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On operator error estimates for homogenization of hyperbolic systems with periodic coefficients

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Abstract. In $L_2(\mathbb{R}^d;\mathbb{C}^n)$, we consider a selfadjoint matrix strongly elliptic second order differential operator $\mathcal{A}_\varepsilon$, $\varepsilon > 0$. The coefficients of the operator $\mathcal{A}_\varepsilon$ are periodic and depend on $\mathbf{x}/\varepsilon$. We study the asymptotic behavior of the operator $\mathcal{A}_\varepsilon^{-1/2}\sin(\tau\mathcal{A}_\varepsilon^{1/2})$, $\tau \in \mathbb{R}$, in the small period limit. The principal term of approximation in the $(H^1 \to L_2)$-norm for this operator is found. Approximation in the $(H^2 \to H^1)$-operator norm with the correction term taken into account is also established. The error estimates are of the sharp order $O(\varepsilon)$. The results are applied to homogenization for the solutions of the hyperbolic equation $\partial_\tau^2\mathbf{u}_\varepsilon = -\mathcal{A}_\varepsilon\mathbf{u}_\varepsilon + \mathbf{F}$. As examples, we consider the acoustics equation, the system of elasticity, and the model equation of electrodynamics.

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Contents

Introduction

The paper is devoted to homogenization of periodic differential operators (DO's). A broad literature is devoted to homogenization theory, see, e.g., the books [7, 8, 39, 50]. We use the spectral approach to homogenization problems based on the Floquet–Bloch theory and the analytic perturbation theory. The main results of the paper are briefly announced in [36].

0.1. The class of operators. In $L_2(\mathbb{R}^d;\mathbb{C}^n)$, we consider a matrix elliptic second order DO $\mathcal{A}_\varepsilon$ admitting a factorization $\mathcal{A}_\varepsilon = b(\mathbf{D})^* g(\mathbf{x}/\varepsilon) b(\mathbf{D})$, $\varepsilon > 0$. Here $b(\mathbf{D}) = \sum_{j=1}^d b_j D_j$ is an $(m \times n)$-matrix-valued first order DO with constant coefficients. Assume that $m \geqslant n$ and that the symbol $b(\boldsymbol{\xi})$ has maximal rank. A periodic $(m \times m)$-matrix-valued function $g(\mathbf{x})$ is such that $g(\mathbf{x}) > 0$; $g, g^{-1} \in L_\infty$. The coefficients of the operator $\mathcal{A}_\varepsilon$ oscillate rapidly as $\varepsilon \to 0$.

0.2. Operator error estimates for elliptic and parabolic problems. In a series of papers [9, 10, 11, 12] by M. Sh. Birman and T. A. Suslina, an abstract operator-theoretic (spectral) approach to homogenization problems in $\mathbb{R}^d$ was developed.

This approach is based on the scaling transformation, the Floquet–Bloch theory, and the analytic perturbation theory. Surely, the spectral approach was applied to homogenization problems before papers by Birman and Suslina, see, e.g., [2, 3, 22, 41, 51]. But the reminder estimates in the operator norm were not discussed in these papers. Moreover, in [2, 3, 22, 41, 51] only scalar problems were considered, while Birman and Suslina deal with systems. (In the scalar case, application of the analytic perturbation theory with respect to the quasimomentum simplifies significantly.)

A typical homogenization problem is to study the behavior of the solution $\mathbf{u}_\varepsilon$ of the equation $\mathcal{A}_\varepsilon \mathbf{u}_\varepsilon + \mathbf{u}_\varepsilon = \mathbf{F}$, where $\mathbf{F} \in L_2(\mathbb{R}^d; \mathbb{C}^n)$, as $\varepsilon \to 0$. It turns out that the solutions $\mathbf{u}_\varepsilon$ converge in some sense to the solution $\mathbf{u}_0$ of the homogenized equation $\mathcal{A}^0 \mathbf{u}_0 + \mathbf{u}_0 = \mathbf{F}$. Here $\mathcal{A}^0 = b(\mathbf{D})^* g^0 b(\mathbf{D})$ is the *effective operator* and g^0 is the constant *effective matrix*. The way to construct g^0 is well known in homogenization theory.

