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Resolvent of the generator of the C_0-group with non-basis family of eigenvectors and sharpness of the XYZ theorem

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Abstract. The paper presents an explicit form of the resolvent and characterisation of the spectrum for the class of generators of C_0-groups with purely imaginary eigenvalues, clustering at $i\infty$, and complete minimal non-basis family of eigenvectors, constructed recently by the authors in [28]. The discrete Hardy inequality serves as the cornerstone for the proofs of the corresponding results. Furthermore, it is shown that the main result on the Riesz basis property for invariant subspaces of the generator of the C_0-group (the XYZ theorem), obtained a decade ago by G. Q. Xu, S. P. Yung and H. Zwart in [31] and [32], is sharp.

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1. Introduction

Problems of spectral theory for nonselfadjoint (NSA) operators attract more and more growing interest of experts in different fields of mathematics and natural sciences, see, e.g., [4], [5], [6], [7], [15], [16], [17] [23], [28], [31], [32] and the references therein. This is primarily caused by the recent progress in theoretical physics of non-Hermitian systems [4] on the one hand, and, on the other, by the fact that many mathematical models of dynamical processes lead to the study of

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linear evolution equations

$$\begin{cases} \dot{x}(t) = Ax(t), & t \geq 0, \\ x(0) = x_0 \in H, \end{cases} \tag{1}$$

in Hilbert spaces H with unbounded NSA operator A.

In last years NSA Schrödinger operators are studied very intensively, see [7], [8], [15], [16], [17], [23] and, especially, [4], [5], and [6]. In 2000 E.B. Davies [7] studied NSA anharmonic oscillators

$$\mathcal{L}_\alpha = -\frac{d^2}{dx^2} + c|x|^\alpha, \tag{2}$$

defined on $L_2(\mathbb{R})$ as the closure of the associated quadratic form defined on $C_0^\infty(\mathbb{R})$, where $\alpha > 0$, $c \in \mathbb{C} \setminus \mathbb{R}$, $|\arg c| < C(\alpha)$. He proved that for all $\alpha > 0$ the spectrum of $\mathcal{L}_\alpha$ consists of discrete simple eigenvalues and, if we denote them in nondecreasing modulus order by λ_n, $|\lambda_n| \to \infty$, and consider corresponding one-dimensional spectral projections P_n, then the norms $\|P_n\|$ grow more rapidly than any polynomial of n as $n \to \infty$, see [6] and [7]. Davies called operators with such spectral behavior by spectrally wild ones. A family of eigenvectors of spectrally wild operator, although can be complete and minimal in a space, cannot constitute a Schauder basis. E.g., the eigenvectors of $\mathcal{L}_\alpha$, where $\Re(c) > 0$, are dense in $L_2(\mathbb{R})$ if either $\alpha \geq 1$, or $0 < \alpha < 1$ and $|\arg c| < \alpha\pi/2$, see [7]. We recall that a sequence $\{\phi_n\}_{n=1}^\infty$ of a Banach space X forms a Schauder basis of X provided each $x \in X$ has a unique norm-convergent expansion $x = \sum_{n=1}^\infty c_n \phi_n$.

In 2004 E.B. Davies and A.B.J. Kuijlaars proved that spectral projections P_n of the operator $\mathcal{L}_2$, where $c = e^{i\theta}$, $0 < |\theta| < \pi$, grow exponentially [8]:

$$\lim_{n\to\infty} \frac{1}{n} \ln \|P_n\| = 2\Re\{f(r(\theta)e^{\frac{i\theta}{4}})\},$$

where $f(z) = \ln(z + g(z)) - zg(z)$, $g(z) = (z^2 - 1)^{1/2}$, $r(\theta) = (2\cos(\theta/2))^{-1/2}$.

These studies were continued by R. Henry, who determined exponential growth rates of spectral projections of the so-called complex Airy operator $\mathcal{L}_1$, where $c = e^{i\theta}$, $0 < |\theta| < \frac{3\pi}{4}$, and anharmonic oscillators $\mathcal{L}_{2k}$, $k \in \mathbb{N}$, where $c = e^{i\theta}$, $0 < |\theta| < \frac{(k+1)\pi}{2k}$, see [15], [16]. Moreover, in [17] Henry studied spectral projections P_n of the complex cubic oscillator $\mathcal{C}_\beta = -\frac{d^2}{dx^2} + ix^3 + i\beta x$, $\beta \geq 0$ with domain $H^2(\mathbb{R}) \cap L_2(\mathbb{R}; x^6 dx) \subset L_2(\mathbb{R})$ and showed that for all $\beta \geq 0$ we have $\lim_{n\to\infty} \frac{1}{n} \ln \|P_n\| = \frac{\pi}{\sqrt{3}}$.

Recently, B. Mityagin et al. considered NSA perturbations of selfadjoint Schrödinger operators with single-well potentials and demonstrated that norms of spectral projections P_n of these operators can grow at intermediate levels, from arbitrary slowly to exponentially fast [23]. In particular, natural classes of operators with projections obeying $\lim_{n\to\infty} \frac{1}{n^\gamma} \ln \|P_n\| = C$, where $C \in (0,\infty)$ and $\gamma \in (0,1)$, were found.

On the other hand, in "good" situation, i.e. when the operator A has a Riesz basis of A-invariant subspaces, the system (1) can be split into countable family of subsystems (each subsystem lives in a corresponding A-invariant subspace) and we can make conclusions on the behavior of (1) on the basis of the study of its subsystems, see, e.g., [22], [24], [25], [26], [27], [31], [32], and the references therein. That is why Riesz bases are convenient tools of infinite-dimensional linear systems theory and the following question is important.

