

We next consider the Rellich compactness property. For the proof we refer to Proposition A.2 in Appendix A.

Lemma 2.1. *For $R > 0$, let χ_R denote the characteristic function of $M_R = B_R(q_-) \cup B_R(q_+) \subset M$. Then the mapping $\mathsf{H}^1(\mathcal{M}) \ni u \mapsto \chi_R u \in \mathsf{L}_2(\mathcal{M})$ is compact.*

We now turn to scattering theory and introduce the comparison dynamics for the scattering channels associated with the two sheets (and two infinities) of M.

Let $M_0 := \mathbb{R}^2 \uplus \mathbb{R}^2 = \mathbb{R}^2 \times \{\ell, \mathrm{u}\}$ denote the disjoint union of two copies of the Euclidean plane $\mathbb{R}^2$, and write $M_{0,\ell} = \mathbb{R}^2 \times \{\ell\}$, $M_{0,\mathrm{u}} = \mathbb{R}^2 \times \{\mathrm{u}\}$. We then let $\mathcal{H}_0 = \mathsf{L}_2(M_0, g_\mathrm{E}) = \mathsf{L}_2(\mathbb{R}^2) \oplus \mathsf{L}_2(\mathbb{R}^2)$ Moreover, we let H_0 denote the Laplacian on M_0. To fix the notation, let A_0 denote the (unique) self-adjoint extension of $-\Delta$ on $\mathsf{C}_\mathrm{c}^\infty(\mathbb{R}^2)$. We may then write $H_0 = H_{0,\ell} \oplus H_{0,\mathrm{u}}$ where $H_{0,\ell}$ and $H_{0,\mathrm{u}}$ act as A_0 in $\mathsf{L}_2(M_{0,\ell}, g_\mathrm{E})$ and in $\mathsf{L}_2(M_{0,\mathrm{u}})$, respectively.

We denote the straight line segment in $\mathbb{R}^2$ connecting the points $q_\pm$ as Γ,

$$\Gamma := [-1, 1] \times \{0\} \subset \mathbb{R}^2, \tag{2.5}$$

a set of measure zero. There is a natural embedding $\iota\colon (\mathbb{R}^2 \setminus \Gamma) \times \{\ell, \mathrm{u}\} \to M$, $\iota = (\iota_\ell, \iota_\mathrm{u})$, where ι_ℓ maps the point $((x, y), \ell) \in M_{0,\ell} \setminus \Gamma$ to $((x, y), \ell) \in M$, and similarly for ι_u. The embedding ι induces a unitary mapping $J\colon \mathcal{H}_0 \to \mathcal{H}$ where $J = J_\ell \oplus J_\mathrm{u}$ in an obvious manner (and with a slight abuse of notation). J_ℓ maps functions $f \in \mathsf{L}_2(M_{0,\ell}, g_\mathrm{E})$ to the same function on the lower sheet of M and extends them by zero to all of M, and similarly for J_u. We then have:

Proposition 2.2. *The wave operators*

$$W_\pm(H, H_0, J) = \operatorname*{s-lim}_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}tH} J \mathrm{e}^{-\mathrm{i}tH_0} \tag{2.6}$$

exist and are unitary.

Remark 2.3. As is often the case in two Hilbert space scattering [16, 22], there is a certain arbitrariness in the choice of the mapping J. By local compactness, the same wave operators and the same results would be obtained with J replaced by $(1 - \chi_R)J$, for some $R > 0$, or by $(1 - \varphi)J$ with an arbitrary $\varphi \in \mathsf{C}_\mathrm{c}^\infty(\mathbb{R}^2)$.

Proof of Proposition 2.2. We decouple both H and H_0 by Dirichlet boundary conditions along two circles defined as follows. Let $C_2 := \{\, (x, y) \in \mathbb{R}^2\,;\, x^2 + y^2 = 4\,\}$, $C_2' := C_2 \times \{\ell, \mathrm{u}\} \subset M_0$, and $C_2'' := \iota(C_2') \subset M$. Introducing Dirichlet boundary conditions on C_2' and on C_2'' decomposes H_0 into a direct sum of four operators while H is decomposed into a direct sum of three operators. More precisely, we introduce the following three "building blocks": in the plane $\mathbb{R}^2$, we have the Dirichlet Laplacian h_int on the disk of radius 2 and the Dirichlet Laplacian h_ext on the exterior of this disk. Furthermore, defining

$$M_{0,\mathrm{ext}} := \{\, (x, y) \in \mathbb{R}^2\,;\, x^2 + y^2 > 4\,\} \times \{\ell, \mathrm{u}\}, \tag{2.7a}$$

$$M_\mathrm{ext} := \iota(M_{0,\mathrm{ext}}), \qquad M_\mathrm{int} := M \setminus \bar{M}_\mathrm{ext} \tag{2.7b}$$

we denote by H_{int} the Dirichlet Laplacian of M_{int}. Note that M_{int} is a branched covering with two sheets of the punctured disk $\{(x, y) \in \mathbb{R}^2 ; x^2 + y^2 < 4\} \setminus \{q_+, q_-\}$. We then write

$$H_{0,\mathrm{dec}} := (h_{\mathrm{int}}, \ell) \oplus (h_{\mathrm{ext}}, \ell) \oplus (h_{\mathrm{int}}, \mathrm{u}) \oplus (h_{\mathrm{ext}}, \mathrm{u}), \tag{2.8a}$$

$$H_{\mathrm{dec}} := H_{\mathrm{int}} \oplus (h_{\mathrm{ext}}, \ell) \oplus (h_{\mathrm{ext}}, \mathrm{u}); \tag{2.8b}$$

note that h_{ext} is purely absolutely continuous while h_{int} and H_{int} (by Lemma 2.1) have compact resolvent.

It is well-known ([2, 6, 11]) that the wave operators

$$W_\pm(H_{0,\mathrm{dec}}, H_0) = \operatorname*{s-lim}_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}tH_{0,\mathrm{dec}}} \mathrm{e}^{-\mathrm{i}tH_0} \tag{2.9}$$

exist, are complete, and isometric with initial subspace $\mathscr{H}_{\mathrm{ac}}(H_0) = \mathscr{H}_0$ and final subspace $\mathscr{H}_{\mathrm{ac}}(H_{0,\mathrm{dec}}) = \mathsf{L}_2(M_{0,\mathrm{ext}}, g_{\mathrm{E}})$. Similarly, it can be shown by standard methods (cf. [6, 10, 11]), that the wave operators

$$W_\pm(H, H_{\mathrm{dec}}) = \operatorname*{s-lim}_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}tH} \mathrm{e}^{-\mathrm{i}tH_{\mathrm{dec}}} P_{\mathrm{ac}}(H_{\mathrm{dec}}) \tag{2.10}$$

exist, are complete, and partially isometric with initial subspace

$$\mathscr{H}_{\mathrm{ac}}(H_{\mathrm{dec}}) = \mathsf{L}_2(M_{\mathrm{ext}}, g_{\mathrm{E}}) = \mathscr{H}_{\mathrm{ac}}(H_{0,\mathrm{dec}}) = \mathsf{L}_2(M_{0,\mathrm{ext}}, g_{\mathrm{E}})$$

and final subspace $\mathscr{H}_{\mathrm{ac}}(H) = \mathsf{L}_2(M, g_{\mathrm{E}})$. Finally, the wave operators

$$W_\pm(H_{\mathrm{dec}}, H_{0,\mathrm{dec}}, J) = \operatorname*{s-lim}_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}tH_{\mathrm{dec}}} J \mathrm{e}^{-\mathrm{i}tH_{0,\mathrm{dec}}} P_{\mathrm{ac}}(H_{0,\mathrm{dec}}) \tag{2.11}$$

simply act as the identity on $\mathsf{L}_2(M_{\mathrm{ext}}, g_{\mathrm{E}})$, and as the zero operator on $\mathsf{L}_2(M_{\mathrm{int}}, g_{\mathrm{E}})$. Therefore, they exist and are complete. It is now clear that the wave operators $W_\pm(H, H_0, J)$ exist and are unitary. □

With $J = J_\ell \oplus J_{\mathrm{u}}$ and $H_{0,\ell}$ as defined above, we furthermore see that the *channel wave operators*

$$W_\pm(H, H_{0,\ell}, J_\ell) = \operatorname*{s-lim}_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}tH} J_\ell \mathrm{e}^{-\mathrm{i}tH_{0,\ell}} \tag{2.12}$$

(and, analogously, $W_\pm(H, H_{0,\mathrm{u}}, J_{\mathrm{u}})$) exist and are isometric with

$$\operatorname{Ran} W_\pm(H, H_{0,\ell}, J_\ell) \oplus \operatorname{Ran} W_\pm(H, H_{0,\mathrm{u}}, J_{\mathrm{u}}) = \operatorname{Ran} W_\pm(H, H_0, J) = \mathscr{H}_{\mathrm{ac}}(H); \tag{2.13}$$

recall that $f \in \operatorname{Ran} W_+(H, H_{0,\ell}, J_\ell)$ means that there exists $g \in \mathsf{L}_2(M_{0,\ell}, g_{\mathrm{E}})$ such that

$$\|\mathrm{e}^{-\mathrm{i}tH} f - J_\ell \mathrm{e}^{-\mathrm{i}tH_{0,\ell}} g\| \longrightarrow 0, \quad t \to \infty; \tag{2.14}$$

in particular, $\mathrm{e}^{-\mathrm{i}tH} f$ is asymptotically on the lower sheet for $t \to \infty$. Eq. (2.13) establishes two orthogonal decompositions of $\mathscr{H}_{\mathrm{ac}}(H) = \mathsf{L}_2(M, g_{\mathrm{E}})$, one for the plus-sign and another one for the minus-sign. We will see later on (cf. Lemma 2.11) that these two decompositions are in fact different.

Remark 2.4. Let us note that $H_0 = H_{0,\ell} \oplus H_{0,\mathrm{u}}$ provides a reference operator for H in the sense of [11, Definition 4.7] with two channels. Strictly speaking, branch points like $q_\pm$ are not directly included in the framework used in [11]. However, this technical difficulty is easy to resolve: we might just take each of the sets $B_{1/2}(q_\pm)$ as an end, albeit an end which does not participate in the scattering process since the Dirichlet Laplacian of $B_{1/2}(q_\pm)$ has compact resolvent by Lemma 2.1. The possibility of allowing such "dead ends" is described in Remark 4.4 of [11]. We thus have (formally) a manifold with 4 ends, with two ends given by a copy of $\mathbb{R}^2 \setminus B_2(0)$ and another two ends given by $B_{1/2}(q_\pm)$.

It is a major goal in scattering theory to obtain information on the *scattering operator*

$$S = S(H, H_0, J) := (W_+(H, H_0, J))^* \circ W_-(H, H_0, J)\colon \mathscr{H}_0 \longrightarrow \mathscr{H}_0, \tag{2.15}$$

a unitary operator, and the closely related *scattering matrix* $(S_{ij})_{i,j\in\{\ell,\mathrm{u}\}}$, with

$$S_{ij} := (W_+(H, H_{0,i}, J_i))^* \circ W_-(H, H_{0,j}, J_j)\colon \mathsf{L}_2(M_{0,j}, g_{\mathrm{E}}) \longrightarrow \mathsf{L}_2(M_{0,i}, g_{\mathrm{E}}), \tag{2.16}$$

for $i, j \in \{\ell, \mathrm{u}\}$. We will show that the four components of (S_{ij}) are non-zero which yields the openness of all scattering channels.

