J. Spectr. Theory 11 (2021), 355–367
DOI 10.4171/JST/343

On a continuum limit of discrete Schrödinger operators on square lattices

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Abstract. The norm resolvent convergence of discrete Schrödinger operators to a continuum Schrödinger operator in the continuum limit is proved under relatively weak assumptions. This result implies, in particular, the convergence of the spectrum with respect to the Hausdorff distance.

Mathematics Subject Classification (2010). Primary: 47A10; Secondary: 47B39, 47A25, 81U05.

Keywords. Discrete Schrödinger operators, continuum limit, convergence of the spectrum.

1. Introduction

We consider a Schrödinger operator

$$H = H_0 + V(x), \quad H_0 = -\triangle, \quad x \in \mathbb{R}^d,$$

on $\mathcal{H} = L^2(\mathbb{R}^d)$, where $d \geq 1$, and corresponding discrete Schrödinger operators: We let $h > 0$ be the mesh size, and we write

$$\mathcal{H}_h = \ell^2(h\mathbb{Z}^d), \quad h\mathbb{Z}^d = \{(hz_1, \dots, hz_d) \mid z \in \mathbb{Z}^d\},$$

with the norm $\|v\|_h^2 = h^d \sum |v(hz)|^2$ for $v \in \mathcal{H}_h$. We denote the standard basis of $\mathbb{R}^d$ by $e_j = (\delta_{jk})_{k=1}^d \in \mathbb{R}^d$, $j = 1, \dots, d$. Our discrete Schrödinger operator is

$$H_h = H_{0,h} + V(z), \quad z \in h\mathbb{Z}^d,$$

[1] This work was supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

[2] Partially supported by JSPS Grant Kiban-B 15H03622

[3] Supported by JSPS Research Fellowship for Young Scientists 17J05051.

where

$$H_{0,h}v(z) = h^{-2}\sum_{j=1}^{d}(2v(z) - v(z + he_j) - v(z - he_j)), \quad v \in \mathcal{H}_h.$$

We suppose:

Assumption A. V is a real-valued continuous function on $\mathbb{R}^d$, and bounded from below. $(V(x) + M)^{-1}$ is uniformly continuous with some $M > 0$, and there is $c_1 > 0$ such that

$$c_1^{-1}(V(x) + M) \le V(y) + M \le c_1(V(x) + M), \quad \text{if } |x - y| \le 1.$$

The above assumption implies V is slowly varying in some sense, and uniformly continuous relative to the size of $V(x)$. Under the assumption, H is essentially self-adjoint, and H_h is self-adjoint (cf. Reed-Simon [16], Theorem X.28). We note that H and H_h may be considered as quadratic forms with natural form domains. The assumption is satisfied if V is bounded and uniformly continuous. $V(x) = a\langle x\rangle^{\mu}$ with $a, \mu > 0$, also satisfies the assumption.

For $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $h > 0$ and $z \in h\mathbb{Z}^d$, we set

$$\varphi_{h,z}(x) = \varphi(h^{-1}(x - z)), \quad x \in \mathbb{R}^d,$$

and we define $P_h = P_{h,\varphi}: \mathcal{H} \to \mathcal{H}_h$ by

$$P_h u(z) := h^{-d}\int_{\mathbb{R}^d} \overline{\varphi_{h,z}(x)}u(x)dx, \quad h > 0,\ z \in h\mathbb{Z}^d.$$

The adjoint operator is given by

$$P_h^* v(x) = \sum_{z \in h\mathbb{Z}^d} \varphi_{h,z}(x)v(z), \quad h > 0,\ v \in \mathcal{H}_h.$$

It is easy to observe that P_h^* is an isometry and hence P_h is a partial isometry, if and only if $\{\varphi_{1,z} \mid z \in \mathbb{Z}^d\}$ is an orthonormal system. This condition is also equivalent to the condition:

$$\sum_{n \in \mathbb{Z}^d} |\hat{\varphi}(\xi + n)|^2 = 1 \quad \text{for } \xi \in \mathbb{R}^d, \tag{1}$$

where $\hat{\varphi}$ is the Fourier transform:

$$\hat{\varphi}(\xi) = \mathcal{F}\varphi(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi}\varphi(x)dx, \quad \xi \in \mathbb{R}^d.$$