Question 1. *Which conditions are sufficient to guarantee that A has a Riesz basis of eigenvectors* (*or A-invariant subspaces*)?

For equivalent definitions and stability properties of Schauder bases of subspaces (Schauder decompositions) and Riesz bases of subspaces we refer to [19], [20], [21], and the references therein.

A number of recent papers are devoted to Question 1 in the case when A is a perturbation of selfadjoint, nonnegative operator with discrete spectrum, including perturbations of harmonic oscillator type operators. We refer to [23], Section 4.3, for the brief overview of the corresponding results.

In the study of (in fact, quite old) Question 1 a breakthrough was made by G. Q. Xu, S. P. Yung and H. Zwart – the XYZ Theorem:

XYZ Theorem ([31] and [32]). *Suppose that the following three conditions hold*:

(1) *the operator A generates the C_0-group on a Hilbert space H*;

(2) *the set of eigenvalues $\{\lambda_n\}_{n=1}^\infty$ of A is a union of $K < \infty$ interpolation sequences Λ_k, $1 \le k \le K$. In other words, $\{\lambda_n\}_{n=1}^\infty = \bigcup_{k=1}^K \Lambda_k$, where*

$$\min_k \inf_{\lambda_n,\lambda_m \in \Lambda_k : n\neq m} |\lambda_n - \lambda_m| > 0; \tag{3}$$

(3) *generalized eigenvectors* (*eigen- and rootvectors*) *of A form a dense set.*

Then there exists a certain sequence of (*multidimensional, if $K > 1$*) *spectral projections $\{P_n\}_{n=1}^\infty$ of A such that $\{P_nH\}_{n=1}^\infty$ forms a Riesz basis of subspaces in H with $\sup_{n\in\mathbb{N}} \dim P_nH \le K$.*

We note that operators satisfying conditions 1–3 of the XYZ Theorem naturally arise from applications, e.g., in the analysis of neutral type systems [24], [25], [26], and [27].

However, it was totally unclear: what if the eigenvalues of A lie in a strip, parallel to imaginary axis, and do not satisfy the condition of separation (3)? In particular:

Question 2. *Is it possible to construct the unbounded generator of the C_0-group with eigenvalues $\{\lambda_n\}_{n=1}^{\infty} \subset i\mathbb{R}$ not satisfying* (3) *and dense family of eigenvectors, which does not form a Riesz basis*?

In [28] the authors obtained an affirmative answer to the Question 2 and presented the class of infinitesimal operators with such eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ and complete minimal family of eigenvectors, which, however, does not form even a Schauder basis. To formulate the corresponding result we need to consider the following classes of sequences.

Definition 1.1 ([28]). Let $k \in \mathbb{N}$ and Δ stands for the difference operator. Then we define classes of sequences

$$\mathcal{S}_k = \{\{f(n)\}_1^{\infty} \subset \mathbb{R}\colon \lim_{n\to\infty} f(n) = +\infty; \{n^j \Delta^j f(n)\}_1^{\infty} \in \ell_{\infty} \text{ for } 1 \le j \le k\}.$$

Clearly for all k we have that $\{\ln n\}_{n=1}^{\infty} \in \mathcal{S}_k$ but $\{\sqrt{n}\}_{n=1}^{\infty} \notin \mathcal{S}_k$.

Theorem 1.2 ([28]). *Let $\{e_n\}_{n=1}^{\infty}$ be a Riesz basis of a Hilbert space H and $k \in \mathbb{N}$.*

(1) *The space $H_k(\{e_n\}) = \{x = (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n\colon \{c_n\}_{n=1}^{\infty} \in \ell_2(\Delta^k)\}$ is a separable Hilbert space. Here $(\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n$ denotes a formal series and $\ell_2(\Delta^k) = \{s = \{\alpha_n\}_{n=1}^{\infty} \subset \mathbb{C}\colon \Delta^k s \in \ell_2\}$.*

(2) *The sequence $\{e_n\}_{n=1}^{\infty}$ is dense and minimal in $H_k(\{e_n\})$, but it is not uniformly minimal in $H_k(\{e_n\})$. Hence $\{e_n\}_{n=1}^{\infty}$ does not form a Schauder basis of $H_k(\{e_n\})$.*

(3) *The operator $A_k\colon H_k(\{e_n\}) \supset D(A_k) \mapsto H_k(\{e_n\})$, defined by*

$$A_k x = A_k (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n = (\mathfrak{f}) \sum_{n=1}^{\infty} i f(n) \cdot c_n e_n,$$

where $\{f(n)\}_{n=1}^{\infty} \in \mathcal{S}_k$, with domain

$$D(A_k) = \left\{x = (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n \in H_k(\{e_n\})\colon \{f(n) \cdot c_n\}_1^{\infty} \in \ell_2(\Delta^k)\right\}, \tag{4}$$

generates the C_0-group $\{e^{A_k t}\}_{t\in\mathbb{R}}$ on $H_k(\{e_n\})$, which acts for every $t \in \mathbb{R}$ by the formula

$$e^{A_k t} x = e^{A_k t} (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n = (\mathfrak{f}) \sum_{n=1}^{\infty} e^{itf(n)} c_n e_n.$$

Similar results take place for the case of operators with the same spectral behaviour on certain Banach spaces $\ell_{p,k}(\{e_n\})$, $p > 1$, $k \in \mathbb{N}$, see [28]. Note that if we take, e.g., $\{f(n) = \sqrt{n}\}_{n=1}^{\infty} \notin \mathcal{S}_k$ for any k, and define the operator A_1 on $H_1(\{e_n\})$ as in Theorem 1.2, then A_1 will not generate a C_0-semigroup on $H_1(\{e_n\})$, see [28], Proposition 7, Proposition 8.