The following lemma establishes the existence of a state w_0 for which $\mathrm{e}^{-\mathrm{i}tH} w_0$ is asymptotically in the lower sheet for $t \to -\infty$ and in the upper sheet for $t \to +\infty$, up to small errors. Recall that A_0 denotes the self-adjoint extension of the Laplacian on $\mathbb{R}^2$. We then have:

Lemma 2.5. *For $\varepsilon > 0$ given, there exist $w_0 \in \mathsf{L}_2(M, g_{\mathrm{E}}) \cap \mathsf{C}^\infty(M)$, $v_0 \in \mathcal{S}(\mathbb{R}^2)$, and $t_0 \geq 0$ such that the following estimates hold:*

$$\|\mathrm{e}^{-\mathrm{i}tH} w_0 - J_{\mathrm{u}} \mathrm{e}^{-\mathrm{i}tA_0} v_0\| < \varepsilon \|w_0\|, \quad t \geq t_0, \tag{2.17}$$

and

$$\|\mathrm{e}^{-\mathrm{i}tH} w_0 - J_\ell \mathrm{e}^{-\mathrm{i}tA_0} v_0\| < \varepsilon \|w_0\|, \quad t \leq -t_0. \tag{2.18}$$

In the proof of Lemma 2.5 we basically construct a state $v_0 \in \mathsf{L}_2(\mathbb{R}^2)$ which passes at high speed between the points $q_\pm$ under the evolution determined by $\mathrm{e}^{-\mathrm{i}tA_0}$ (up to small errors) and whose spreading can be controlled by stationary phase estimates, for $|t|$ large. Note that we have complete control of the unitary group $(\mathrm{e}^{-\mathrm{i}tA_0}\,;\, t \in \mathbb{R})$, acting in $\mathsf{L}_2(\mathbb{R}^2)$, while we know much less about $(\mathrm{e}^{-\mathrm{i}tH}\,;\, t \in \mathbb{R})$, acting in $\mathsf{L}_2(M, g_{\mathrm{E}})$. By a simple lifting, v_0 is transformed into a function w_0 on M. Here we wish to gain information on the evolution of $\mathrm{e}^{-\mathrm{i}tH} w_0$ from the properties of $\mathrm{e}^{-\mathrm{i}tA_0} v_0$ using the fact that both operators act locally as the Laplacian.

Recall that $H_{0,\mathrm{u}}$ and $H_{0,\ell}$ denote the self-adjoint Laplacian in $\mathsf{L}_2(M_{0,\mathrm{u}}, g_{\mathrm{E}})$ and in $\mathsf{L}_2(M_{0,\ell}, g_{\mathrm{E}})$, respectively. We let $\mathcal{F}$ denote the Fourier transform on the Schwartz spaces $\mathcal{S}(\mathbb{R}^d)$ for $d \in \mathbb{N}$. It is well known that $\mathcal{F}$ acts bijectively on $\mathcal{S}(\mathbb{R}^d)$ and extends to a unitary map $\mathcal{F}\colon \mathsf{L}_2(\mathbb{R}^d) \to \mathsf{L}_2(\mathbb{R}^d)$.

Our construction starts with a function $u_0 \in \mathcal{S}(\mathbb{R}^2)$ of the form $u_0 = u_0(x, y)$, given as the product of two functions $\psi_1 = \psi_1(x)$ and $\psi_2 = \psi_2(y)$ enjoying certain properties, which we describe now.

Let $\varepsilon \in (0, 1)$ be given and let $\varepsilon' := \varepsilon/5$. We first pick a function $\varphi_1 \in \mathsf{C}_\mathrm{c}^\infty(\mathbb{R})$ of norm 1 and we let $\psi_1 := \mathcal{F}^{-1}\varphi_1 \in \mathcal{S}(\mathbb{R})$ where we assume that

$$\|\chi_{(-\frac{1}{4},\frac{1}{4})}\psi_1\| > 1 - \varepsilon'. \tag{2.19}$$

We let $a = a_\varepsilon > 0$ be such that $\operatorname{supp}\varphi_1 \subset (-a, a)$. Next, let $\varphi_2 \in C_c^\infty(0, 1)$, of norm 1 again, and let $\psi_2 := \mathcal{F}^{-1}[\varphi_2(\,.\, - s)] \in \mathcal{S}(\mathbb{R})$, where $s > 0$ will be chosen later. Let

$$u_0 = u_0(x, y) = \psi_1(x)\psi_2(y), \quad (x, y) \in \mathbb{R}^2. \tag{2.20}$$

Then $u_0 \in \mathcal{S}(\mathbb{R}^2) \subset \mathrm{Dom}(A_0)$ and $u(t) := \mathrm{e}^{-\mathrm{i}tA_0}u_0$ is a classical solution of the initial value problem for the Schrödinger equation in $\mathsf{L}_2(\mathbb{R}^2)$, i.e.,

$$\dot{u}(t) = -\mathrm{i}A_0u(t) \quad \text{for} \quad t \in (0, \infty),\ u(0) = u_0.$$

The following construction is illustrated in Figure 3. We write

$$Q_{s,t} := \begin{cases} (-st, st) \times (st, \infty), & t > 0, \\ (st, -st) \times (-\infty, st), & t < 0, \end{cases}$$

for $s > 0$, and we let $\chi_{s,t}$ denote the characteristic function of $Q_{s,t}$. Lemma B.2 implies that for any $m \in \mathbb{N}$ there exists a constant $\tilde{c}_m \geq 0$ such that

$$\|(1 - \chi_{s,t})\mathrm{e}^{-\mathrm{i}tA_0}u_0\| \leq \tilde{c}_m(1 + st)^{1-2m}, \quad s \geq 2a,\ t > 0,$$

so that for $s \geq 2a$ and t large, $t \geq t_0$ say,

$$\|(1 - \chi_{s,t})\mathrm{e}^{-\mathrm{i}tA_0}u_0\| \leq \varepsilon'. \tag{2.21}$$

Now let

$$\Omega := \{ (x, y) \in \mathbb{R}^2 ;\ |x| < 1/2 + |y| \}, \tag{2.22}$$

let χ_Ω denote the characteristic function of Ω, and, finally,

$$\chi := j_{\frac{1}{4}} * \chi_\Omega, \tag{2.23}$$

where $(j_\delta)_{\delta>0}$ is the kernel of the usual Friedrichs mollifier on $\mathbb{R}^2$; in particular, $0 \le j_\delta \in \mathsf{C}_\mathrm{c}^\infty(\mathbb{R}^2)$ with support in the closed disk of radius δ, and $\int j_\delta = 1$. Also let $\mathcal{X}$ denote the support of χ and X the characteristic function of $\mathcal{X}$, i.e, $X = \chi_{\mathcal{X}}$. Note that χ is independent of t.

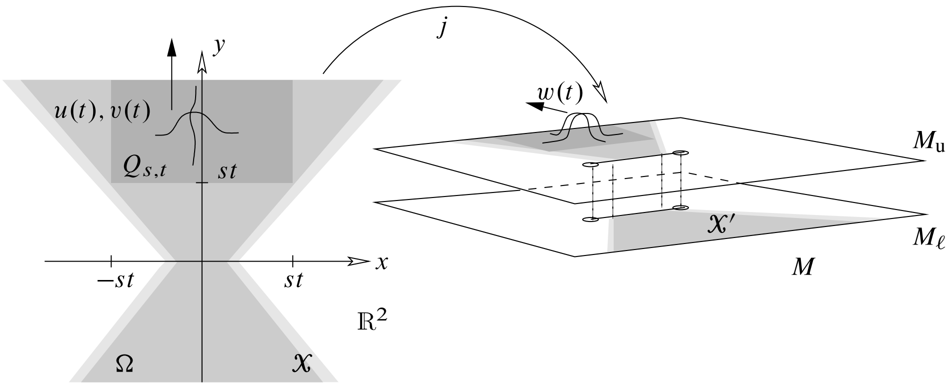

Figure 3. *Left*: the wave packet u_0 at time 0 has speed s in y-direction and is concentrated in x-direction near $x = 0$; the support of the wave packet $u(t) = \mathrm{e}^{itA_0}u_0$ at time $t > 0$ is essentially contained in the dark grey area $Q_{s,t}$. Moreover, when considering the time evolution $v(t)$ of the initial state $v_0 = \chi u_0$ (with a cut-off function χ defined as a smooth version of the indicator function χ_Ω) with support in $\mathcal{X}$, the deviation from $u(t)$ is small. *Right*: the corresponding sets and the wave packet $w(t)$ corresponding to $v(t)$ on M. The initial state here is $w_0 = w(0)$.

We next consider $v_0 := \chi u_0$ and observe that the (smooth) function $v := \chi u$ is a solution of the inhomogeneous initial value problem

$$\dot{v}(t) = -\mathrm{i}A_0 v(t) + f(t), \quad v(0) = v_0, \tag{2.24}$$

with $f = f(t) = f(x, y; t)$ given by

$$f = -2\mathrm{i}\nabla\chi \cdot \nabla u - \mathrm{i}u\Delta\chi. \tag{2.25}$$

We also have $\|v_0 - u_0\| < \varepsilon'$ and $\|v_0\| > 1 - \varepsilon'$. Stationary phase estimates (cf. Lemma B.3 in the appendix) imply that there exists $s_0 \geq 0$ such that

$$\int_{-\infty}^{\infty} \|f(\tau)\| \,\mathrm{d}\tau < \varepsilon', \quad s \geq s_0. \tag{2.26}$$

The solution $v = v(t)$ of eq. (2.24) can be written as

$$v(t) = \mathrm{e}^{-\mathrm{i}tA_0} v_0 + \int_0^t \mathrm{e}^{\mathrm{i}(t-\tau)A_0} f(\tau) \,\mathrm{d}\tau. \tag{2.27}$$

Notice that there is no reason to expect that for $t \neq 0$ the individual terms $\mathrm{e}^{-\mathrm{i}tA_0} v_0$ and $\int_0^t \mathrm{e}^{\mathrm{i}(t-\tau)A_0} f(\tau) \,\mathrm{d}\tau$ on the right-hand side of (2.27) should vanish outside of $\mathcal{X}$; it is only the sum of the two terms which has support contained in $\mathcal{X}$. It is immediate from eq. (2.21), $\|u_0 - v_0\| < \varepsilon'$, and $\|u_0\| = 1$ that

$$\|(1 - \chi_{s,t})\mathrm{e}^{-\mathrm{i}tA_0} v_0\| \leq 2\varepsilon', \quad \|\chi_{s,t} \mathrm{e}^{-\mathrm{i}tA_0} v_0\| \geq 1 - 2\varepsilon'. \tag{2.28}$$

We have now gathered all the information we need on $\mathrm{e}^{-itA_0} v_0$ and are ready for the proof of Lemma 2.5.

Proof of Lemma 2.5. (i) In order to make the transition from $\mathbb{R}^2$ to M we define a map $j\colon \mathcal{X} \to M$ which assigns to $(x, y) \in \mathcal{X}$ the point $((x, y), \ell) \in M$ for $y < 0$, and the point $((x, y), \mathrm{u}) \in M$ for $y > 0$. The points in $\mathcal{X}$ with $y = 0$ are mapped to the line segment where the lower and the upper sheets of M are connected as we move in the direction of increasing values of y. Let $\mathcal{X}' := j(\mathcal{X})$. For functions $\eta\colon \mathcal{X} \to \mathbb{C}$, we obtain a lifting $\widetilde{J}\eta\colon \mathcal{X}' \to \mathbb{C}$ defined by

$$(\widetilde{J}\eta)(j(x, y)) := \eta(x, y), \quad (x, y) \in \mathcal{X}. \tag{2.29}$$

We may extend $\widetilde{J}\eta$ by zero to all of M. Obviously, we have $w(t) := \widetilde{J}v(t) \in \mathrm{Dom}(H)$ for $t > 0$ and $H(w(t)) = \widetilde{J}A_0(v(t))$. Hence w is a classical solution in $\mathsf{L}_2(\mathcal{M})$ of the initial value problem

$$\dot{w}(t) = -\mathrm{i}Hw(t) + \widetilde{J}f(t), \quad w(0) = \widetilde{J}v_0, \tag{2.30}$$

so that

$$w(t) = \mathrm{e}^{-\mathrm{i}tH} \widetilde{J}v_0 + \int_0^t \mathrm{e}^{\mathrm{i}(t-\tau)H} \widetilde{J}f(\tau) \,\mathrm{d}\tau. \tag{2.31}$$

We conclude from eqns. (2.27) and (2.31) that

$$\begin{aligned}0 &= w(t) - \widetilde{J}v(t)\\ &= \mathrm{e}^{-\mathrm{i}tH}\widetilde{J}v_0 + \int_0^t \mathrm{e}^{\mathrm{i}(t-\tau)H}\widetilde{J}f(\tau)\,\mathrm{d}\tau - \widetilde{J}X\mathrm{e}^{-\mathrm{i}tA_0}v_0 - \widetilde{J}X\int_0^t \mathrm{e}^{\mathrm{i}(t-\tau)A_0}f(\tau)\,\mathrm{d}\tau,\end{aligned} \tag{2.32}$$

whence

$$\left\|\mathrm{e}^{-\mathrm{i}tH}\widetilde{J}v_0 - \widetilde{J}X\mathrm{e}^{-\mathrm{i}tA_0}v_0\right\|_{\mathsf{L}_2(\mathcal{M})} \le 2\int_0^t \|f(s)\|\,\mathrm{d}s < 2\varepsilon'. \tag{2.33}$$

We finally define $w_0 := \widetilde{J}v_0$ and note that $\|w_0\| > 1-\varepsilon'$.

