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Quantum graphs on radially symmetric antitrees

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Abstract. We investigate spectral properties of Kirchhoff Laplacians on radially symmetric antitrees. This class of metric graphs admits a lot of symmetries, which enables us to obtain a decomposition of the corresponding Laplacian into the orthogonal sum of Sturm–Liouville operators. In contrast to the case of radially symmetric trees, the deficiency indices of the Laplacian defined on the minimal domain are at most one and they are equal to one exactly when the corresponding metric antitree has finite total volume. In this case, we provide an explicit description of all self-adjoint extensions including the Friedrichs extension.

Furthermore, using the spectral theory of Krein strings, we perform a thorough spectral analysis of this model. In particular, we obtain discreteness and trace class criteria, a criterion for the Kirchhoff Laplacian to be uniformly positive and provide spectral gap estimates. We show that the absolutely continuous spectrum is in a certain sense a rare event, however, we also present several classes of antitrees such that the absolutely continuous spectrum of the corresponding Laplacian is $[0, \infty)$.

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Contents

1. Introduction

This paper is devoted to one particular class of infinite quantum graphs, namely *Kirchhoff Laplacians on radially symmetric antitrees*. Antitrees appear in the study of *discrete Laplacians* on graphs at least since the 1980's (see [12] and also [11, Section 2]) and they attracted a considerable attention after the work of Wojciechowski [47]. More precisely, Wojciechowski used them in [47] (see also [30, §6] and [23]) to construct graphs of polynomial volume growth for which the combinatorial Laplacian is stochastically incomplete and the bottom of the essential spectrum is strictly positive, which is in sharp contrast to the manifold setting (cf. [9], [21], [22]). These apparent discrepancies were resolved later using the notion of intrinsic metrics, with antitrees appearing as key examples for certain thresholds (see [18, 24, 25, 29]). During the recent years, antitrees were also actively studied from other perspectives and we only refer to a brief selection of articles [1], [8], [11], [20], [42], where further references can be found.

In this paper, we consider antitrees from the perspective of quantum graphs and perform a detailed spectral analysis of the Kirchhoff Laplacian on radially symmetric antitrees. Our discussion includes characterization of self-adjointness and a complete description of self-adjoint extensions, spectral gap estimates and spectral types (discrete, singular and absolutely continuous spectrum).

Before proceeding further, let us first recall necessary definitions. Let $\mathcal{G}_d = (\mathcal{V}, \mathcal{E})$ be a connected, simple (no loops or multiple edges) combinatorial graph. Fix a root vertex $o \in \mathcal{V}$ and then order the graph with respect to the combinatorial spheres S_n, $n \in \mathbb{Z}_{\geq 0}$ (notice that $S_0 = \{o\}$).

Definition 1.1. A connected simple rooted (infinite) graph $\mathcal{G}_d$ is called an *antitree* if every vertex in S_n, $n \geq 1$,[1] is connected to all vertices in S_{n-1} and S_{n+1} and no vertices in S_k for all $|k-n| \neq 1$.

Notice that combinatorial antitrees admit radial symmetry and every antitree is uniquely determined by its sphere numbers $s_n = \#S_n$, $n \geq 0$ (see Figure 1).

If every edge of $\mathcal{G}_d$ is assigned a length $|e| \in (0,\infty)$, then $\mathcal{G} = (\mathcal{G}_d, |\cdot|)$ is called a *metric graph*. Upon identifying each edge e with the interval of length $|e|$, $\mathcal{G}$ may be considered as a "network" of intervals glued together at the vertices. In the following we shall denote combinatorial and metric antitrees by $\mathcal{A}_d$ and, respectively, $\mathcal{A}$. The analog of the Laplace–Beltrami operator for metric graphs is the *Kirchhoff Laplacian* $\mathbf{H}$ (or Kirchhoff–Neumann Laplacian, see Section 3.1), also called a *quantum graph*. It acts as an edgewise (negative) second derivative $f_e \mapsto -\frac{d^2}{dx_e^2} f_e$, $e \in \mathcal{E}$, and is defined on edgewise H^2-functions satisfying continuity and Kirchhoff conditions at the vertices (we refer to [2, 3, 15, 17, 32, 39] for more information and references).

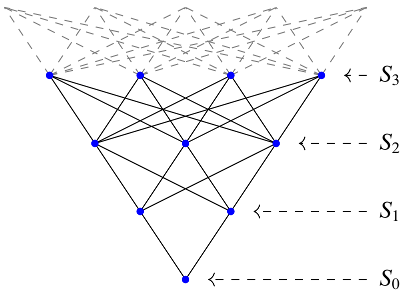

Figure 1. Antitree with sphere numbers $s_n = n + 1$.

Our approach employs the high degree of symmetry and this naturally demands symmetry assumptions also on the choice of edge lengths:

Hypothesis 1.2. We shall assume that the metric antitree $\mathcal{A}$ is *radially symmetric*, that is, for each $n \geq 0$, all edges connecting combinatorial spheres S_n and S_{n+1} have the same length, say $\ell_n > 0$.

One of our main motivations is Lemma 8.9 in [32]. More precisely, the symmetry of antitrees structure turned out useful in studying isoperimetric estimates

[1] By definition, the root o is connected to all vertices in S_1 and no vertices in S_k, $k \geq 2$.

and we were even able to compute explicitly the bottom of the essential spectrum of some (non-equilateral) quantum graphs (see [32, §8.2]). Despite an enormous interest in quantum graphs during the last two decades, to the best of our knowledge a detailed discussion of their spectral properties without further restrictions on edges lengths (for instance, one of the most common assumptions is $\inf_{e\in\mathcal{E}} |e| > 0$) has so far been obtained only for a few models and the most studied ones are *radially symmetric trees* (see e.g. [6, 10, 16, 36, 37, 44]). However, the assumption that $\mathcal{G}$ is a tree prevents many interesting phenomena to happen (for instance, by [32, Lemma 8.1], in this case the Kirchhoff Laplacian, actually, its Friedrichs extension, is boundedly invertible if and only if $\sup_{e\in\mathcal{E}} |e| < \infty$; in fact, this condition is only necessary in general [43]). Hence our goal in this work is to present a model which can be thoroughly analyzed but still exhibits in some sense rich spectral behavior.

Let us now briefly describe the content of the paper and our main results. To some extent we follow the approach developed by Naimark and Solomyak for radially symmetric trees (see [36, 37] and also [10, 43, 44]) and use some ideas from [8], where discrete Laplacians on radially symmetric "weighted" graphs have been analyzed. However, some modifications are necessary since comparing to [10, 37, 44] we are dealing with a completely different class of graphs (antitrees have a lot of cycles) and, in contrast to discrete Laplacians [8], we have to deal with unbounded operators (even when restricting to compact subsets of a metric graph) and in this case a search for *reducing subspaces* is a rather complicated task.[2]

First of all, the radial symmetry of $\mathcal{A}$ naturally hints to consider the space $\mathcal{F}_{\rm sym}$ of radially symmetric functions (w.r.t. the root $o \in \mathcal{V}$). It turns out that $\mathcal{F}_{\rm sym}$ is indeed reducing for the pre-minimal Kirchhoff Laplacian $\mathbf{H}_0$ (this means that $\mathbf{H}_0$ as well as its closure $\mathbf{H} = \overline{\mathbf{H}}_0$, the minimal Kirchhoff Laplacian, commutes with the orthogonal projection onto $\mathcal{F}_{\rm sym}$) and its restriction $\mathbf{H}_0 \upharpoonright \mathcal{F}_{\rm sym}$ is unitarily equivalent to a pre-minimal Sturm–Liouville operator H_0 defined in $L^2((0,\mathcal{L});\mu)$ by the differential expression

$$\tau := -\frac{1}{\mu(t)}\frac{d}{dt}\mu(t)\frac{d}{dt}, \qquad \mu(t) = \sum_{n\geq 0} s_n s_{n+1} \mathbf{1}_{[t_n,t_{n+1})}(t), \tag{1.1}$$

[2] After the submission of our paper we learned about the preprint [7] dealing with a similar decomposition in the general case of family preserving metric graphs, which includes antitrees as a particular example. However, the main focus of [7] is on the existence of a decomposition in a rather general situation, whereas in our work we use it mainly as a starting point for the spectral analysis.

and subject to the Neumann boundary condition at $x = 0$. Here $t_0 = 0$, $t_n = \sum_{k\le n-1} \ell_k$ for all $n \ge 1$ and $\mathcal{L} = \sum_{n\ge 0} \ell_n$ (see Section 3.2). Moreover, the remaining part of $\mathbf{H} = \overline{\mathbf{H}}_0$ decomposes into an infinite sum of self-adjoint (regular) Sturm–Liouville operators (see Theorem 3.5; its proof is given in Sections 2 and 3). This decomposition is the starting point of our analysis since it enables us to investigate $\mathbf{H}$ using the well-developed spectral theory of Sturm–Liouville operators. For example, this immediately provides a self-adjointness criterion together with a complete description of self-adjoint extensions of $\mathbf{H}$ (see Section 4). Namely, since all the summands in (3.18) except $\mathrm{H} = \overline{\mathrm{H}}_0$ are self-adjoint operators, we reduce the problem to the study of the operator H_0. Employing Weyl's limit point/limit circle classification, we obtain in Theorem 4.1 that deficiency indices of $\mathbf{H}$ are at most 1. Moreover, $\mathbf{H}$ is self-adjoint if and only if $\mathcal{A}$ has *infinite total volume*, i.e.

$$\operatorname{vol}(\mathcal{A}) := \sum_{e\in\mathcal{E}} |e| = \sum_{n\ge 0} s_n s_{n+1} \ell_n = \int_0^{\mathcal{L}} \mu(t)dt = \infty.$$

If $\mathcal{A}$ has finite total volume, $\operatorname{vol}(\mathcal{A}) < \infty$, all self-adjoint extensions can be described through a single boundary condition (in particular, this also provides a description of the domain of the Friedrichs extension). Moreover, all of their spectra are purely discrete and eigenvalues satisfy Weyl's law (see Corollary 5.1).

If $\operatorname{vol}(\mathcal{A}) = \infty$, i.e., $\mathbf{H}$ is self-adjoint, it was already observed in [32, Section 8.2] that $\sigma(\mathbf{H})$ is not necessarily discrete. In Section 5, we characterize the cases when $\mathbf{H}$ has purely discrete spectrum and when its resolvent $\mathbf{H}^{-1}$ belongs to the trace class (see Theorem 5.4 and Theorem 5.6). Let us stress that our main tool is the spectral theory of Krein strings [27] (see also [13]). More precisely, by a simple change of variables H can be transformed into the string form (see (5.12)) and then one simply needs to use the corresponding results from [26, 27]. Section 6 is devoted to *spectral estimates*, i.e., the investigation of the bottom of the spectrum $\lambda_0(\mathbf{H})$ of $\mathbf{H}$, $\lambda_0(\mathbf{H}) := \inf \sigma(\mathbf{H})$. This can be solved again by using the results of Kac and Krein from [26]. More precisely, we characterize the positivity of $\lambda_0(\mathbf{H})$ (Theorem 6.1 and Theorem 6.3) and derive two-sided estimates (Remark 6.2). Let us also mention at this point that the decomposition (3.18) indicates the way to compute the isoperimetric constant of a radially symmetric antitree (see Theorem 7.1) and hence it is interesting to compare Theorem 6.1 and Theorem 6.3 with the estimates obtained recently in [32] (see Remark 7.2).

To our best knowledge, the theory of Krein strings is applied in the context of quantum graphs for the first time. In fact, most of the analysis in Sections 5 and 6 can be performed with the help of Muckenhoupt inequalities [35] since the questions addressed in these sections allow a variational reformulation (in particular, Solomyak used this approach in [44] to investigate quantum graphs on radially symmetric trees). However, spectral theory of strings enables us to treat more subtle problems (like the study of the structure of the essential spectrum of $\mathbf{H}$). In Section 9, we employ the recent results from [4] and [14] on the absolutely continuous spectrum of strings to construct several classes of antitrees with absolutely continuous spectrum supported on $[0,\infty)$. For instance, if

$$\inf_{n\geq 0} \ell_n > 0, \qquad \sum_{n=1}^{\infty} \Big(\frac{s_{n+2}}{s_n} - 1\Big)^2 < \infty, \tag{1.2}$$

then $\sigma_{\mathrm{ac}}(\mathbf{H}) = [0,\infty)$ (see Theorem 9.6). Notice that to prove this claim we employ the analog of the Szegő theorem for strings recently established by Bessonov and Denisov [4]. Antitrees with polynomially growing sphere numbers satisfy the last assumption, however, it can be shown that in this case the usual trace class arguments do not apply (see Remark 9.4). Let us also emphasize that similar to the case of trees quantum graphs typically have purely singular spectrum in the case of antitrees (see Section 8). However, to the best of our knowledge, the only known examples of quantum graphs on trees having nonempty absolutely continuous spectrum are *eventually periodic radially symmetric trees* (see [16, Theorem 5.1]).

In the final section we demonstrate our results by considering two special classes of antitrees and complement the results of [32, Section 8.2]. In Section 10.1 we consider antitrees with exponentially increasing sphere numbers and demonstrate that in this case there are a lot of similarities with the spectral properties of quantum graphs on radially symmetric trees. Antitrees with polynomially increasing sphere numbers are treated in Section 10.2 and this class of quantum graphs exhibits a number of interesting phenomena. For example, one can show a transition from absolutely continuous spectrum supported on $[0,\infty)$ to purely discrete spectrum (see Corollary 10.7).

2. Decomposition of $L^2(\mathcal{A})$

2.1. Auxiliary subspaces. Let $\mathcal{A}$ be a metric radially symmetric antitree with sphere numbers $\{s_n\}_{n\geq 0}$ and lengths $\{\ell_n\}_{n\geq 0}$. Upon identifying every edge e with a copy of the interval $\mathcal{I}_e = [0, |e|]$ and considering $\mathcal{A}$ as the union of all edges glued together at certain endpoints, one can introduce the Hilbert space $L^2(\mathcal{A})$ of functions $f\colon \mathcal{A} \to \mathbb{C}$ as $L^2(\mathcal{A}) = \bigoplus_e L^2(e)$. Next, denote

$$t_n := \sum_{j=0}^{n-1} \ell_j, \quad I_n := [t_n, t_{n+1}),$$

and let $\mathcal{H}_n := \mathbb{C}^{s_n s_{n+1}}$, $n \geq 0$. Notice that $s_n s_{n+1}$ is the number of edges in $\mathcal{E}_n^+$, where $\mathcal{E}_n^+$ is the set of edges connecting S_n with S_{n+1}. Enumerating the vertices in each sphere, let each entry a_{ij} of some $\mathbf{a} = (a_{ij})_{i,j} \in \mathcal{H}_n$ correspond to a coefficient of the edge $e \in \mathcal{E}_n^+$ connecting the i-th vertex of S_n with the j-th vertex of S_{n+1}. Moreover, we can identify each function $f\colon \mathcal{A} \to \mathbb{C}$ in a natural way with the sequence of functions $\mathbf{f} = (\mathbf{f}^n)_{n\geq 0}$ such that $\mathbf{f}^n\colon I_n \to \mathcal{H}_n$. In fact, $\mathbf{f}^n$ is given by

$$\mathbf{f}^n_{i,j}(t) := f(x_{ij}(t)), \quad t \in I_n, \tag{2.1}$$

where $x_{ij}(t)$ is the unique $x \in \mathcal{A}$, such that $|x| = t$ and x lies on the edge connecting the i-th vertex in S_n with the j-th vertex of S_{n+1}. Notice that the map

$$\begin{aligned} U\colon L^2(\mathcal{A}) &\longrightarrow \bigoplus_{n\geq 0} L^2(I_n; \mathcal{H}_n), \\ f &\longmapsto \mathbf{f} = (\mathbf{f}^n)_{n\geq 0}, \end{aligned} \tag{2.2}$$

is an isometric isomorphism since

$$(f,g)_{L^2(\mathcal{A})} = \sum_{n\geq 0} \int_{I_n} (\mathbf{f}^n(t), \mathbf{g}^n(t))_{\mathcal{H}_n}\, dt \tag{2.3}$$

for all $f, g \in L^2(\mathcal{A})$. Next we introduce the following subspaces:

$$\mathcal{H}_n^{\mathrm{sym}} := \{\mathbf{a} \in \mathcal{H}_n \mid a_{ij} = a_{11} \text{ for all } i,j\},$$

$$\mathcal{H}_n^{+} := \Big\{\mathbf{a} \in \mathcal{H}_n \;\Big|\; a_{ij} = a_{i1} \text{ for all } i,j, \text{ and } \sum_{i,j} a_{ij} = \sum_i a_{i1} = 0\Big\},$$

$$\mathcal{H}_n^{-} := \Big\{\mathbf{a} \in \mathcal{H}_n \;\Big|\; a_{ij} = a_{1j} \text{ for all } i,j, \text{ and } \sum_{i,j} a_{ij} = \sum_j a_{1j} = 0\Big\},$$

$$\mathcal{H}_n^{0} := \Big\{\mathbf{a} \in \mathcal{H}_n \;\Big|\; \sum_j a_{ij} = 0 \text{ for all } i \text{ and } \sum_i a_{ij} = 0 \text{ for all } j\Big\}.$$

It is straightforward to check that the above spaces are mutually orthogonal and their dimensions are given by

$$\dim(\mathcal{H}_n^{\mathrm{sym}}) = 1, \qquad \dim(\mathcal{H}_n^{0}) = (s_n - 1)(s_{n+1} - 1),$$

$$\dim(\mathcal{H}_n^{+}) = s_n - 1, \quad \dim(\mathcal{H}_n^{-}) = s_{n+1} - 1.$$

Hence $\mathcal{H}_n$ admits the decomposition

$$\mathcal{H}_n = \begin{cases} \mathcal{H}_n^{\mathrm{sym}} \oplus \mathcal{H}_n^-, & n = 0, \\ \mathcal{H}_n^{\mathrm{sym}} \oplus \mathcal{H}_n^+ \oplus \mathcal{H}_n^- \oplus \mathcal{H}_n^0, & n \geq 1. \end{cases} \tag{2.4}$$

Notice that if $s_n = 1$ for some $n \geq 1$, then $\mathcal{H}_n^+ = \mathcal{H}_n^0 = \mathcal{H}_{n-1}^0 = \mathcal{H}_{n-1}^- = \{0\}$.