The paper has two purposes. The first is to obtain an explicit form of the resolvent and to characterise the spectrum of the class of generators A_k of C_0-groups from Theorem 1.2. We note that Theorem 1.2 together with the XYZ Theorem show that Theorem 1.1 from [32] dealing with the case of simple eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ in the XYZ Theorem, satisfying

$$\inf_{n \neq m} |\lambda_n - \lambda_m| > 0,$$

is sharp, see also Example 1.3 in [32]. The second purpose of the paper is to demonstrate that Theorem 1.2 means that the XYZ Theorem is also sharp, see Section 2.

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2. The XYZ Theorem is sharp

We will use the notation from [28], see also Theorem 1.2. By Proposition 3 of [28] we have that for any $k \in \mathbb{N}$ the sequence $\{e_n\}_{n=1}^{\infty}$ is dense and minimal in $H_k(\{e_n\})$, but it is not uniformly minimal in $H_k(\{e_n\})$. It means that for each $n \in \mathbb{N}$

$$\varrho(e_n, \overline{\operatorname{Lin}}\,\{e_j\}_{j \neq n}) > 0,$$

but

$$\inf_{n \in \mathbb{N}} \varrho(e_n, \overline{\operatorname{Lin}}\,\{e_j\}_{j \neq n}) = 0,$$

where $\varrho(x, Y)$ denotes a standard distance from the point x to a set Y, defined by

$$\varrho(x, Y) = \inf_{y \in Y} \|x - y\|.$$

Let $\{\phi_n\}_{n=1}^{\infty}$ be dense and minimal sequence in a Hilbert space H, but is not uniformly minimal in H. Then it can happen that there exists a splitting of $\{\phi_n\}_{n=1}^{\infty}$ into infinite number of disjoint groups with at most $K < \infty$ elements in each of them, i.e.

$$\{\phi_n\}_{n=1}^{\infty} = \{\{\phi_j\}_{j \in E_n}\}_{n=1}^{\infty},$$

where

$$\bigcup_{n=1}^{\infty} E_n = \mathbb{N}, \quad E_n \cap E_m = \emptyset \text{ if } n \neq m, \quad |E_n| \leq K \text{ for all } n, \tag{5}$$

such that the corresponding sequence of subspaces $\{\text{Lin}\{\phi_n\}_{n \in E_n}\}_{n=1}^{\infty}$ constitute a Riesz basis of subspaces of H with uniform bound of dimensions of all subspaces not exceeding K. See e.g. Example 1.3 in [32] for details. In order to show that the XYZ Theorem is sharp we will prove that this situation is impossible for our construction from Theorem 1.2. More precisely, thereby we demonstrate that if the eigenvalues of the generator of the C_0-group in a Hilbert space do not satisfy (3), then the conclusion of the XYZ Theorem can be false. Furthermore, we will prove a little more.

Theorem 2.1. *Let $k \in \mathbb{N}$ and $\{e_n\}_{n=1}^{\infty} \subset H_k(\{e_n\})$ be a sequence from Theorem 1.2. Suppose that $\{E_n\}_{n=1}^{\infty}$ is an arbitrary decomposition of $\mathbb{N}$ into disjoint sets, i.e.*

$$\bigcup_{n=1}^{\infty} E_n = \mathbb{N}, \quad E_n \cap E_m = \emptyset, n \neq m.$$

Then $\{\overline{\text{Lin}}\{e_j\}_{j \in E_n}\}_{n=1}^{\infty}$ does not form a Schauder decomposition of $H_k(\{e_n\})$.

Proof. Fix $k \in \mathbb{N}$ and assume the opposite, i.e. let there exists a decomposition of $\mathbb{N}$ into disjoint sets,

$$\bigcup_{n=1}^{\infty} E_n = \mathbb{N}, \quad E_n \cap E_m = \emptyset, \; n \neq m,$$

such that $\{\mathfrak{M}_n = \overline{\text{Lin}}\{e_j\}_{j \in E_n}\}_{n=1}^{\infty}$ constitutes a Schauder decomposition of $H_k(\{e_n\})$. Then, by the definition of the Schauder decomposition, every $x \in H_k(\{e_n\})$ can be uniquely represented in a series

$$x = \sum_{n=1}^{\infty} x_n,$$

where $x_n \in \mathfrak{M}_n$ for each $n \in \mathbb{N}$, and there exists an associated sequence of coordinate linear projections $\{P_n\}_{n=1}^{\infty}$ defined by $P_n x = P_n \sum_{m=1}^{\infty} x_m = x_n$, where $x_n \in \mathfrak{M}_n$, $n \in \mathbb{N}$. It follows that for every $n, j \in \mathbb{N}$

$$P_n e_j = \begin{cases} e_j, & j \in E_n, \\ 0, & j \notin E_n. \end{cases} \tag{6}$$

Consider an element $x^* = (\mathfrak{f}) \sum_{j=1}^{\infty} e_j \in H_k(\{e_n\})$. Then, taking into account (6), we have that for every $n \in \mathbb{N}$

$$P_n x^* = P_n\Big((\mathfrak{f}) \sum_{j=1}^{\infty} e_j\Big) = (\mathfrak{f}) \sum_{j \in E_n} e_j. \tag{7}$$