(ii) We now prove eq. (2.17). Combining (2.33) and (2.28) we see that

$$\begin{aligned}\|\mathrm{e}^{-\mathrm{i}tH}w_0 - J_u\mathrm{e}^{-\mathrm{i}tA_0}v_0\| &\le \|\mathrm{e}^{-\mathrm{i}tH}w_0 - \widetilde{J}X\mathrm{e}^{-\mathrm{i}tA_0}v_0\| + \|(\widetilde{J}X - J_u)\mathrm{e}^{-\mathrm{i}tA_0}v_0\|\\ &\le 2\varepsilon' + \|(1-\chi_{s,t})\mathrm{e}^{-\mathrm{i}tA_0}v_0\|\\ &\le 4\varepsilon' < \frac{4\varepsilon'}{1-\varepsilon'}\|w_0\| < \varepsilon\|w_0\|,\end{aligned}$$

since $(\widetilde{J}X - J_u)\chi_{s,t}\mathrm{e}^{-\mathrm{i}tA_0}v_0 = 0$ for $t > 0$, $0 < \varepsilon < 1$, and $\varepsilon' = \varepsilon/5$.

The proof of (2.18) is similar and omitted. □

It is now easy to prove the main result of this section.

Theorem 2.6. *The entries of the scattering matrix* $(S_{ij})_{i,j\in\{\ell,\mathrm{u}\}}$*, as defined in eq.* (2.16)*, are all non-zero operators.*

Proof. (i) We first show that the operator $S_{\ell\mathrm{u}}$ is non-zero. Let $0 < \varepsilon < 1/4$ and let v_0 and w_0 be as in Lemma 2.5. Without loss of generality we may assume, in addition, that $\|w_0\| = \|v_0\| = 1$. Then

$$\langle S_{\ell\mathrm{u}}v_0, v_0\rangle = \langle W_-(H, H_{0,\ell}, J_\ell)v_0, W_+(H, H_{0,\mathrm{u}}, J_\mathrm{u})v_0\rangle$$

where, by Lemma 2.5,

$$\|W_-(H, H_{0,\ell}, J_\ell)v_0 - w_0\| < \varepsilon, \quad \|W_+(H, H_{0,\mathrm{u}}, J_\mathrm{u})v_0 - w_0\| < \varepsilon.$$

It now follows that

$$|\langle S_{\ell\mathrm{u}}v_0, v_0\rangle - 1| \le 3\varepsilon < 3/4.$$

This shows that $S_{\ell\mathrm{u}}$ is non-zero; but then, by symmetry, we also have $S_{\mathrm{u}\ell} \neq 0$.

(ii) In order to show that $S_{\ell\ell}$ (and, analogously, S_{uu}) is non-zero, it is enough to construct wave packets which come in on the lower sheet (limit $t \to -\infty$) and which go out on the lower sheet as well (limit $t \to +\infty$), up to a small error. It is easy to modify v_0 and w_0 as in Lemma 2.5 to achieve this goal; cf. also Remark 2.8 below. E.g., we may replace the function ψ_1 in the proof of Lemma 2.5 with $\psi_1(\cdot - k)$ with $|k| > 1$ so that the associated wave packet is located away from the slit at time $t = 0$. We then translate Ω, χ, and $\mathcal{X}$ in the x-direction accordingly. The maps j and $\widetilde{J}$ can be simply defined as an embedding of $\mathcal{X}$ into $M_{0,\ell}$. We leave the details to the reader. □

Remark 2.7. In fact, what we obtain here is a particularly strong version of openness of the channels in the sense that the norm of the wave packet going out on one sheet is close to the norm of the incoming state on the other sheet, for suitably chosen states. For example, for any $\varepsilon > 0$ there are states where the norm of the outgoing wave packet on the upper sheet is greater than $(1 - \varepsilon)$ times the norm of what is coming in on the lower sheet, etc. One might say then that the channels are *strongly open* (see Lemma 2.11).

Remark 2.8. In dealing with $S_{\ell\ell}$ we might as well exchange the variables x and y and translate in the y-direction to avoid the slit. In the end, all one needs is a rigid motion of $\mathcal{X}$ which avoids the slit and one gets the impression that "most" initial states will belong to the range of $S_{\ell\ell}$ or S_{uu} while only a tiny fraction of initial states communicates between the two sheets under the evolution $\mathrm{e}^{-\mathrm{i}tH}$. Thus, if one wishes to be heard on the upper plane as a member of the lower plane one should shout in the right direction (and also rather at a high pitch).

Remark 2.9. Here we give some indications on coverings of the Euclidean plane with three or more sheets. In the case of three sheets and two branch points the southern rim of the cut in the sheets numbered I, II, and III is identified with the northern rim of the sheets numbered II, III, and I. Then the situation is basically the same as with two sheets and all channels are open. In the case of four sheets and two branch points the identification of the rims proceeds as above. Here we can show that neighboring sheets are open to one another while our method fails to decide whether the sheets I and III are open one to another; the same holds for the sheets II and IV. We suspect that the transmission is very weak (or zero) in the latter cases.

For three and more sheets there are of course also other possibilities to connect the sheets along cuts. For three sheets we might look at two different cuts (and thus four branch points) with sheets I and II connected along the first cut and sheets II and III connected along the second cut. If the two cuts are not aligned we may still construct wave packets that move from sheet I up to sheet III, up to small errors. If the two cuts are aligned (i.e., both lie on the real axis and have positive distance) our method fails. In this last case we would expect that there is only very weak (or no) transmission from sheet I to sheet III.

Also note that we are dealing with two (or more) branch points because a manifold with two sheets and a single branch point—like the Riemann surface of $\sqrt{z}$—constitutes just one scattering channel in our setup. In this case there is no simple comparison with the free Laplacian on the Euclidean plane.

Remark 2.10. The singularities at the branch points are only a side issue in our investigations. For most of our results, it wouldn't make much of a difference if we would "punch out" two small holes around the branch points and consider the Laplacian with Dirichlet boundary conditions on the (smooth) boundaries of these balls. However, the radius of these balls would introduce a parameter which is not well motivated and one would have to investigate questions of convergence etc. as this radius goes to zero.

For the record, we complement the estimates of Lemma 2.5 with some further basic properties of w_0.

Lemma 2.11. *Let $P_{\pm,\mathrm{u}}$ and $P_{\pm,\ell}$ denote the projections onto the ranges of the wave operators $W_\pm(H, H_{0,\mathrm{u}}, J_\mathrm{u})$ and $W_\pm(H, H_{0,\ell}, J_\ell)$, respectively. For $\varepsilon > 0$ let w_0 be as in Lemma 2.5. We then have*:

$$\|P_{+,\mathrm{u}} w_0\| > (1-\varepsilon)\|w_0\|, \quad \|P_{+,\ell} w_0\| < \varepsilon\|w_0\|, \tag{2.34}$$

and

$$\|P_{-,\ell} w_0\| > (1-\varepsilon)\|w_0\|, \quad \|P_{-,\mathrm{u}} w_0\| < \varepsilon\|w_0\|. \tag{2.35}$$

Proof. We only show (2.34); the proof of (2.35) is analogous and omitted. By the Projection Theorem, we have

$$\begin{aligned}\|P_{\pm,\mathrm{u}} w_0\| &= \sup\bigl\{\,\bigl|\langle w_0, \psi\rangle\bigr|\,;\ \psi \in \operatorname{Ran} W_\pm(H, H_{0,\mathrm{u}}, J_\mathrm{u}),\ \|\psi\| = 1\,\bigr\}\\ &= \sup\bigl\{\,\bigl|\langle w_0, W_\pm(H, H_{0,\mathrm{u}}, J_\mathrm{u})\varphi\rangle\bigr|\,;\ \varphi \in \mathcal{H}_\mathrm{u},\ \|\varphi\| = 1\,\bigr\},\end{aligned} \tag{2.36}$$

since $\|W_\pm(H, H_{0,\mathrm{u}}, J_\mathrm{u})\varphi\| = \|\varphi\|$ for all $\varphi \in \mathcal{H}_\mathrm{u}$. In the RHS of eq. (2.36) we have

$$\langle w_0, W_\pm(H, H_{0,\mathrm{u}}, J_\mathrm{u})\varphi\rangle_{\mathsf{L}_2(\mathcal{M})} = \lim_{t\to\pm\infty} \langle \mathrm{e}^{-\mathrm{i}tH} w_0, J_\mathrm{u}\mathrm{e}^{-\mathrm{i}tH_{0,\mathrm{u}}}\varphi\rangle_{\mathsf{L}_2(\mathcal{M})}. \tag{2.37}$$

In order to obtain a lower bound on $\|P_{+,\mathrm{u}} w_0\|$ we choose $\varphi := v_0$ in Eq. (2.36) and use Lemma 2.5 to find

$$|\langle \mathrm{e}^{-\mathrm{i}tH} w_0, J_\mathrm{u}\mathrm{e}^{-\mathrm{i}tH_{0,\mathrm{u}}} v_0\rangle_{\mathsf{L}_2(\mathcal{M})} - \|w_0\|^2| < \varepsilon. \tag{2.38}$$

For an upper bound on $\|P_{+,\ell} w_0\|$ we use $J_{\mathrm{u}} J_\ell = 0$, combined with Lemma 2.5, to see that $\|P_{+,\ell} w_0\| < \varepsilon$. □

Of course, one could as well work with the usual formula for the projection onto the range of a partial isometry. In our case this formula reads

$$P_{\pm,\mathrm{u}} = W_\pm(H, H_{0,\mathrm{u}}, J_{\mathrm{u}}) \circ (W_\pm(H, H_{0,\mathrm{u}}, J_{\mathrm{u}}))^*. \tag{2.39}$$

Let us first show that the adjoints $(W_\pm(H, H_{0,\mathrm{u}}, J_{\mathrm{u}}))^*$ of the wave operators $W_\pm(H, H_{0,\mathrm{u}}, J_{\mathrm{u}})$ are given by strong limits,

$$(W_\pm(H, H_{0,\mathrm{u}}, J_{\mathrm{u}}))^* = \operatorname*{s-lim}_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH_{0,\mathrm{u}}} J_{\mathrm{u}}^* \mathrm{e}^{-\mathrm{i}tH}, \tag{2.40}$$

with $P_{\mathrm{ac}}(H) = I$. Since the wave operators $W_\pm(H, H_0, J)$ exist and are complete (and because J satisfies the requirements of [22, p. 36, Proposition 5(c)]), it follows that the wave operators

$$W_\pm(H_0, H, J^*) = \operatorname*{s-lim}_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH_0} J^* \mathrm{e}^{-\mathrm{i}tH} \tag{2.41}$$

exist. Here $H_0 = H_{0,\ell} \oplus H_{0,\mathrm{u}}$ and $J^* = (P_\ell, P_{\mathrm{u}})$ and we see that

$$\mathrm{e}^{\mathrm{i}tH_0} J^* \mathrm{e}^{-\mathrm{i}tH} = (\mathrm{e}^{\mathrm{i}tH_{0,\ell}} P_\ell \mathrm{e}^{-\mathrm{i}tH}, \mathrm{e}^{\mathrm{i}tH_{0,\mathrm{u}}} P_{\mathrm{u}} \mathrm{e}^{-\mathrm{i}tH}). \tag{2.42}$$

The ranges of $\mathrm{e}^{\mathrm{i}tH_{0,\ell}} P_\ell \mathrm{e}^{-\mathrm{i}tH}$ and $\mathrm{e}^{\mathrm{i}tH_{0,\mathrm{u}}} P_{\mathrm{u}} \mathrm{e}^{-\mathrm{i}tH}$ being orthogonal, it is clear that the strong limit of the left hand side of (2.42) can only exist if the strong limits of both terms on the right hand side exist (as $t \to \pm\infty$).