This claim is well-known, but we give its proof in Appendix for the completeness (Lemma A.1). We note operators of this form have been extensively studied in the context of signal analysis and wavelet analysis (see, e.g., [13], [7]). By this observation, we learn that there is a large class of φ's satisfying the above condition. In this paper, we use P_h to identify $\mathcal{H}_h$ with a subspace of $\mathcal{H}$. We suppose:

Assumption B. φ satisfies the condition (1), and $\mathrm{supp}[\hat{\varphi}] \subset (-1, 1)^d$.

Theorem 1.1. *Suppose Assumptions A and B. Then, for any $\mu \in \mathbb{C}\setminus\mathbb{R}$,*

$$\| P_h^*(H_h - \mu)^{-1} P_h - (H - \mu)^{-1} \|_{\mathcal{B}(\mathcal{H})} \longrightarrow 0 \quad \text{as } h \to 0.$$

Furthermore, if $(V(x) + M)^{-1}$ is uniformly Hölder continuous of order $\alpha \in (0, 1]$ with some $M > 0$, then for any $0 < \beta < \alpha$,

$$\| P_h^*(H_h - \mu)^{-1} P_h - (H - \mu)^{-1} \|_{\mathcal{B}(\mathcal{H})} \le C_\mu h^\beta \quad \text{for } h \in (0, 1].$$

Here $\mathcal{B}(X)$ denotes the Banach space of the operators on a Banach space X. Combining this with the argument of Theorem VIII.23 (b) in [17], we obtain the following corollary. We denote the spectrum of a self-adjoint operator A by $\sigma(A)$, and the spectral projection by $E_A(\Omega)$ for $\Omega \subset \mathbb{R}$.

Corollary 1.2. *Suppose Assumptions A and B. Let $a, b \in \mathbb{R}$, $a < b$, be not in $\sigma(H)$. Then $a, b \notin \sigma(H_h)$ for sufficiently small $h > 0$ and*

$$\| P_h^* E_{H_h}((a, b)) P_h - E_H((a, b)) \|_{\mathcal{B}(\mathcal{H})} \longrightarrow 0 \quad \text{as } h \to 0.$$

We denote the Hausdorff distance between sets $X, Y \subset \mathbb{C}$ by

$$d_{\mathrm{H}}(X, Y) = \max\{\sup_{x \in X} d(x, Y), \sup_{y \in Y} d(y, X)\},$$

where $d(\cdot, \cdot)$ denotes the standard distance in $\mathbb{C}$. It is not difficult to show $d_{\mathrm{H}}(\sigma(A), \sigma(B)) \le \|A - B\|$ for normal operators A and B (see Lemma A.2 in Appendix). Thus we also have the following result. We note this result is independent of Assumption B, since the identification operators P_h do not appear in the statement.

Corollary 1.3. *Suppose Assumptions A. Then for $M > -\inf \sigma(H)$,*

$$d_H(\sigma((H_h + M)^{-1}), \sigma((H + M)^{-1})) \longrightarrow 0, \quad \text{as } h \to 0.$$

Similar convergences in the norm resolvent sense are studied by Exner and Post, where convergence of operators on manifolds and graphs is considered. See [8] and references therein. In numerical analysis, the convergence of spectrum is studied extensively in the context of domain truncations, and general notions of convergence, e.g., *generalized norm resolvent convergence* and *generalized strong resolvent convergence* (see [3] and references therein). The general framework of convergence is analogous to our result, but the models are quite different and hence analytic methods also differ considerably. We also note that for spectral consequence of our main result (Corollary 1.3), we simply adopt the argument of the Reed-Simon textbook [16].