We recall that the norm in a Hilbert space $H_k(\{e_n\})$ is defined by

$$\begin{aligned} \|x\|_k &= \Big\|(\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n\Big\|_k \\ &= \Big\| \sum_{n=1}^{\infty} (c_n - C_k^1 c_{n-1} + \cdots + (-1)^{k+1} C_k^{k-1} c_{n-k+1} + (-1)^k c_{n-k}) e_n \Big\|, \end{aligned}$$

where $x = (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n \in H_k(\{e_n\})$, C_k^m are binomial coefficients, $\|\cdot\|$ denotes the norm in an initial Hilbert space H and $c_{1-j} = 0$ for all $j \in \mathbb{N}$, see [28]. Since $\{e_n\}_{n=1}^{\infty}$ is a Riesz basis of H (see Theorem 1.2), there exist two constants $M \geq m > 0$ such that for every $y = \sum_{n=1}^{\infty} \alpha_n e_n \in H$ we have

$$m\|y\|^2 \leq \sum_{n=1}^{\infty} |\alpha_n|^2 \leq M\|y\|^2. \tag{8}$$

By virtue of (7) and (8) we obtain that for every $n \in \mathbb{N}$

$$\|P_n x^*\|_k^2 = \Big\|(\mathfrak{f}) \sum_{j \in E_n} e_j\Big\|_k^2 = \Big\|(\mathfrak{f}) \sum_{j=1}^{\infty} \xi_j(n) e_j\Big\|_k^2 \geq \frac{(2^{k-1})^2}{M},$$

where for every $n, j \in \mathbb{N}$

$$\xi_j(n) = \begin{cases} 1, & j \in E_n, \\ 0, & j \notin E_n. \end{cases}$$

Thus $\|P_n x^*\|_k \nrightarrow 0$ as $n \to \infty$, which means that x^* can not be represented in a convergent series

$$\sum_{n=1}^{\infty} x_n^* = \sum_{n=1}^{\infty} P_n x^*,$$

where $x_n^* \in \mathfrak{M}_n$ for each $n \in \mathbb{N}$. So we arrived at a contradiction with the definition of the Schauder decomposition. □

Theorem 2.1 leads to the following.

Corollary 2.2. *The XYZ Theorem is sharp. None of its conditions can be removed.*

Proof. Indeed, condition 3 of the XYZ Theorem obviously can not be removed. If one removes condition 2 but conditions 1 and 3 are fulfilled, then, by virtue of Theorem 2.1, the class of counterexamples are given by Theorem 1.2.

Let us remove condition 1. Suppose that conditions 2 and 3 are satisfied, operator A does not generate the C_0-group on H but A generates the C_0-semigroup on H. Then the counterexample is given as follows.

Let $\{\phi_n\}_{n=1}^{\infty}$ be a bounded non-Riesz basis of H, i.e. bounded conditional basis. It means that $\{\phi_n\}_{n=1}^{\infty}$ constitutes a Schauder basis of H, but does not form a Riesz basis of H, and we have

$$0 < \inf_n \|\phi_n\|, \quad \sup_n \|\phi_n\| < \infty.$$

Since $\{\phi_n\}_{n=1}^{\infty}$ is a Schauder basis of H, every $x \in H$ has a unique norm-convergent expansion

$$x = \sum_{n=1}^{\infty} c_n \phi_n.$$

Then we define the operator $A : H \supset D(A) \mapsto H$ as follows,

$$Ax = A \sum_{n=1}^{\infty} c_n \phi_n = -\sum_{n=1}^{\infty} n c_n \phi_n,$$

where

$$D(A) = \left\{ x = \sum_{n=1}^{\infty} c_n \phi_n \in H : \sum_{n=1}^{\infty} n c_n \phi_n \in H \right\}.$$

It can be easily shown that A generates the C_0-semigroup on H, the spectrum of A is pure point and consists of simple eigenvalues $-n$, $n \in \mathbb{N}$, with corresponding eigenvectors $\{\phi_n\}_{n=1}^{\infty}$, see, e.g., [14]. Finally, it is not hard to prove that, if $\{E_n\}_{n=1}^{\infty}$ is a decomposition of $\mathbb{N}$ into disjoint sets with at most K elements in each of them, such that (5) holds, then

$$\{\overline{\mathrm{Lin}}\, \{\phi_j\}_{j \in E_n}\}_{n=1}^{\infty}$$

does not form a Riesz basis of subspaces of H. □

For our construction of generators of C_0-groups with complete minimal non-basis family of eigenvectors in special classes of Banach spaces $\ell_{p,k}(\{e_n\})$, $p > 1$, $k \in \mathbb{N}$ (see Theorem 16 in [28]), we have a result similar to the Theorem 2.1. Here $\{e_n\}_{n=1}^{\infty}$ denotes an arbitrary symmetric basis of an initial Banach space ℓ_p, $p \geq 1$. Recall that Schauder basis $\{e_n\}_{n=1}^{\infty}$ is called symmetric provided any its permutation $\{e_{\theta(n)}\}_{n=1}^{\infty}$, $\theta(n)\colon \mathbb{N} \mapsto \mathbb{N}$, also forms a Schauder basis, equivalent to $\{e_n\}_{n=1}^{\infty}$. For any $p \geq 1$ and $k \in \mathbb{N}$ the space