For w_0 as in Lemma 2.5 we now compute

$$\langle P_{\pm,\mathrm{u}} w_0, w_0\rangle = \lim_{t\to\pm\infty} \|\mathrm{e}^{\mathrm{i}tH_{0,\mathrm{u}}} J_{\mathrm{u}}^* \mathrm{e}^{-\mathrm{i}tH} w_0\|^2 = \lim_{t\to\pm\infty} \|P_{\mathrm{u}} \mathrm{e}^{-\mathrm{i}tH} w_0\|^2$$

and the desired result follows by Lemma 2.5.

3. Perturbations of the metric

We first recall some notions and definitions in Differential Geometry as used in [11]. Given a (smooth) Riemannian metric $g = (g_{ij})$ on the C^∞-manifold M, we denote by $\mathcal{M} = (M, g)$ the Riemannian manifold and we let $B_\delta(p) = B_{\delta,\mathcal{M}}(p)$ denote the geodesic open ball centered at $p \in M$ with radius $\delta > 0$. For simplicity, we only consider *smooth* metrics g on M; cf., however, the discussion in [11, Remark 2.6] on the non-smooth case. Our assumptions on g will mainly involve the (sectional or Gauß) curvature of g and the injectivity radius. The *homogenized injectivity radius* $\iota_{\mathcal{M}}(p)$ at $p \in M$ is defined as in [1] or [11, eq. (2.7)] by

$$\iota_{\mathcal{M}}(p) := \sup_{\delta>0} \min\{\delta, \inf\{\operatorname{inj}_{\mathcal{M}}(y)\,;\, y \in B_{\delta,\mathcal{M}}(p)\}\} \tag{3.1}$$

where $\operatorname{inj}_{\mathcal{M}}(y)$ denotes the usual injectivity radius at the point y. The number $\iota_{\mathcal{M}}(p)$ is the largest number $\delta > 0$ for which the injectivity radius at any $y \in B_\delta(p)$ is not smaller than δ.

The following definition (cf. [11, Definition 2.4]) is of basic importance for our investigations:

Definition 3.1. For a continuous positive function $r_0 \colon M \to (0, 1]$ we denote by $\mathsf{Met}_{r_0}(M)$ the set of smooth metrics g on M that satisfy the lower bounds

$$\iota_{\mathcal{M}}(p) \ge r_0(p), \quad \text{and} \quad \inf\{\operatorname{Ric}^-_{\mathcal{M}}(y)\,;\, y \in B_{r_0(p),\mathcal{M}}(p)\} \ge -\frac{1}{r_0(p)^2}, \tag{3.2}$$

for all $p \in M$, where $\mathcal{M} = (M, g)$ and where $\operatorname{Ric}^-_{\mathcal{M}}(y)$ denotes the lowest eigenvalue of the Ricci tensor as endomorphism on $T_y M$.

Since we are in two dimensions, the Ricci curvature equals the Gauß curvature (times the metric tensor g). The second condition in eq. (3.2) is a lower bound for the homogenized Ricci curvature. Notice that the Euclidean metric $g_{\mathrm{E}} = (\delta_{ij})$ on M belongs to $\mathsf{Met}_{r_0}(M)$ if and only if r_0 satisfies the condition

$$r_0(p) \le \frac{1}{2} \min\{\operatorname{dist}(p, q_-), \operatorname{dist}(p, q_+)\}. \tag{3.3}$$

We denote by $\mathsf{L}_2(\mathcal{M})$ the usual space of (equivalence classes of) L_2-integrable functions on the Riemannian manifold $\mathcal{M} = (M, g)$ with respect to the Riemannian measure $\mathrm{d}\operatorname{vol}_g$. The following definition is standard.

Definition 3.2 (cf. [11, Definition 3.1]). We say that the Riemannian metrics g_1, g_2 are *quasi-isometric* if there exists a constant $\eta > 0$ such that

$$\eta g_1(p)(\xi, \xi) \le g_2(p)(\xi, \xi) \le \eta^{-1} g_1(p)(\xi, \xi), \tag{3.4}$$

for all $\xi \in T_p M$ and $p \in M$.

In our case T_pM can be identified with $\mathbb{R}^2$. The Hilbert spaces $\mathsf{L}_2(\mathcal{M}_1)$ and $\mathsf{L}_2(\mathcal{M}_2)$ coincide as sets if $\mathcal{M}_i = (M, g_i)$ with g_1 quasi-isometric to g_2. In this case we let I denote the natural identification operator mapping a function $f \in \mathsf{L}_2(\mathcal{M}_1)$ to the same function f in $\mathsf{L}_2(\mathcal{M}_2)$.

We now take a closer look at the property that the metrics g and g_E are quasi-isometric. Let $A(p)$ be the endomorphism on T_pM given by $g(p)(\xi,\xi) = g_\mathrm{E}(A(p)\xi,\xi)$ for all $\xi \in T_pM$ and $p \in M$ and let $\alpha_k(p)$, $k = 1,2$, denote the eigenvalues of $A(p)$. If $(g_{ij}(p))$ denotes the matrix representation of g on T_pM in the standard coordinates, then the $\alpha_k(p)$ are also the eigenvalues of $(g_{ij}(p))$. Thus g and g_E are quasi-isometric if and only if there is a number $\eta > 0$ such that $\eta \le \alpha_k(p) \le \eta^{-1}$, for $k = 1,2$ and for all $p \in M$.

We are now ready to define the basic distance function $\tilde{d}_1$: Let $\alpha_1(p)$, $\alpha_2(p)$ denote the eigenvalues of $A(p)$. We then define as in [11, eq. (3.2) and eq. (3.5)]

$$\tilde{d}(g_\mathrm{E}, g)(p) := \max_k |\alpha_k(p)^{1/2} - \alpha_k(p)^{-1/2}|, \tag{3.5}$$
$$\tilde{d}_\infty(g_\mathrm{E}, g) := \sup_{p\in M} \tilde{d}(g_\mathrm{E}, g)(p),$$

and

$$\tilde{d}_1(g_\mathrm{E}, g) := \int_M \tilde{d}(g_\mathrm{E}, g)(p) r_0(p)^{-4}\,\mathrm{d}p. \tag{3.6}$$

We call $\tilde{d}_1(g_\mathrm{E}, g)$ the *weighted* L_1*-quasi-distance of* g *and* g_E; we have dropped the symmetrizing factor $1 + \varrho_{g,g_\mathrm{E}}(p) = 1 + (\alpha_1(p)\alpha_2(p))^{-1/2}$ of $\tilde{d}_1$ appearing in [11, eq. (3.5)] (which has no influence on our estimates because it is a bounded function).

Let us assume now that g is quasi-isometric to the Euclidean metric g_E (this is equivalent with $\tilde{d}_\infty(g, g_\mathrm{E}) < \infty$), and denote by $\mathcal{M} = (M, g)$ the corresponding Riemannian manifold. Then there is a (unique) self-adjoint Laplacian H_g, acting in the Hilbert space $\mathsf{L}_2(\mathcal{M})$, with quadratic form domain given by the Sobolev space $\mathring{\mathsf{H}}^1(\mathcal{M})$, and defined by

$$\langle H_g u, v\rangle = \int_M \langle \nabla u, \nabla v\rangle_g \,\mathrm{d}\,\mathrm{vol}_g = \sum_{ij} \int_M g^{ij}\, \partial_i u \partial_j \bar{v} \sqrt{\det g}\,\mathrm{d}p, \tag{3.7}$$

for any $u \in \mathrm{Dom}(H_g) \subset \mathring{\mathsf{H}}^1(\mathcal{M})$ and $v \in \mathring{\mathsf{H}}^1(\mathcal{M})$, where (g^{ij}) is the inverse of (g_{ij}). In the Euclidean case ($g = g_\mathrm{E}$) the operator H_{g_E} agrees with the operator H defined in Section 2; recall that H_{g_E} is purely a.c. From Theorem 3.7 of [11] we now obtain the following result on the existence and completeness of the wave operators.

Theorem 3.3. *Suppose we are given a continuous function* $r_0 \colon M \longrightarrow (0,1]$ *satisfying condition* (3.3) *and a metric* $g \in \mathsf{Met}_{r_0}(M)$ *which is quasi-isometric to the Euclidean metric* g_{E} *on* M. *We also assume that the difference between* g *and* g_{E} *satisfies the* r_0*-dependent weighted integral condition* $\tilde{d}_1(g, g_{\mathrm{E}}) < \infty$ *with* $\tilde{d}_1$ *as in* (3.6).

Then the wave operators

$$W_\pm(H_g, H_{g_{\mathrm{E}}}, I) = \operatorname*{s\text{-}lim}_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}tH_g} I \mathrm{e}^{-\mathrm{i}tH_{g_{\mathrm{E}}}} \tag{3.8}$$

and

$$W_\pm(H_g, H_0, IJ) = \operatorname*{s\text{-}lim}_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}tH_g} IJ \mathrm{e}^{-\mathrm{i}tH_0} \tag{3.9}$$

exist and are complete with final subspace $\mathcal{H}_{\mathrm{ac}}(H_g)$.

Remark. Under suitable conditions on g the operator H_g will be absolutely continuous (cf. Donnelly [5], Kumura [17, 18]). In this case the wave operators in (3.8) are even unitary.

Remark. In applying the fundamental perturbation theorems in [11] we can deal with the branch points $q_\pm$ in the way described in Remark 2.4, i.e., we have (formally) a manifold with four ends, with two ends given by M_{ext} as in Eq. (2.7) and two ends given by $B_{1/2}(q_\pm)$. Again, the ends $B_{1/2}(q_\pm)$ do not participate in the scattering.

Following the development in Section 5 of [11] we next consider the question of continuity of the scattering matrix and the openness of the scattering channels for small perturbations of the Euclidean metric. As in [11] we define for r_0 as above and $\gamma, \varepsilon > 0$

$$\mathsf{Met}_{r_0}(M, g_{\mathrm{E}}, \gamma, \varepsilon) := \{\, g \in \mathsf{Met}_{r_0}(M) \,;\, \tilde{d}_\infty(g, g_{\mathrm{E}}) \le \gamma,\ \tilde{d}_1(g, g_{\mathrm{E}}) \le \varepsilon \,\},$$

i.e., $\mathsf{Met}_{r_0}(M, g_{\mathrm{E}}, \gamma, \varepsilon)$ is the set of smooth metrics g on M enjoying the following properties:

(i) the homogenized injectivity radius and the homogenized curvature of g at $p \in M$ are bounded from below by $r_0(p)$ and by $-1/r_0(p)^2$, respectively;

(ii) the metric g is quasi-isometric to g_{E} with the bound $\tilde{d}_\infty(g, g_{\mathrm{E}}) \le \gamma$;

(iii) the weighted L_1-quasi-distance $\tilde{d}_1(g, g_{\mathrm{E}})$ is not larger than ε.

Note that condition (iii) requires a quantitative smallness of the deviation of g from the Euclidean metric in the sense that $\tilde{d}_1(g, g_{\mathrm{E}}) \le \varepsilon$ while the main assumption in Theorem 3.3 only stipulates $\tilde{d}_1(g, g_{\mathrm{E}}) < \infty$.

Then Theorem 5.1 of [11] yields the strong convergence of the scattering operators as $\varepsilon \downarrow 0$, and Cor. 5.3 of [11] establishes the openness of the scattering channels, for small $\varepsilon > 0$. We are now going to make this precise.