One can also describe the above subspaces by identifying $\mathcal{H}_n$ with the tensor product $\mathbb{C}^{s_n} \otimes \mathbb{C}^{s_{n+1}}$. For example, setting

$$\mathbf{1}_{s_n} := (\underbrace{1, 1, \ldots, 1}_{s_n}) \in \mathbb{C}^{s_n}, \quad \mathbf{1}^n := \mathbf{1}_{s_n} \otimes \mathbf{1}_{s_{n+1}} \in \mathcal{H}_n, \tag{2.5}$$

for all $n \geq 0$, we get

$$\mathcal{H}_n^{\mathrm{sym}} = \operatorname{span}\{\mathbf{1}^n\}. \tag{2.6}$$

Moreover, denote

$$\omega_n := \mathrm{e}^{2\pi \mathrm{i}/s_n}, \quad n \geq 0,$$

and set

$$\mathbf{a}_{s_n}^j := \{\omega_n^j, \ldots, \omega_n^{j(s_n-1)}, 1\} \in \mathbb{C}^{s_n}, \quad j \in \{1, \ldots, s_n\}. \tag{2.7}$$

Notice that $\{\mathbf{a}_{s_n}^j\}_{j=1}^{s_n}$ forms an orthogonal basis in $\mathbb{C}^{s_n}$ for all $n \geq 0$. In particular, $\mathbf{a}_{s_n}^{s_n} = \mathbf{1}_{s_n}$ and $\|\mathbf{a}_{s_n}^j\|^2 = s_n$. Hence setting

$$\mathbf{a}_n^{i,j} := \mathbf{a}_{s_n}^i \otimes \mathbf{a}_{s_{n+1}}^j \in \mathcal{H}_n, \tag{2.8}$$

where $1 \leq i \leq s_n$ and $1 \leq j \leq s_{n+1}$, we easily get

$$\begin{aligned} \mathcal{H}_n^+ &= \operatorname{span}\{\mathbf{a}_{s_n}^i \otimes \mathbf{1}_{s_{n+1}} \mid 1 \leq i < s_n\} = \operatorname{span}\{\mathbf{a}_n^{i,s_{n+1}} \mid 1 \leq i < s_n\}, \\ \mathcal{H}_n^- &= \operatorname{span}\{\mathbf{1}_{s_n} \otimes \mathbf{a}_{s_{n+1}}^j \mid 1 \leq j < s_{n+1}\} = \operatorname{span}\{\mathbf{a}_n^{s_n,j} \mid 1 \leq j < s_{n+1}\}, \\ \mathcal{H}_n^0 &= \operatorname{span}\{\mathbf{a}_n^{i,j} \mid 1 \leq i < s_n,\ 1 \leq j < s_{n+1}\}. \end{aligned} \tag{2.9}$$

Finally, observe that

$$\|\mathbf{a}_n^{i,j}\|^2 = s_n s_{n+1} \tag{2.10}$$

for all $1 \leq i \leq s_n$, $1 \leq j \leq s_{n+1}$ and $n \geq 0$.

2.2. Definition of the subspaces. The decomposition (2.4) naturally induces a decomposition of the Hilbert space $L^2(\mathcal{A})$. First consider the subspace

$$\mathcal{F}_{\mathrm{sym}} := \{f \in L^2(\mathcal{A}) \mid \mathbf{f}^n \colon I_n \longrightarrow \mathcal{H}_n^{\mathrm{sym}},\ n \geq 0\}. \tag{2.11}$$

Clearly, it consists of functions which depend only on the distance to the root:

$$\mathcal{F}_{\mathrm{sym}} = \{f \in L^2(\mathcal{A}) \mid f(x) = f(y) \text{ if } |x| = |y|\}. \tag{2.12}$$

Moreover, its orthogonal complement is given by

$$\begin{aligned}\mathcal{F}_{\mathrm{sym}}^{\perp} &= \{f \in L^2(\mathcal{A}) \mid \mathbf{f}^n\colon I_n \longrightarrow (\mathcal{H}_n^{\mathrm{sym}})^{\perp},\ n \geq 0\}\\ &= \Big\{f \in L^2(\mathcal{A}) \,\Big|\, \sum_{e\in\mathcal{E}_n^+} f_e \equiv 0,\ n \geq 0\Big\}.\end{aligned} \tag{2.13}$$

Next we need to decompose $\mathcal{F}_{\mathrm{sym}}^{\perp}$. Set

$$\mathcal{F}_n^0 := \{f \in L^2(\mathcal{A}) \mid \mathbf{f}^n\colon I_n \longrightarrow \mathcal{H}_n^0;\ \mathbf{f}^k \equiv 0,\ k \neq n\} \tag{2.14}$$

for all $n \geq 1$. Taking into account the definition of $\mathcal{H}_n^0$, it is not difficult to see that

$$\mathcal{F}_n^0 = \left\{ f \in L^2(\mathcal{A}) \,\middle|\, \begin{array}{c} f \equiv 0 \text{ on } \mathcal{A}\setminus\mathcal{E}_n^+ \\ \sum_{e\in\mathcal{E}_v^+} f_e = \sum_{e\in\mathcal{E}_u^-} f_e \equiv 0 \text{ for all } v \in S_n, u \in S_{n+1} \end{array}\right\}.$$

Here, for every $v \in \mathcal{V}$, $\mathcal{E}_v^+$ and $\mathcal{E}_v^-$ denote the edges connecting v with the next and, respectively, previous combinatorial spheres.

We need to be more careful with the remaining part since our aim is to find reducing subspaces for the quantum graph operator $\mathbf{H}$. For every $v \in \mathcal{V}\setminus\{o\}$, define the subspace $\widetilde{\mathcal{F}}_v$ consisting of functions which vanish away of $\mathcal{E}_v$, where $\mathcal{E}_v$ is the set of edges emanating from v. Moreover, on the corresponding star $\mathcal{E}_v$ they depend only on the distance to the root, that is,

$$\widetilde{\mathcal{F}}_v := \left\{ f \in L^2(\mathcal{A}) \,\middle|\, \begin{array}{c} f \equiv 0 \text{ on } \mathcal{A}\setminus\mathcal{E}_v \\ f(x) = f(y) \text{ for a.e. } x, y \in \mathcal{E}_v, |x| = |y| \end{array}\right\}. \tag{2.15}$$

Notice that $\widetilde{\mathcal{F}}_v$ and $\widetilde{\mathcal{F}}_u$ are orthogonal for $u \neq v$ if u and v are not adjacent vertices. Next for all $n \geq 1$ consider the spaces

$$\widetilde{\mathcal{F}}_n := \bigoplus_{v\in S_n} \widetilde{\mathcal{F}}_v, \quad n \geq 1, \tag{2.16}$$

and

$$\mathcal{F}_n := \widetilde{\mathcal{F}}_n \ominus \mathcal{F}_{\mathrm{sym}} = \Big\{ f \in \widetilde{\mathcal{F}}_n \,\Big|\, \sum_{e\in\mathcal{E}_m^+} f_e \equiv 0,\ m \geq 0\Big\}. \tag{2.17}$$

Notice that with respect to the decomposition (2.4), we have

$$\mathcal{F}_n = \left\{ f \in L^2(\mathcal{A}) \,\middle|\, \begin{array}{c} \mathbf{f}^{n-1}\colon I_{n-1} \longrightarrow \mathcal{H}_{n-1}^-,\ \mathbf{f}^n\colon I_n \longrightarrow \mathcal{H}_n^+ \\ \mathbf{f}^m \equiv 0,\ m \neq n-1, n \end{array}\right\}. \tag{2.18}$$

Thus, we arrive at the following result.

Lemma 2.1. *The Hilbert space $L^2(\mathcal{A})$ admits the decomposition*

$$L^2(\mathcal{A}) = \mathcal{F}_{\mathrm{sym}} \oplus \bigoplus_{n\geq 1} \mathcal{F}_n \oplus \bigoplus_{n\geq 1} \mathcal{F}_n^0. \tag{2.19}$$

Proof. The orthogonality of the subspaces in (2.19) follows directly from (2.3) and (2.4). Hence we only need to show that every $f \in L^2(\mathcal{A})$ is contained in the right-hand side of (2.19). Since $L^2(\mathcal{A}) = \bigoplus_{e\in\mathcal{E}} L^2(e)$, it suffices to prove this claim in the case when f is zero except on a single edge $e \in \mathcal{E}$. Suppose that $e \in \mathcal{E}_n^+$ for some $n \geq 0$. Then for almost every $t \in I_n$ we have

$$\mathbf{f}^n(t) = \mathcal{P}_n^{\mathrm{sym}}(\mathbf{f}^n(t)) + \mathcal{P}_n^+(\mathbf{f}^n(t)) + \mathcal{P}_n^-(\mathbf{f}^n(t)) + \mathcal{P}_n^0(\mathbf{f}^n(t)) \in \mathcal{H}_n,$$

where $\mathcal{P}_n^j$ is the orthogonal projection in $\mathcal{H}_n$ onto $\mathcal{H}_n^j$, $j \in \{\mathrm{sym}, +, -, 0\}$. Define $f_j : \mathcal{A} \to \mathbb{C}$ as the function identified with the sequence of functions $\mathbf{f}_j = (\mathbf{f}_j^k)_{k\geq 0}$ given by

$$\mathbf{f}_j^k(t) := P_k^j(\mathbf{f}^k(t)), \quad j \in \{\mathrm{sym}, +, -, 0\},$$

for a.e. $t \in I_k$. Then $f_j \in L^2(\mathcal{A})$ for all $j \in \{\mathrm{sym}, +, -, 0\}$ and

$$f = f_{\mathrm{sym}} + f_+ + f_- + f_0.$$

Since $\mathbf{f}_j^k(t) \in \mathcal{H}_k^j$ for a.e. $t \in I_k$, we conclude that $f_{\mathrm{sym}} \in \mathcal{F}_{\mathrm{sym}}$, $f_0 \in \mathcal{F}_n^0$, $f_+ \in \mathcal{F}_n$ and $f_- \in \mathcal{F}_{n+1}$. □

Our next aim is to write down explicit formulas for projections onto the subspaces in the decomposition (2.19). First, for any $\tilde{f} \in L^2(I_n)$ and $\mathbf{a} \in \mathcal{H}_n$, we set $\tilde{\mathbf{f}} := \tilde{f} \otimes \mathbf{a}$. Recalling that every function $f : \mathcal{A} \to \mathbb{C}$ can be identified via (2.2) with the sequence of vector-valued functions $\mathbf{f} = (\mathbf{f}^n)_{n\geq 0}$, we denote

$$\mathcal{F}_{\mathbf{a}}^n := \{f \in L^2(\mathcal{A}) \mid \mathbf{f}^n = f^n \otimes \mathbf{a},\ f^n \in L^2(I_n);\ \mathbf{f}^k \equiv 0,\ k \neq n\}. \tag{2.20}$$

Note that the orthogonal projection $P_{\mathbf{a}}^n$ of $L^2(\mathcal{A})$ onto $\mathcal{F}_{\mathbf{a}}^n$ is given by

$$(U(P_{\mathbf{a}}^n f))(t) := \begin{cases} 0, & t \notin I_n, \\ \dfrac{1}{\|\mathbf{a}\|^2}(\mathbf{f}^n(t), \mathbf{a})_{\mathcal{H}_n}\mathbf{a}, & t \in I_n, \end{cases} \tag{2.21}$$

where U is the isometric isomorphism (2.2).

Combining the form of $P_{\mathbf{a}}^n$ with the decomposition (2.4) and (2.6), (2.9), we easily obtain the following result.

Lemma 2.2. *Let* $\mathbf{1}^n \in \mathcal{H}_n$ *and* $\mathbf{a}_n^{i,j} \in \mathcal{H}_n$*,* $n \geq 0$ *be given by* (2.5) *and* (2.8)*. Then the orthogonal projections in the decomposition* (2.19) *are given by*

$$P_{\mathrm{sym}} = \sum_{n\geq 0} P^n_{\mathbf{1}^n}, \tag{2.22}$$

$$P_n^0 = \sum_{\substack{1\leq i<s_n\\ 1\leq j<s_{n+1}}} P^n_{\mathbf{a}_n^{i,j}}, \qquad n \geq 1, \tag{2.23}$$

$$P_n = \sum_{j=1}^{s_n-1} P^{n-1}_{\mathbf{a}_{n-1}^{s_{n-1},j}} + \sum_{i=1}^{s_n-1} P^n_{\mathbf{a}_n^{i,s_{n+1}}}, \quad n \geq 1. \tag{2.24}$$

3. Reduction of the quantum graph operator

In this section, we show that each of the spaces in the above decomposition (2.19) is reducing for the quantum graph operator with Kirchhoff conditions and also obtain a description of the corresponding restrictions.

3.1. Kirchhoff's Laplacian.

Let us briefly recall the definition of the Laplacian on a metric graph (for details we refer to [3, 17, 32]). Let $L^2(\mathcal{A})$ be the corresponding Hilbert space and the subspace of compactly supported L^2-functions will be denoted by $L^2_c(\mathcal{A})$. Moreover, denote by $H^2(\mathcal{A}\setminus\mathcal{V})$ the subspace of $L^2(\mathcal{A})$ consisting of edgewise H^2-functions, that is, $f \in H^2(\mathcal{A}\setminus\mathcal{V})$ if $f \in H^2(e)$ for every $e \in \mathcal{E}$, where $H^2(e)$ is the usual Sobolev space. The Kirchhoff (or Kirchhoff–Neumann) boundary conditions at every vertex $v \in \mathcal{V}$ are then given by

$$\begin{cases} f \text{ is continuous at } v, \\ \sum_{e\in\mathcal{E}_v} f_e'(v) = 0, \end{cases} \tag{3.1}$$

where

$$f_e(v) := \lim_{x_e\to v} f(x_e), \qquad f_e'(v) := \lim_{x_e\to v} \frac{f(x_e)-f_e(v)}{|x_e - v|}, \tag{3.2}$$

are well defined for all $f \in H^2(\mathcal{A}\setminus\mathcal{V})$ and every vertex $v \in \mathcal{V}$. Imposing these boundary conditions and restricting to compactly supported functions we get the pre-minimal operator $\mathbf{H}_0$ acting edgewise as the (negative) second derivative $f_e \mapsto -\frac{d^2}{dx_e^2} f_e$, $e \in \mathcal{E}$ on the domain

$$\mathrm{dom}(\mathbf{H}_0) = \{f \in H^2(\mathcal{A}\setminus\mathcal{V}) \cap L^2_c(\mathcal{G}) \mid f \text{ satisfies (3.1)},\ v \in \mathcal{V}\}. \tag{3.3}$$

The operator $\mathbf{H}_0$ is symmetric and its closure $\mathbf{H} = \overline{\mathbf{H}}_0$ is called *the minimal Kirchhoff Laplacian*.

First, we need the following simple but useful fact.

Lemma 3.1. *Let* $f \in L^2(\mathcal{A})$ *and* $\mathbf{f} = Uf$ *be given by* (2.2)*. Then* $f \in \operatorname{dom}(\mathbf{H}_0)$ *if and only if* $\mathbf{f} = (\mathbf{f}^n)_{n\geq 0}$ *satisfies the following conditions*:

(i) $\mathbf{f}^n \equiv 0$ *for all sufficiently large* n,

(ii) $\mathbf{f}^n_{i,j} \in H^2(I_n)$ *for all* $n \geq 0$,

(iii) *for all* $j \in \{1, \dots, s_1\}$

$$\mathbf{f}^0_{1,j}(0+) = \mathbf{f}^0_{1,1}(0+), \qquad \sum_{j=1}^{s_1} (\mathbf{f}^0_{1,j})'(0+) = 0,$$

(iv) *for all* $n \geq 1$, $i \in \{1, \dots, s_n\}$,

$$\mathbf{f}^n_{i,j}(t_n+) = \mathbf{f}^{n-1}_{k,i}(t_n-),$$

$$\sum_{j=1}^{s_{n+1}} (\mathbf{f}^n_{i,j})'(t_n+) = \sum_{k=1}^{s_{n-1}} (\mathbf{f}^{n-1}_{k,i})'(t_n-).$$

Proof. The proof is straightforward. We only need to mention that (i) is equivalent to the fact that f is compactly supported; (ii) means that f belongs to the Sobolev space H^2 on each edge $e \in \mathcal{E}$; (iii) and (iv) are continuity and Kirchhoff's conditions at the vertices. □

3.2. The subspace $\mathfrak{F}_{\mathrm{sym}}$. Set $\mathcal{I}_{\mathcal{L}} = [0, \mathcal{L})$, and define the length $\mathcal{L}$ and the weight function $\mu: \mathcal{I}_{\mathcal{L}} \to \mathbb{R}_{\geq 0}$ by

$$\mu(t) = \sum_{n\geq 0} s_n s_{n+1} \mathbf{1}_{I_n}(t), \quad t \in [0, \mathcal{L}); \qquad \mathcal{L} = \sum_{n\geq 0} \ell_n. \tag{3.4}$$

Consider the (pre-minimal) operator H_0 defined in $L^2(\mathcal{I}_{\mathcal{L}}; \mu)$ by the Sturm–Liouville differential expression

$$\tau = -\frac{1}{\mu(t)} \frac{d}{dt} \mu(t) \frac{d}{dt}, \tag{3.5}$$

on the domain

$$\operatorname{dom}(\mathrm{H}_0) = \{f \in L^2_c(\mathcal{I}_{\mathcal{L}}; \mu) \mid f, \mu f' \in AC(\mathcal{I}_{\mathcal{L}}),\ \tau f \in L^2(\mathcal{I}_{\mathcal{L}}; \mu);\ f'(0) = 0\}. \tag{3.6}$$

More concretely, H_0 acts as a negative second derivative and its domain $\mathrm{dom}(\mathrm{H}_0)$ consists of functions $f \in L^2(\mathcal{I}_{\mathcal{L}};\mu)$ having compact support in $\mathcal{I}_{\mathcal{L}}$, belonging to H^2 on every interval I_n and at each point t_n satisfying the boundary conditions

$$\begin{cases} f \text{ is continuous at } t_n, \\ s_{n-1} f'(t_n-) = s_{n+1} f'(t_n+). \end{cases} \tag{3.7}$$

Here we set $s_{-1} := 0$ in the case $n = 0$ for notational simplicity and the corresponding condition (3.7) reads as the Neumann boundary condition at $t = 0$.