There are studies concerning continuum limits of NLS equations, in many cases, mainly with applications to numerical analysis. We refer Bambusi and Penati [2], Hong and Yang [9] and references therein. For linear discrete Schrödinger operators, Rabinovich [15] has studied the relation between the essential and discrete spectra of the discrete and continuum Schrödinger operators, provided V is bounded and uniformly continuous. Approximations of spectrum of continuum Schrödinger operators are studied by the Galerkin method ([5, 11, 18, 19]) and domain truncations ([4]).

In Section 2, we give the proof of our main theorem, and proofs of several technical lemmas are given in Appendix.

2. Proof

We denote the discrete Fourier transform

$$F_h : \mathcal{H}_h \longrightarrow \widehat{\mathcal{H}}_h = L^2(h^{-1}\mathbb{T}^d), \quad \mathbb{T} = \mathbb{R}/\mathbb{Z},$$

by

$$F_h v(\zeta) = h^d \sum_{z\in h\mathbb{Z}^d} e^{-2\pi i z\cdot\zeta} v(z), \quad \zeta \in h^{-1}\mathbb{T}^d,\ v \in \mathcal{H}_h.$$

F_h is unitary, and its adjoint is given by

$$F_h^* g(z) = \int_{h^{-1}\mathbb{T}^d} e^{2\pi i z\cdot\zeta} g(\zeta) d\zeta, \quad z \in h\mathbb{Z}^d,\ g \in \widehat{\mathcal{H}}_h.$$

2.1. Convergence of the free Hamiltonian. If we set $H_0(\xi) = |2\pi\xi|^2$, it is well-known that $H_0 = \mathcal{F}^* H_0(\cdot)\mathcal{F}$ on $\mathcal{H}$. Similarly, if we set

$$H_{0,h}(\zeta) = 2h^{-2}\sum_{j=1}^{d}(1-\cos(2\pi h\zeta_j)), \quad \zeta \in h^{-1}\mathbb{T}^d,$$

then $H_{0,h} = F_h^* H_{0,h}(\cdot) F_h$. We denote

$$Q_h := F_h P_h \mathcal{F}^* : \widehat{\mathcal{H}} \longrightarrow \widehat{\mathcal{H}}_h.$$

The following formula is convenient in the following argument. It is well-known in signal analysis (see, e.g., [13]), but we give a proof in Appendix for the completeness.

Lemma 2.1. *For $f \in \mathcal{S}(\mathbb{R}^d)$,*

$$Q_h f(\zeta) = \sum_{n\in\mathbb{Z}^d} \overline{\hat{\varphi}(h\zeta + n)} f(\zeta + h^{-1}n), \quad \zeta \in h^{-1}\mathbb{T}. \tag{2}$$

For $g \in \widehat{\mathcal{H}}_h$,

$$Q_h^* g(\xi) = \hat{\varphi}(h\xi)\tilde{g}(\xi), \quad \xi \in \mathbb{R}^d, \tag{3}$$

where $\tilde{g}$ is the periodic extension of g on $\mathbb{R}^d$.

Lemma 2.2. *For $\mu \in \mathbb{C} \setminus \mathbb{R}_+$ there is $C > 0$ such that*

$$\|(1 - P_h^* P_h)(H_0 - \mu)^{-1}\|_{\mathcal{B}(\mathcal{H})} \le C h^2, \quad h > 0.$$

Proof. We first note

$$\|(1 - P_h^* P_h)(H_0 - \mu)^{-1}\|_{\mathcal{B}(\mathcal{H})} = \|(1 - Q_h^* Q_h)(|2\pi\xi|^2 - \mu)^{-1}\|_{\mathcal{B}(\widehat{\mathcal{H}})},$$

where $\widehat{\mathcal{H}} = \mathcal{F}[\mathcal{H}] = L^2(\mathbb{R}^d)$. Let $f \in \widehat{\mathcal{H}}$ and $g = (|2\pi\xi|^2 - \mu)^{-1} f$. Then we have, by using the above lemma,

$$(1 - Q_h^* Q_h)g(\xi) = (1 - |\hat{\varphi}(h\xi)|^2)g(\xi) - \hat{\varphi}(h\xi) \sum_{n\neq 0} \overline{\hat{\varphi}(h\xi + n)} g(\xi + h^{-1}n).$$