Let $\gamma > 0$ be fixed. For $\varepsilon > 0$, we consider $g_\varepsilon \in \mathsf{Met}_{r_0}(M, g_{\mathrm{E}}, \gamma, \varepsilon)$ and we let H_{g_ε} denote the Laplacian of (M, g_ε). The natural identification operator from $\mathsf{L}_2(M, g_{\mathrm{E}})$ to $\mathsf{L}_2(M, g_\varepsilon)$ is written I_{g_ε}. Then the scattering operator is given by

$$S_{g_\varepsilon} = S(H_{g_\varepsilon}, H_0, I_{g_\varepsilon} J) = (W_+(H_{g_\varepsilon}, H_0, I_{g_\varepsilon} J))^* \circ W_-(H_{g_\varepsilon}, H_0, I_{g_\varepsilon} J),$$

with H_0 and J as in Section 2, Proposition 2.2, and the scattering matrix $(S_{ij})_{i,j \in \{\ell, \mathrm{u}\}}$ is defined by

$$S_{ij}(H_{g_\varepsilon}, H_0, I_{g_\varepsilon} J) := (W_+(H_{g_\varepsilon}, H_{0,i}, I_{g_\varepsilon} J_i))^* \circ W_-(H_{g_\varepsilon}, H_{0,j}, I_{g_\varepsilon} J_j),$$

for $i, j \in \{\ell, \mathrm{u}\}$. Then [11, Theorem 5.1] yields the following result:

Theorem 3.4. *Let g_{E}, $H_{g_{\mathrm{E}}}$, H_0, J as above, let $S(H_{g_{\mathrm{E}}}, H_0, J)$ as in eq. (2.15), and let $\gamma > 0$. For $\varepsilon > 0$ and $g_\varepsilon \in \mathsf{Met}_{r_0}(M, g_{\mathrm{E}}, \gamma, \varepsilon)$ we denote by H_{g_ε} the Laplacian of $\mathcal{M}_\varepsilon = (M, g_\varepsilon)$ and by $I_\varepsilon \colon \mathsf{L}_2(\mathcal{M}) \to \mathsf{L}_2(\mathcal{M}_\varepsilon)$ the natural identification. Then the scattering operators $S(H_{g_\varepsilon}, H_0, I_\varepsilon J)$ converge strongly to $S(H, H_0, J)$, as $\varepsilon \to 0$.*

As in [11, Corollary 5.3], we immediately obtain a stability result for the scattering matrix where we also use the fact, established in Theorem 2.6, that the operators $S_{ik}(H, H_0, J)$, $i, k \in \{\ell, \mathrm{u}\}$, are non-zero, i.e., all scattering channels are open.

Corollary 3.5. *For any $\gamma > 0$, there is $\varepsilon_0 > 0$ such that $S_{ik}(H_{g_\varepsilon}, H_0, I_\varepsilon J) \neq 0$ for all metrics $g_\varepsilon \in \mathsf{Met}_{r_0}(M, g_{\mathrm{E}}, \gamma, \varepsilon)$ and all $0 < \varepsilon \leq \varepsilon_0$.*

4. Examples

We first illustrate Theorem 3.3 in the special case where the perturbed metric on M is associated with the graph of a function $f \colon M \to \mathbb{R}$ of class C^2. As usual, we define $\Phi \colon M \to \mathbb{R}^3$ by $\Phi(p) := (p, f(p))$ and

$$g = g_f = J_\Phi^{\mathrm{T}} \cdot J_\Phi = \begin{pmatrix} 1 + f_x^2 & f_x f_y \\ f_x f_y & 1 + f_y^2 \end{pmatrix},$$

where J_Φ is the Jacobian of Φ. The eigenvalues of g are 1 and $\det g = 1 + f_x^2 + f_y^2$.

The curvature κ of $\mathcal{M} := (M, g)$ is given by the well-known formula

$$\kappa = \frac{f_{xx} \cdot f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2} = \frac{\det H_f}{\det^2 g} \tag{4.1}$$

(cf. [4, p. 163], [8, eq. (14.105)]), where H_f is the Hessian of f. We let

$$d_0(p) := \min\{1, \operatorname{dist}(p, q_-), \operatorname{dist}(p, q_+)\}, \qquad p \in M, \tag{4.2}$$

where the distances are measured in (M, g_{E}). We have the following proposition.

Proposition 4.1. *Let $f : M \longrightarrow \mathbb{R}$ be of class C^2 with bounded first and second order derivatives and suppose that*

$$\int_M |\nabla f|^2 d_0^{-4} \,\mathrm{d}p < \infty. \tag{4.3}$$

Then the wave operators $W_\pm(H_g, H_{g_{\mathrm{E}}}, J)$ exist and are complete.

Proof. Since $g = g_f$ has the eigenvalues 1 and $1 + f_x^2 + f_y^2$ with f_x, f_y bounded, the metric g is quasi-isometric to the Euclidean metric on M. We now choose a suitable function r_0 which then defines the class $\mathsf{Met}_{r_0}(M)$. Note that the choice of r_0 is not unique, and one may obtain different results for different choices. In view of eq. (3.3) the simplest choice appears to be $r_0 := \varrho\, d_0$ with a constant $\varrho \in (0, 1/2]$ which we are going to fix now.

Since f has bounded second order derivatives, the curvature of (M, g_f) is bounded in absolute value by some constant $K \geq 0$ and the second condition in eq. (3.2) is satisfied provided $\varrho \leq 1/\sqrt{K}$. According to Proposition C.5 there exists a constant $c_0 > 0$ such that the (homogenized) injectivity radius of (M, g) at $p \in M$ is bounded from below by $c_0 d_0(p)$. We may thus pick any $\varrho > 0$ satisfying $\varrho \leq \min\{1/2, 1/\sqrt{K}, c_0\}$.

In remains to show that $\tilde{d}_1(g_{\mathrm{E}}, g)$ as in eq. (2.6) is finite. Here we first estimate

$$\tilde{d}(g_{\mathrm{E}}, g) = \sqrt{1 + f_x^2 + f_y^2} - \frac{1}{\sqrt{1 + f_x^2 + f_y^2}} \leq |\nabla f|^2;$$

therefore condition (4.3) implies $\tilde{d}_1(g_{\mathrm{E}}, g) < \infty$. Since the assumptions of Theorem 3.3 are satisfied, we may conclude that the wave operators for the pair $(H_{g_{\mathrm{E}}}, H_g)$ exist and are complete. □

Remarks 4.2. (i) Condition (4.3) is satisfied if ∇f is square integrable at infinity and decays near q_- and q_+ like

$$|\nabla f(p)| \leq \min\{\operatorname{dist}(p, q_-), \operatorname{dist}(p, q_+)\}^{1+\beta}$$

for some $\beta > 0$.

(ii) It is illuminating to take a look at other choices of r_0 where r_0 tends to zero at infinity. The class of admissible functions $f: M \to \mathbb{R}$ that define the perturbed metric changes in the following way. On the one hand, the injectivity radius associated with the metric g may now go to zero at infinity and the (Gauß) curvature need no longer be bounded from below by a constant; on the other hand, it is now more difficult to satisfy the weighted integral condition (4.3).

In an analogous way one can indicate simple conditions on f which allow the application of Theorem 3.4. We consider functions $f: M \to \mathbb{R}$ of class C^2 with first and second order derivatives bounded by some constant C and which are such that $g_f \in \mathsf{Met}_{r_0}(M)$ with r_0 as above. Then $\tilde{d}_\infty(g_f, g_\mathrm{E}) \le C^2$, and we may now choose $\gamma := C^2$. For $\varepsilon > 0$, the condition $\tilde{d}_1(g_f, g_\mathrm{E}) \le \varepsilon$ is satisfied if

$$\int_M |\nabla f|^2 d_0^{-4}\, \mathrm{d}p \le \varepsilon; \tag{4.4}$$

in this case, we have $g_f \in \mathsf{Met}_{r_0}(M, g_\mathrm{E}, \gamma, \varepsilon)$ and the results of Theorem 3.4 and Corollary 3.5 apply.

Proposition 4.3. *Suppose we are given a sequence $(f_n) \subset \mathsf{C}^2(M)$ enjoying the following properties*:

(i) *there is a constant $C \ge 0$ such that $|\partial_j f_n(p)| \le C$ and $|\partial_{ij} f_n(p)| \le C$ for all $p \in M$ and all $n \in \mathbb{N}$*;

(ii) *we have*

$$\int_M |\nabla f_n|^2 d_0^{-4}\, \mathrm{d}p \longrightarrow 0, \quad n \to \infty. \tag{4.5}$$

Let g_n denote the metric induced by f_n and let I_n the associated natural identification operator, as above. Then the scattering operators $S(H_{g_n}, H_0, I_n J)$ exist and converge strongly to $S(H_{g_\mathrm{E}}, H_0, J)$, as $n \to \infty$.

Appendices

A. Self-adjointness and spectral properties

In this appendix we study the Sobolev spaces $\mathring{\mathsf{H}}^1$ and Laplace–Beltrami operators on branched coverings of the Euclidean plane. Here we are mainly interested in self-adjointness, compactness properties, and the question of absolute continuity of the Laplacian.

A.1. Double covering with a single branch point. It is convenient to begin the analysis of the Laplacian on branched coverings with the case of a single branch point, i.e., we look at a real version of the Riemann surface of $\sqrt{z}$. In the case of a single branch point one can use separation of variables in polar coordinates. We take the liberty of using the same symbols M_0, $\mathcal{M}_0$, H_0 etc. as in the case of two branch points. For most of our results the corresponding analogue for the case of two branch points will be immediate; cf. Section A.2.

Let M_0 denote the C^∞-manifold obtained by joining two copies of $\mathbb{R}^2$ along the line $\{(x,0) \in \mathbb{R}^2\,;\, x \le 0\}$ in the usual crosswise fashion. Equipped with the Euclidean metric tensor $g_{\mathrm{E}} = (\delta_{ij})$ we obtain the Riemannian manifold $\mathcal{M}_0 = (M_0, g_{\mathrm{E}})$ with the single branch point $(0,0)$. The origin $(0,0)$ does not belong to M_0 and $\mathcal{M}_0$ is not complete. For $r > 0$ we let $B_r \subset M_0$ denote the set of points in M_0 with distance less than r from the origin; the "disks" B_r form a two-sheeted covering of the punctured disk $\{\,(x,y) \in \mathbb{R}^2\,;\, 0 < x^2 + y^2 < r^2\,\}$.

In order to define the Laplacian H_0 of $\mathcal{M}_0$ we consider the Hilbert space $\mathcal{H}_0 := \mathsf{L}_2(\mathcal{M}_0)$, with scalar product denoted by $\langle\cdot,\cdot\rangle$, and the Sobolev space $\mathring{\mathsf{H}}^1(\mathcal{M}_0)$, given as the completion of $\mathsf{C}_{\mathrm{c}}^\infty(M_0)$ with respect to the norm $\|\cdot\|_1$ defined by

$$\|\psi\|_1^2 := \int_{M_0} |\psi(x)|^2 + |\nabla\psi(x)|^2 \,\mathrm{d}x, \qquad \psi \in \mathsf{C}_{\mathrm{c}}^\infty(M_0). \tag{A.1}$$

Then H_0 is defined as the unique self-adjoint operator satisfying $\mathrm{Dom}(H) \subset \mathring{\mathsf{H}}^1(M_0)$ and

$$\langle H_0 u, v\rangle = \int_{M_0} \nabla u \cdot \nabla \bar{v} \,\mathrm{d}x, \qquad u \in \mathrm{Dom}(H_0),\ v \in \mathring{\mathsf{H}}^1(M_0). \tag{A.2}$$

By elliptic regularity, we have $\mathrm{Dom}(H_0) \subset \mathsf{H}^2_{\mathrm{loc}}(M_0)$ and $H_0 u = -\Delta u$ for all $u \in \mathrm{Dom}(H_0)$. More precisely, if u belongs to $\mathrm{Dom}(H_0)$, then the restriction of u to $M_0 \setminus \overline{B}_\varepsilon$ belongs to $\mathsf{H}^2(M_0 \setminus \overline{B}_\varepsilon)$, for any $\varepsilon > 0$. We note as an aside that $\mathrm{Dom}(H_0) \not\subset \mathsf{H}^2(M_0)$. Indeed, the function $u\colon M_0 \longrightarrow \mathbb{R}$, defined in polar coordinates by $u(r,\vartheta) := \frac{1}{\sqrt{r}} \sin r \cos\frac{\vartheta}{2}$, satisfies $-\Delta u = u$ in M_0. If we now take any smooth function $\varphi\colon M_0 \longrightarrow \mathbb{R}$ which is 1 on B_1 and vanishes outside of B_2, say, then $\varphi u \in \mathrm{Dom}(H_0)$ but, by a straight-forward calculation, $(\varphi u)_{xx} \notin \mathsf{L}_2(M_0)$.

Another natural Sobolev space is the space $\mathsf{H}^1(M_0) = \mathsf{W}^1_2(M_0)$, consisting of all functions in $\mathsf{L}_2(M_0)$ that have first order distributional derivatives in $\mathsf{L}_2(M_0)$. For an open set $\Omega \subset \mathbb{R}^d$ with smooth boundary, $\mathring{\mathsf{H}}^1(\Omega)$ is associated with a (weak form of) Dirichlet boundary conditions while the Laplacian with form domain $\mathsf{H}^1(\Omega)$ is called the Neumann Laplacian of Ω. In the case at hand, however, the

Sobolev spaces $\mathsf{H}^1(M_0)$ and $\mathring{\mathsf{H}}^1(M_0)$ coincide. For completeness, we include the (standard) proof.