Lemma 3.2. *The subspace $\mathcal{F}_{\mathrm{sym}}$ reduces the operator $\mathbf{H}_0$. Moreover, its restriction $\mathbf{H}_0 \restriction \mathcal{F}_{\mathrm{sym}}$ onto $\mathcal{F}_{\mathrm{sym}}$ is unitarily equivalent to the operator* H_0.

Proof. First let us show that $f_{\mathrm{sym}} := P_{\mathrm{sym}} f \in \mathrm{dom}(\mathbf{H}_0)$ for every $f \in \mathrm{dom}(\mathbf{H}_0)$. In fact, we need to show that $\mathbf{f}_{\mathrm{sym}} = U f_{\mathrm{sym}}$ satisfies conditions (i)–(iv) of Lemma 3.1. Clearly, by continuity of f and (2.21), (2.22), $\mathbf{f}_{\mathrm{sym}}$ satisfies (i) and (ii). Moreover, both $(\mathbf{f}_{\mathrm{sym}})^n_{i,j}(t_n+)$ and $(\mathbf{f}_{\mathrm{sym}})^n_{k,m}(t_{n+1}-)$ depend only on $n \geq 0$. Since $\mathbf{f}$ satisfies both (iii) and (iv), we obtain that $(\mathbf{f}_{\mathrm{sym}})^0_{1,j}(0+)$ does not depend on j and

$$\begin{aligned} (\mathbf{f}_{\mathrm{sym}})^n_{i,j}(t_n+) &= \frac{1}{s_n s_{n+1}} (\mathbf{f}^n(t_n+), \mathbf{1}^n)_{\mathcal{H}_n} \\ &= \frac{1}{s_{n-1} s_n} (\mathbf{f}^{n-1}(t_n-), \mathbf{1}^{n-1})_{\mathcal{H}_{n-1}} \\ &= (\mathbf{f}_{\mathrm{sym}})^{n-1}_{k,i}(t_n-) \end{aligned}$$

for all $i \in \{1, \dots, s_n\}$ and $n \geq 1$. Similarly,

$$\begin{aligned} \sum_{j=1}^{s_{n+1}} (\mathbf{f}'_{\mathrm{sym}})^n_{i,j}(t_n+) &= \frac{1}{s_n} ((\mathbf{f}^n)'(t_n+), \mathbf{1}^n)_{\mathcal{H}_n} = \frac{1}{s_n} \sum_{i,j} (\mathbf{f}^n_{i,j})'(t_n+) \\ &= \frac{1}{s_n} \sum_{i=1}^{s_n} \sum_{j=1}^{s_{n+1}} (\mathbf{f}^n_{i,j})'(t_n+) = \frac{1}{s_n} \sum_{i=1}^{s_n} \sum_{k=1}^{s_{n-1}} (\mathbf{f}^{n-1}_{k,i})'(t_n-) \\ &= \frac{1}{s_n} ((\mathbf{f}^{n-1})'(t_n-), \mathbf{1}^{n-1})_{\mathcal{H}_{n-1}} = \sum_{k=1}^{s_{n-1}} (\mathbf{f}'_{\mathrm{sym}})^{n-1}_{k,i}(t_n-), \end{aligned} \tag{3.8}$$

which holds for all $i \in \{1, \dots, s_n\}$, $n \geq 1$. Moreover, for $n = 0$ we have

$$(\mathbf{f}'_{\mathrm{sym}})^0_{1,j}(0+) = \frac{1}{s_1} \sum_{m=1}^{s_1} (\mathbf{f}^0_{1,m})'(0+) = 0$$

for all $j \in \{1, \dots, s_1\}$. Hence $f_{\mathrm{sym}} = P_{\mathrm{sym}} f \in \mathrm{dom}(\mathbf{H}_0)$ for all $f \in \mathrm{dom}(\mathbf{H}_0)$. Noting that $\mathbf{H}_0$ is symmetric and $\mathcal{F}_{\mathrm{sym}}$ is clearly invariant for $\mathbf{H}_0$ we conclude that $\mathcal{F}_{\mathrm{sym}}$ is reducing for $\mathbf{H}_0$.

To prove the last claim, observe that the subspace $\mathcal{F}_{\mathrm{sym}}$ is isometrically isomorphic to the Hilbert space $L^2(\mathcal{I}_{\mathcal{L}};\mu)$. Indeed, for every $f \in \mathcal{F}_{\mathrm{sym}}$, set

$$\tilde{f}(t) := \frac{1}{s_n s_{n+1}} \sum_{e\in\mathcal{E}_n^+} f(x_e(t)) = \frac{1}{\|\mathbf{1}^n\|^2} (\mathbf{f}^n(t), \mathbf{1}^n)_{\mathcal{H}_n}, \quad t \in I_n, n \geq 0, \tag{3.9}$$

where $x_e(t)$ is the unique point on e satisfying $|x_e(t)| = t$. Consider the map

$$\begin{aligned} U_s \colon \mathcal{F}_{\mathrm{sym}} &\longrightarrow L^2(\mathcal{I}_{\mathcal{L}};\mu), \\ f &\longmapsto \tilde{f}. \end{aligned} \tag{3.10}$$

Clearly, for every $f \in \mathcal{F}_{\mathrm{sym}}$, $\mathbf{f}^n(t) = \tilde{f}(t) \otimes \mathbf{1}^n$ for a.e. $t \in I_n$ and hence

$$\|\tilde{f}\|^2_{L^2(\mathcal{I}_{\mathcal{L}};\mu)} = \sum_{n\geq 0} s_n s_{n+1} \|\tilde{f}\|^2_{L^2(I_n)} = \sum_{n\geq 0} \|\mathbf{f}^n\|^2_{L^2(I_n;\mathcal{H}_n)} = \|\mathbf{f}\|^2 = \|f\|^2_{L^2(\mathcal{A})}.$$

It turns out that

$$\mathrm{H}_0 = U_s(\mathbf{H}_0 \restriction \mathcal{F}_{\mathrm{sym}}) U_s^{-1}. \tag{3.11}$$

Indeed, $\mathbf{H}_0$ acts as the negative second derivative on every edge $e \in \mathcal{E}$ and hence for every $f \in \mathcal{F}_{\mathrm{sym}}$ we get

$$(U_s(\mathbf{H}_0 f))(t) = -\tilde{f}''(t), \quad t \in I_n,$$

for all $n \geq 0$. Therefore, it remains to show that $U_s(\mathcal{F}_{\mathrm{sym}} \cap \mathrm{dom}(\mathbf{H}_0)) = \mathrm{dom}(\mathrm{H}_0)$. In fact, we only need to show that every $\tilde{f} = U_s f$ with $f \in \mathcal{F}_{\mathrm{sym}}$ satisfies (3.7) if and only if $f \in \mathrm{dom}(\mathbf{H}_0)$. Indeed, by (3.9) and continuity of f, $\tilde{f}(t_n+) = \tilde{f}(t_n-)$ for all $n \geq 1$ if $f \in \mathcal{F}_{\mathrm{sym}} \cap \mathrm{dom}(\mathbf{H}_0)$. Moreover, similar to (3.8) one checks that

$$s_{n+1}\tilde{f}'(t_n+) = s_{n-1}\tilde{f}'(t_n-), \quad n \geq 0,$$

exactly when $f \in \mathcal{F}_{\mathrm{sym}} \cap \mathrm{dom}(\mathbf{H}_0)$. This finishes the proof of Lemma 3.2. □

3.3. Restriction to $\mathcal{F}_n^0$. Our next aim is to show that each $\mathcal{F}_n^0$, $n \geq 1$, is a reducing subspace for $\mathbf{H}_0$ and its restriction is unitarily equivalent to $(s_n-1)(s_{n+1}-1)$ copies of $\mathbf{h}_n$, the second derivative with the Dirichlet boundary conditions on $L^2(I_n)$,

$$\mathbf{h}_n := -\frac{d^2}{dt^2}, \quad \mathrm{dom}(\mathbf{h}_n) = \{f \in H^2(I_n) \mid f(t_n+) = f(t_{n+1}-) = 0\}. \tag{3.12}$$

By Lemma 2.2, this will be a consequence of the following lemma.

Lemma 3.3. *Let $n \geq 1$ be such that $s_n > 1$ and $s_{n+1} > 1$. Then each of the subspaces $\mathcal{F}^n_{\mathbf{a}}$, where $\mathbf{a} = \mathbf{a}_n^{i,j}$ with $1 \leq i < s_n$ and $1 \leq j < s_{n+1}$, is reducing for the operator $\mathbf{H}_0$. The restricted operator $\mathbf{H}_0 \upharpoonright \mathcal{F}^n_{\mathbf{a}}$ is unitarily equivalent to the operator $\mathbf{h}_n$ defined by (3.12).*

Proof. Clearly, $\mathcal{F}^n_{\mathbf{a}}$ is invariant for $\mathbf{H}_0$. Since $\mathbf{H}_0$ is symmetric, we only have to prove that $\tilde{f} := P^n_{\mathbf{a}} f \in \operatorname{dom}(\mathbf{H}_0)$ whenever $f \in \operatorname{dom}(\mathbf{H}_0)$. In fact, we need to show that $\tilde{\mathbf{f}} := U(P^n_{\mathbf{a}} f)$ given by (2.21) satisfies conditions (i)–(iv) of Lemma 3.1. Conditions (i) and (ii) are obviously satisfied since $f \in \operatorname{dom}(\mathbf{H}_0)$ and by the definition of $U(P^n_{\mathbf{a}} f)$. Since $\tilde{\mathbf{f}}^m = 0$ for all $m \neq n$ and $n \geq 1$, (iii) clearly holds and, moreover, we need to verify (iv) only at t_n and t_{n+1}.

Let us start with continuity. Suppose $\mathbf{a} = \mathbf{a}_n^{i,j}$ for some $1 \leq i < s_n$ and $1 \leq j < s_{n+1}$. First observe that

$$\tilde{\mathbf{f}}^n_{k,m}(t_n+) = \tilde{\mathbf{f}}^n_{k,m}(t_{n+1}-) = 0$$

for all $k \in \{1, \dots, s_n\}$ and $m \in \{1, \dots, s_{n+1}\}$. Indeed,

$$\lim_{t \to t_n+} (\mathbf{f}^n(t), \mathbf{a})_{\mathcal{H}_n} = (\mathbf{f}^n(t_n+), \mathbf{a})_{\mathcal{H}_n} = \sum_{k=1}^{s_n} \mathbf{f}^n_{k,1}(t_n+)\omega_n^{-ik} \sum_{m=1}^{s_{n+1}} \omega_{n+1}^{-jm} = 0.$$

Here we employed the continuity of f, $\mathbf{f}^n_{k,j}(t_n+) = \mathbf{f}^n_{k,1}(t_n+)$ for all $j \in \{1, \dots, s_{n+1}\}$, together with (2.8). This shows that $\tilde{\mathbf{f}}$ satisfies the first condition in (iv).

Next observe that

$$\sum_{m=1}^{s_{n+1}} (\tilde{\mathbf{f}}^n_{k,m})'(t_n+) = \frac{\omega_n^{ik}}{s_n s_{n+1}} ((\mathbf{f}^n)'(t_n+), \mathbf{a})_{\mathcal{H}_n} \sum_{m=1}^{s_{n+1}} \omega_{n+1}^{jm} = 0$$

for all $k \in \{1, \dots, s_n\}$. Since $(\tilde{\mathbf{f}}^{n-1})' = 0$, $\tilde{\mathbf{f}}$ satisfies (iv) at t_n. Similar arguments shows that (iv) holds true at t_{n+1} as well. This finishes the proof of the inclusion $\tilde{f} = P^n_{\mathbf{a}} f \in \operatorname{dom}(\mathbf{H}_0)$.

Finally, noting that

$$\begin{aligned} U^n_{\mathbf{a}} \colon L^2(I_n) &\longrightarrow \mathcal{F}^n_{\mathbf{a}}, \\ f &\longmapsto f \cdot \tfrac{\mathbf{a}}{\|\mathbf{a}\|}, \end{aligned} \tag{3.13}$$

establishes an isometric isomorphism of $L^2(I_n)$ onto $\mathcal{F}^n_{\mathbf{a}}$, it is straightforward to verify the last claim and we leave it to the reader. □

3.4. Restriction to $\mathfrak{F}_n$. Next, we show that $\mathfrak{F}_n$, $n \geq 1$ is reducing for $\mathbf{H}_0$ as well and the corresponding restriction is unitarily equivalent to $s_n - 1$ copies of the operator $\tilde{\mathbf{h}}_n$ defined by

$$\tilde{\tau}_n = -\frac{1}{\mu(t)}\frac{d}{dt}\mu(t)\frac{d}{dt},$$

on $L^2((t_{n-1}, t_{n+1}); \mu)$ and equipped with Dirichlet conditions at the endpoints. Here the weight function μ is defined by (3.4). The domain of $\tilde{\mathbf{h}}_n$ admits a very simple description since inside I_{n-1} and I_n the differential expression $\tilde{\tau}_n$ reduces to the negative second derivative and hence $\mathrm{dom}(\tilde{\mathbf{h}}_n)$ consists of functions which are H^2 in I_{n-1} and I_n, satisfy the Dirichlet conditions at t_{n-1} and t_{n+1} and also the following coupling conditions at t_n:

$$\begin{cases} f(t_n+) = f(t_n-), \\ s_{n-1} f'(t_n-) = s_{n+1} f'(t_n+). \end{cases} \tag{3.14}$$

Recall that $\mathfrak{F}_n = \mathrm{ran}(P_n)$, where the projection P_n is given by (2.24). By (2.8) and (2.5),

$$\mathbf{a}_{n-1}^{s_{n-1},j} = \mathbf{1}_{s_{n-1}} \otimes \mathbf{a}_{s_n}^j, \quad \mathbf{a}_n^{j,s_{n+1}} = \mathbf{a}_{s_n}^j \otimes \mathbf{1}_{s_{n+1}},$$

and hence

$$P_n = \sum_{j=1}^{s_n - 1} (P^{n-1}_{\mathbf{1}_{s_{n-1}} \otimes \mathbf{a}_{s_n}^j} + P^n_{\mathbf{a}_{s_n}^j \otimes \mathbf{1}_{s_{n+1}}}). \tag{3.15}$$

Denoting the summands in (3.15) by $\widetilde{P}_n^j$, $j \in \{1, \dots, s_n - 1\}$, we set

$$\widetilde{\mathfrak{F}}_n^j := \mathrm{ran}(\widetilde{P}_n^j) = \mathfrak{F}^{n-1}_{\mathbf{1}_{s_{n-1}} \otimes \mathbf{a}_{s_n}^j} \oplus \mathfrak{F}^n_{\mathbf{a}_{s_n}^j \otimes \mathbf{1}_{s_{n+1}}}. \tag{3.16}$$

Since $\mathfrak{F}_n = \bigoplus_{j=1}^{s_n-1} \widetilde{\mathfrak{F}}_n^j$, these claims will follow from the following lemma:

Lemma 3.4. *Every subspace $\widetilde{\mathfrak{F}}_n^j$ with $n \geq 1$ and $j \in \{1, \dots, s_n - 1\}$, is reducing for the operator $\mathbf{H}_0$. Moreover, its restriction onto $\widetilde{\mathfrak{F}}_n^j$ is unitarily equivalent to $\tilde{\mathbf{h}}_n$.*

Proof. Since $\widetilde{\mathfrak{F}}_n^j$ is invariant for $\mathbf{H}_0$ and $\mathbf{H}_0$ is symmetric, we only need to show that for every $f \in \mathrm{dom}(\mathbf{H}_0)$ its projection $\tilde{f} := \widetilde{P}_n^j f$ onto $\widetilde{\mathfrak{F}}_n^j$ also belongs to $\mathrm{dom}(\mathbf{H}_0)$. Following step by step the proof of Lemma 3.3, we only need to show that $\tilde{\mathbf{f}} := U\tilde{f}$ satisfies condition (iv) of Lemma 3.1 at t_n.

First observe that by (2.21)

$$\tilde{\mathbf{f}}(t) = \begin{cases} \tilde{f}_{n-1}(t)(\mathbf{1}_{s_{n-1}} \otimes \mathbf{a}_{s_n}^j), & t \in I_{n-1}, \\ \tilde{f}_n(t)(\mathbf{a}_{s_n}^j \otimes \mathbf{1}_{s_{n+1}}), & t \in I_n, \end{cases} \tag{3.17}$$

where

$$\tilde{f}_{n-1}(t) = \frac{1}{s_{n-1}s_n}(\mathbf{f}^{n-1}(t), \mathbf{a}_{n-1}^{s_{n-1},j})_{\mathcal{H}_{n-1}}, \quad \tilde{f}_n(t) = \frac{1}{s_n s_{n+1}}(\mathbf{f}^n(t), \mathbf{a}_n^{j,s_{n+1}})_{\mathcal{H}_n}.$$

Notice that

$$\tilde{f}_{n-1}(t_n-) = \frac{1}{s_{n-1}s_n}\sum_{k=1}^{s_{n-1}}\sum_{m=1}^{s_n} \mathbf{f}_{k,m}^{n-1}(t_n-)\omega_n^{-jm} = \frac{1}{s_n}\sum_{m=1}^{s_n} \mathbf{f}_{1,m}^{n-1}(t_n-)\omega_n^{-jm}$$

and

$$\tilde{f}_n(t_n+) = \frac{1}{s_n s_{n+1}}\sum_{m=1}^{s_n}\sum_{k=1}^{s_{n+1}} \mathbf{f}_{m,k}^{n}(t_n+)\omega_n^{-jm} = \frac{1}{s_n}\sum_{m=1}^{s_n} \mathbf{f}_{m,1}^{n}(t_n+)\omega_n^{-jm}.$$

However, by Lemma 3.1,

$$\mathbf{f}_{1,m}^{n-1}(t_n-) = \mathbf{f}_{m,1}^{n}(t_n+), \quad m \in \{1, \ldots, s_n\},$$

and hence we get

$$\begin{aligned}\tilde{\mathbf{f}}_{1,k}^{n-1}(t_n-) &= \frac{\omega_n^{jk}}{s_n}\sum_{m=1}^{s_n} \mathbf{f}_{1,m}^{n-1}(t_n-)\omega_n^{-jm} \\ &= \frac{\omega_n^{jk}}{s_n}\sum_{m=1}^{s_n} \mathbf{f}_{m,1}^{n}(t_n+)\omega_n^{-jm} \\ &= \tilde{\mathbf{f}}_{k,1}^{n}(t_n+)\end{aligned}$$

for all $k \in \{1, \ldots, s_n\}$. This shows that $\tilde{\mathbf{f}}$ satisfies the first equality in condition (iv) of Lemma 3.1. Let us check the second one. However, we have

$$\begin{aligned}\sum_{k=1}^{s_{n-1}} (\tilde{\mathbf{f}}_{k,m}^{n-1})'(t_n-) &= \sum_{k=1}^{s_{n-1}} \tilde{f}_{n-1}'(t_n-)\omega_n^{jm} = s_{n-1}\tilde{f}_{n-1}'(t_n-)\omega_n^{jm} \\ &= \frac{\omega_n^{jm}}{s_n}\sum_{l=1}^{s_n}\omega_n^{-jl}\sum_{k=1}^{s_{n-1}} (\mathbf{f}_{k,l}^{n-1})'(t_n-) \\ &= \frac{\omega_n^{jm}}{s_n}\sum_{l=1}^{s_n}\omega_n^{-jl}\sum_{k=1}^{s_{n+1}} (\mathbf{f}_{l,k}^{n})'(t_n+) = s_{n+1}\tilde{f}_n'(t_n+)\omega_n^{jm} \\ &= \sum_{k=1}^{s_{n+1}} \tilde{f}_n'(t_n+)\omega_n^{jm} \\ &= \sum_{k=1}^{s_{n+1}} (\tilde{\mathbf{f}}_{m,k}^{n})'(t_n+).\end{aligned}$$

This shows that $\tilde{\mathbf{f}}$ satisfies all the conditions of Lemma 3.1 and hence that $\tilde{f} \in \mathrm{dom}(\mathbf{H}_0)$.