In [9], it was shown that

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{L_2(\mathbb{R}^d)} \leqslant C\varepsilon \|\mathbf{F}\|_{L_2(\mathbb{R}^d)}. \tag{0.1}$$

This estimate is order-sharp. The constant C is controlled explicitly in terms of the problem data. Inequality (0.1) means that the resolvent $(\mathcal{A}_\varepsilon + I)^{-1}$ converges to the resolvent of the effective operator in the $L_2(\mathbb{R}^d; \mathbb{C}^n)$-operator norm, as $\varepsilon \to 0$. Moreover,

$$\|(\mathcal{A}_\varepsilon + I)^{-1} - (\mathcal{A}^0 + I)^{-1}\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant C\varepsilon.$$

Results of this type are called *operator error estimates* in homogenization theory.

In [12], approximation of the resolvent $(\mathcal{A}_\varepsilon + I)^{-1}$ in the $(L_2 \to H^1)$-operator norm was found:

$$\|(\mathcal{A}_\varepsilon + I)^{-1} - (\mathcal{A}^0 + I)^{-1} - \varepsilon K(\varepsilon)\|_{L_2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leqslant C\varepsilon.$$

Here the *correction term* $K(\varepsilon)$ is taken into account. It contains a rapidly oscillating factor and so depends on ε. Herewith, $\|\varepsilon K(\varepsilon)\|_{L_2 \to H^1} = O(1)$. In contrast to the traditional corrector of homogenization theory, the operator $K(\varepsilon)$ contains an auxiliary smoothing operator Π_ε (see (9.6) below).

To parabolic homogenization problems the spectral approach was applied in [42, 43, 44]. The principal term of approximation was found in [42, 43]:

$$\|e^{-\tau \mathcal{A}_\varepsilon} - e^{-\tau \mathcal{A}^0}\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant C\varepsilon \tau^{-1/2}, \quad \tau > 0.$$

Approximation with the corrector taken into account was obtained in [44]:

$$\|e^{-\tau \mathcal{A}_\varepsilon} - e^{-\tau \mathcal{A}^0} - \varepsilon \mathcal{K}(\varepsilon, \tau)\|_{L_2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leqslant C\varepsilon(\tau^{-1} + \tau^{-1/2}), \quad 0 < \varepsilon \leqslant \tau^{1/2}.$$

Another approach to deriving operator error estimates in $\mathbb{R}^d$ (the so-called *modified method of first order approximation* or the *shift method*) was suggested by V. V. Zhikov [52, 53] and developed by V. V. Zhikov and S. E. Pastukhova [54]. In these papers the elliptic problems for the operators of acoustics and elasticity theory were studied. To parabolic problems the shift method was applied in [55]. Further results of V. V. Zhikov and S. E. Pastukhova are discussed in the recent survey [56].

Operator error estimates were also studied for problems in a bounded domain. The first results were obtained by G. Griso [28, 29] who studied the scalar elliptic problems by using the unfolding method [19]. Close results were obtained in [52, 54]. The periodic elliptic systems were considered in [32] and [37, 46, 47]. Parabolic problems in a bounded domain were discussed in [27, 35].

Now, operator error estimates (and close results) are a hot topic in homogenization. Recently, a progress in the high-contrast case was achieved by K. D. Cherednichenko and S. Cooper [18], in the locally periodic case—by S. E. Pastukhova and R. N. Tikhomirov [38], D. I. Borisov [15], and N. N. Senik [40]. For almost periodic case, some advances were obtained by S. N. Armstrong and Z. Shen [6]. For stochastic problems, some results were obtained in [4, 5]. Note, finally, that the unfolding method was very recently transferred to the stochastic case, see [30]. Surely, this survey is incomplete.