Lemma A.1. *We have* $\mathsf{H}^1(M_0) = \mathring{\mathsf{H}}^1(M_0)$.

Proof. Let $0 \le u \in \mathsf{H}^1(M_0)$ and let $u_n := \min\{u, n\}$ for $n \in \mathbb{N}$. Then $u_n \to u$ in $\mathsf{H}^1(M_0)$ (cf. [8]) and we see that $\mathsf{H}^1(M_0) \cap \mathsf{L}_\infty(M_0)$ is dense in $\mathsf{H}^1(M_0)$. Consider a sequence of Lipschitz continuous functions $\varphi_k \colon M_0 \to [0,1]$ with the following properties: φ_k vanishes on $B_{1/k}$ and $\varphi_k(x) = 1$ for $x \notin B_{2/k}$; furthermore, there exists a constant c such that $|\nabla\varphi_k(x)| \le c/k$, for all $k \in \mathbb{N}$. For any $n \in \mathbb{N}$ fixed, we have $\varphi_k u_n \to u_n$ in $\mathsf{L}_2(M_0)$ and $\nabla(\varphi_k u_n) \to \nabla u_n$ weakly in $(\mathsf{L}_2(M_0))^2$, as $k \to \infty$. Thus, for any $\varepsilon > 0$, there exist $n_0 \in \mathbb{N}$ and a (finite) convex combination v_ε of the $\varphi_k u_{n_0}$ such that $\|v_\varepsilon - u_{n_0}\|_1 < \varepsilon$. But $v_\varepsilon \in \mathring{\mathsf{H}}^1(M_0)$, and the result follows. □

By Lemma A.1 there is only one self-adjoint extension of the Laplacian on $\mathsf{C}_c^\infty(M_0)$ with form-domain contained in the Sobolev space $\mathsf{H}^1(M_0)$. On the other hand, it is easy to see that H_0 is *not* essentially self-adjoint on $\mathsf{C}_c^\infty(M_0)$. Indeed, we may just follow the line of arguments leading to [20, Theorem X.11]) for the Laplacian in $\mathbb{R}^2$. In the present situation, we use separation of variables in polar coordinates (r, ϑ), with $r > 0$ and the angle variable ϑ running through $[0, 4\pi)$ instead of $[0, 2\pi)$. The eigenvalues of the angular operator are now given by $\kappa_\ell = -\frac{1}{4}\ell^2$ with $\ell \in \mathbb{N}_0$. As a consequence, the corresponding radial operators (cf. eq. (X.18) in [20, loc. cit.])

$$h_\ell := -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\ell^2 - 1}{4r^2}, \qquad \ell \in \mathbb{N}_0, \tag{A.3}$$

are not essentially self-adjoint on $\mathsf{C}_c^\infty(0,\infty)$ for $\ell = 0$ and for $\ell = 1$.

We next consider the Rellich compactness property. In the following lemma we let χ_r denote the characteristic function of B_r.

Proposition A.2. *For all* $R > 0$ *the operators* $\chi_R(H_0+1)^{-1/2}$ *and* $(H_0+1)^{-1/2}\chi_R$ *are compact.*

Proof. It is clearly enough to show that the mapping $\mathsf{H}^1(M_0) \ni u \mapsto \chi_R u \in \mathsf{L}_2(M_0)$ is compact. Away from the origin we may apply the standard Rellich Compactness Theorem, but we need a different argument in a neighborhood of the origin.

(i) Let us first show that the embedding $\mathsf{H}^1(M_0) \hookrightarrow \mathsf{L}_{2,\mathrm{loc}}(M_0)$ is compact. Indeed, any compact subset $K \subset M_0$ can be covered by a finite number of disks

$B_r(p_i)$, $i = 1, \ldots, n$ with suitable $n \in \mathbb{N}$, $0 < r < \operatorname{dist}\{K, (0,0)\}$, and $p_i \in M_0$. Then $\overline{B}_r(p_i) \subset M_0$ and each disk $B_r(p_i)$ is (equivalent to) a Euclidean disk in $\mathbb{R}^2$. We may then use a partition of unity subordinate to this covering of K and we may apply the usual Rellich Compactness Theorem in each $B_r(p_i)$.

(ii) Let us define the Dirichlet Laplacian $H_{0;1}$ of B_1 as the (unique) self-adjoint operator with quadratic form domain $\mathring{\mathsf{H}}^1(B_1)$ and with quadratic form (A.1). Using again separation of variables in polar coordinates as above, we have to deal with the Friedrichs extension of the operators h_ℓ on $\mathsf{C}_c^\infty(0,1)$, for $\ell \in \mathbb{N}_0$. Each of the operators h_ℓ has purely discrete spectrum with the lowest eigenvalue tending to ∞ as $\ell \to \infty$. It follows that $H_{0;1}$ has compact resolvent.

(iii) Let $(u_k) \subset \mathring{\mathsf{H}}^1(M_0)$ and suppose that $u_k \to 0$ weakly in $\mathsf{H}^1(M_0)$. It is enough to show that $\chi_R u_k \to 0$ in $\mathsf{L}_2(M_0)$ strongly, for all $R > 0$.

Choose a (smooth) cutoff-function φ with support in B_1 and which is equal to 1 in $B_{1/2}$. We then have $\varphi u_k \in \mathring{\mathsf{H}}^1(B_1)$ and $\varphi u_k \to 0$ weakly in $\mathring{\mathsf{H}}^1(B_1)$. By the second part of this proof $H_{0;1}$ has compact resolvent. This implies that $\varphi u_k \to 0$ in $\mathsf{L}_2(B_1)$ since

$$\|\varphi u_k\|^2 = \langle H_{0;1}^{-1}(\nabla(\varphi u_k)), \nabla(\varphi u_k)\rangle$$

where $\nabla(\varphi u_k) \to 0$ weakly and $H_{0;1}^{-1}(\nabla(\varphi u_k)) \to 0$ strongly in $\mathsf{L}_2(B_{1/2})$.

On the other hand, $B_R \setminus B_{1/2}$ is a relatively compact subset of M_0, and therefore $(1-\varphi)\chi_R u_k \to 0$ in $\mathsf{L}_2(M_0)$ by part (i) of this proof. □

We next comment on the spectral properties of H_0. As $H_0 \geq 0$ we have $\sigma(H_0) \subset [0, \infty)$. Clearly, $\sigma_{\mathrm{ess}}(H_0) \supset [0,\infty)$ and so $\sigma(H_0) = \sigma_{\mathrm{ess}}(H_0) = [0, \infty)$. All operators h_ℓ in (A.3) have purely absolutely continuous spectrum since 0 is not an eigenvalue and the operators h_ℓ are purely a.c. in $(0, \infty)$; cf., e.g., [25, Satz 14.25]. It is then clear that H_0 is also purely a.c.; in other words, H_0 has no singular continuous spectrum and has no eigenvalues.

A.2. Double coverings with two branch points. We now return to the manifold $\mathcal{M}$ with two branch points and the associated Laplacian H as in Section 2. As in the case of a single branch point the Sobolev spaces $\mathsf{H}^1(\mathcal{M})$ and $\mathring{\mathsf{H}}^1(\mathcal{M})$ coincide. Again, H is not essentially self-adjoint on $\mathsf{C}_c^\infty(M)$. Also $(H+1)^{-1}\chi_R$ is compact for all $R > 0$ with χ_R as in Section 2. The proofs require only some obvious modifications as compared to the case of a single branch point. As for the spectral properties of H it is again clear that $\sigma(H) = \sigma_{\mathrm{ess}}(H) = [0, \infty)$ and it remains to deal with the question of absolute continuity. Here we refer to some work of Donnelly [5] and Kumura [17, 18] who have pertinent statements for complete manifolds which are asymptotically Euclidean. It is clear from their proofs that the presence of a finite number of branch points can be accommodated.

As an alternative, it is easy to adapt the Enß method of scattering (cf. e.g. [22]) to exclude singular continuous spectrum of H. The absence of eigenvalues can be obtained as in the Kato-Agmon-Simon theorem in [21]:

Proposition A.3. *The Laplacian H of (M, g_{E}) has no eigenvalues.*

Proof. Clearly, 0 cannot be an eigenvalue of H since an eigenfunction for the eigenvalue 0 would have to be constant. Positive eigenvalues can be excluded by following the proof of the Kato-Agmon-Simon Theorem [21, Theorem XIII.58] with some obvious modifications and simplifications. In the case at hand, the operator H is not essentially self-adjoint on $\mathsf{C}_{\mathrm{c}}^{\infty}(M)$, but any eigenfunction ψ of H is clearly in $\mathsf{C}^{\infty}(M)$ and there is a sequence of smooth functions $\psi_n \in \mathrm{Dom}(H)$, vanishing outside the radius n, such that $\psi_n \to \psi$ and $H\psi_n \to H\psi$ in $\mathsf{L}_2(M)$, as $n \to \infty$. □

For the present paper it is quite useful—albeit not essential—to know that the Laplacian of (M, g_{E}) is purely absolutely continuous. Of course, it is also natural to ask whether the operators H_g on M with metric g as in Section 3 are purely absolutely continuous. Here the papers [5] and [17, 18] mentioned above give sufficient conditions.

B. Stationary phase estimates

We refer to [22] for the basics of stationary phase estimates. In this appendix we consider two functions $\psi_1, \psi_2 \in \mathcal{S}(\mathbb{R})$ with $\|\psi_1\| = \|\psi_2\| = 1$, and we let $u_0 := \psi_1 \otimes \psi_2 \in \mathcal{S}(\mathbb{R}^2)$. We let h_0 denote the (unique) self-adjoint extension of $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ on $\mathsf{C}_{\mathrm{c}}^{\infty}(\mathbb{R})$ and we let A_0 denote the (unique) self-adjoint extension of $-\Delta$ on $\mathsf{C}_{\mathrm{c}}^{\infty}(\mathbb{R}^2)$ so that $A_0 = h_0 \otimes I_{\{y\}} + I_{\{x\}} \otimes h_0$. We then write

$$\Psi_i = \Psi_i(x,t) := \mathrm{e}^{-\mathrm{i}th_0}\psi_i, \quad i = 1, 2,\ t \in \mathbb{R}; \tag{B.1}$$

in particular, we have

$$\mathrm{e}^{-\mathrm{i}tA_0}u_0 = (\mathrm{e}^{-\mathrm{i}th_0}\psi_1) \otimes (\mathrm{e}^{-\mathrm{i}th_0}\psi_2) = \Psi_1(\cdot,t) \otimes \Psi_2(\cdot,t), \quad t \in \mathbb{R}. \tag{B.2}$$

We will be using the following basic estimate on the real line where $\hat{\psi} = \mathcal{F}\psi$ denotes the Fourier transform for $\psi \in \mathcal{S}(\mathbb{R})$. It is clearly enough to consider $t \geq 0$, in the sequel.

Lemma B.1. *Let* $\psi \in \mathcal{S}(\mathbb{R})$ *with* $\|\psi\| = 1$ *and let* $\Psi = \Psi(\cdot, t) := \mathrm{e}^{-\mathrm{i}th_0}\psi$.

(i) *Suppose* $\hat{\psi} \in \mathsf{C}_\mathrm{c}^\infty(\mathbb{R})$ *and let* $a \geq 0$ *be such that* $\operatorname{supp}\hat{\psi} \subset [-a, a]$. *We then have*: *For any* $m \in \mathbb{N}$ *there exists a constant* $c_m \geq 0$ *such that*

$$|\Psi(x,t)| \leq c_m(1 + |x| + t)^{-m}, \quad |x| \geq 2at. \tag{B.3}$$

(ii) *Suppose there exists* $s \geq 0$ *such that* $\operatorname{supp}\hat{\psi} \subset [s, s+1]$. *We then have*: *For any* $m \in \mathbb{N}$ *there exists a constant* $c_m \geq 0$ *such that*

$$|\Psi(x,t)| \leq c_m(1 + |x - 2st| + t)^{-m}, \quad x \leq 2st, \tag{B.4}$$

for all $m \in \mathbb{N}$, *where the constant* c_m *can be chosen independently of* s.