It remains to notice that the map $U_n^j : L^2((t_{n-1}, t_{n+1}); \mu) \to \widetilde{\mathfrak{F}}_n^j$ defined by (3.17) is an isometric isomorphism and $(U_n^j)^{-1}(\mathbf{H}_0 \restriction \widetilde{\mathfrak{F}}_n^j)U_n^j = \tilde{\mathbf{h}}_n$. □

3.5. The decomposition of the operator H. Combining the results of Sections 3.2–3.4, we arrive at the following decomposition of quantum graph operators on radially symmetric anti-trees.

Theorem 3.5. *Let $\mathcal{A}$ be an infinite radially symmetric antitree. The decomposition (2.19) reduces the operator* $\mathbf{H}$. *Moreover, with respect to this decomposition,* $\mathbf{H}$ *is unitarily equivalent to the following orthogonal sum of Sturm–Liouville operators*

$$\mathrm{H} \oplus \bigoplus_{n\ge 1} \Big(\bigoplus_{j=1}^{(s_n-1)(s_{n+1}-1)} \mathbf{h}_n \Big) \oplus \bigoplus_{n\ge 1} \Big(\bigoplus_{j=1}^{s_n-1} \tilde{\mathbf{h}}_n \Big), \tag{3.18}$$

where $\mathrm{H} = \overline{\mathrm{H}_0}$ *and the operators* H_0, $\mathbf{h}_n$, *and* $\tilde{\mathbf{h}}_n$ *are defined in Sections* 3.2, 3.3, *and* 3.4, *respectively.*

4. Self-adjointness

Theorem 3.5 reduces the spectral analysis of quantum graph operators on radially symmetric antitrees to the analysis of certain classes of Sturm–Liouville operators. Moreover, the Sturm–Liouville operators $\mathbf{h}_n$ and $\tilde{\mathbf{h}}_n$ in the decomposition (3.18) are self-adjoint for all $n \ge 1$ and their spectra can be computed explicitly. This enables us to perform a rather detailed study of spectral properties of the operator $\mathbf{H} = \overline{\mathbf{H}_0}$. We begin with the characterization of self-adjoint extensions of the operator $\mathbf{H}$.

Theorem 4.1. *Let $\mathcal{A}$ be an infinite radially symmetric antitree.*

(i) *The operator* $\mathbf{H}$ *is self-adjoint if and only if the total volume of $\mathcal{A}$ is infinite,*

$$\mathrm{vol}(\mathcal{A}) := \sum_{e\in\mathcal{E}(\mathcal{A})} |e| = \sum_{n\ge 0} s_n s_{n+1} \ell_n = \infty. \tag{4.1}$$

(ii) *If* $\mathrm{vol}(\mathcal{A}) < \infty$, *then the deficiency indices of* $\mathbf{H}$ *equal* 1 *and self-adjoint extensions of* $\mathbf{H}$ *form a one-parameter family* $\mathbf{H}_\theta := \mathbf{H}^* \restriction \mathrm{dom}(\mathbf{H}_\theta)$, $\theta \in [0, \pi)$, *and*

$$\mathrm{dom}(\mathbf{H}_\theta) = \{f \in \mathrm{dom}(\mathbf{H}^*) \mid \cos(\theta) f(\mathcal{L}) + \sin(\theta) f'(\mathcal{L}) = 0\},$$

where

$$f(\mathcal{L}) := \lim_{t\to\mathcal{L}} (U_s P_{\mathrm{sym}} f)(t), \tag{4.2}$$

$$f'(\mathcal{L}) := \lim_{t\to\mathcal{L}} \mu(t)(U_s P_{\mathrm{sym}} f)'(t), \tag{4.3}$$

and the operators P_{sym} and U_s are given, respectively, by (2.22) *and* (3.10).

Proof. (i) By Theorem 3.5, the operator $\mathbf{H}$ is self-adjoint only if so are the operators on the right-hand side in the decomposition (3.18). However, both $\mathbf{h}_n$ and $\tilde{\mathbf{h}}_n$ are self-adjoint for all $n \ge 1$. The self-adjointness criterion for $\mathrm{H} = \overline{\mathrm{H}_0}$ follows from the standard limit point/limit circle classification (see, e.g., [46]). Namely, the equation $\tau y = 0$ with τ given by (3.5), has two linearly independent solutions

$$y_1(t) \equiv 1, \quad y_2(t) = \int_0^t \frac{ds}{\mu(s)}.$$

Now one simply needs to verify whether or not both solutions y_1 and y_2 belong to $L^2(\mathcal{I}_\mathcal{L};\mu)$. Clearly, $y_1 \in L^2(\mathcal{I}_\mathcal{L};\mu)$ exactly when the series in (4.1) converges. Moreover, it is straightforward to check that $y_2 \in L^2(\mathcal{I}_\mathcal{L};\mu)$ if and only if the series

$$\sum_{n\ge 0} s_n s_{n+1} \ell_n \Big(\sum_{k\le n} \frac{\ell_k}{s_k s_{k+1}}\Big)^2 \tag{4.4}$$

converges. Since $s_n s_{n+1} \ge 1$ for all $n \ge 0$, this series converges exactly when the series in (4.1) converges. The Weyl alternative finishes the proof of (i).

(ii) The above considerations imply that the deficiency indices of $\mathbf{H}$ and H coincide. However, the deficiency indices of H are at most 1. Thus, if the operator H is not self-adjoint, its deficiency indices equal 1. Moreover, one can easily describe all self-adjoint extensions of H. First of all, for every $g \in \mathrm{dom}(\mathrm{H}_0^*) = \mathrm{dom}(\mathrm{H}^*)$ the following limits

$$\lim_{t\to\mathcal{L}} W_t(g, y_1), \quad \lim_{t\to\mathcal{L}} W_t(g, y_2)$$

exist and are finite (see, e.g., [46]). Here $W_t(g,h) = g(t)(\mu h')(t) - (\mu g')(t)h(t)$ is the modified Wronskian. Thus for every $g \in \mathrm{dom}(\mathrm{H}_0^*)$ the following limits

$$g(\mathcal{L}) := \lim_{t\to\mathcal{L}} g(t), \quad g'_\mu(\mathcal{L}) := \lim_{t\to\mathcal{L}} \mu(t) g'(t) \tag{4.5}$$

exist and are finite. Hence self-adjoint extensions of H form a one-parameter family

$$\mathrm{dom}(\mathrm{H}(\theta)) = \{g \in \mathrm{dom}(\mathrm{H}_0^*) \mid \cos(\theta) g(\mathcal{L}) + \sin(\theta) g'_\mu(\mathcal{L}) = 0\}, \quad \theta \in [0,\pi).$$

It remains to use (3.11) and (2.22). □

Remark 4.2. Let us mention that in the case $\mathrm{vol}(\mathcal{A}) < \infty$ the Friedrichs extension of $\mathbf{H}$ coincides with the operator $\mathbf{H}_\theta$ with $\theta = 0$. Moreover, it is possible to show that in fact the limits in (4.2) and (4.3) coincide with

$$\lim_{|x|\to\mathcal{L}} f(x), \quad \lim_{t\to\mathcal{L}} \sum_{|x|=t} f'(x)$$

for every f in the domain of $\mathbf{H}^*$. In particular, this would imply that the Friedrichs extension of $\mathbf{H}$ is simply given as the restriction of $\mathbf{H}^*$ to functions vanishing at $\mathcal{L}$. Let us also mention that $\mathbf{H}^* = \mathbf{H}_0^*$ in fact coincides with the maximal operator, that is $\mathrm{dom}(\mathbf{H}^*)$ consists of functions $f \in L^2(\mathcal{A}) \cap H^2(\mathcal{A}\setminus\mathcal{V})$ satisfying boundary conditions (3.1) for all $v \in \mathcal{V}$ and such that $f'' \in L^2(\mathcal{A})$.

5. Discreteness

As an immediate corollary of Theorem 4.1 we obtain the following result.

Corollary 5.1. *If* $\mathrm{vol}(\mathcal{A}) < \infty$*, then the spectrum of each self-adjoint extension* $\mathbf{H}_\theta$ *of* $\mathbf{H}$ *is purely discrete and, moreover,*

$$N(\lambda; \mathbf{H}_\theta) = \frac{\mathrm{vol}(\mathcal{A})}{\pi}\sqrt{\lambda}(1 + o(1)), \quad \lambda \to \infty, \tag{5.1}$$

for all $\theta \in [0, \pi)$.

Here $N(\lambda; A)$ is the eigenvalue counting function of a (bounded from below) self-adjoint operator A with purely discrete spectrum. Namely,

$$N(\lambda; A) = \#\{k : \lambda_k(A) \le \lambda\},$$

where $\{\lambda_k(A)\}_{k\ge0}$ are the eigenvalues of A (counting multiplicities) ordered in the increasing order.

Proof. By Theorem 3.5,

$$\sigma(\mathbf{H}_\theta) = \sigma(\mathrm{H}(\theta)) \cup \overline{\bigcup_{n\ge1} \sigma(\mathbf{h}_n)} \cup \overline{\bigcup_{n\ge1} \sigma(\tilde{\mathbf{h}}_n)}. \tag{5.2}$$

Since $s_n \ge 1$ for all $n \ge 1$, $\mathrm{vol}(\mathcal{A}) < \infty$ implies that $\ell_n = o(1)$ as $n \to \infty$ and hence both sets $\bigcup_{n\ge1} \sigma(\mathbf{h}_n)$ and $\bigcup_{n\ge1} \sigma(\tilde{\mathbf{h}}_n)$ have no finite accumulation points. It remains to note that the spectrum of $\mathrm{H}(\theta)$ is discrete in this case as well.

According to the decomposition (3.18), we clearly have

$$N(\lambda;\mathbf{H}_\theta) = N(\lambda;\mathrm{H}(\theta)) + \sum_{n\geq 1}(s_n-1)(s_{n+1}-1)N(\lambda;\mathbf{h}_n) + \sum_{n\geq 1}(s_n-1)N(\lambda;\tilde{\mathbf{h}}_n).$$

It is well known that (cf., e.g., [19, Chapter 6.7])

$$N(\lambda;\mathrm{H}(\theta)) = \frac{\mathcal{L}}{\pi}\sqrt{\lambda}(1+o(1)), \quad \lambda\to\infty,$$

for all $\theta\in[0,\pi)$. Taking into account that

$$\sigma(\mathbf{h}_n) = \left\{\frac{\pi^2k^2}{\ell_n^2}\right\}_{k\geq 1}, \tag{5.3}$$

we clearly have

$$N(\lambda;\mathbf{h}_n) = \left\lfloor \frac{\ell_n}{\pi}\sqrt{\lambda}\right\rfloor \tag{5.4}$$

for all $\lambda\geq 0$, where $\lfloor\cdot\rfloor$ is the usual floor function. Moreover,

$$\left\lfloor \frac{\ell_{n-1}}{\pi}\sqrt{\lambda}\right\rfloor + \left\lfloor \frac{\ell_n}{\pi}\sqrt{\lambda}\right\rfloor \leq N(\lambda;\tilde{\mathbf{h}}_n) \leq \left\lfloor \frac{\ell_{n-1}}{\pi}\sqrt{\lambda}+\frac{1}{2}\right\rfloor + \left\lfloor \frac{\ell_n}{\pi}\sqrt{\lambda}+\frac{1}{2}\right\rfloor, \tag{5.5}$$

for all $\lambda>0$. The latter follows by employing the standard Dirichlet–Neumann bracketing, that is, one can estimate the eigenvalues of $\tilde{\mathbf{h}}_n$ via the eigenvalues of the operators $\tilde{\mathbf{h}}_n^D$ and $\tilde{\mathbf{h}}_n^N$ subject to Dirichlet, respectively, Neumann boundary conditions at t_n:

$$\lambda_k(\tilde{\mathbf{h}}_n^N) \leq \lambda_k(\tilde{\mathbf{h}}_n) \leq \lambda_k(\tilde{\mathbf{h}}_n^D), \quad k\geq 1. \tag{5.6}$$

Combining (5.4) with (5.5) and using a very simple estimate (see Lemma 5.2 below), we immediately arrive at (5.1). □

Lemma 5.2. *Let* $\{a_n\}_{n\geq 1}$ *and* $\{b_n\}_{n\geq 1}$ *be nonnegative sequences such that*

$$\lim_n b_n = 0 \quad \textit{and} \quad \sum_n a_n b_n < \infty.$$

Then for every $\alpha\in[0,1)$,

$$\lim_{\lambda\to\infty}\sum_{n\geq 1} a_n \frac{|b_n\lambda - \lfloor b_n\lambda+\alpha\rfloor|}{\lambda} = 0. \tag{5.7}$$

Proof. Indeed,

$$\sum_{n\geq 1} a_n \frac{|b_n\lambda - \lfloor b_n\lambda + \alpha\rfloor|}{\lambda} = \sum_{n:b_n<\frac{1-\alpha}{\lambda}} + \sum_{n:b_n\geq\frac{1-\alpha}{\lambda}} a_n \frac{|b_n\lambda - \lfloor b_n\lambda + \alpha\rfloor|}{\lambda}.$$

The first summand can be estimated as follows

$$\sum_{n:b_n<\frac{1-\alpha}{\lambda}} a_n \frac{|b_n\lambda - \lfloor b_n\lambda + \alpha\rfloor|}{\lambda} = \sum_{n:b_n<\frac{1-\alpha}{\lambda}} a_n b_n = o(1),$$

as $\lambda \to \infty$. Moreover, we have

$$\sum_{n:b_n\geq\frac{1-\alpha}{\lambda}} a_n \frac{|b_n\lambda - \lfloor b_n\lambda + \alpha\rfloor|}{\lambda} \leq \sum_{n:b_n\geq\frac{1-\alpha}{\lambda}} a_n \frac{1}{\lambda} = o(1),$$

as $\lambda \to \infty$, which proves the claim. □

Remark 5.3. We are not aware (except a few special cases) of a closed form of eigenvalues of $\tilde{\mathbf{h}}_n$. It is not difficult to show that $\sigma(\tilde{\mathbf{h}}_n)$ consists of simple positive eigenvalues $\{\tilde{\lambda}_k\}_{k\geq 1}$ satisfying (5.5) and even to express $\sigma(\tilde{\mathbf{h}}_n)$ with the help of the *arctangent function with two arguments*, although this does not lead to a closed formula.

In the case $\mathrm{vol}(\mathcal{A}) = \infty$, the spectrum of $\mathbf{H}$ may have a rather complicated structure. In particular, it may not be purely discrete. The next result provides a criterion for $\mathbf{H}$ to have purely discrete spectrum. Set

$$\mathcal{L}_\mu := \int_0^{\mathcal{L}} \frac{dx}{\mu(x)} = \sum_{n\geq 0} \frac{\ell_n}{s_n s_{n+1}}. \tag{5.8}$$

Theorem 5.4. *Let $\mathcal{A}$ be an infinite radially symmetric antitree with* $\mathrm{vol}(\mathcal{A}) = \infty$. *Then the spectrum of* $\mathbf{H}$ *is discrete if and only if the following conditions are satisfied*:

(i) $\ell_n \to 0$ *as* $n \to \infty$;

(ii) $\mathcal{L}_\mu < \infty$;

(iii) *we have*

$$\lim_{n\to\infty} \sum_{k=0}^{n} s_k s_{k+1} \ell_k \sum_{k\geq n} \frac{\ell_k}{s_k s_{k+1}} = 0. \tag{5.9}$$

Proof. Denote

$$\mathbf{H}^1 := \bigoplus_{n\ge 1}\Big(\bigoplus_{j=1}^{(s_n-1)(s_{n+1}-1)} \mathbf{h}_n\Big), \qquad \mathbf{H}^2 := \bigoplus_{n\ge 1}\Big(\bigoplus_{j=1}^{s_n-1} \tilde{\mathbf{h}}_n\Big). \tag{5.10}$$

By Theorem 4.1(i), **H** is self-adjoint and hence (3.18) implies that

$$\sigma(\mathbf{H}) = \sigma(\mathrm{H}) \cup \sigma(\mathbf{H}^1) \cup \sigma(\mathbf{H}^2) = \sigma(\mathrm{H}) \cup \overline{\bigcup_{n\ge 1}\sigma(\mathbf{h}_n)} \cup \overline{\bigcup_{n\ge 1}\sigma(\tilde{\mathbf{h}}_n)}. \tag{5.11}$$

Thus the spectrum of **H** is discrete if and only if the spectra of all three operators H, $\mathbf{H}^1$ and $\mathbf{H}^2$ are discrete.