0.3. Operator error estimates for homogenization of hyperbolic equations and nonstationary Schrödinger-type equations. For elliptic and parabolic problems operator error estimates are well studied. The situation with homogenization of nonstationary Schrödinger-type and hyperbolic equations is different. In [13], the operators $e^{-i\mathcal{A}_\varepsilon}$ and $\cos(\tau\mathcal{A}_\varepsilon^{1/2})$ were studied. It turned out that for these operators it is impossible to find approximations in the $(L_2 \to L_2)$-norm. Approximations in the $(H^s \to L_2)$-norms with suitable s were found in [13]:

$$\|e^{-i\tau\mathcal{A}_\varepsilon} - e^{-i\tau\mathcal{A}^0}\|_{H^3(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant C\varepsilon(1+|\tau|), \tag{0.2}$$

$$\|\cos(\tau\mathcal{A}_\varepsilon^{1/2}) - \cos(\tau(\mathcal{A}^0)^{1/2})\|_{H^2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant C\varepsilon(1+|\tau|). \tag{0.3}$$

Later T. A. Suslina [48], by using the analytic perturbation theory, proved that estimate (0.2) in the general case cannot be refined with respect to the type of the operator norm. Developing the method of [48], M. A. Dorodnyi and T. A. Suslina [23, 24] showed that estimate (0.3) is sharp in the same sense. In [23, 24, 48], under some additional assumptions on the operator, the results (0.2) and (0.3) were improved with respect to the type of the operator norm. In [13, 24], by virtue of the identity $\mathcal{A}_\varepsilon^{-1/2}\sin(\tau\mathcal{A}_\varepsilon^{1/2}) = \int_0^\tau \cos(\tilde{\tau}\mathcal{A}_\varepsilon^{1/2})\,d\tilde{\tau}$ and the similar identity for the effective operator, the estimate

$$\|\mathcal{A}_\varepsilon^{-1/2}\sin(\tau\mathcal{A}_\varepsilon^{1/2}) - (\mathcal{A}^0)^{-1/2}\sin(\tau(\mathcal{A}^0)^{1/2})\|_{H^2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant C\varepsilon(1+|\tau|)^2 \tag{0.4}$$

(with $\tau \in \mathbb{R}$) was deduced from (0.3) as a (rough) consequence. The sharpness of estimate (0.4) with respect to the type of the operator norm was not discussed. Estimates (0.3) and (0.4) were applied to homogenization for the solution of the Cauchy problem

$$\begin{cases} \partial_\tau^2 \mathbf{u}_\varepsilon(\mathbf{x},\tau) = -\mathcal{A}_\varepsilon \mathbf{u}_\varepsilon(\mathbf{x},\tau) + \mathbf{F}(\mathbf{x},\tau), \\ \mathbf{u}_\varepsilon(\mathbf{x},0) = \boldsymbol{\varphi}(\mathbf{x}), \quad \partial_\tau \mathbf{u}_\varepsilon(\mathbf{x},0) = \boldsymbol{\psi}(\mathbf{x}). \end{cases} \tag{0.5}$$

0.4. Approximation for the solutions of hyperbolic systems with the correction term taken into account. Operator error estimates with the correction term for nonstationary equations of Schrödinger type and hyperbolic type previously have not been established. So, we discuss the known "classical" homogenization results that cannot be written in the uniform operator topology. These results concern the operators in a bounded domain $\mathcal{O} \subset \mathbb{R}^d$. Approximation for the solution of the hyperbolic equation with the zero initial data and a non-zero right-hand side was obtained in [8, Chapter 2, Subsection 3.6]. In [8], it was shown that the difference of the solution and the first order approximation strongly converges to zero in $L_2((0,T);H^1(\mathcal{O}))$. The error estimate was not established. The case of zero initial data and non-zero right-hand side was also considered in [7, Chapter 4, Section 5]. In [7], the complete asymptotic expansion of the solution was constructed and the estimate of order $O(\varepsilon^{1/2})$ for the difference of the solution and the first order approximation in the H^1-norm on the cylinder $\mathcal{O}\times(0,T)$ was obtained. Herewith, the right-hand side was assumed to be C^∞-smooth.