For the first term in the right hand side, we observe by Assumption B that $|\hat{\varphi}(h\xi)| = 1$ if $|\xi| \le h^{-1}\delta$ with some $\delta > 0$. Then we learn

$$\|(1 - |\hat{\varphi}(h\xi)|^2)g(\xi)\|_{\widehat{\mathcal{H}}} \le \sup_{|\xi| > h^{-1}\delta} ||2\pi\xi|^2 - \mu|^{-1} \|f\|_{\widehat{\mathcal{H}}} \le C h^2 \|f\|_{\widehat{\mathcal{H}}}.$$

For the second term, we note that the terms in the summation vanish except for $n \in \{0, \pm 1\}^d \setminus 0$. Using the support condition of $\hat{\varphi}$ again, we learn that $\hat{\varphi}(h\xi)\overline{\hat{\varphi}(h\xi + n)} = 0$ if $|\xi + h^{-1}n| \le h^{-1}\delta$ with some $\delta > 0$. Thus we can use the same argument to show that the second term is bounded by Ch^2. □

Lemma 2.3. *For $\mu \in \mathbb{C} \setminus \mathbb{R}_+$ there is $C > 0$ such that*

$$\|(H_{0,h} - \mu)^{-1} P_h - P_h (H_0 - \mu)^{-1}\|_{\mathcal{B}(\mathcal{H}, \mathcal{H}_h)} \le C h^2, \quad h > 0.$$

Proof. Since P_h^* is isometric, it suffices to estimate

$$\begin{aligned}
&\|(H_{0,h}-\mu)^{-1}P_h - P_h(H_0-\mu)^{-1}\| \\
&\quad = \|P_h^*(H_{0,h}-\mu)^{-1}P_h - P_h^*P_h(H_0-\mu)^{-1}\| \\
&\quad = \|Q_h^*(H_{0,h}(\cdot)-\mu)^{-1}Q_h - Q_h^*Q_h(H_0(\cdot)-\mu)^{-1}\|.
\end{aligned}$$

Then we compute, for $f \in \mathcal{S}(\mathbb{R}^d)$,

$$\begin{aligned}
&(Q_h^*(H_{0,h}(\cdot)-\mu)^{-1}Q_h - Q_h^*Q_h(H_0(\cdot)-\mu)^{-1})f(\xi) \\
&\quad = \sum_{n\in\mathbb{Z}^d} \hat\varphi(h\xi)\overline{\hat\varphi(h\xi+n)}B_h(\xi+h^{-1}n)f(\xi+h^{-1}n),
\end{aligned}$$

where $B_h(\xi) := (H_{0,h}(\xi)-\mu)^{-1} - (H_0(\xi)-\mu)^{-1}$. We note, as well as in the proof of Lemma 2.2, $\hat\varphi(h\xi)\overline{\hat\varphi(h\xi-n)}$ vanishes except for $n \in \{0,\pm1\}^d$.

By the Taylor expansion, we have

$$|H_{0,h}(\xi) - H_0(\xi)| \le Ch^{-2}(h|\xi|)^4 = Ch^2|\xi|^4, \quad h>0,\ \xi\in\mathbb{R}^d.$$

On the other hand, if $h\xi \in \mathrm{supp}[\hat\varphi]$, we have $H_{0,h}(\xi) \ge c_0|\xi|^2$ with some $c_0 > 0$. These imply

$$|\hat\varphi(h\xi)|^2|B_h(\zeta)| \le Ch^2|\hat\varphi(h\xi)|^2, \quad h>0,\ \xi\in\mathbb{R}^d,$$