In order to investigate the operator H, we need to transform it to the Krein string form by using a suitable change of variables ($x \mapsto \int_0^x \frac{ds}{\mu(s)}$) and then to apply the Kac–Krein criterion [26]. To be more precise, it is straightforward to verify that H is unitarily equivalent to the operator $\tilde{\mathrm{h}}$ defined in the Hilbert space $L^2([0,\mathcal{L}_\mu);\tilde{\mu})$ by the differential expression

$$\tilde{\tau} = -\frac{1}{\tilde{\mu}(x)}\frac{d^2}{dx^2} \tag{5.12}$$

and subject to the Neumann boundary condition at $x = 0$. Here

$$\tilde{\mu} := \mu^2 \circ g^{-1}, \tag{5.13}$$

where g^{-1} is the inverse of the function $g\colon [0,\mathcal{L}) \to [0,\mathcal{L}_\mu)$ given by

$$g(x) = \int_0^x \frac{ds}{\mu(s)}, \quad \mathcal{L}_\mu := g(\mathcal{L}) = \int_0^{\mathcal{L}} \frac{ds}{\mu(s)}. \tag{5.14}$$

Notice that g is strictly increasing and locally absolutely continuous on $[0,\mathcal{L})$ and maps $[0,\mathcal{L})$ onto $[0,\mathcal{L}_\mu)$. Hence its inverse $g^{-1}\colon [0,\mathcal{L}_\mu) \to [0,\mathcal{L})$ is also strictly increasing and locally absolutely continuous on $[0,\mathcal{L}_\mu)$.

Applying the Kac–Krein criterion (see [26], [27, §11.9]), we conclude that H has purely discrete spectrum if and only if $\mathcal{L}_\mu < \infty$ and

$$\lim_{x\to\mathcal{L}} \Phi(x) = 0, \tag{5.15}$$

where $\Phi\colon [0,\mathcal{L}) \to \mathbb{R}_{\ge 0}$ is given by

$$\Phi(x) := \int_0^x \mu(s)ds \cdot \int_x^{\mathcal{L}} \frac{ds}{\mu(s)}, \quad x \in [0,\mathcal{L}). \tag{5.16}$$

First of all, observe that

$$\Phi(x) \le \int_0^{t_{n+1}} \mu(s)ds \cdot \int_{t_n}^{\mathcal{L}} \frac{ds}{\mu(s)} = \sum_{k=0}^{n} s_k s_{k+1}\ell_k \sum_{k\ge n} \frac{\ell_k}{s_k s_{k+1}}$$

for all $x \in [t_n, t_{n+1})$ and hence sufficiency of (5.9) follows. Moreover, straightforward calculations show that

$$\Phi\Big(\frac{t_n + t_{n+1}}{2}\Big) = \Big(\sum_{k=0}^{n-1} s_k s_{k+1}\ell_k + s_n s_{n+1}\frac{\ell_n}{2}\Big)\Big(\sum_{k\ge n+1} \frac{\ell_k}{s_k s_{k+1}} + \frac{\ell_n}{2 s_n s_{n+1}}\Big)$$
$$\ge \frac{1}{4}\sum_{k=0}^{n} s_k s_{k+1}\ell_k \sum_{k\ge n} \frac{\ell_k}{s_k s_{k+1}},$$

which implies the necessity of (5.9). Notice also that the right-hand side in the last inequality is strictly greater than $\frac{1}{4}\ell_n^2$, which also implies (i).

It remains to note that the spectra of the operators $\mathbf{H}^1$ and $\mathbf{H}^2$ are discrete if condition (i) is satisfied (see (5.3) and (5.4)). □

Remark 5.5. Let us mention that in fact both conditions (i) and (ii) in Theorem 5.4 follow from (iii).

If $\mathrm{vol}(\mathcal{A}) = \infty$ and the corresponding Hamiltonian $\mathbf{H}$ has purely discrete spectrum, it follows from the proof of Weyl's law (5.1) that $\frac{N(\lambda;\mathbf{H})}{\sqrt{\lambda}} \to \infty$ as $\lambda \to \infty$. However, we can characterize radially symmetric antitress such that the resolvent of the corresponding quantum graph operator $\mathbf{H}$ belongs to the trace class.

Theorem 5.6. *Let $\mathcal{A}$ be an infinite radially symmetric antitree with* $\mathrm{vol}(\mathcal{A}) = \infty$. *Also, let the spectrum of* $\mathbf{H}$ *be purely discrete. Then*[3]

$$\sum_{\lambda\in\sigma(\mathbf{H})} \frac{1}{\lambda} < \infty \tag{5.17}$$

if and only if

$$\sum_{n\ge 1} s_n s_{n+1}\ell_n^2 < \infty, \tag{5.18}$$

and

$$\sum_{n\ge 0} \frac{\ell_n}{s_n s_{n+1}} \sum_{k=0}^{n-1} s_k s_{k+1}\ell_k < \infty. \tag{5.19}$$

[3] The summation in (5.17) is according to multiplicities.

Proof. As in the proof of Theorem 5.4, observe that the spectrum of $\mathbf{H}$ consists of three sets of eigenvalues. Let us denote the second and the third summands in (3.18) by $\mathbf{H}^1$ and $\mathbf{H}^2$, respectively. The spectrum of the self-adjoint operator $\mathbf{h}_n$ is given by (5.3) and hence

$$\sum_{\lambda\in\sigma(\mathbf{H}^2)}\frac{1}{\lambda}=\sum_{n\geq 1}(s_n-1)(s_{n+1}-1)\sum_{k\geq 1}\frac{\ell_n^2}{\pi^2k^2}=\frac{1}{6}\sum_{n\geq 1}(s_n-1)(s_{n+1}-1)\ell_n^2. \quad (5.20)$$

Similarly, using the Dirichlet–Neumann bracketing (5.6), we get

$$\begin{aligned}\sum_{\lambda\in\sigma(\mathbf{H}^1)}\frac{1}{\lambda}&\leq\sum_{n\geq 1}(s_n-1)\sum_{\lambda\in\sigma(\tilde{\mathbf{h}}_n^N)}\frac{1}{\lambda}\\&=\sum_{n\geq 1}(s_n-1)\sum_{k\geq 1}\frac{\ell_{n-1}^2}{\pi^2(k-1/2)^2}+\frac{\ell_n^2}{\pi^2(k-1/2)^2}\\&=\frac{1}{2}\sum_{n\geq 1}(s_n-1)(\ell_{n-1}^2+\ell_n^2)\leq\frac{1}{2}\sum_{n\geq 0}(s_n+s_{n+1}-2)\ell_n^2.\end{aligned}$$

Using the Dirichlet eigenvalues, one can prove a similar bound from below. Moreover, combining the latter with (5.18) implies that the resolvents of both $\mathbf{H}_1$ and $\mathbf{H}_2$ belong to the trace class exactly when

$$\sum_{n\geq 1}(s_ns_{n+1}-1)\ell_n^2<\infty. \quad (5.21)$$

Next observe that $0\in\sigma(\mathrm{H})$ exactly when $\mathbf{1}\in L^2(\mathcal{I}_{\mathcal{L}};\mu)$, which is equivalent to $\mathrm{vol}(\mathcal{A})<\infty$. Thus 0 is not an eigenvalue of $\mathbf{H}$ whenever $\mathrm{vol}(\mathcal{A})=\infty$. Finally, applying M. G. Krein's theorem to the operator H (see [26], [27, §11.10]), we conclude that H^{-1} is trace class if and only if $\mathcal{L}_\mu<\infty$ and

$$\int_0^{\mathcal{L}}\frac{1}{\mu(x)}\int_0^x\mu(s)ds\,dx<\infty. \quad (5.22)$$

However, using (3.4), we get

$$\begin{aligned}\int_0^{\mathcal{L}}\frac{1}{\mu(x)}\int_0^x\mu(s)ds\,dx&=\sum_{n\geq 0}\int_{t_n}^{t_{n+1}}\frac{1}{\mu(x)}\int_0^x\mu(s)ds\,dx\\&=\sum_{n\geq 0}\frac{1}{s_ns_{n+1}}\int_{t_n}^{t_{n+1}}\Big(\sum_{k=0}^{n-1}s_ks_{k+1}\ell_k+s_ns_{n+1}(x-t_n)\Big)\,dx\\&=\sum_{n\geq 0}\frac{\ell_n}{s_ns_{n+1}}\sum_{k=0}^{n-1}s_ks_{k+1}\ell_k+\frac{1}{2}\sum_{n\geq 0}\ell_n^2.\end{aligned}$$

Notice that the latter in particular shows that $\{\ell_n\}_{n\geq 0} \in \ell^2$ and combining this fact with (5.21) we arrive at (5.18). This completes the proof. $\square$

Remark 5.7. Using the same arguments and the results from [28, 41] one would be able to characterize radially symmetric antitrees such that the resolvent of the corresponding Kirchhoff Laplacian belongs to the Schatten–von Neumann ideal $\mathfrak{S}_p$, $p \in (1/2, \infty)$ (and even to other trace ideals), however, these results look cumbersome and we decided not to include them.

6. Spectral gap estimates

We restrict our discussion to the case $\mathrm{vol}(\mathcal{A}) = \infty$ for several reasons. Of course, the main one is the fact that in this case $\mathbf{H}_0$ is essentially self-adjoint and this simplifies some considerations. However, for finite volume metric graphs the corresponding estimates remain to be true for the Friedrichs extension of $\mathbf{H}_0$.

Theorem 6.1. *Let $\mathcal{A}$ be an infinite radially symmetric antitree with* $\mathrm{vol}(\mathcal{A}) = \infty$. *Then the bottom of the spectrum $\lambda_0(\mathbf{H}) = \inf \sigma(\mathbf{H})$ of $\mathbf{H}$ is strictly positive if and only if the following conditions are satisfied*:

(i) $\ell^*(\mathcal{A}) = \sup_{n\geq 0} \ell_n < \infty$;

(ii) $\mathcal{L}_\mu = \sum_{n\geq 0} \frac{\ell_n}{s_n s_{n+1}} < \infty$;

(iii) *we have*

$$C(\mathcal{L}) := \sup_{x\in(0,\mathcal{L})} \int_0^x \mu(s)ds \cdot \int_x^{\mathcal{L}} \frac{ds}{\mu(s)} < \infty. \tag{6.1}$$

Moreover, we have the following estimate

$$\frac{1}{4C(\mathcal{L})} \leq \lambda_0(\mathbf{H}) \leq \frac{1}{C(\mathcal{L})}. \tag{6.2}$$

Proof. Since $\mathrm{vol}(\mathcal{A}) = \infty$, the operator $\mathbf{H}$ is self-adjoint by Theorem 4.1. Moreover, by Theorem 3.5, we have

$$\lambda_0(\mathbf{H}) = \min\{\lambda_0(\mathrm{H}), \lambda_0(\mathbf{H}^1), \lambda_0(\mathbf{H}^2)\}, \tag{6.3}$$

where $\mathbf{H}^1$ and $\mathbf{H}^2$ are given by (5.10). Observe that

$$\lambda_0(\mathbf{H}) = \lambda_0(\mathrm{H}). \tag{6.4}$$

Indeed, it suffices to compare the domains of H_0 and $\mathbf{h}_n$, $\tilde{\mathbf{h}}_n$ and then exploit the Rayleigh quotient. For instance,

$$\begin{aligned}\lambda_0(\mathrm{H}) &= \inf_{\substack{f\in\mathrm{dom}(\mathrm{H}_0)\\ f\neq 0}} \frac{(\mathrm{H}f,f)_{L^2(\mathcal{I}_{\mathcal{L}};\mu)}}{\|f\|^2_{L^2(\mathcal{I}_{\mathcal{L}};\mu)}}\\ &\le \inf_{\substack{f\in\mathrm{dom}(\mathrm{H}_0)\\ \mathrm{supp}(f)\subset[t_{n-1},t_{n+1}]}} \frac{(\mathrm{H}f,f)_{L^2(\mathcal{I}_{\mathcal{L}};\mu)}}{\|f\|^2_{L^2(\mathcal{I}_{\mathcal{L}};\mu)}}\\ &\le \inf_{\substack{f\in\mathrm{dom}(\tilde{\mathbf{h}}_n)\\ f\neq 0}} \frac{(\tilde{\mathbf{h}}_n f,f)_{L^2(I_{n-1}\cup I_n;\mu)}}{\|f\|^2_{L^2(I_{n-1}\cup I_n;\mu)}}\\ &= \lambda_0(\tilde{\mathbf{h}}_n).\end{aligned}$$

The operator H can be studied in the framework of Krein strings, however, we need to apply the Kac–Krein criteria [26] to the *dual string* since both Corollary 1.1 and Remark 2.2 in [26] are stated subject to the Dirichlet boundary condition at $x = 0$. For a detailed discussion of dual strings we refer to [27, §12] and the desired connection is [27, equality (12.6)][4]. More precisely, assuming $\mathcal{L}_\mu < \infty$ and then applying Theorem 1 from [26], we get the estimate

$$x(M^{-1}(\infty) - M^{-1}(x)) \le \frac{1}{\lambda_0(\mathrm{H})}, \tag{6.5}$$

which holds for all $x > 0$. Here M^{-1} denotes the inverse to the function $M\colon [0,\mathcal{L}_\mu) \to [0,\infty)$ defined by (see also (5.13) and (5.14))

$$M(x) := \int_0^x \tilde{\mu}(s)ds = \int_0^x (\mu^2 \circ g^{-1})(s)ds = \int_0^{g^{-1}(x)} \mu(s)ds. \tag{6.6}$$

Notice that M is a strictly increasing absolutely continuous function mapping $[0,\mathcal{L}_\mu)$ onto $[0,\infty)$ (the latter follows from the assumption $\mathrm{vol}(\mathcal{A}) = \infty$). Thus equation (6.5) is equivalent to

$$M(x)(\mathcal{L}_\mu - x) \le \frac{1}{\lambda_0(\mathrm{H})}, \quad x \in (0,\mathcal{L}_\mu). \tag{6.7}$$

[4] This statement can be seen as the analog of the *abstract commutation*: for a closed operator A acting in a Hilbert space $\mathfrak{H}$, the operators $(A^*A)\restriction_{\ker(A)^\perp}$ and $(AA^*)\restriction_{\ker(A^*)^\perp}$ are unitarily equivalent.

By changing variables, we end up with the following estimate

$$\sup_{x\in(0,\mathcal{L})} \int_0^x \mu(s)ds \cdot \int_x^{\mathcal{L}} \frac{ds}{\mu(s)} \le \frac{1}{\lambda_0(\mathrm{H})}. \tag{6.8}$$

Applying Theorem 3 from [26] and using the same arguments, we end up with the lower bound

$$\frac{1}{4\lambda_0(\mathrm{H})} \le \sup_{x\in(0,\mathcal{L})} \int_0^x \mu(s)ds \cdot \int_x^{\mathcal{L}} \frac{ds}{\mu(s)}. \tag{6.9}$$

Taking into account [26, Remark 2.2], we conclude that the condition $\mathcal{L}_\mu < \infty$ is also necessary for the positivity of $\lambda_0(\mathrm{H})$. It remains to note that the necessity of (i) follows from (iii). Indeed, assuming the converse, that is, there is a sequence of lengths ℓ_{n_k} tending to infinity, and then choosing x_{n_k} as the middle points of the corresponding intervals, one immediately concludes that $C(\mathcal{L}) = \infty$ by evaluating (6.1) at x_{n_k}. □

Remark 6.2. Arguing as in the proof of Theorem 5.4 one can show that conditions (i)–(iii) in Theorem 6.1 can be replaced by the single condition

$$\sup_{n\ge 0} \sum_{k=0}^{n} s_k s_{k+1} \ell_k \sum_{k\ge n} \frac{\ell_k}{s_k s_{k+1}} < \infty. \tag{6.10}$$

However, this expression provides only an upper bound on $C(\mathcal{L})$:

$$\sup_{n\ge 0} \sum_{k=0}^{n} s_k s_{k+1} \ell_k \sum_{k\ge n+1} \frac{\ell_k}{s_k s_{k+1}} \le C(\mathcal{L}) \le \sup_{n\ge 0} \sum_{k=0}^{n} s_k s_{k+1} \ell_k \sum_{k\ge n} \frac{\ell_k}{s_k s_{k+1}}. \tag{6.11}$$

Since 0 is not an eigenvalue of **H** if $\mathrm{vol}(\mathcal{A}) = \infty$, $\lambda_0(\mathbf{H}) > 0$ is equivalent to $\lambda_0^{\mathrm{ess}}(\mathbf{H}) > 0$, where $\lambda_0^{\mathrm{ess}}(\mathbf{H})$ denotes the bottom of the essential spectrum of **H**, $\lambda_0^{\mathrm{ess}}(\mathbf{H}) := \inf \sigma_{\mathrm{ess}}(\mathbf{H})$. Thus Theorem 6.1 also provides a criterion for $\lambda_0^{\mathrm{ess}}(\mathbf{H})$ to be strictly positive. Moreover, by employing Glazman's decomposition principle one can prove a similar to (6.1) bound on $\lambda_0^{\mathrm{ess}}(\mathbf{H})$.

Theorem 6.3. *Let $\mathcal{A}$ be an infinite radially symmetric antitree with* $\mathrm{vol}(\mathcal{A}) = \infty$. *Then $\lambda_0^{\mathrm{ess}}(\mathbf{H}) > 0$ if and only if (6.10) holds true. Moreover,*

$$\frac{1}{4C_{\mathrm{ess}}(\mathcal{L})} \le \lambda_0^{\mathrm{ess}}(\mathbf{H}) \le \frac{1}{C_{\mathrm{ess}}(\mathcal{L})}, \tag{6.12}$$

where the constant $C_{\mathrm{ess}}(\mathcal{L})$ *is given by*

$$C_{\mathrm{ess}}(\mathcal{L}) = \lim_{x\to\mathcal{L}} \sup_{y\in(x,\mathcal{L})} \int_x^y \mu(s)ds \cdot \int_y^{\mathcal{L}} \frac{ds}{\mu(s)}. \tag{6.13}$$

A few remarks are in order.

Remark 6.4. (i) The equality $C_{\mathrm{ess}}(\mathcal{L}) = 0$ implies Theorem 5.4.