It is natural to be interested in the approximation with the correction term for the solutions of hyperbolic systems with non-zero initial data, i.e., in approximation of the operator cosine $\cos(\tau\mathcal{A}_\varepsilon^{1/2})$ in some suitable sense. One could expect the correction term in this case to be of similar structure as for elliptic and parabolic problems. However, in [14] it was observed that this is true only for very special class of initial data. In the general case, approximation with the corrector was found in [16, 17], but the correction term was non-local because of the dispersion of waves in inhomogeneous media. Dispersion effects for homogenization of the wave equation were discussed in [1, 20, 21] via the Floquet–Bloch theory and the analytic perturbation theory. Operator error estimates have not been obtained.

0.5. Main results. *Our goal* is to refine estimate (0.4) with respect to the type of the operator norm without any additional assumptions and to find an approximation for the operator $\mathcal{A}_\varepsilon^{-1/2}\sin(\tau\mathcal{A}_\varepsilon^{1/2})$ in the $(H^2 \to H^1)$-norm. We wish to apply the results to problem (0.5) with $\boldsymbol{\varphi} = 0$ and non-zero $\mathbf{F}$ and $\boldsymbol{\psi}$.

Our first main result is the estimate

$$\|\mathcal{A}_\varepsilon^{-1/2}\sin(\tau\mathcal{A}_\varepsilon^{1/2}) - (\mathcal{A}^0)^{-1/2}\sin(\tau(\mathcal{A}^0)^{1/2})\|_{H^1(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant C\varepsilon(1+|\tau|), \tag{0.6}$$

with $\varepsilon > 0$, $\tau \in \mathbb{R}$. (Under additional assumptions on the operator, improvement of estimate (0.6) with respect to the type of the norm was obtained by M. A. Dorodnyi and T. A. Suslina in the paper [25] that is, actually, major revision of [24].) Our second main result is the approximation

$$\begin{aligned}\|\mathcal{A}_\varepsilon^{-1/2}\sin(\tau\mathcal{A}_\varepsilon^{1/2}) - (\mathcal{A}^0)^{-1/2}\sin(\tau(\mathcal{A}^0)^{1/2}) - \varepsilon\,\mathrm{K}(\varepsilon,\tau)\|_{H^2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)}\\ \leqslant C\varepsilon(1+|\tau|),\end{aligned} \tag{0.7}$$

with $\varepsilon > 0$, $\tau \in \mathbb{R}$. In the general case, the corrector contains the smoothing operator. We distinguish the cases when the smoothing operator can be removed.

The results are applied to homogenization of the system (0.5) with $\boldsymbol{\varphi} = 0$. A more general equation $Q(\mathbf{x}/\varepsilon)\partial_\tau^2\mathbf{u}_\varepsilon(\mathbf{x},\tau) = -\mathcal{A}_\varepsilon\mathbf{u}_\varepsilon(\mathbf{x},\tau) + Q(\mathbf{x}/\varepsilon)\mathbf{F}(\mathbf{x},\tau)$ is also considered. Here $Q(\mathbf{x})$ is a Γ-periodic $(n\times n)$-matrix-valued function such that $Q(\mathbf{x}) > 0$ and $Q, Q^{-1} \in L_\infty$. In Introduction, we discuss only the case $Q = \mathbf{1}_n$ for simplicity.