with some $C > 0$. On the support of $\hat\varphi(h\xi)\overline{\hat\varphi(h\xi+n)}$, $n \ne 0$, we have $H_{0,h}(\xi + h^{-1}n) \ge c_1h^{-2}$, $H_0(\xi+h^{-1}n) \ge c_1h^{-2}$ with some $c_1 > 0$, and hence $|B_h(\xi)| = O(h^2)$ as $h \to 0$. Combining these, we learn

$$\begin{aligned}
&|(Q_h^*(H_{0,h}(\cdot)-\mu)^{-1}Q_h - Q_h^*Q_h(H_0(\cdot)-\mu)^{-1})f(\xi)| \\
&\quad \le Ch^2 \sum_{n\in\{0,\pm1\}^d} |f(\xi+h^{-1}n)|, \quad \xi\in\mathbb{R}^d,
\end{aligned}$$

and the assertion follows. □

2.2. Relative boundedness. For simplicity, in this subsection, we suppose $V \ge 1$ without loss of generality. In particular, $V(x)^{-1}$ is uniformly bounded, and

$$c_1^{-1}V(x) \le V(y) \le c_1V(x) \quad \text{for } x,y\in\mathbb{R}^d,\ |x-y|\le 1. \tag{4}$$

Lemma 2.4. *Suppose Assumption A. Then V is H-bounded, and hence H_0 is also H-bounded.*

Proof. By the quadratic inequality, it is easy to observe $V^{1/2}$ and $(H_0+1)^{1/2}$ are $H^{1/2}$-bounded. Let $\eta \in C_0^\infty(\mathbb{R}^d)$ be a smooth cut-off function such that $\eta(x) \geq 0$, $\mathrm{supp}[\eta] \subset \{|x| \leq 1\}$ and $\int \eta(x)dx = 1$. Then we set $\widetilde{V} = \eta * V$, and we use $\widetilde{V} \geq 1$ as a smooth weight function comparable to V. By (4), we have

$$c_1^{-1} V(x) \leq \widetilde{V}(x) \leq c_1 V(x), \quad x \in \mathbb{R}^d.$$

By elementary computation, we also have

$$|\partial_x^\alpha \widetilde{V}(x)| \leq C_\alpha \widetilde{V}(x), \quad x \in \mathbb{R}^d$$

with some $C_\alpha > 0$, where $\alpha \in \mathbb{Z}_+^d$. It suffices to show $\widetilde{V}$ is H-bounded.

We write $W(x) = \widetilde{V}(x)^{1/2} \geq 1$, and compute

$$\begin{aligned}\widetilde{V} H^{-1} &= W H^{-1} W + W[W, H^{-1}] \\ &= (W H^{-1/2})(W H^{-1/2})^* + W H^{-1}[H, W] H^{-1}.\end{aligned}$$

The first term in the right hand side is bounded since W is $H^{1/2}$-bounded. We note

$$[H, W] = -\partial_x \cdot \partial_x W(x) - \partial_x W(x) \cdot \partial_x,$$

and ∂_x is $H^{1/2}$-bounded. We also note

$$|\partial_x W(x)| = \frac{1}{2} \widetilde{V}^{-1/2}(x) |\partial_x \widetilde{V}(x)| \leq C W(x)$$

with some $C > 0$, and hence $\partial_x W$ is $H^{1/2}$-bounded. Thus we learn

$$\begin{aligned}W H^{-1}[H, W] H^{-1} &= (W H^{-1/2})(\partial_x H^{-1/2})^* ((\partial_x W) H^{-1/2}) H^{-1/2} \\ &\quad - (W H^{-1/2})((\partial_x W) H^{-1/2})^* (\partial_x H^{-1/2}) H^{-1/2}\end{aligned}$$

is bounded, and hence $\widetilde{V}$ is H-bounded. □

Lemma 2.5. *Suppose Assumption A. Then V is H_h-bounded uniformly in $h > 0$, and hence $H_{0,h}$ is also H_h-bounded uniformly in $h > 0$.*

Proof. The proof is analogous to that of Lemma 2.4. We note $W = \widetilde{V}^{1/2}$ and $H_{0,h}^{1/2}$ are uniformly $H_h^{1/2}$-bounded. We similarly have

$$\widetilde{V} H_h^{-1} = (W H_h^{-1/2})(W H_h^{-1/2})^* + W H_h^{-1}[H_h, W] H_h^{-1},$$

and the first term in the right hand side is uniformly bounded.