(ii) One can prove Theorem 6.1 avoiding the use of the Kac–Krein results [26]. Namely, with the help of the Rayleigh quotient, one can rewrite the inequality $\lambda_0(\mathrm{H}) > 0$ as a variational problem and then apply Muckenhoupt's inequalities (see, e.g., [33, §1.3.1], [35]). In particular, M. Solomyak employed this approach in the study of quantum graph operators on radially symmetric trees (see [44, §5]).

(iii) It is interesting to compare Theorems 6.1 and 6.3 with volume growth estimates (cf. [45]). For instance, by [32, Theorem 7.1],

$$\lambda_0(\mathbf{H}) \le \lambda_0^{\mathrm{ess}}(\mathbf{H}) \le \frac{1}{4}\mathrm{v}(\mathcal{A})^2, \tag{6.14}$$

where

$$\mathrm{v}(\mathcal{A}) := \liminf_{n\to\infty} \frac{1}{\sum_{k=0}^n \ell_k} \log\Big(\sum_{k=0}^n s_k s_{k+1}\ell_k\Big). \tag{6.15}$$

However, this result applies only if $\mathcal{L} = \sum_{n\ge 0}\ell_n = \infty$.

7. Isoperimetric constant

Recall that [32, §3] the *isoperimetric constant* $\alpha(\mathcal{G})$ of a metric graph $\mathcal{G}$ is

$$\alpha(\mathcal{G}) := \inf_{\widetilde{\mathcal{G}}} \frac{\deg_{\mathcal{G}}(\partial\widetilde{\mathcal{G}})}{\mathrm{vol}(\widetilde{\mathcal{G}})}, \tag{7.1}$$

where the infimum is taken over all finite connected subgraphs $\widetilde{\mathcal{G}} = (\widetilde{\mathcal{V}}, \widetilde{\mathcal{E}})$. Here

$$\partial\widetilde{\mathcal{G}} = \{v \in \widetilde{\mathcal{V}} \mid \deg_{\widetilde{\mathcal{G}}}(v) < \deg_{\mathcal{G}}(v)\},$$

is the *boundary* of $\widetilde{\mathcal{G}}$ and

$$\deg_{\mathcal{G}}(\partial\widetilde{\mathcal{G}}) := \sum_{v\in\partial\widetilde{\mathcal{G}}} \deg_{\widetilde{\mathcal{G}}}(v), \quad \mathrm{vol}(\widetilde{\mathcal{G}}) := \sum_{e\in\widetilde{\mathcal{E}}} |e|. \tag{7.2}$$

Computation of the isoperimetric constant is known to be an NP-hard problem, however, due to the presence of symmetries, we are able to find $\alpha(\mathcal{A})$ for radially symmetric antitrees.

Theorem 7.1. *The isoperimetric constant of a radially symmetric antitree $\mathcal{A}$ is*

$$\alpha(\mathcal{A}) = \inf_{n\geq 0} \frac{s_n s_{n+1}}{\sum_{k=0}^{n} s_k s_{k+1}\ell_k}. \tag{7.3}$$

Proof. The decomposition obtained in Theorem 3.5 suggests to take the infimum in (7.1) only over radially symmetric subgraphs. Namely, choosing $\mathcal{A}_n$ for every $n \geq 0$ as the subgraph consisting of all edges between the root o and the combinatorial sphere S_{n+1}, we have $\partial\mathcal{A}_n = S_{n+1}$ and $\deg_{\mathcal{A}_n}(v) = s_n$ for all vertices $v \in S_{n+1}$. Hence by (7.1) we get

$$\alpha(\mathcal{A}) \leq \frac{\deg(\partial\mathcal{A}_n)}{\mathrm{vol}(\mathcal{A}_n)} = \frac{s_n s_{n+1}}{\sum_{k\leq n} s_k s_{k+1}\ell_k}. \tag{7.4}$$

Thus it remains to show that indeed it suffices to restrict the infimum in (7.1) to the family $\{\mathcal{A}_n\}_{n\geq 0}$. Observe that $\{\mathcal{A}_n\}_{n\geq 0}$ is a net, that is, for every finite connected subgraph $\widetilde{\mathcal{A}}$ of $\mathcal{A}$ there is $n \geq 0$ such that $\widetilde{\mathcal{A}}$ is a subgraph of $\mathcal{A}_n$. Hence we will proceed by induction in n.

Let us start with subgraphs $\widetilde{\mathcal{A}} \subsetneq \mathcal{A}_0$. Then $\widetilde{\mathcal{A}}$ consists of $m < s_0 s_1$ edges of $\mathcal{E}_0^+$ and $\mathrm{vol}(\widetilde{\mathcal{A}}) = m\ell_0$. Moreover, for all vertices of $\widetilde{\mathcal{A}}$, $\deg_{\widetilde{\mathcal{A}}}(v) < \deg_{\mathcal{A}}(v)$ and hence $\deg(\partial\widetilde{\mathcal{A}}) = 2m$, which implies

$$\frac{\deg(\partial\widetilde{\mathcal{A}})}{\mathrm{vol}(\widetilde{\mathcal{A}})} = \frac{2m}{m\ell_0} = \frac{2}{\ell_0} > \frac{\deg(\partial\mathcal{A}_0)}{\mathrm{vol}(\mathcal{A}_0)} = \frac{1}{\ell_0}.$$

Take $n \geq 1$ and assume that

$$\frac{\deg(\partial\widetilde{\mathcal{A}})}{\mathrm{vol}(\widetilde{\mathcal{A}})} \geq \inf_{k\leq n-1} \frac{\deg(\partial\mathcal{A}_k)}{\mathrm{vol}(\mathcal{A}_k)} = \inf_{k\leq n-1} \frac{s_k s_{k+1}}{\sum_{j\leq k} s_j s_{j+1}\ell_j} \tag{7.5}$$

holds for all connected subgraphs $\widetilde{\mathcal{A}} \subseteq \mathcal{A}_{n-1}$. Take now a connected subgraph $\widetilde{\mathcal{A}} \subseteq \mathcal{A}_n$ such that $\widetilde{\mathcal{A}} \not\subseteq \mathcal{A}_{n-1}$. The latter in particular implies that $\mathcal{V}(\widetilde{\mathcal{A}}) \cap S_n \neq \emptyset$ and $\mathcal{V}(\widetilde{\mathcal{A}}) \cap S_{n+1} \neq \emptyset$. We can also assume that $\mathcal{V}(\widetilde{\mathcal{A}}) \cap S_{n-1} \neq \emptyset$ since otherwise $\mathcal{E}(\widetilde{\mathcal{A}}) \subseteq \mathcal{E}_n^+$ and hence in this case

$$\frac{\deg(\partial\widetilde{\mathcal{A}})}{\mathrm{vol}(\widetilde{\mathcal{A}})} = \frac{2}{\ell_n} > \frac{s_n s_{n+1}}{\sum_{k\leq n} s_k s_{k+1}\ell_k} = \frac{\deg(\partial\mathcal{A}_n)}{\mathrm{vol}(\mathcal{A}_n)}. \tag{7.6}$$

Let us first show that without loss of generality we can take $\widetilde{\mathcal{A}}$ such that each edge $e \in \mathcal{E}(\widetilde{\mathcal{A}})$ contains at least one vertex in $\mathcal{V}_{\mathrm{int}}(\widetilde{\mathcal{A}}) := \mathcal{V}(\widetilde{\mathcal{A}}) \setminus \partial\widetilde{\mathcal{A}}$. Indeed, if not, consider the induced subgraph $\widetilde{\mathcal{A}}_{\mathrm{int}}$, which we can split into a finite disjoint union of connected subgraphs $\{\widetilde{\mathcal{A}}_j\}$. In particular, $\widetilde{\mathcal{V}}_{\mathrm{int}} = \bigcup_j \mathcal{V}(\widetilde{\mathcal{A}}_j)$. Let $\mathcal{G}_j$ be the star-like subgraphs of $\mathcal{A}$ with edge sets $\mathcal{E}(\mathcal{G}_j) = \bigcup_{v\in\mathcal{V}(\widetilde{\mathcal{A}}_j)} \mathcal{E}_v$. By construction, $\mathcal{G}_j \subseteq \mathcal{A}_n$ and each edge of $\mathcal{G}_j$ contains a vertex from $\mathcal{V}(\mathcal{G}_j) \setminus \partial\mathcal{G}_j = \mathcal{V}(\widetilde{\mathcal{A}}_j)$. Moreover, let $\mathcal{E}_r = \mathcal{E}(\widetilde{\mathcal{A}}) \setminus \bigcup_j \mathcal{E}(\mathcal{G}_j)$ be the remaining edges of $\widetilde{\mathcal{A}}$. Then it is straightforward to verify (see also [38, proof of Lemma 3.5]) that

$$\frac{\deg(\partial\widetilde{\mathcal{A}})}{\mathrm{vol}(\widetilde{\mathcal{A}})} = \frac{\sum_j \deg(\partial\mathcal{G}_j) + 2\#\mathcal{E}_r}{\sum_j \mathrm{vol}(\mathcal{G}_j) + \sum_{e\in\mathcal{E}_r} |e|} \geq \min_{j,e\in\mathcal{E}_r} \Big\{\frac{\deg(\partial\mathcal{G}_j)}{\mathrm{vol}(\mathcal{G}_j)}, \frac{2}{|e|}\Big\}.$$

Taking into account (7.6), this proves the claim.

Consider a new graph $\widetilde{\mathcal{A}}'$ obtained from $\widetilde{\mathcal{A}}$ by adding all possible edges connecting S_n with S_{n-1} and S_{n+1} such that the new graph $\widetilde{\mathcal{A}}'$ is connected. By construction, $\widetilde{\mathcal{A}}' \subseteq \mathcal{A}_n$. Moreover, $S_{n+1} \subseteq \partial\widetilde{\mathcal{A}}'$ and $\deg_{\widetilde{\mathcal{A}}'}(v) = s_n$ for all $v \in S_{n+1}$. Hence

$$\frac{\deg(\partial\widetilde{\mathcal{A}}')}{\mathrm{vol}(\widetilde{\mathcal{A}}')} \geq \frac{s_n s_{n+1}}{\mathrm{vol}(\mathcal{A}_n)} = \frac{\deg(\partial\mathcal{A}_n)}{\mathrm{vol}(\mathcal{A}_n)}.$$

We also need another subgraph $\widetilde{\mathcal{A}}''$ of $\widetilde{\mathcal{A}}$ obtained by removing the edges of $\widetilde{\mathcal{A}}$ connecting S_{n+1} with $S_n \setminus \partial\widetilde{\mathcal{A}}$ and also $S_n \setminus \partial\widetilde{\mathcal{A}}$ with the vertices in $S_{n-1} \cap \partial\widetilde{\mathcal{A}}$. The obtained graph $\widetilde{\mathcal{A}}''$ is a connected subgraph of $\mathcal{A}_{n-1}$ and hence satisfies the induction hypothesis (7.5). Our aim is to show that

$$\frac{\deg(\partial\widetilde{\mathcal{A}})}{\mathrm{vol}(\widetilde{\mathcal{A}})} \geq \min\Big\{\frac{\deg(\partial\widetilde{\mathcal{A}}')}{\mathrm{vol}(\widetilde{\mathcal{A}}')}, \frac{\deg(\partial\widetilde{\mathcal{A}}'')}{\mathrm{vol}(\widetilde{\mathcal{A}}'')}\Big\}, \tag{7.7}$$

Denoting $M := \#(S_n \cap \widetilde{\mathcal{V}}_{\mathrm{int}})$ and $N := \#(S_{n-1} \cap \partial\widetilde{\mathcal{A}})$, we get

$$\mathrm{vol}(\widetilde{\mathcal{A}}') = \mathrm{vol}(\widetilde{\mathcal{A}}) + (s_n - M)s_{n+1}\ell_n + (s_n - M)N\ell_{n-1}, \tag{7.8}$$

and

$$\mathrm{vol}(\widetilde{\mathcal{A}}'') = \mathrm{vol}(\widetilde{\mathcal{A}}) - Ms_{n+1}\ell_n - MN\ell_{n-1}. \tag{7.9}$$

Moreover, a careful inspection shows that

$$\deg(\partial\widetilde{\mathcal{A}}') \leq \deg(\partial\widetilde{\mathcal{A}}) + (s_n - M)(s_{n+1} - s_{n-1} + 2N), \tag{7.10}$$

and

$$\deg(\partial\widetilde{\mathcal{A}}) = \deg(\partial\widetilde{\mathcal{A}}'') + M(s_{n+1} - s_{n-1} + 2N). \tag{7.11}$$

Now observe that if (7.7) fails to hold, then (7.9) and (7.11) would imply

$$\frac{s_{n+1} + 2N - s_{n-1}}{s_{n+1}\ell_n + N\ell_{n-1}} < \frac{\deg(\partial\widetilde{\mathcal{A}})}{\mathrm{vol}(\widetilde{\mathcal{A}})}, \tag{7.12}$$

and, moroever, (7.8) and (7.10) lead to

$$\frac{s_{n+1} + 2N - s_{n-1}}{s_{n+1}\ell_n + N\ell_{n-1}} > \frac{\deg(\partial\widetilde{\mathcal{A}})}{\operatorname{vol}(\widetilde{\mathcal{A}})}. \tag{7.13}$$

This contradiction proves (7.7) and hence finishes the proof of (7.3). □

Remark 7.2. A few remarks are in order.

(i) By the Cheeger-type estimate [32, Theorem 3.4], we have

$$\lambda_0(\mathbf{H}) \geq \frac{1}{4}\alpha(\mathcal{A})^2. \tag{7.14}$$

Comparing (7.14) and (7.3) with (6.2) and (6.11), we conclude that positivity of the isoperimetric constant is indeed only sufficient for $\lambda_0(\mathbf{H}) > 0$. For example, $\alpha(\mathcal{A}) = 0$ whenever $\operatorname{vol}(\mathcal{A}) = \infty$ and $\{s_n s_{n+1}\}_{n\geq 0}$ has a bounded subsequence.

(ii) The isoperimetric constant $\alpha(\mathcal{A})$ measures the ratio of the number of boundary points of $\mathcal{A}_n$ to the volume of $\mathcal{A}_n$ and thus provides a lower bound for $\lambda_0(\mathbf{H})$. The volume growth estimate (6.14) provides an upper bound by relating the exponential growth of the volume of $\mathcal{A}_n$ with its diameter. Notice that the volume of the subgraph $\mathcal{A}_n$ also appears in (6.10)–(6.11). The meaning of the other quantity in (6.11), namely, of $\sum_{k\geq n} \frac{\ell_k}{s_k s_{k+1}}$, which however provides two-sided estimates, remains unclear to us.

8. Singular spectrum

Using the isometric isomorphism

$$U_\mu \colon f \longmapsto \sqrt{\mu} f$$

between Hilbert spaces $L^2(\mathcal{I}_\mathcal{L};\mu)$ and $L^2(\mathcal{I}_\mathcal{L})$, it is straightforward to check that the pre-minimal operator H_0 defined in Section 3.2 is unitarily equivalent to the operator $\tilde{\mathbf{h}}_0$ defined in $L^2(\mathcal{I}_\mathcal{L})$ by

$$\tilde{\mathbf{h}}_0 f = -f'', \quad f \in \operatorname{dom}(\tilde{\mathbf{h}}_0) = U_\mu(\operatorname{dom}(\mathrm{H}_0)),$$

$$\operatorname{dom}(\tilde{\mathbf{h}}_0) = \Big\{ f \in L^2_c(\mathcal{I}_\mathcal{L}) \,\Big|\, \frac{1}{\sqrt{\mu}} f, \sqrt{\mu} f' \in AC(\mathcal{I}_\mathcal{L}),\ f'(0) = 0,\ f'' \in L^2(\mathcal{I}_\mathcal{L}) \Big\}.$$

Since μ is piece-wise constant on $(0,\mathcal{L})$, the domain of $\tilde{\mathbf{h}}_0$ consists of compactly supported functions $f \in L^2_c(\mathcal{I}_\mathcal{L})$ such that $f \in H^2(I_n)$ for all $n \geq 0$ and also satisfying the following boundary conditions

$$f'(0) = 0; \qquad f(t_n+) = \sqrt{\frac{s_{n+1}}{s_{n-1}}} f(t_n-), \quad f'(t_n+) = \sqrt{\frac{s_{n-1}}{s_{n+1}}} f'(t_n-),$$

for all $n \geq 1$. Denote the closure of $\tilde{\mathbf{h}}_0$ by $\widetilde{\mathrm{H}}$. The operator $\widetilde{\mathrm{H}}$ has actively been studied since its spectral properties play a crucial role in understanding spectral properties of Kirchhoff Laplacians on radial metric trees (let us only mention [6, 16]). It turns out that one can immediately apply most of the results from [6] and [16] in order to prove the corresponding spectral properties of Kirchhoff Laplacians on radially symmetric antitrees. However, we need the following assumptions on the geometry of metric antitrees:

Hypothesis 8.1. There is a positive lower bound on the edge lengths,

$$\ell_*(\mathcal{A}) := \inf_{n\geq 0} \ell_n > 0,$$

and sphere numbers are such that

$$\liminf_{n\geq 0} \frac{s_{n+2}}{s_n} > 1. \tag{8.1}$$

In this case clearly $\mathcal{L} = \sum_{n\geq 0} \ell_n = \infty$ and hence both operators H and $\tilde{\mathbf{h}}$ are self-adjoint. The next result is the analog of [6, Theorem 2].

Theorem 8.2. *Assume Hypothesis 8.1. If in addition*

$$\sup_{n\geq 0} \ell_n = \infty, \tag{8.2}$$

then $\sigma(\mathbf{H}) = \mathbb{R}_{\geq 0}$ *and* $\sigma_{\mathrm{ac}}(\mathbf{H}) = \emptyset$.

Proof. By Theorem 3.5, it suffices to show that $\sigma(\widetilde{\mathrm{H}}) = \mathbb{R}_{\geq 0}$ and $\sigma_{\mathrm{ac}}(\widetilde{\mathrm{H}}) = \emptyset$ since $\widetilde{\mathrm{H}} = U_\mu \mathrm{H} U_\mu^{-1}$. However, the latter follows from [6, Theorem 6]. $\square$

Moreover, using the results from [31, §4] and arguing as in the proof of [34, Theorem 1] (see also [17, Theorem 5.20]), one can prove the following statement.