0.6. Method. We apply the method of [13, 24] carrying out all the constructions for the operator $\mathcal{A}_\varepsilon^{-1/2}\sin(\tau\mathcal{A}_\varepsilon^{1/2})$. To obtain the result with the correction term, we borrow some technical tools from [44]. By the *scaling transformation*, inequality (0.6) is equivalent to

$$\begin{aligned}&\|(\mathcal{A}^{-1/2}\sin(\varepsilon^{-1}\tau\mathcal{A}^{1/2}) - (\mathcal{A}^0)^{-1/2}\sin(\varepsilon^{-1}\tau(\mathcal{A}^0)^{1/2}))\\ &\quad\cdot\varepsilon(-\Delta+\varepsilon^2 I)^{-1/2}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\\ &\leqslant C(1+|\tau|),\end{aligned} \tag{0.8}$$

with $\tau \in \mathbb{R}$, $\varepsilon > 0$. Here $\mathcal{A} = b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D})$. Because of the presence of differentiation in the definition of H^1-norm, by the scaling transformation, inequality (0.7) reduces to the estimate of order $O(\varepsilon)$:

$$\begin{aligned}&\|\mathbf{D}(\mathcal{A}^{-1/2}\sin(\varepsilon^{-1}\tau\mathcal{A}^{1/2}) - (\mathcal{A}^0)^{-1/2}\sin(\varepsilon^{-1}\tau(\mathcal{A}^0)^{1/2}) - \mathrm{K}(1,\varepsilon^{-1}\tau))\\ &\quad\cdot\varepsilon^2(-\Delta+\varepsilon^2 I)^{-1}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\\ &\leqslant C\varepsilon(1+|\tau|),\end{aligned} \tag{0.9}$$

with $\tau \in \mathbb{R}$, $\varepsilon > 0$. For this reason, in estimate (0.9), we use the "smoothing operator" $\varepsilon^2(-\Delta + \varepsilon^2 I)^{-1}$ instead of the operator $\varepsilon(-\Delta + \varepsilon^2 I)^{-1/2}$ which was used in estimate (0.8) of order $O(1)$. Thus, the principal term of approximation of the operator $\mathcal{A}_\varepsilon^{-1/2} \sin(\tau \mathcal{A}_\varepsilon^{1/2})$ is obtained in the $(H^1 \to L_2)$-norm, but approximation in the energy class is given in the $(H^2 \to H^1)$-norm.

To obtain estimates (0.8) and (0.9), using the unitary *Gelfand transformation* (see Section 4.2 below), we decompose the operator $\mathcal{A}$ into the direct integral of operators $\mathcal{A}(\mathbf{k})$ acting in the space L_2 on the cell of periodicity and depending on the parameter $\mathbf{k} \in \mathbb{R}^d$ called the *quasimomentum*. We study the family $\mathcal{A}(\mathbf{k})$ by means of the analytic perturbation theory with respect to the onedimensional parameter $|\mathbf{k}|$. Then we should make our constructions and estimates uniform in the additional parameter $\boldsymbol{\theta} := \mathbf{k}/|\mathbf{k}|$. Herewith, a good deal of considerations can be done in the framework of an abstract operator-theoretic scheme.

0.7. Plan of the paper. The paper consists of three chapters. Chapter I (Sections 1–3) contains necessary operator-theoretic material.

Chapter II (Sections 4–8) is devoted to periodic DO's. In Sections 4–6, the class of operators under consideration is introduced, the direct integral decomposition is described, and the effective characteristics are found. In Section 7 and 8, the approximations for the operator-valued function $\mathcal{A}^{-1/2} \sin(\varepsilon^{-1}\tau \mathcal{A}^{1/2})$ are obtained and estimates (0.8) and (0.9) are proven.

In Chapter III (Sections 9–11), homogenization for hyperbolic systems is considered. In Section 9, the main results of the paper in operator terms (estimates (0.6) and (0.7)) are obtained. Afterwards, in Section 10, these results are applied to homogenization for solutions of the hyperbolic systems. Section 11 is devoted to applications of the general results to the acoustics equation, the operator of elasticity theory, and the model equation of electrodynamics.