$$\ell_{p,k}(\{e_n\}) = \Big\{x = (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n \colon \{c_n\}_{n=1}^{\infty} \in \ell_p(\Delta^k)\Big\},$$

where $(\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n$ also denotes a formal series and

$$\ell_p(\Delta^k) = \{s = \{\alpha_n\}_{n=1}^{\infty} \subset \mathbb{C}\colon \Delta^k s \in \ell_p\},$$

is a separable Banach space, isomorphic to ℓ_p, see [28]. If $p > 1$, then, according to Proposition 5 of [28], the sequence $\{e_n\}_{n=1}^{\infty}$ is dense and minimal in $\ell_{p,k}(\{e_n\})$, $p > 1$, $k \in \mathbb{N}$, but it is not uniformly minimal in $\ell_{p,k}(\{e_n\})$, so $\{e_n\}_{n=1}^{\infty}$ does not form a Schauder basis of $\ell_{p,k}(\{e_n\})$. Using similar arguments we obtain the following result, analogous to Theorem 2.1.

Theorem 2.3. *Let $k \in \mathbb{N}$ and $\{e_n\}_{n=1}^{\infty} \subset \ell_{p,k}(\{e_n\})$, $p \geq 1$, be a sequence defined above. Suppose that $\{E_n\}_{n=1}^{\infty}$ is an arbitrary decomposition of $\mathbb{N}$ into disjoint sets. Then $\{\overline{\mathrm{Lin}}\{e_j\}_{j\in E_n}\}_{n=1}^{\infty}$ does not form a Schauder decomposition of $\ell_{p,k}(\{e_n\})$.*

3. The resolvent and spectrum of generators of C_0-groups with non-basis family of eigenvectors

Recall that if $p > 1$ and $a_n \geq 0$ for $n \in \mathbb{N}$, then the discrete Hardy inequality states that

$$\sum_{n=1}^{\infty} \Big(\frac{1}{n} \sum_{k=1}^{n} a_k\Big)^p < \Big(\frac{p}{p-1}\Big)^p \sum_{n=1}^{\infty} a_n^p \tag{9}$$

with the exception of the case when $a_n = 0$ for all $n \in \mathbb{N}$. Moreover, the constant $(\frac{p}{p-1})^p$ is the best possible.

The following theorem provides an explicit form of the resolvent for the class of generators $A_k\colon H_k(\{e_n\}) \supset D(A_k) \mapsto H_k(\{e_n\})$, $k \in \mathbb{N}$, of C_0-groups from Theorem 1.2 and the description of the spectrum $\sigma(A_k)$ of generators A_k.

Theorem 3.1. *Let $k \in \mathbb{N}$ and A_k be the operator from Theorem 1.2. Then*:

(i) $\sigma(A_k) = \sigma_p(A_k) = \{if(n)\}_1^{\infty}$.

(ii) *The resolvent of A_k is given by the following formula*:

$$(A_k - \lambda I)^{-1} x = (\mathfrak{f}) \sum_{n=1}^{\infty} \frac{c_n e_n}{if(n) - \lambda}, \quad \lambda \in \rho(A_k) = \mathbb{C} \setminus \{if(n)\}_1^{\infty}, \tag{10}$$

where $x = (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n \in H_k(\{e_n\})$.

Proof. First we prove the Theorem for the case $k = 1$.

Let us prove that $\rho(A_1) = \mathbb{C} \setminus \{if(n)\}_1^\infty$ is the resolvent set of the operator A_1 and the operator

$$A(\lambda)x = (\mathfrak{f}) \sum_{n=1}^{\infty} \frac{1}{if(n) - \lambda} c_n e_n,$$

where $\lambda \neq if(n)$ for all $n \in \mathbb{N}$ and $x = (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n \in H_1(\{e_n\})$, is the resolvent of A_1.

To this end denote $\lambda_n = if(n)$, $n \in \mathbb{N}$. Recall that the norm in Hilbert space $H_1(\{e_n\})$ is

$$\|x\|_1 = \left\| (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n \right\|_1 = \left\| \sum_{n=1}^{\infty} (c_n - c_{n-1}) e_n \right\|,$$

where $\|\cdot\|$ denotes the norm in an initial Hilbert space H and $c_0 = 0$, see [28]. Observe that

$$\begin{aligned}
\|A(\lambda)x\|_1^2 &= \left\| (\mathfrak{f}) \sum_{n=1}^{\infty} \frac{c_n e_n}{\lambda_n - \lambda} \right\|_1^2 = \left\| \frac{c_1 e_1}{\lambda_1 - \lambda} + \sum_{n=2}^{\infty} \left(\frac{c_n}{\lambda_n - \lambda} - \frac{c_{n-1}}{\lambda_{n-1} - \lambda} \right) e_n \right\|^2 \\
&= \left\| \frac{c_1 e_1}{\lambda_1 - \lambda} + \sum_{n=2}^{\infty} \left(\frac{c_n}{\lambda_n - \lambda} - \frac{c_{n-1}}{\lambda_n - \lambda} + \frac{c_{n-1}}{\lambda_n - \lambda} - \frac{c_{n-1}}{\lambda_{n-1} - \lambda} \right) e_n \right\|^2 \\
&\leq \left(\left\| \sum_{n=1}^{\infty} \frac{c_n - c_{n-1}}{\lambda_n - \lambda} e_n \right\| + \left\| \sum_{n=2}^{\infty} \left(\frac{1}{\lambda_n - \lambda} - \frac{1}{\lambda_{n-1} - \lambda} \right) c_{n-1} e_n \right\| \right)^2 \\
&= (\Sigma_1 + \Sigma_2)^2 \leq 2\Sigma_1^2 + 2\Sigma_2^2.
\end{aligned}$$