Theorem 8.3. *Assume Hypothesis 8.1. If in addition*

$$\sup_{n\geq 0} \frac{s_{n+2}}{s_n} = \infty, \tag{8.3}$$

then $\sigma_{\mathrm{ac}}(\mathbf{H}) = \emptyset$.

In contrast to radially symmetric trees, antitrees always have a rather rich point spectrum (see Theorem 3.5). Moreover, under the assumptions of Hypothesis 8.1 this point spectrum is not a discrete subset, that is, it has finite accumulation points (see Remark 5.3). On the other hand, similar to [6, Theorem 7], we can construct a class of antitrees such that $\sigma(\mathrm{H})$ is purely singular continuous. Moreover, it is possible to show that under the assumption $\ell_*(\mathcal{A}) > 0$ this situation is in a certain sense typical (cf. [6, Theorems 4 and 8]). Let us only mention the following Remling-type result (cf. [40, Theorem 1.1]).

Theorem 8.4. *Assume Hypothesis 8.1. Also, assume that the sets $\{\ell_n\}_{n\ge 0}$ and $\{\frac{s_{n+2}}{s_n}\}_{n\ge 0}$ are finite. Then $\sigma_{\mathrm{ac}}(\mathbf{H}) \neq \emptyset$ if and only if the sequence $\{(\ell_n, \frac{s_{n+2}}{s_n})\}_{n\ge 0}$ is eventually periodic.*

The proof is again omitted since it is analogous to that of [16, Theorem 5.1].

9. Absolutely continuous spectrum

The decomposition (3.18) shows that

$$\sigma_{\mathrm{ac}}(\mathbf{H}) = \sigma_{\mathrm{ac}}(\mathrm{H}) \tag{9.1}$$

and both have multiplicity at most 1. The results of the previous section show that antitrees with nonempty absolutely continuous spectrum is a rare event. Our main aim in this section is to apply two recent results from [4] and [14] on the absolutely continuous spectrum of Krein and generalized indefinite strings, respectively, in order to construct several classes of antitrees with rich absolutely continuous spectra, however, which are not eventually periodic in the sense of Theorem 8.4. We begin with the following result.

Theorem 9.1. *Let $\mathcal{A}$ be an infinite radially symmetric antitree such that*

$$\mathcal{L} = \sum_{n\ge 0} \ell_n = \infty.$$

Also, let μ be the function given by (3.4). If

$$\sum_{n\ge 0} \Big(\int_n^{n+2} \mu(x)dx \int_n^{n+2} \frac{dx}{\mu(x)} - 4 \Big) < \infty, \tag{9.2}$$

then $\sigma_{\mathrm{ac}}(\mathbf{H}) = \mathbb{R}_{\ge 0}$.

Proof. We only need to use Theorem 2 from [4]. Indeed, as we know (see the proof of Theorem 6.1), the operator H is unitarily equivalent to the Krein string operator $\tilde{\mathbf{h}}$ given by (5.12)–(5.14). Applying now Theorem 2 from [4] to the operator $\widetilde{\mathrm{H}}$, after straightforward calculations the corresponding condition (1.9) from [4] turns into (9.2). □

Remark 9.2. Let us mention that in Theorem 9.1, upon suitable modifications of [4, Theorem 2], one can replace the intervals $(n, n+2)$ by intervals $\mathcal{I}_n$, $n \geq 0$ which "asymptotically" behave like $(n, n+2)$ (actually, by intervals with lengths uniformly bounded from above as well as by a positive constant from below and satisfying a suitable overlapping property [5]), however, one has to replace 4 by a square of the length of the corresponding interval:

$$\sum_{n\geq 0}\Big(\int_{\mathcal{I}_n}\mu(x)dx\int_{\mathcal{I}_n}\frac{dx}{\mu(x)} - |\mathcal{I}_n|^2\Big) < \infty. \tag{9.3}$$

Let us first demonstrate the above result by considering an example of equilateral antitrees and then we shall extend it to a much wider setting (see Theorem 9.6 below).

Corollary 9.3 (equilateral antitrees). *Let $\mathcal{A}$ be an infinite radially symmetric antitree with $\ell_n = \ell > 0$ for all $n \geq 0$. If*

$$\sum_{n\geq 0}\Big(\frac{s_{n+2}}{s_n} - 1\Big)^2 < \infty, \tag{9.4}$$

then $\sigma_{\mathrm{ac}}(\mathbf{H}) = \mathbb{R}_{\geq 0}$.

Proof. Setting $\mathcal{I}_n = (\ell n, \ell(n+2))$, $n \geq 0$, straightforward calculations show that

$$\begin{aligned}\int_{\mathcal{I}_n}\mu(x)dx\int_{\mathcal{I}_n}\frac{dx}{\mu(x)} - |\mathcal{I}_n|^2 \\ &= (s_n s_{n+1} + s_{n+1}s_{n+2})\Big(\frac{1}{s_n s_{n+1}} + \frac{1}{s_{n+1}s_{n+2}}\Big)\ell^2 - 4\ell^2 \\ &= \frac{(s_{n+2}+s_n)^2}{s_n s_{n+2}}\ell^2 - 4\ell^2 = \ell^2\frac{(s_{n+2}-s_n)^2}{s_n s_{n+2}} = \ell^2\frac{s_n}{s_{n+2}}\Big(\frac{s_{n+2}}{s_n} - 1\Big)^2.\end{aligned}$$

Theorem 9.1 and Remark 9.2 complete the proof. □

Remark 9.4. First of all, Corollary 9.3 demonstrates that (8.1) is essential for the results of Section 8. Let us also mention that it is possible to show by using the results of [31, §4.2] that the stronger condition

$$\sum_{n\geq 0}\left|\frac{s_{n+2}}{s_n}-1\right|<\infty \tag{9.5}$$

holds exactly when the operator $\tilde{\mathbf{h}}$ considered in Section 8 is a trace class perturbation (in the resolvent sense) of the free Hamiltonian $-\frac{d^2}{dx^2}$ acting in $L^2(\mathbb{R}_+)$ and hence in this case the Birman–Krein theorem implies $\sigma_{\mathrm{ac}}(\mathrm{H}) = \mathbb{R}_{\geq 0}$. However, equation (9.5) does not hold already for polynomially growing equilateral antitrees, e.g., take $s_n = n+1$ (see also Section 10.2). Moreover, (9.4) is equivalent to the fact that $\tilde{\mathbf{h}}$ is a Hilbert–Schmidt class perturbation (in the resolvent sense) of the free Hamiltonian.

The rather strong assumption that $\mathcal{A}$ is equilateral can indeed be replaced by $\ell_*(\mathcal{A}) > 0$. In order to do this, it will turn out useful to rewrite (9.2). Let

$$\mathcal{M} := \mathrm{ran}(\mu) = \{s_n s_{n+1}\colon\ n \in \mathbb{Z}_{\geq 0}\} \tag{9.6}$$

be the image of the function μ defined in (3.4). For every $s \in \mathcal{M}$, we set

$$\mathcal{I}_s := \mu^{-1}(\{s\}) = \{x \in [0,\infty)\colon\ \mu(x) = s\}, \tag{9.7}$$

that is, $\mathcal{I}_s$ is the preimage of $\{s\} \in \mathcal{M}$ with respect to μ.

Lemma 9.5. *Let $\mathcal{A}$ be an infinite radially symmetric antitree with $\mathcal{L}=\infty$. Then*

$$\sum_{n\geq 0}\Big(\int_n^{n+2}\mu(x)dx\int_n^{n+2}\frac{dx}{\mu(x)}-4\Big) = \frac{1}{2}\sum_{n\geq 0}\sum_{s\in\mathcal{M}}\sum_{\xi\neq s}|\mathcal{I}_s^n||\mathcal{I}_\xi^n|\frac{(s-\xi)^2}{s\xi}, \tag{9.8}$$

where $|\mathcal{I}_s^n|$ is the Lebesgue measure of $\mathcal{I}_s^n := \mathcal{I}_s \cap (n, n+2)$.

Proof. For every fixed $n \in \mathbb{Z}_{\geq 0}$, we clearly have

$$\begin{aligned}\int_n^{n+2}\mu(x)dx\int_n^{n+2}\frac{dx}{\mu(x)} &= \Big(\sum_{s\in\mathcal{M}} s|\mathcal{I}_s^n|\Big)\Big(\sum_{\xi\in\mathcal{M}}\frac{1}{\xi}|\mathcal{I}_\xi^n|\Big)\\ &= \sum_{s\in\mathcal{M}}\sum_{\xi\neq s}|\mathcal{I}_s^n||\mathcal{I}_\xi^n|\frac{s}{\xi} + \sum_{s\in\mathcal{M}}|\mathcal{I}_s^n|^2\\ &= \frac{1}{2}\sum_{s\in\mathcal{M}}\sum_{\xi\neq s}|\mathcal{I}_s^n||\mathcal{I}_\xi^n|\Big(\frac{\xi}{s}+\frac{s}{\xi}\Big) + \sum_{s\in\mathcal{M}}|\mathcal{I}_s^n|^2.\end{aligned}$$

Moreover, by construction

$$\sum_{s\in\mathcal{M}} |\mathcal{J}_s^n| = 2, \tag{9.9}$$

and hence

$$\sum_{s\in\mathcal{M}} |\mathcal{J}_s^n|^2 - 4 = \sum_{s\in\mathcal{M}} |\mathcal{J}_s^n|(|\mathcal{J}_s^n| - 2) = -\sum_{s\in\mathcal{M}}\sum_{\xi\neq s} |\mathcal{J}_s^n||\mathcal{J}_\xi^n|.$$

Combining the last two equalities, we get

$$\begin{aligned}\int_n^{n+2} \mu(x)dx \int_n^{n+2} \frac{dx}{\mu(x)} - 4 &= \frac{1}{2}\sum_{s\in\mathcal{M}}\sum_{\xi\neq s} |\mathcal{J}_s^n||\mathcal{J}_\xi^n|\Big(\frac{\xi}{s} + \frac{s}{\xi} - 2\Big)\\ &= \frac{1}{2}\sum_{s\in\mathcal{M}}\sum_{\xi\neq s} |\mathcal{J}_s^n||\mathcal{J}_\xi^n|\frac{(s-\xi)^2}{s\xi},\end{aligned}$$

which completes the proof. □

Theorem 9.6. *Let $\mathcal{A}$ be an infinite radially symmetric antitree with sphere numbers satisfying* (9.4). *If*

$$\ell_*(\mathcal{A}) = \inf_{n\geq 0} \ell_n > 0,$$

then $\sigma_{\mathrm{ac}}(\mathbf{H}) = \mathbb{R}_{\geq 0}$.

Proof. Suppose $\ell_*(\mathcal{A}) \geq 2$. Then, by Lemma 9.5, for every $n \in \mathbb{Z}_{\geq 0}$, we get

$$\begin{aligned}\int_n^{n+2} \mu(x)dx \int_n^{n+2} \frac{dx}{\mu(x)} - 4 &= \frac{1}{2}\sum_{s\in\mathcal{M}}\sum_{\xi\neq s} |\mathcal{J}_s^n||\mathcal{J}_\xi^n|\frac{(s-\xi)^2}{s\xi}\\ &\leq \sum_{s\in\mathcal{M}_n}\sum_{\xi\neq s} |\mathcal{J}_\xi^n|\frac{(s-\xi)^2}{s\xi},\end{aligned}$$

where $\mathcal{M}_n := \mu((n, n+2)) = \{s_k s_{k+1} : (n, n+2) \cap I_k \neq \varnothing\}$. Since $\ell_k \geq 2$ for all $k \geq 0$ by assumption, μ is either constant on $(n, n+2)$ or attains precisely two different values. In the first case, the right-hand side is equal to zero. In the second, we obviously get the estimate

$$\int_n^{n+2} \mu(x)dx \int_n^{n+2} \frac{dx}{\mu(x)} - 4 \leq 2\sum_{t_k\in(n,n+2)} \frac{(s_{k+1} - s_{k-1})^2}{s_{k-1}s_{k+1}}.$$

Thus we end up with the following bound

$$\sum_{n\ge 0}\Big(\int_n^{n+2}\mu(x)dx\int_n^{n+2}\frac{dx}{\mu(x)}-4\Big)\le 2\sum_{n\ge 0}\sum_{t_k\in(n,n+2)}\frac{(s_{k+1}-s_{k-1})^2}{s_{k-1}s_{k+1}}$$
$$\le 4\sum_{n\ge 0}\frac{(s_{n+2}-s_n)^2}{s_ns_{n+2}}<\infty,$$

which proves the claim by applying Theorem 9.1.

It remains to note that the general case $\ell_*(\mathcal{A})>0$ can be reduced to the one with $\ell_*(\mathcal{A})\ge 2$ by using the standard scaling argument (see also Remark 9.2). □

In fact, one can extend the above result to the case when lengths do not admit a strictly positive lower bound. However, in this case one has to modify (9.4) in an appropriate way.

Lemma 9.7. *Let $\mathcal{A}$ be an infinite radially symmetric antitree with $\mathcal{L}=\infty$. Also, let $\ell_n\le 1$ for all $n\ge 0$ and $\ell_n=o(1)$ as $n\to\infty$. If $\{s_n\}_{n\ge 0}$ is a nondecreasing sequence such that*

$$\sum_{n\ge 0}\Big(\frac{s_{m(n+2)}}{s_{m(n)}}-1\Big)^2<\infty, \tag{9.10}$$

then $\sigma_{\mathrm{ac}}(\mathbf{H})=\mathbb{R}_{\ge 0}$.

Here for each $n\in\mathbb{Z}_{\ge 0}$ the natural number $m(n)$ is defined by

$$t_{m(n)}\le n<t_{m(n)+1},\qquad t_n=\sum_{k=0}^{n-1}\ell_k. \tag{9.11}$$

Proof. Set

$$\mathcal{I}_n:=(t_{m(n)},t_{m(n+2)+1}),\qquad n\ge 0.$$

By construction $(n,n+2)\subseteq\mathcal{I}_n$ for all $n\ge 0$ and $|\mathcal{I}_n\setminus(n,n+2)|=o(1)$ as $n\to\infty$. Thus, by Theorem 9.1 and Remark 9.2, it suffices to show that

$$\sum_{n\ge 0}\underbrace{\Big(\int_{t_{m(n)}}^{t_{m(n+2)+1}}\mu(x)dx\int_{t_{m(n)}}^{t_{m(n+2)+1}}\frac{dx}{\mu(x)}-(t_{m(n+2)+1}-t_{m(n)})^2\Big)}_{=:R_n}<\infty. \tag{9.12}$$

Since μ is given by (3.4), we get

$$
\begin{aligned}
R_n &= \sum_{k=m(n)}^{m(n+2)} s_k s_{k+1} \ell_k \sum_{k=m(n)}^{m(n+2)} \frac{\ell_k}{s_k s_{k+1}} - \Big(\sum_{k=m(n)}^{m(n+2)} \ell_k \Big)^2 \\
&= \sum_{k,j=m(n)}^{m(n+2)} \ell_k \ell_j \Big(\frac{s_j s_{j+1}}{s_k s_{k+1}} - 1 \Big) \\
&= \sum_{m(n)\le k<j\le m(n+2)} \ell_k \ell_j \frac{(s_j s_{j+1} - s_k s_{k+1})^2}{s_k s_{k+1} s_j s_{j+1}} \\
&\le \sum_{m(n)\le k<j\le m(n+2)} \ell_k \ell_j \frac{(s^2_{m(n+2)+1} - s^2_{m(n)})^2}{s^4_{m(n)}} \\
&\lesssim \sup_{k\ge 0} |\mathcal{I}_k|^2 \Big(\frac{s^2_{m(n+2)}}{s^2_{m(n)}} - 1 \Big)^2 \lesssim \Big(\frac{s_{m(n+2)}}{s_{m(n)}} - 1 \Big)^2
\end{aligned}
$$

for all $n \ge 0$ if $\frac{s_{m(n+2)}}{s_{m(n)}} = 1 + o(1)$. □

Remark 9.8. In fact, the assumptions on lengths that $\ell_n \le 1$ for all $n \ge 0$ and $\ell_n = o(1)$ as $n \to \infty$ as well as monotonicity of sphere numbers are superfluous and we need them for simplicity only. Of course, one can considerably weaken them, however, the analysis becomes more involved and cumbersome.

We finish this section with another result based on [14], which also allows to construct antitrees with absolutely continuous spectrum supported on $\mathbb{R}_{\ge 0}$.

Theorem 9.9. *Let $\mathcal{A}$ be an infinite radially symmetric antitree such that both* $\mathrm{vol}(\mathcal{A}) = \infty$ *and $\mathcal{L}_\mu = \infty$. If there are constants $a \in \mathbb{R}$ and $b \in \mathbb{R}_{>0}$ such that*

$$
\int_0^{\mathcal{L}} \frac{1}{\mu(x)} \Big| \int_0^x \Big(\mu(s) - \frac{b}{\mu(s)} \Big) ds - a \Big|^2 dx < \infty, \tag{9.13}
$$

where μ is given by (3.4)*, then $\sigma_{\mathrm{ac}}(\mathbf{H}) = \mathbb{R}_{\ge 0}$.*

Proof. As in the proof of Theorem 9.1, we know that the operator H is unitarily equivalent to the operator $\widetilde{\mathrm{H}}$. By Theorem 3.1 from [14], $\sigma_{\mathrm{ac}}(\widetilde{\mathrm{H}}) = [0, \infty)$ if there are constants $a \in \mathbb{R}$ and $b \in \mathbb{R}_{>0}$ such that

$$
\int_0^\infty |M(x) - a - bx|^2 \, dx < \infty,
$$

where M is defined by (6.6). Straightforward calculations finish the proof. □

Remark 9.10. For a string operator defined by (5.12), Theorem 9.1 and Theorem 9.9 also imply that the entropy, respectively, some sort of relative entropy of the corresponding spectral measure is finite (see [4] for details). However, the meaning of this fact for the corresponding quantum graph operator $\mathbf{H}$ is unclear to us.