For the second term, we recall that $H_{0,h} = \sum_{j=1}^{d} \nabla_j^* \nabla_j$, where

$$\nabla_j v(z) := \frac{1}{h}(v(z + he_j) - v(z)), \quad v \in \mathcal{H}_h.$$

Then we learn

$$[W, H_h] = \sum_{j=1}^{d}([\nabla_j, W]^* \nabla_j - \nabla_j^* [\nabla_j, W]).$$

By elementary computations, we can show $[\nabla_j, W]W^{-1}$ is bounded uniformly in h, and hence $WH_h^{-1}[H_h, W]H_h^{-1}$ is bounded uniformly in h.

Finally, $H_{0,h} = H_h - V$ is also H_h-bounded uniformly in h. □

2.3. Proof of Theorem 1.1

Lemma 2.6. *If G is a bounded uniformly continuous function, then*

$$\|GP_h - P_h G\|_{\mathcal{B}(\mathcal{H},\mathcal{H}_h)} \longrightarrow 0, \quad h \to 0.$$

If, in addition, G is uniformly Hölder continuous of order $\alpha \in (0, 1]$, then

$$\|GP_h - P_h G\|_{\mathcal{B}(\mathcal{H},\mathcal{H}_h)} \le C_\varepsilon h^{\alpha-\varepsilon}, \quad h > 0,$$

with any $\varepsilon > 0$.

Proof. We note

$$(GP_h - P_h G)u(z) = \int_{\mathbb{R}^d} K(x, z; h)u(x)dx,$$

where

$$K(x, z; h) := h^{-d}(G(z) - G(x))\overline{\varphi(h^{-1}(x - z))}.$$

By Schur's lemma, we have

$$\|GP_h - P_h G\| \le \sqrt{K_1 K_2},$$

where

$$K_1 = \sup_{z \in h\mathbb{Z}^d} \int_{\mathbb{R}^d} |K(x, z)|dx, \quad K_2 = \operatorname*{ess\,sup}_{x \in \mathbb{R}^d} h^d \sum_{z \in h\mathbb{Z}^d} |K(x, z)|.$$

We set

$$R(\delta) := \sup_{\substack{x,y \in \mathbb{R}^d, \\ |x-y|<\delta}} |G(x) - G(y)|$$

and we choose $n > d$. Then we have

$$\int_{\mathbb{R}^d} |K(x,z)|dx = \int_{|x-z|<\delta} |K(x,z)|dx + \int_{|x-z|\geq\delta} |K(x,z)|dx$$
$$\leq CR(\delta) \int_{|y|<\delta} \langle hy\rangle^{-n} h^{-d} dy + C \int_{|y|\geq\delta} \langle hy\rangle^{-n} h^{-d} dy$$
$$\leq C' R(\delta) + C' \langle h^{-1}\delta\rangle^{-(n-d)}.$$

By the same computation, we also have

$$h^d \sum_{z\in h\mathbb{Z}^d} |K(x,z)| \leq CR(\delta) + C\langle h^{-1}\delta\rangle^{-(n-d)}.$$

Combining these and setting $\delta = h^\gamma$ with $\gamma \in (0,1)$, we obtain

$$\|GP_h - P_h G\| \leq CR(h^\gamma) + Ch^{(1-\gamma)(n-d)}.$$

By the assumption, $R(\delta) \to 0$ as $\delta \to 0$, and we conclude the first assertion.