Lemma B.1 is an immediate consequence of classical stationary phase estimates, as discussed, e.g., in Appendix 1 to Section XI.3 of [22]. A motivation for these estimates is that the "classically allowed" region for $\mathrm{e}^{-\mathrm{i}tA_0}(\psi_1 \otimes \psi_2)$ at time $t \geq 0$ is contained in the rectangle $[-2at, 2at] \times [2st, 2(s+1)t]$ if ψ_1 and ψ_2 are as in Lemma B.1(i) and (ii), respectively.

We use the estimates (B.1) and (B.2) in the following lemma where

$$Q_{s,t} := [-st, st] \times [st, \infty) \subset \mathbb{R}^2, \quad t \geq 0, \tag{B.5}$$

and $\chi_{s,t}$ is the characteristic function of $Q_{s,t}$.

Lemma B.2. *Let* $u_0 = \psi_1 \otimes \psi_2$ *as above where* ψ_1 *satisfies the assumptions of Lemma* B.1(i) *and* ψ_2 *satisfies the assumptions of Lemma* B.1(ii).

We then have: *for any* $m \in \mathbb{N}$ *there exists a constant* $\tilde{c}_m \geq 0$ *such that for* $t \geq 0$ *and* $s \geq 2a$

$$\|(1 - \chi_{s,t})\mathrm{e}^{-\mathrm{i}tA_0}u_0\|^2 \leq \tilde{c}_m(1 + st)^{1-2m}, \tag{B.6}$$

and

$$\|(1 - \chi_{s,t})\nabla\mathrm{e}^{-\mathrm{i}tA_0}u_0\|^2 \leq \tilde{c}_m(1 + st)^{1-2m}. \tag{B.7}$$

Proof. By Lemma B.1 we have

$$\int\limits_{|x|>st} |\Psi_1(x,t)|^2 \, \mathrm{d}x \leq 2c_m^2 \int\limits_{st}^{\infty} (1 + |x|)^{-2m} \, \mathrm{d}x = \frac{2c_m^2}{2m-1}(1 + st)^{1-2m}$$

and, similarly,

$$\int\limits_{y<st} |\Psi_2(y,t)|^2 \, \mathrm{d}y \leq \frac{c_m^2}{2m-1}(1 + st)^{1-2m},$$

for all $t > 0$ and $s \geq 2a$. Using eq. (B.2), we therefore obtain

$$\|(1-\chi_{s,t})\mathrm{e}^{-\mathrm{i}tA_0}u_0\|^2 \leq C_0\|\Psi_2(\cdot,t)\|^2 \int\limits_{|x|>st} |\Psi_1(x,t)|^2\,\mathrm{d}x + C_0\|\Psi_1(\cdot,t)\|^2 \int\limits_{y<st} |\Psi_2(y,t)|^2\,\mathrm{d}y \leq \tilde{c}_m(1+st)^{1-2m}$$

with (non-negative) constants C_0, $\tilde{c}_m$ that are independent of s. This proves (B.6). For the estimate (B.7), we use the well-known fact that $p := -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}$ and $\mathrm{e}^{-\mathrm{i}th_0} = \mathrm{e}^{-\mathrm{i}tp^2}$ commute, whence

$$\nabla\mathrm{e}^{-\mathrm{i}tA_0}u_0 = (\mathrm{e}^{-\mathrm{i}th_0}\psi_1' \otimes \psi_2, \psi_1 \otimes \mathrm{e}^{-\mathrm{i}th_0}\psi_2').$$

Proceeding as above, we obtain (B.7) with a constant depending on $\|\psi_1'\|$ and $\|\psi_2'\|$. □

We are now ready to provide the basic estimate for the "localization error" as in eq. (2.26).

Lemma B.3. *Suppose* ψ_1 *and* ψ_2 *are as in Lemma B.2. In addition, let* $\chi \in \mathsf{C}^\infty(\mathbb{R}^2)$ *with* χ, $\nabla\chi$, $\Delta\chi$ *bounded, and such that* $\operatorname{supp}\chi \subset \{(x,y) \in \mathbb{R}^2\,;\,|x| \leq 1+|y|\}$. *Let* $f = f(x,y,t) = -2\mathrm{i}(\nabla\chi)\cdot\nabla\mathrm{e}^{-\mathrm{i}tA_0}u_0 - \mathrm{i}(\Delta\chi)\mathrm{e}^{-\mathrm{i}tA_0}u_0$ *with* $u_0 = \psi_1 \otimes \psi_2$. *We then have: For any* $\varepsilon > 0$ *there exists* $s_\varepsilon \geq 0$ *such that*

$$\int\limits_{-\infty}^{\infty} \|f(\cdot,\cdot,t)\|\,\mathrm{d}t < \varepsilon, \quad s \geq s_\varepsilon. \tag{B.8}$$

Proof. Without restriction we may assume $s \geq 2a$. We only consider $t > 0$, the case $t < 0$ being almost identical.

The set $Q_{s,t} = (-st, st) \times (st, \infty)$ does not intersect the double cone $\{(x,y)\,;\,|y| < |x|\} \subset \mathbb{R}^2$ and $\nabla\chi$ and $\Delta\chi$ vanish on $Q_{s,t}$. Then the estimates (B.6) and (B.7) immediately imply that for any $m \in \mathbb{N}$ there exists a constant $C_m \geq 0$ such that

$$\|f(\cdot,\cdot,t)\| \leq C_m(1+st)^{(1-2m)/2}, \quad s \geq 2a.$$

We now fix some $m \geq 2$ and integrate with respect to t to obtain

$$\int\limits_0^\infty \|f(\cdot,\cdot,t)\|\,\mathrm{d}t \leq C_m \int\limits_0^\infty (1+ts)^{(1-2m)/2}\,\mathrm{d}t < \infty,$$

where the integral on the right hand side tends to zero, as $s \to \infty$. □

C. Lower bounds for the injectivity radius

Lower bounds for the injectivity radius are crucial for the applicability of our results to concrete examples. We are now going to explain how a comparison result of Müller and Salomonsen [19] can be used to deal with various situations where the metric is associated with the graph of a function on $\mathbb{R}^2$ or on $\mathbb{R}^2 \setminus \{(0,0)\}$. These estimates may be of independent interest. Appendix D of [11] contains related results for radially symmetric manifolds. Let us first recall the basic comparison result:

Proposition C.1 ([19, Proposition 2.1], [11, Proposition D.1]). *Let M denote a smooth n-dimensional manifold. Suppose that the Riemannian manifolds $\mathcal{M}_0 := (M, g_0)$ and $\mathcal{M}_1 := (M, g_1)$ are complete with quasi-isometric metrics g_0 and g_1, i.e.,*

$$\eta g_0 \le g_1 \le \eta^{-1} g_0,$$

for some constant $\eta \in (0,1]$; cf. Definition 3.2. Furthermore, suppose that the sectional curvature of $\mathcal{M}_0$ and $\mathcal{M}_1$ is bounded (in absolute value) by some constant $K \ge 0$. Let $\operatorname{inj}_{\mathcal{M}_0}(p)$ and $\operatorname{inj}_{\mathcal{M}_1}(p)$ denote the injectivity radius of $\mathcal{M}_0$ and $\mathcal{M}_1$, respectively, at the point $p \in M$. We then have

$$\operatorname{inj}_{\mathcal{M}_1}(p) \ge \frac{1}{2} \min\Big\{ \frac{\eta^2 \pi}{\sqrt{K}}, \eta \operatorname{inj}_{\mathcal{M}_0}(p) \Big\}, \quad p \in M. \tag{C.1}$$

Note that the assumptions of Proposition C.1 are global and that the manifolds are assumed to be complete. We will use simple cut-offs and also an extension procedure for functions of class C^2 to obtain local versions.

In the sequel, we will deal with the special case $n = 2$, $\mathcal{M}_0 = (\mathbb{R}^2, g_{\mathrm{E}})$ and $\mathcal{M}_1 = (\mathbb{R}^2, g_f)$ where the metric g_f comes from a function $f : \mathbb{R}^2 \to \mathbb{R}$ of class C^2, as in Section 4. We start with the particularly simple case where the first and second order derivatives of f are bounded.

Proposition C.2. *Let $f \in \mathsf{C}^2(\mathbb{R}^2, \mathbb{R})$ with bounded derivatives of the first and second order. Let $g = g_f$ as defined above, and let $\mathcal{M}_1 := (\mathbb{R}^2, g_f)$. If $\beta \ge 0$ and $\gamma > 0$ are constants such that*

$$|D_i f(p)| \le \beta, \quad |D_{ij} f(p)| \le \gamma,$$

for all $p \in \mathbb{R}^2$ and $i, j \in \{1,2\}$, then the radius of injectivity of $\mathcal{M}_1$ at $p \in \mathbb{R}^2$ satisfies

$$\operatorname{inj}_{\mathcal{M}_1}(p) \ge \frac{\pi}{2\sqrt{2}} \cdot \frac{1}{(1 + 2\beta^2)^2 \gamma}, \quad p \in \mathbb{R}^2. \tag{C.2}$$

Proof. The injectivity radius of $\mathcal{M}_0 = (\mathbb{R}^2, g_E)$ is infinite at all $p \in \mathbb{R}^2$. As for the constants η and K in Proposition C.1 we may take $\eta := 1/(1 + 2\beta^2)$ and $K := 2\gamma^2$ since the curvature κ satisfies $|\kappa(p)| \le |\det H_f(p)|$ by eq. (4.1). The desired estimate now follows from eq. (C.1). ☐

Henceforth we will drop the factor $\pi/(2\sqrt{2}) > 1$ for better readability. We next consider $f \in \mathsf{C}^2(\mathbb{R}^2, \mathbb{R})$ without assuming a bound for the derivatives of f.

Proposition C.3. *Let $M = \mathbb{R}^2$ and let $f \in \mathsf{C}^2(\mathbb{R}^2, \mathbb{R})$. Let $g = g_f$ as defined above, and let $\mathcal{M}_1 := (\mathbb{R}^2, g_f)$. For $p_0 \in M$, let*

$$\beta(p_0) := \max\{\, |D_i f(p)|\,;\, |p - p_0| \le 2,\ i \in \{1, 2\}\,\}, \tag{C.3}$$

$$\gamma(p_0) := \max\{\, |D_{ij} f(p)|\,;\, |p - p_0| \le 2,\ i, j \in \{1, 2\}\,\}. \tag{C.4}$$

Then there is a constant $c \ge 0$, which is independent of f, such that the radius of injectivity of $\mathcal{M}_1$ at $p_0 \in \mathbb{R}^2$ satisfies

$$\mathrm{inj}_{\mathcal{M}_1}(p_0) \ge \min\{\, 1, (1 + 2c^2\beta(p_0)^2)^{-2}(\gamma(p_0) + c\beta(p_0))^{-1}\,\}. \tag{C.5}$$

Proof. Let $p_0 \in \mathbb{R}^2$. We may assume $f(p_0) = 0$ without restriction of generality since the lower bound of (C.1) depends only on derivatives of f. Let $\varphi \in \mathsf{C}^\infty_{\mathsf{c}}(B_2(p_0))$ satisfy $0 \le \varphi \le 1$ and $\varphi(p) = 1$ for $p \in B_1(p_0)$. The function $\tilde{f} := \varphi f$ has support contained in $B_2(p_0)$. Since $f(p_0) = 0$ we have $|\tilde{f}(p)| \le 2\sqrt{2}\beta(p_0)$ for all $p \in B_2(p_0)$ by the mean value theorem. Routine calculations then lead to the estimates

$$|D_i \tilde{f}(p)| \le c_\varphi \beta(p_0), \quad |D_{ij} \tilde{f}(p)| \le \gamma(p_0) + c_\varphi \beta(p_0),$$

for all $p \in \mathbb{R}^2$ and $i, j \in \{1, 2\}$, where c_φ is a constant depending only on a bound for the first and second order derivatives of φ. We may also assume that these bounds are independent of $p_0 \in \mathbb{R}^2$. Applying the estimate (C.2) with $c_\varphi\beta(p_0)$ and $\gamma(p_0) + c_\varphi\beta(p_0)$ replacing β and γ, respectively, we obtain the desired result. ☐

In order to deal with branch points or other singularities, we now consider functions f on the punctured plane $\mathbb{R}^2_* := \mathbb{R}^2 \setminus \{(0, 0)\}$. The method used in the proof of Proposition C.3 could be easily adapted to the case where p_0 is close to the origin. However, this would require working with cut-offs φ which are supported in $B_{2\varrho}(p_0)$ and which are equal to 1 on $B_\varrho(p_0)$, for some $0 < \varrho < \frac{1}{2}|p_0|$. In this case the constant c_φ in the proof of the estimate (C.5) would blow up like $|p_0|^{-2}$, as $p_0 \to (0, 0)$. We therefore first restrict f to a suitable half-disk (with positive distance to the origin) and then use a C^2-extension method.