Now consider

$$\lambda \colon \inf_{n \in \mathbb{N}} |\lambda_n - \lambda| \geq a > 0.$$

Since $\{e_n\}_{n=1}^\infty$ is a Riesz basis of a Hilbert space H (see Theorem 1.2), there exist two constants $M \geq m > 0$ such that for every $y = \sum_{n=1}^{\infty} \alpha_n e_n \in H$ we have

$$m\|y\|^2 \leq \sum_{n=1}^{\infty} |\alpha_n|^2 \leq M\|y\|^2. \tag{11}$$

Applying (11) we obtain that

$$\Sigma_1^2 \leq \frac{1}{m} \sum_{n=1}^{\infty} \frac{|c_n - c_{n-1}|^2}{|\lambda_n - \lambda|^2} \leq \frac{1}{ma^2} \sum_{n=1}^{\infty} |c_n - c_{n-1}|^2 \leq \frac{M}{ma^2} \|x\|_1^2. \tag{12}$$

Since

$$\frac{1}{\lambda_n - \lambda} - \frac{1}{\lambda_{n-1} - \lambda} = \frac{\lambda_{n-1} - \lambda_n}{(\lambda_n - \lambda)(\lambda_{n-1} - \lambda)}$$

for $n \geq 2$, by virtue of (11) we conclude that

$$\Sigma_2^2 \leq \frac{1}{m} \sum_{n=2}^{\infty} \frac{|\lambda_{n-1} - \lambda_n|^2 |c_{n-1}|^2}{|\lambda_n - \lambda|^2 |\lambda_{n-1} - \lambda|^2} \leq \frac{1}{ma^4} \sum_{n=2}^{\infty} \frac{|c_{n-1}|^2}{n^2} n^2 |\Delta f(n)|^2.$$

Note that $\{f(n)\}_{n=1}^{\infty} \in \mathcal{S}_1$, hence $n|\Delta f(n)| \in \ell_\infty$ by the definition of the class $\mathcal{S}_1$, see Definition 1.1. Denote

$$C = \sup_{n \in \mathbb{N}} n|\Delta f(n)|.$$

Then, since for $n \geq 2$

$$c_{n-1} = \sum_{j=1}^{n-1} (c_j - c_{j-1}),$$

we obtain that

$$\begin{aligned} \Sigma_2^2 &\leq \frac{C^2}{ma^4} \sum_{n=2}^{\infty} \frac{|c_{n-1}|^2}{n^2} \\ &= \frac{C^2}{ma^4} \sum_{n=2}^{\infty} \left(\frac{1}{n} \left| \sum_{j=1}^{n-1} (c_j - c_{j-1}) \right| \right)^2 \\ &\leq \frac{C^2}{ma^4} \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{j=1}^{n} |c_j - c_{j-1}| \right)^2. \end{aligned}$$

By virtue of the Hardy inequality (9) for $p = 2$ and (11) we obtain

$$\Sigma_2^2 \leq \frac{4C^2}{ma^4} \sum_{n=1}^{\infty} |c_n - c_{n-1}|^2 \leq \frac{4MC^2}{ma^4} \|x\|_1^2.$$

Combining this with (12) we finally arrive at the estimate

$$\|A(\lambda)x\|_1^2 \leq \left(\frac{2}{a^2} + \frac{8C^2}{a^4} \right) \frac{M}{m} \|x\|_1^2 \tag{13}$$

and $A(\lambda) \in [H_1(\{e_n\})]$, i.e. $A(\lambda)$ is a linear bounded operator.

Further let $V_N = \mathrm{Lin}\{e_n\}_{n=1}^{N}$. Then for $x_N \in V_N$ we have

$$(A_1 - \lambda I)A(\lambda)x_N = A(\lambda)(A_1 - \lambda I)x_N = x_N. \tag{14}$$

Note that A_1 is closed as the generator of the C_0-group by Theorem 1.2 and

$$\overline{\mathrm{Lin}}\{V_n\}_{n=1}^{\infty} = \overline{\mathrm{Lin}}\{e_n\}_{n=1}^{\infty} = H_1(\{e_n\}).$$

Fix arbitrary $x \in H_1(\{e_n\})$. Then we can find the sequence $\{x_N\}_{N=1}^{\infty}$ with $x_N \in V_N$ such that $x_N \to x$ as $N \to \infty$. Since $A(\lambda)$ is bounded, we have that

$$z_N = A(\lambda)x_N \to A(\lambda)x = z.$$

By virtue of (14) we obtain that

$$(A_1 - \lambda I)z_N = x_N \to x \quad \text{and} \quad z_N \to z$$

as $N \to \infty$. Since A_1 is closed this implies that $z \in D(A_1)$ and

$$(A_1 - \lambda I)z = x.$$

Thus for all $x \in H_1(\{e_n\})$ we have that

$$A(\lambda)x \in D(A_1) \quad \text{and} \quad (A_1 - \lambda I)A(\lambda)x = x, \tag{15}$$

so $A(\lambda)$ is the right inverse of $A_1 - \lambda I$.