If G is uniformly Hölder continuous of order α, then $R(\delta) \leq C\delta^\alpha$, and hence the right hand side of the above estimate is $O(h^{\alpha\gamma}) + O(h^{(1-\gamma)(n-d)})$. We can choose γ very close to 1, and n very large so that $\alpha\gamma \geq \alpha - \varepsilon$ and $(1-\gamma)(n-d) \geq \alpha - \varepsilon$, and we have the second assertion. □

Proof of Theorem 1.1. We compute

$$\begin{aligned} &P_h^*(H_h - \mu)^{-1} P_h - (H-\mu)^{-1} \\ &\quad = P_h^*(H_h-\mu)^{-1} P_h - P_h^* P_h (H-\mu)^{-1} - (1 - P_h^* P_h)(H-\mu)^{-1} \\ &\quad = P_h^*(H_h-\mu)^{-1}(P_h H - H_h P_h)(H-\mu)^{-1} - (1-P_h^* P_h)(H-\mu)^{-1}. \end{aligned}$$

By Lemmas 2.2 and 2.4, we learn

$$\|(1 - P_h^* P_h)(H-\mu)^{-1}\| \leq Ch^2.$$

The other term is estimated as follows:

$$\begin{aligned} &\|(H_h-\mu)^{-1}(P_h H - H_h P_h)(H-\mu)^{-1}\| \\ &\quad \leq \|(H_h-\mu)^{-1}(P_h H_0 - H_{0,h} P_h)(H-\mu)^{-1}\| \\ &\qquad + \|(H_h-\mu)^{-1}(P_h V - V_h P_h)(H-\mu)^{-1}\| \\ &\quad \leq C\|(H_{0,h}-\mu)^{-1}(P_h H_0 - H_{0,h} P_h)(H_0-\mu)^{-1}\| \\ &\qquad + C\|(V-\mu)^{-1}(P_h V - V P_h)(V-\mu)^{-1}\| \\ &\quad = C\|(H_{0,h}-\mu)^{-1} P_h - P_h (H_0-\mu)^{-1}\| \\ &\qquad + C\|(V-\mu)^{-1} P_h - P_h (V-\mu)^{-1}\|, \end{aligned}$$

where we have used Lemmas 2.4 and 2.5 for the second inequality. The two terms in the right hand side are estimated using Lemmas 2.3 and 2.6, respectively, to complete the proof. □

Appendix A. Proof of some lemmas

Here we give the proofs of several technical lemmas.

Lemma A.1. *Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Then, the following are equivalent.*

(1) P_h^* *is isometric.*

(2) P_h *is a partial isometry onto* $\mathcal{H}_h$.

(3) $\int_{\mathbb{R}^d} \varphi(x)\overline{\varphi(x-n)}dx = \delta_{n,0}$ *for* $n \in \mathbb{Z}^d$.

(4) $\sum_{n\in\mathbb{Z}^d} |\hat{\varphi}(\xi+n)|^2 = 1$ *for* $\xi \in \mathbb{R}^d$, *where* $\hat{\varphi} = \mathcal{F}\varphi$.

Proof. (1) and (2) are equivalent by the standard properties of adjoint operators. Since (2) implies the orthonormality of the basis $\{h^{-\frac{d}{2}}\varphi_{h,z}\}_{z\in h\mathbb{Z}^d}$, we learn

$$\int_{\mathbb{R}^d} \varphi(x)\overline{\varphi(x-n)}dx = h^d \int_{\mathbb{R}^d} \varphi_{h,0}(x)\overline{\varphi_{h,hn}(x)}dx = \delta_{0,n},$$

which implies (3). For the equivalence of (3) and (4), we learn by Parseval's identity

$$\begin{aligned}\int_{\mathbb{R}^d} \varphi(x)\overline{\varphi(x-n)}dx &= \int_{\mathbb{R}^d} \hat{\varphi}(\xi)\overline{e^{-2\pi i n\cdot\xi}\hat{\varphi}(\xi)}d\xi \\ &= \int_{\mathbb{R}^d} e^{2\pi i n\cdot\xi}|\hat{\varphi}(\xi)|^2 d\xi \\ &= \int_{\mathbb{T}^d} \sum_{m\in\mathbb{Z}^d} e^{2\pi i n\cdot(\xi+m)}|\hat{\varphi}(\xi+m)|^2 d\xi \\ &= \int_{\mathbb{T}^d} e^{2\pi i n\cdot\xi} \sum_{m\in\mathbb{Z}^d} |\hat{\varphi}(\xi+m)|^2 d\xi,\end{aligned}$$

where $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d \simeq [0,1)^d$. Since $\{e^{2\pi i n\cdot\xi}\}_{n\in\mathbb{Z}^d}$ is a complete orthonormal basis of $L^2(\mathbb{T}^d)$, we conclude that (3) is equivalent to (4). □