10. Examples

10.1. Exponentially growing antitrees. Fix $\beta \in \mathbb{Z}_{\geq 2}$ and let $\mathcal{A}_\beta$ be the antitree with sphere numbers $s_n = \beta^n$, $n \geq 0$ (cf. [32, Example 8.6]). Suppose that $\{\ell_n\}_{n\geq 0}$ are the lengths. Notice that

$$\mathrm{vol}(\mathcal{A}_\beta) = \sum_{n\geq 0} \beta^{2n+1}\ell_n. \tag{10.1}$$

Then the basic spectral properties of the corresponding quantum graph operator are contained in the following proposition.

Proposition 10.1. *Let $\mathbf{H}^\beta$ be the quantum graph operator associated with the antitree $\mathcal{A}_\beta$.*

(i) *The operator $\mathbf{H}^\beta$ is self-adjoint if and only if the series in (10.1) diverges.*

(ii) *If $\mathrm{vol}(\mathcal{A}_\beta) < \infty$, then deficiency indices of $\mathbf{H}^\beta$ are equal to 1. Moreover, the spectra of self-adjoint extensions of $\mathbf{H}^\beta$ are purely discrete and eigenvalues admit the standard Weyl asymptotic (5.1).*

Assume in addition that $\mathrm{vol}(\mathcal{A}_\beta) = \infty$.

(iii) *The spectrum of $\mathbf{H}^\beta$ is purely discrete if and only if $\ell_n = o(1)$ as $n \to \infty$.*

(iv) *The resolvent of $\mathbf{H}^\beta$ belongs to the trace class if and only if*

$$\sum_{n\geq 0} \beta^{2n}\ell_n^2 < \infty. \tag{10.2}$$

(v) *$\mathbf{H}^\beta$ is positive definite if and only if $\sup_{n\geq 0} \ell_n < \infty$. Moreover, in this case*

$$\frac{1}{4C} \leq \lambda_0(\mathbf{H}^\beta) \leq \frac{1}{C}, \qquad \frac{1}{4C_{\mathrm{ess}}} \leq \lambda_0^{\mathrm{ess}}(\mathbf{H}^\beta) \leq \frac{1}{C_{\mathrm{ess}}}, \tag{10.3}$$

where

$$\sup_{n\geq 0} \sum_{k=0}^{n} \beta^{2k}\ell_k \sum_{k\geq n+1} \frac{\ell_k}{\beta^{2k}} \leq C \leq \sup_{n\geq 0} \sum_{k=0}^{n} \beta^{2k}\ell_k \sum_{k\geq n} \frac{\ell_k}{\beta^{2k}}, \tag{10.4}$$

and

$$\lim_{m\to\infty}\sup_{n\ge m}\sum_{k=m}^{n}\beta^{2k}\ell_k\sum_{k\ge n+1}\frac{\ell_k}{\beta^{2k}} \le C_{\mathrm{ess}} \le \lim_{m\to\infty}\sup_{n\ge m}\sum_{k=m}^{n}\beta^{2k}\ell_k\sum_{k\ge n}\frac{\ell_k}{\beta^{2k}}. \tag{10.5}$$

Proof. Items (i) and (ii) follow from Theorem 4.1 and Corollary 5.1.

(iii) Applying Theorem 5.4 (see also Remark 5.5), we only need to show that $\ell_n = o(1)$ as $n \to \infty$ is sufficient for the discreteness. Indeed, we can estimate

$$\begin{aligned}
&\sum_{k=0}^{n}\beta^{2k}\ell_k\sum_{k\ge n}\frac{\ell_k}{\beta^{2k}}\\
&\quad\le \ell^*(\mathcal{A}_\beta)\sup_{k\ge n}\ell_k\sum_{k=0}^{n}\beta^{2k}\sum_{k\ge n}\frac{1}{\beta^{2k}}\\
&\quad= \ell^*(\mathcal{A}_\beta)\sup_{k\ge n}\ell_k\frac{\beta^{2n+2}-1}{\beta^{2n+2}}\Big(\frac{\beta^2}{\beta^2-1}\Big)^2\\
&\quad< \frac{\ell^*(\mathcal{A}_\beta)}{(1-\beta^{-2})^2}\sup_{k\ge n}\ell_k,
\end{aligned} \tag{10.6}$$

where $\ell^*(\mathcal{A}_\beta) = \sup_{n\ge 0}\ell_n$. Hence (5.9) is satisfied if $\ell_n = o(1)$.

(iv) Clearly, (10.2) coincides with condition (i) of Theorem 5.6 and hence it is necessary. Applying the Cauchy–Schwarz inequality, we get the following estimate:

$$\begin{aligned}
\sum_{n\ge 0}\frac{\ell_n}{s_n s_{n+1}}\sum_{k=0}^{n-1}s_k s_{k+1}\ell_k &= \sum_{n\ge 0}\frac{\ell_n}{\beta^{2n}}\sum_{k=0}^{n-1}\beta^{2k}\ell_k\\
&\le \sum_{n\ge 0}\frac{\ell_n}{\beta^{2n}}\Big(\sum_{k=0}^{n-1}\beta^{2k}\ell_k^2\sum_{k=0}^{n-1}\beta^{2k}\Big)^{1/2}\\
&= \sum_{n\ge 0}\frac{\ell_n}{\beta^{2n}}\Big(\frac{\beta^{2n}-1}{\beta^2-1}\sum_{k=0}^{n-1}\beta^{2k}\ell_k^2\Big)^{1/2}\\
&< \sum_{n\ge 0}\frac{\ell_n}{\beta^{n}}\Big(\sum_{k=0}^{n-1}\beta^{2k}\ell_k^2\Big)^{1/2}\\
&< \frac{\ell^*(\mathcal{A}_\beta)}{1-\beta^{-1}}\Big(\sum_{k\ge 0}\beta^{2k}\ell_k^2\Big)^{1/2}.
\end{aligned}$$

Therefore, (10.2) implies condition (ii) of Theorem 5.6, which proves the claim.

(v) immediately follows from (10.6), Theorem 6.1, Theorem 6.3 and Remark 6.2. □

Remark 10.2. (i) Both the discreteness and uniform positivity criteria for $\mathbf{H}^\beta$ were obtained in [32, Example 8.6]. Notice that these results are a consequence of the positivity of the combinatorial isoperimetric constant in this case (see [32]). Moreover, using the rough estimate (10.6), one would be able to recover the lower bounds (8.9) and (8.10) from [32].

(ii) It is impossible to apply Theorem 9.1 and Theorem 9.9 to $\mathcal{A}_\beta$ (this either can be seen from Proposition 10.1(v) or one can prove that both conditions (9.2) and (9.13) are always violated if sphere numbers grow exponentially).

(iii) Since the sphere numbers of $\mathcal{A}_\beta$ satisfy

$$\frac{s_{n+2}}{s_n} = \beta^2$$

for all $n \geq 0$, we can apply the results of Section 8. Namely, under the additional assumption $\ell_*(\mathcal{A}_\beta) > 0$, we conclude that the absolutely continuous spectrum of $\mathbf{H}$ is in general empty. In particular, it is always the case if $\ell^*(\mathcal{A}_\beta) = \infty$ (Theorem 8.2). Moreover, assuming that $\{\ell_n\}_{n\geq 0}$ is a finite set, by Theorem 8.4, $\sigma_{\mathrm{ac}}(\mathbf{H}) \neq \emptyset$ would imply that the sequence $\{\ell_n\}_{n\geq 0}$ is eventually periodic.

(iv) Notice that the isoperimetric constant is given by (see (7.3))

$$\frac{1}{\alpha(\mathcal{A}_\beta)} = \sup_{n\geq 0} \frac{1}{\beta^{2n}} \sum_{k=0}^{n} \beta^{2k} \ell_k.$$

10.2. Polynomially growing antitrees. Fix $q \in \mathbb{Z}_{\geq 1}$ and let $\mathcal{A}^q$ be the antitree with sphere numbers $s_n = (n+1)^q$, $n \geq 0$ (the case $q = 1$ is depicted in Figure 1). Suppose that $\{\ell_n\}_{n\geq 0}$ are the lengths. Notice that

$$\mathrm{vol}(\mathcal{A}^q) = \sum_{n\geq 0} (n+1)^q (n+2)^q \ell_n. \tag{10.7}$$

Then the basic spectral properties of the corresponding quantum graph operator are contained in the following proposition.

Proposition 10.3. *Let* $\mathbf{H}^q$ *be the quantum graph operator associated with the antitree* $\mathcal{A}^q$*.*

(i) *The operator* $\mathbf{H}^q$ *is self-adjoint if and only if*

$$\sum_{n\geq 0} n^{2q} \ell_n = \infty. \tag{10.8}$$

(ii) *If the series in* (10.8) *converges, then deficiency indices of* $\mathbf{H}^q$ *are equal to* 1*. Moreover, the spectra of self-adjoint extensions of* $\mathbf{H}^q$ *are purely discrete and eigenvalues admit the standard Weyl asymptotic* (5.1)*.*

Assume in addition that (10.8) is satisfied, that is, $\mathbf{H}^q$ *is self-adjoint.*

(iii) *The spectrum of* $\mathbf{H}^q$ *is purely discrete if and only if*

$$\lim_{n\to\infty}\sum_{k=0}^{n} k^{2q}\ell_k \sum_{k\geq n}\frac{\ell_k}{k^{2q}} = 0. \tag{10.9}$$

In particular, the spectrum is purely discrete if $\ell_n = o(n^{-1})$ *as* $n \to \infty$.

(iv) *The resolvent of* $\mathbf{H}^q$ *belongs to the trace class if and only if*

$$\sum_{n\geq 0} n^{2q}\ell_n^2 < \infty. \tag{10.10}$$

(v) $\mathbf{H}^q$ *is positive definite if and only if*

$$\sup_{n\geq 1}\sum_{k=0}^{n} k^{2q}\ell_k \sum_{k\geq n}\frac{\ell_k}{k^{2q}} < \infty. \tag{10.11}$$

In particular, $\lambda_0(\mathbf{H}^q) > 0$ *if* $\ell_n = \mathcal{O}(n^{-1})$ *as* $n \to \infty$.

(vi) *If* $\ell_*(\mathcal{A}^q) > 0$, *then* $\sigma_{\mathrm{ac}}(\mathbf{H}^q) = \mathbb{R}_{\geq 0}$.

Proof. (i) and (ii) follow immediately from Theorem 4.1 and Corollary 5.1 since $\mathrm{vol}(\mathcal{A}^q) = \infty$ exactly when (10.8) is satisfied.

(iii) Applying Theorem 5.4 (see also Remark 5.5), we conclude that in the case (10.8), the operator $\mathbf{H}$ has purely discrete spectrum if and only if

$$\lim_{n\to\infty}\sum_{k=0}^{n}(k^2+3k+2)^q\ell_k \sum_{k\geq n}\frac{\ell_k}{(k^2+3k+2)^q} = 0.$$

It is not difficult to show that the latter is equivalent to (10.9). Moreover, (10.9) holds true whenever $\ell_n = o(n^{-1})$ as $n \to \infty$ since

$$\sum_{k=0}^{n} k^{2q-1} = \frac{n^{2q}}{2q}(1+o(1)), \qquad \sum_{k\geq n}\frac{1}{k^{2q+1}} = \frac{n^{-2q}}{2q}(1+o(1)).$$

(iv) First observe that (5.18) is equivalent to (10.10). Moreover, (10.10) implies also (5.19). Indeed, we get

$$\begin{aligned}
&\sum_{n\geq 0}\frac{\ell_n}{(n^2+3n+2)^q}\sum_{k=0}^{n-1}(k^2+3k+2)^q\ell_k\\
&< \sum_{n\geq 0}\frac{\ell_n}{(n+1)^{2q}}\sum_{k=0}^{n-1}(k+2)^{2q}\ell_k\\
&\leq \sum_{n\geq 0}\frac{\ell_n}{(n+1)^{2q}}\Big(\sum_{k=0}^{n-1}(k+2)^{2q}\ell_k^2\sum_{k=0}^{n-1}(k+2)^{2q}\Big)^{1/2}\\
&\lesssim \sum_{n\geq 0}\frac{\ell_n}{(n+1)^{2q}}\Big((n+1)^{2q+1}\sum_{k=0}^{n-1}(k+2)^{2q}\ell_k^2\Big)^{1/2}\\
&< \Big(\sum_{k\geq 0}(k+2)^{2q}\ell_k^2\Big)^{1/2}\sum_{n\geq 0}\frac{\ell_n}{(n+1)^{q-1/2}}\\
&< \sum_{k\geq 0}(k+2)^{2q}\ell_k^2\Big(\sum_{n\geq 1}\frac{1}{n^{4q-1}}\Big)^{1/2},
\end{aligned}$$

where the second and the last inequalities we obtained by applying the Cauchy–Schwarz inequality. It remains to use Theorem 5.6.

(v) follows by applying Theorem 6.1 (see also Remark 6.2).

(vi) Since

$$\sum_{n\geq 0}\Big(\frac{s_{n+2}}{s_n}-1\Big)^2=\sum_{n\geq 1}\Big(\frac{(n+2)^q}{n^q}-1\Big)^2\lesssim\sum_{n\geq 1}\frac{1}{n^2}=\frac{\pi^2}{6},$$

the claim is immediate from Theorem 9.6. □

Remark 10.4. A few remarks are in order.

(i) The antitree $\mathcal{A}^q$ and the corresponding Kirchhoff Laplacian $\mathbf{H}$ have been considered in [32, Example 8.7]. The analysis of spectral properties (in particular, spectral estimates) is a rather delicate task in this case since the combinatorial isoperimetric constant of $\mathcal{A}^q$ is equal to 0. We were able to describe basic spectral properties of $\mathbf{H}^q$ only due to the presence of radial symmetry. Spectral properties of Kirchhoff Laplacians without radial symmetry seems to be a rather complicated problem – even the self-adjointness problem (modulo some recent criteria obtained in [17]) is unclear to us at the moment.

(ii) It can be demonstrated by examples that the conditions $\ell_n = o(n^{-1})$ (resp., $\ell_n = \mathcal{O}(n^{-1})$) as $n \to \infty$ are not necessary for the discreteness (resp., positivity). However, they are in a certain sense sharp (see [32, Lemma 8.9] and also Example 10.6 below).

(iii) Since $s_{n+2} = s_n(1 + o(1))$, we can't apply the results of Section 8 (see Hypothesis 8.1). Moreover, Proposition 10.3(vi) shows that in general $\mathbf{H}^q$ has absolutely continuous spectrum supported on $\mathbb{R}_{\geq 0}$. However, Theorem 9.1 is a consequence of [4, Theorem 2], which allows a presence of a rather rich singular (continuous) spectrum.

We can also improve Proposition 10.3(vi) by allowing arbitrarily small lengths.

Corollary 10.5. *Suppose $\ell_n \leq 1$ for all $n \geq 0$ and $\ell_n = o(1)$ as $n \to \infty$. If*

$$\sum_{n\geq 0} \Big(\frac{m(n+2)}{m(n)} - 1\Big)^2 < \infty, \tag{10.12}$$

then $\sigma_{\mathrm{ac}}(\mathbf{H}^q) = \mathbb{R}_{\geq 0}$. Here $m(n)$ is defined as in Lemma 9.7.

Proof. We need to apply Lemma 9.7 and notice that in this case

$$\frac{s_{m(n+2)}}{s_{m(n)}} - 1 = \Big(\frac{m(n+2)+1}{m(n)+1}\Big)^q - 1 \approx \frac{m(n+2)}{m(n)} - 1,$$

as $n \to \infty$. □

Example 10.6. Fix $s \geq 0$. Let the lengths of the metric antitree $\mathcal{A}^q$ be given by

$$\ell_n = \frac{1}{(n+1)^s}, \quad n \geq 0. \tag{10.13}$$

Denote the corresponding Kirchhoff Laplacian by $\mathbf{H}^{q,s}$. Applying Proposition 10.3 and Corollary 10.5, we end up with the following description of the spectral properties of $\mathbf{H}^{q,s}$.

Corollary 10.7. (i) *$\mathbf{H}^{q,s}$ is self-adjoint if and only if $s \in [0, 2q+1]$. If $s > 2q+1$, then then deficiency indices of $\mathbf{H}^{q,s}$ are equal to 1. Moreover, in this case the spectra of self-adjoint extensions $\mathbf{H}^{q,s}_\theta$ of $\mathbf{H}^{q,s}$ are purely discrete and eigenvalues admit the standard Weyl asymptotic*

$$\lim_{\lambda\to\infty} \frac{N(\lambda; \mathbf{H}^{q,s}_\theta)}{\sqrt{\lambda}} = \frac{1}{\pi}\sum_{k=0}^{q}\binom{q}{k}\zeta(s-2q+k), \tag{10.14}$$

where ζ is the Riemann zeta function.

Assume in addition that $s \in [0, 2q+1]$, *that is,* $\mathbf{H}^q$ *is self-adjoint.*

(ii) *The spectrum of* $\mathbf{H}^{q,s}$ *is purely discrete if and only if* $s \in (1, 2q+1]$. *Moreover, the resolvent of* $\mathbf{H}^{q,s}$ *belongs to the trace class if and only if* $s \in (q+1/2, 2q+1]$.

(iii) $\mathbf{H}^{q,s}$ *is positive definite if and only if* $s \in [1, 2q+1]$.

(iv) *If* $s \in [0,1)$, *then* $\sigma_{\mathrm{ac}}(\mathbf{H}^{q,s}) = \mathbb{R}_{\geq 0}$.

We leave its proof to the reader and finish this section with a few remarks.

Remark 10.8. Corollary 10.7 complements the results obtained in [32, Example 8.7]. Moreover, items (ii) and (iii) demonstrate sharpness of sufficient conditions obtained in Proposition 10.3(iii) and (v). Let us only mention that the question on the structure of the essential spectrum of $\mathbf{H}^{q,1}$ as well as on the structure of the singular spectrum of $\mathbf{H}^{q,s}$ with $s \in [0,1]$ remains open.

Remark 10.9. In conclusion let us mention that choosing slightly different lengths

$$\ell_n = \frac{(n+1)^{q-s}}{(n+2)^q}, \quad n \geq 0,$$

and denoting the corresponding operator by $\widetilde{\mathbf{H}}^{q,s}$, we obtain

$$\lim_{\lambda\to\infty} \frac{N(\lambda; \widetilde{\mathbf{H}}^{q,s}_\theta)}{\sqrt{\lambda}} = \frac{1}{\pi}\zeta(s-2q), \quad s > 2q+1. \tag{10.15}$$

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