Now take $z \in D(A_1)$ and consider $x = (A_1 - \lambda I)z$. Then by (15) we have that

$$x = (A_1 - \lambda I)A(\lambda)x = (A_1 - \lambda I)A(\lambda)(A_1 - \lambda I)z.$$

Consequently,

$$(A_1 - \lambda I)(z - A(\lambda)(A_1 - \lambda I)z) = x - x = 0. \tag{16}$$

To show that $\ker(A_1 - \lambda I) = \{0\}$ for $\lambda \neq \lambda_n$, $n \in \mathbb{N}$, we suppose that

$$(A_1 - \lambda I)x = (A_1 - \lambda I)(\mathfrak{f})\sum_{n=1}^{\infty} c_n e_n = (\mathfrak{f})\sum_{n=1}^{\infty}(\lambda_n - \lambda)c_n e_n = 0.$$

It follows that

$$|(\lambda_1 - \lambda)c_1|^2 + \sum_{n=2}^{\infty} |(\lambda_n - \lambda)c_n - (\lambda_{n-1} - \lambda)c_{n-1}|^2 = 0.$$

Since $\lambda \neq \lambda_1$, we obtain that $c_1 = 0$ and for any $n \geq 2$,

$$(\lambda_n - \lambda)c_n - (\lambda_{n-1} - \lambda)c_{n-1} = 0. \tag{17}$$

Since $\lambda \neq \lambda_n$, $n \in \mathbb{N}$, from (17) we consistently get $c_n = 0$ for all $n \geq 2$, and hence $x = 0$.

Since $\ker(A_1 - \lambda I) = \{0\}$ for $\lambda \neq \lambda_n$, $n \in \mathbb{N}$, then by (16) we have for every $z \in D(A_1)$

$$z = A(\lambda)(A_1 - \lambda I)z,$$

and, combining this equality with (15), we infer that $\lambda \in \rho(A_1)$ and $A(\lambda) = (A_1 - \lambda I)^{-1}$ is the resolvent of A_1. Besides, we proved that

$$\{\lambda \in \mathbb{C}: \lambda \neq if(n),\, n \in \mathbb{N}\} \subset \rho(A_1).$$

Finally we observe that since $\lambda_n \in \sigma(A_1)$, $n \in \mathbb{N}$, A_1 is closed, the spectrum of closed operator is closed set and the set $\{if(n)\}_1^\infty$ contains all its limit points, then $\sigma(A_1) = \sigma_p(A_1) = \{if(n)\}_1^\infty$ and

$$\rho(A_1) = \{\lambda \in \mathbb{C}: \lambda \neq if(n),\, n \in \mathbb{N}\} = \mathbb{C} \setminus \sigma(A_1).$$

The proof in the case $k \geq 2$ is based on a combination of ideas of the proof for the case $k = 1$ with technical combinatorial elements like in the proof of Theorem 11 from [28] and can be performed similarly to the above. □

Remark 3.2. Operators with simple eigenvalues $\{\lambda_n\}_{n=1}^\infty$ not satisfying the condition 2 of the XYZ Theorem and non-basis family of eigenvectors are considered in recent applications. In [2] the author study the stability of the normal state of superconductors in the presence of electric currents in the large domain limit using the time-dependent Ginzburg–Landau model. The study involves spectral analysis of the operator $\mathcal{L}: D(\mathcal{L}) \mapsto L_2(\mathbb{R}, \mathbb{C})$, defined by

$$\mathcal{L} = -\frac{d^2}{dx^2} + ix,$$

where $D(\mathcal{L}) = \{\psi \in L_2(\mathbb{R}_+, \mathbb{C}): x\psi \in L_2(\mathbb{R}_+, \mathbb{C}), \psi \in H_0^2(\mathbb{R}_+, \mathbb{C})\}$. Let $\{\mu_n\}_{n=1}^\infty \subset \mathbb{R}$ denotes the non-increasing sequence of zeroes of $Ai(z)$, Airy function. Then $\{\lambda_n\}_{n=1}^\infty$, where $\lambda_n = e^{-\frac{2\pi}{3}i}\mu_n$, $n \in \mathbb{N}$, is a sequence of eigenvalues of $\mathcal{L}$ [2]. Since $\lim_{n\to\infty} \mu_n = -\infty$ and $\lim_{n\to\infty} |\mu_{n+1} - \mu_n| = 0$ (see [30]), the eigenvalues $\{\lambda_n\}_{n=1}^\infty$ of $\mathcal{L}$ obey the condition

$$\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$$

and, hence, the set $\{\lambda_n\}_{n=1}^\infty$ cannot be decomposed into a finite number of sets Λ_k satisfying (3).

The eigenfunctions of $\mathcal{L}$ are

$$\tilde{\psi}_n = Ai(e^{\frac{\pi}{6}i}x + \mu_n) \in H_0^2(\mathbb{R}_+, \mathbb{C}),\, n \in \mathbb{N}.$$

Normalized eigenfunctions $\psi_n = \frac{\tilde{\psi}_n}{\|\tilde{\psi}_n\|}$, $n \in \mathbb{N}$, are dense in $L_2(\mathbb{R}, \mathbb{C})$, as it is proved in [2], but do not form a Schauder basis of $L_2(\mathbb{R}, \mathbb{C})$, since $\mathcal{L}$ is spectrally wild [7].