Proposition C.4. *For $f \in \mathsf{C}^2(\mathbb{R}^2_*, \mathbb{R})$ we define the metric $g = g_f$ on $\mathbb{R}^2_*$ as before and we let $\mathcal{M}_1 := (\mathbb{R}^2_*, g_f)$. For $p_0 \in \mathbb{R}^2_*$ we consider the annulus*

$$A(p_0) := \left\{ p \in \mathbb{R}^2 \,;\, \frac{1}{2}|p_0| \le |p| \le \frac{1}{2}|p_0| + 2 \right\}$$

and we define

$$\begin{aligned} \beta(p_0) &:= \max\{\, |D_i f(p)| \,;\, p \in A(p_0), i \in \{1,2\} \,\}, \\ \gamma(p_0) &:= \max\{\, |D_{ij} f(p)| \,;\, p \in A(p_0), i, j \in \{1,2\} \,\}. \end{aligned}$$

Then the radius of injectivity of $\mathcal{M}_1$ at $p_0 \in \mathbb{R}^2_*$ with $|p_0| \le 1$ satisfies the lower bound

$$\mathrm{inj}_{\mathcal{M}_1}(p_0) \ge \min\{|p_0|/2, (1 + 2c^2\beta(p_0)^2)^{-2}(\gamma(p_0) + c\beta(p_0))^{-1}\}, \qquad \text{(C.6)}$$

where $c \ge 0$ is a constant which can be chosen to be independent of f and p_0.

Proof. Without restriction of generality we may assume that $p_0 = (x_0, 0)$ with $0 < x_0 \le 1$. For the following construction we refer to Figure 4. We write $r_0 := \frac{1}{2}x_0$ and we let $p_1 := (\frac{1}{2}x_0, 0) = \frac{1}{2}p_0$. Then the circle $\partial B_{r_0}(p_0)$ passes through the point p_1. Let $H_+(p_1) \subset \mathbb{R}^2$ denote the half-plane to the right of p_1, i.e.,

$$H_+(p_1) := \{p = (x, y) \in \mathbb{R}^2 \,;\, 2x > x_0\}.$$

It is easy to see that $B_{r_0}(p_0)$ is contained in $B_{1,+}(p_1) := H_+(p_1) \cap B_1(p_1)$. Furthermore, $B_{2,+}(p_1) := H_+(p_1) \cap B_2(p_1)$ is contained in the annulus $A(p_0)$.

We now apply the well-known formula for the extension of a function of class C^2 across a hyperplane as in [8, Lemma 6.37] to obtain an extension F of f from the (closure of) the half-disk $B_{2,+}(p_1)$ into the disk $B_2(p_1)$ satisfying the following estimates, valid for all $p \in B_2(p_1)$:

$$\begin{aligned} |D_i F(p)| &\le C_{\mathrm{ext}} \max\{\, |D_i f(q)| \,;\, q \in \overline{B}_{2,+}(p_1) \,\} \le C_{\mathrm{ext}}\beta(p_0), \\ |D_{ij} F(p)| &\le C_{\mathrm{ext}} \max\{\, |D_{ij} f(q)| \,;\, q \in \overline{B}_{2,+}(p_1) \,\} \le C_{\mathrm{ext}}\gamma(p_0), \end{aligned}$$

for some constant $C_{\mathrm{ext}} \ge 0$ as in [8, loc. cit.]. We may now proceed as in the proof of Proposition C.3: choose a cut-off function $\varphi \in \mathsf{C}^\infty_{\mathrm{c}}(B_2(p_1))$ satisfying $\varphi(p) = 1$ for all $p \in B_1(p_1)$ and let $\tilde{f} := \varphi F$. We then take $c := c_\varphi C_{\mathrm{ext}}$ with c_φ as in the proof of Proposition C.3, and the desired estimate follows as before. □

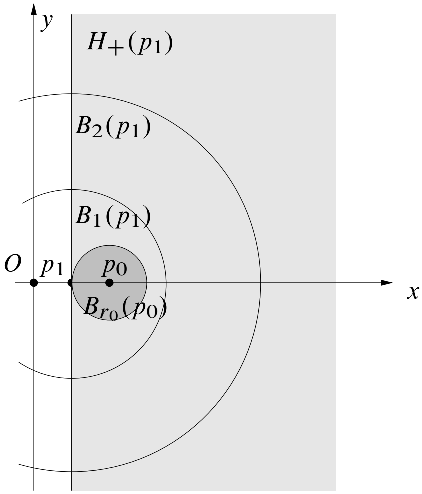

Figure 4. The reflection method.

Remark. Higher order reflections are just one method of obtaining extensions of functions of class C^2. In the case of Proposition C.4 the geometry is particularly simple and we can use reflection at a line. Here the orders of differentiation are not mixed in the sense that the bounds for the k-th order derivatives of the extended function F depend solely on bounds for the k-th order derivatives of f, for $k = 1, 2$.

In a more complicated geometric setting, one could work with extension from the closed disk $\overline{B}_{r_0}(p_0)$ using [8, Lemma 6.37] or employing an extension theorem of Whitney type as in [24, Section VI.2.3]. An advantage of Whitney extension lies in the fact that the constant C_{ext} can be chosen to be independent of the size of the disk $\overline{B}_{r_0}(p_0)$; on the other hand, Whitney extension would involve bounds on some Hölder-norm for the second order derivatives.

We finally return to M as in the body of the paper, with the branch points $q_\pm$. This is the case which is needed in Section 3. We have the following result.

Proposition C.5. *Let M be the double covering of $\mathbb{R}^2$ with the branch points $q_\pm$ and let $f \in \mathsf{C}^2(M, \mathbb{R})$ with bounded first and second order derivatives. We define the metric $g = g_f$ on M as before and we let $\mathcal{M}_1 := (M, g_f)$. Then there is a constant $c_f > 0$ such that the radius of injectivity of $\mathcal{M}_1$ at $p \in M$ satisfies the lower bound*

$$\operatorname{inj}_{\mathcal{M}_1}(p) \geq c_f \min\{1, \operatorname{dist}(p, q_+), \operatorname{dist}(p, q_-)\}. \tag{C.7}$$

Proof. If $p_0 \in M$ has distance at least 2 to $q_\pm$, the estimate (C.5) applies. In the other cases we may proceed as in the proof of Proposition C.4 with some more or less obvious modifications which we indicate now:

(i) since the distance between $q_\pm$ is 2, we need to scale down all sizes in the proof of Proposition C.4 by a factor smaller than 1;

(ii) the annulus $A(p_0)$ will now run through both sheets;

(iii) since the first and second order derivatives of f are bounded, the numbers $\beta(p_0)$ and $\gamma(p_0)$ can be estimated uniformly by a fixed constant. □

References

[1] M. T. Anderson and J. Cheeger, *C^α-compactness for manifolds with Ricci curvature and injectivity radius bounded below. J. Differential Geom.* **35** (1992), no. 2, 265–281. MR 1158336 Zbl 0774.53021

[2] M. Š. Birman, *Existence conditions for wave operators. Izv. Akad. Nauk SSSR Ser. Mat.* **27** (1963), 883–906. In Russian. MR 0161150 Zbl 0123.31404

[3] V. Bonaillie-Noël, B. Helffer, and T. Hoffmann-Ostenhof, Aharonov–Bohm Hamiltonians, isospectrality and minimal partitions. *J. Phys. A* **42** (2009), no. 18, 185203, 20 pp. MR 2591197 Zbl 1196.35031

[4] M. P. do Carmo, Differential geometry of curves and surfaces. Translated from the Portuguese. Prentice-Hall, Englewood Cliffs, N.J., 1976. MR 0394451 Zbl 0326.53001

[5] H. Donnelly, Spectrum of the Laplacian on asymptotically Euclidean spaces. *Michigan Math. J.* **46** (1999), no. 1, 101–111. MR 1682891 Zbl 0963.58010

[6] P. Deift and B. Simon, On the decoupling of finite singularities from the question of asymptotic completeness in two body quantum systems. *J. Functional Analysis* **23** (1976), no. 3, 218–238. MR 0432051 Zbl 0344.47007

[7] G. Ford, A. Hassell, and L. Hillairet, Wave propagation on Euclidean surfaces with conical singularities. I: Geometric diffraction. *J. Spectr. Theory* **8** (2018), no. 2, 605–667. MR 3812811 Zbl 1403.35015

[8] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order.* Second edition. Grundlehren der Mathematischen Wissenschaften, 224. Springer-Verlag, Berlin etc., 1983. MR 0737190 Zbl 0562.35001

[9] B. Güneysu and A. Thalmaier, Scattering theory without injectivity radius assumptions, and spectral stability for the Ricci flow. *Ann. Inst. Fourier* (*Grenoble*) **70** (2020), no. 1, 437–456. MR 4105943 Zbl 07206866

[10] R. Hempel and R. Weder, On the completeness of wave operators under loss of local compactness. *J. Funct. Anal.* **113** (1993), no. 2, 391–412. MR 1218101 Zbl 0811.35089

[11] R. Hempel, O. Post, and R. Weder, On open scattering channels for manifolds with ends. *J. Funct. Anal.* **266** (2014), no. 9, 5526–5583. MR 3182951 Zbl 1298.35129

[12] L. Hillairet, Spectral theory of translation surfaces: a short introduction. In *Actes du Séminaire de Théorie Spectrale et Géometrie*. Volume 28. Année 2009–2010. Université de Grenoble I, Institut Fourier, Saint-Martin-d'Hères, 2010, 51–62. MR 2848211 Zbl 1404.58050

[13] K. Ito and E. Skibsted, Stationary scattering theory on manifolds II. Preprint, 2016. arXiv:1602.07487 [math-ph]

[14] H. T. Ito and H. Tamura, Aharonov–Bohm effect in scattering by point-like magnetic fields at large separation. *Ann. Henri Poincaré* **2** (2001), no. 2, 309–359. MR 1832970 Zbl 0992.81084

[15] T. Kato, *Perturbation theory for linear operators.* Die Grundlehren der mathematischen Wissenschaften, 132. Springer-Verlag New York, New York, 1966. MR 0203473 Zbl 0148.12601

[16] T. Kato, Scattering theory with two Hilbert spaces. *J. Functional Analysis* **1** (1967), 342–369. MR 0220097 Zbl 0171.12303

[17] H. Kumura, The radial curvature of an end that makes eigenvalues vanish in the essential spectrum. I. *Math. Ann.* **346** (2010), no. 4, 795–828. MR 2587093 Zbl 1188.53031

[18] H. Kumura, Limiting absorption principle on manifolds having ends with various measure growth rate limits. *Proc. Lond. Math. Soc.* (3) **107** (2013), no. 3, 517–548. MR 3100776 Zbl 1331.58035

[19] W. Müller and G. Salomonsen, Scattering theory for the Laplacian on manifolds with bounded curvature. *J. Funct. Anal.* **253** (2007), no. 1, 158–206. MR 2362420 Zbl 1133.58026

[20] M. Reed and B. Simon, *Methods of modern mathematical physics.* II. Fourier analysis, self-adjointness. Academic Press, New York and London, 1975. MR 493420 Zbl 0308.47002

[21] M. Reed and B. Simon, *Methods of modern mathematical physics.* IV. Analysis of operators. Academic Press, New York and London, 1978. MR 0493421 Zbl 0401.47001

[22] M. Reed and B. Simon, *Methods of modern mathematical physics.* III. Scattering theory. Academic Press, New York and London, 1979. MR 0529429 Zbl 0405.47007

[23] M. Reed and B. Simon, *Methods of modern mathematical physics.* I. Functional analysis. Second edition. Academic Press, New York and London, 1980. MR 0751959 Zbl 0459.46001

[24] E. M. Stein, *Singular integrals and differentiability properties of functions.* Princeton Mathematical Series, 30. Princeton University Press, Princeton, N.J., 1970. MR 0290095 Zbl 0207.13501

[25] J. Weidmann, Lineare Operatoren in Hilberträumen. Teil II: Anwendungen. Mathematische Leitfäden. B. G. Teubner, Stuttgart, 2003. MR 2382320 Zbl 1025.47001

Received October 1, 2018

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