Lemma A.2. *For normal operators A and B, $d_H(\sigma(A), \sigma(B)) \le \|A - B\|$.*

Proof. It suffices to show that $\mathrm{dist}(\mu, \sigma(B)) > \|A - B\|$ implies $\mu \notin \sigma(A)$. This condition implies $\|(A-B)(B-\mu)^{-1}\| < 1$, since $\|(B-\mu)^{-1}\| \le 1/\mathrm{dist}(\mu, \sigma(B))$ provided B is normal. Hence the Neumann series

$$\begin{aligned}(A-\mu)^{-1} &= (B-\mu+A-B)^{-1}\\ &= (B-\mu)^{-1}(1+(A-B)(B-\mu)^{-1})^{-1}\\ &= (B-\mu)^{-1}\sum_{n=0}^{\infty}(-1)^n((A-B)(B-\mu)^{-1})^n\end{aligned}$$

converges, and thus we learn $\mu \notin \sigma(A)$. □

Proof of Lemma 2.1. We compute

$$\begin{aligned}Q_h f(\zeta) &= h^d \sum_{z\in h\mathbb{Z}^d} e^{-2\pi i z\cdot\zeta}\Big(h^{-d}\int_{\mathbb{R}^d}\overline{\varphi_{h,z}(x)}\int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} f(\xi)d\xi dx\Big)\\ &= \sum_{z\in h\mathbb{Z}^d} e^{-2\pi i z\cdot\zeta}\int_{\mathbb{R}^d}\overline{\varphi_{h,z}(x)}\int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} f(\xi)d\xi dx\\ &= h^d\sum_{z\in h\mathbb{Z}^d}\int_{\mathbb{R}^d} e^{2\pi i z\cdot(\xi-\zeta)}\overline{\hat{\varphi}(h\xi)}f(\xi)d\xi\\ &= h^d\sum_{z\in h\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d}\int_{h^{-1}(\mathbb{T}^d+n)} e^{2\pi i z\cdot(\xi-\zeta)}\overline{\hat{\varphi}(h\xi)}f(\xi)d\xi\\ &= h^d\sum_{z\in h\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d}\int_{h^{-1}\mathbb{T}^d} e^{2\pi i z\cdot(\xi-\zeta)}\overline{\hat{\varphi}(h\xi+n)}f(\xi+h^{-1}n)d\xi\\ &= h^d\sum_{z\in h\mathbb{Z}^d}\int_{h^{-1}\mathbb{T}^d} e^{2\pi i z\cdot(\xi-\zeta)}\sum_{n\in\mathbb{Z}^d}\overline{\hat{\varphi}(h\xi+n)}f(\xi+h^{-1}n)d\xi\\ &= \sum_{n\in\mathbb{Z}^d}\overline{\hat{\varphi}(h\zeta+n)}f(\zeta+h^{-1}n).\end{aligned}$$

We have used the Fourier inversion formula for the last equality. We also have

$$\begin{aligned}\langle Q_h^* g, f\rangle &= \int_{h^{-1}\mathbb{T}^d}\sum_{n\in\mathbb{Z}^d} g(\zeta)\hat{\varphi}(h(\zeta+h^{-1}n))\overline{f(\zeta+h^{-1}n)}d\zeta\\ &= \int_{\mathbb{R}^d}\tilde{g}(\xi)\hat{\varphi}(h\xi)\overline{f(\xi)}d\xi,\end{aligned}$$

and this implies (3). □

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Received April 18, 2019

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