Define operators $\widetilde{A_k} : \ell_{p,k}(\{e_n\}) \supset D(\widetilde{A_k}) \mapsto \ell_{p,k}(\{e_n\})$ on a class of Banach spaces $\ell_{p,k}(\{e_n\})$, $p > 1$, $k \in \mathbb{N}$, see [28] or Section 2, as follows:

$$\widetilde{A_k} x = \widetilde{A_k} (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n = (\mathfrak{f}) \sum_{n=1}^{\infty} i f(n) \cdot c_n e_n, \tag{18}$$

where $\{f(n)\}_{n=1}^{\infty} \in \mathcal{S}_k$, with domain

$$D(\widetilde{A_k}) = \left\{ x = (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n \in \ell_{p,k}(\{e_n\}) : \{f(n) \cdot c_n\}_{n=1}^{\infty} \in \ell_p(\Delta^k) \right\}. \tag{19}$$

Then, by virtue of Theorem 16 in [28], $\widetilde{A_k}$ generates the C_0-group $\{e^{\widetilde{A_k} t}\}_{t \in \mathbb{R}}$ on $\ell_{p,k}(\{e_n\})$, which acts on $\ell_{p,k}(\{e_n\})$ for every $t \in \mathbb{R}$ by the formula

$$e^{\widetilde{A_k} t} x = e^{\widetilde{A_k} t} (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n = (\mathfrak{f}) \sum_{n=1}^{\infty} e^{i t f(n)} c_n e_n. \tag{20}$$

An explicit form of the resolvent and the description of the spectrum $\sigma(\widetilde{A_k})$ of generators $\widetilde{A_k}$ are provided by the following theorem, similar to the Theorem 3.1.

Theorem 3.3. *Let $k \in \mathbb{N}$, $p > 1$, and $\widetilde{A_k}$ be the operator defined above. Then,*

(i) $\sigma(\widetilde{A_k}) = \sigma_p(\widetilde{A_k}) = \{i f(n)\}_1^{\infty}$;

(ii) *the resolvent of $\widetilde{A_k}$ is given by the formula*

$$(\widetilde{A_k} - \lambda I)^{-1} x = (\mathfrak{f}) \sum_{n=1}^{\infty} \frac{c_n e_n}{i f(n) - \lambda}, \quad \lambda \in \rho(\widetilde{A_k}) = \mathbb{C} \setminus \{i f(n)\}_1^{\infty}, \tag{21}$$

where $x = (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n \in \ell_{p,k}(\{e_n\})$.

Proof. If $\{e_n\}_{n=1}^{\infty}$ is a symmetric basis of ℓ_p, then there exist two constants $\tilde{M} \geq \tilde{m} > 0$ such that for every $\tilde{y} = \sum_{n=1}^{\infty} \alpha_n e_n \in \ell_p$ we have

$$\tilde{m} \|\tilde{y}\|^p \leq \sum_{n=1}^{\infty} |\alpha_n|^p \leq \tilde{M} \|\tilde{y}\|^p,$$

see [28] for details. Thus the proof repeats ideas and lines of the proof of Theorem 3.1. □

It is well known that the spectrum does not contain much information about the behaviour of NSA operator A, see also [5], [6]. For this reason the notion of pseudospectra was introduced and came into play [29]. The pseudospectra of A is the family of sets

$$\Big\{\lambda \in \mathbb{C} \colon \|(A-\lambda I)^{-1}\| \geq \frac{1}{\varepsilon}\Big\}_{\varepsilon>0}$$

and it describes the behaviour of NSA operator A much more effectively than the spectrum, see [5], [6], [29].

Another way to control the resolvent is to obtain for it estimates from above, see [5], [7], and works [18], [10], [11], [12], where direct links between the polynomial growth of the C_0-semigroup $\{e^{At}\}_{t\geq 0}$ in t and asymptotic behaviour of the corresponding resolvent $(A-\lambda I)^{-1}$ along vertical lines were established. Note that C_0-semigroups and C_0-groups with polynomial growth condition naturally appear in theory and applications of evolution equations, see, e.g., [3], [13], [27].

Remark 3.4. It was established in [28] that the constructed C_0-groups $\{e^{A_k t}\}_{t\in\mathbb{R}}$ on $H_k(\{e_n\})$ from Theorem 1.2 are polynomially bounded. More precisely, C_0-groups $\{e^{A_k t}\}_{t\in\mathbb{R}}$ grow in norm as $t \to \pm\infty$ but there exists a polynomial $\mathfrak{p}_k$ with positive coefficients such that $\deg \mathfrak{p}_k = k$ and for every $t \in \mathbb{R}$ we have

$$\|e^{A_k t}\| \leq \mathfrak{p}_k(|t|).$$

For details see Proposition 12 from [28]. The similar result is true for the case of C_0-groups $\{e^{\widetilde{A}_k t}\}_{t\in\mathbb{R}}$ on Banach spaces $\ell_{p,k}(\{e_n\})$, $p>1$, $k\in\mathbb{N}$, with generators from Theorem 3.3, see Proposition 17 from [28]. Thus we can apply results from [10], [11], [18] to describe the asymptotic behaviour of the corresponding resolvents.

In the forthcoming work we will prove the sharpness of polynomial growth of C_0-groups $\{e^{A_k t}\}_{t\in\mathbb{R}}$, $\{e^{\widetilde{A}_k t}\}_{t\in\mathbb{R}}$.

Finally we note that, by virtue of Theorem 7.4 from [1] (p. 91), the weak spectral mapping theorem holds for our classes of C_0-groups, since they are polynomially bounded. In other words it means that for all $t \in \mathbb{R}$

$$\sigma(e^{A_k t}) = \overline{e^{t\sigma(A_k)}}, \quad \sigma(e^{\widetilde{A}_k t}) = \overline{e^{t\sigma(\widetilde{A}_k)}}.$$

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