0.8. Acknowledgement. The author is grateful to T. A. Suslina for attention to work and numerous comments that helped to improve the quality of presentation.

0.9. Notation. Let $\mathfrak{H}$ and $\mathfrak{H}_*$ be separable Hilbert spaces. The symbols $(\cdot,\cdot)_{\mathfrak{H}}$ and $\|\cdot\|_{\mathfrak{H}}$ mean the inner product and the norm in $\mathfrak{H}$, respectively; the symbol $\|\cdot\|_{\mathfrak{H}\to\mathfrak{H}_*}$ denotes the norm of a bounded linear operator acting from $\mathfrak{H}$ to $\mathfrak{H}_*$. Sometimes we omit the indices if this does not lead to confusion. By $I = I_{\mathfrak{H}}$ we denote the identity operator in $\mathfrak{H}$. If $A\colon \mathfrak{H} \to \mathfrak{H}_*$ is a linear operator, then $\operatorname{Dom} A$ denotes the domain of A. If $\mathfrak{N}$ is a subspace of $\mathfrak{H}$, then

$$\mathfrak{N}^\perp := \mathfrak{H} \ominus \mathfrak{N}.$$

The symbol $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{C}^n$, $|\cdot|$ means the norm of a vector in $\mathbb{C}^n$; $\mathbf{1}_n$ is the unit matrix of size $n \times n$. If a is an $(m \times n)$-matrix, then $|a|$ denotes its norm as a linear operator from $\mathbb{C}^n$ to $\mathbb{C}^m$; a^* means the Hermitian conjugate $(n \times m)$-matrix.

The classes L_p of $\mathbb{C}^n$-valued functions on a domain $\mathcal{O} \subset \mathbb{R}^d$ are denoted by $L_p(\mathcal{O};\mathbb{C}^n)$, $1 \leqslant p \leqslant \infty$. The Sobolev spaces of order s of $\mathbb{C}^n$-valued functions on a domain $\mathcal{O} \subset \mathbb{R}^d$ are denoted by $H^s(\mathcal{O};\mathbb{C}^n)$. By $\mathcal{S}(\mathbb{R}^d;\mathbb{C}^n)$ we denote the Schwartz class of $\mathbb{C}^n$-valued functions in $\mathbb{R}^d$. If $n = 1$, then we simply write $L_p(\mathcal{O})$, $H^s(\mathcal{O})$ and so on, but sometimes we use such simplified notation also for the spaces of vector-valued or matrix-valued functions. The symbol $L_p((0,T);\mathfrak{H})$, $1 \leqslant p \leqslant \infty$, stands for L_p-space of $\mathfrak{H}$-valued functions on the interval $(0,T)$.

Next, $\mathbf{x} = (x_1,\dots,x_d) \in \mathbb{R}^d$, $iD_j = \partial_j = \partial/\partial x_j$, $j = 1,\dots,d$, $\mathbf{D} = -i\nabla = (D_1,\dots,D_d)$. The Laplace operator is denoted by $\Delta = \partial^2/\partial x_1^2 + \dots + \partial^2/\partial x_d^2$.

By C, $\mathcal{C}$, $\mathfrak{C}$, c, $\mathfrak{c}$ (probably, with indices and marks) we denote various constants in estimates. The absolute constants are denoted by β with various indices.

Chapter I
Abstract scheme

1. Preliminaries

1.1. Quadratic operator pencils. Let $\mathfrak{H}$ and $\mathfrak{H}_*$ be separable complex Hilbert spaces. Suppose that $X_0 \colon \mathfrak{H} \to \mathfrak{H}_*$ is a densely defined and closed operator, and that $X_1 \colon \mathfrak{H} \to \mathfrak{H}_*$ is a bounded operator. On the domain $\operatorname{Dom} X(t) = \operatorname{Dom} X_0$, consider the operator

$$X(t) := X_0 + tX_1, \quad t \in \mathbb{R}.$$

Our main object is a family of operators

$$A(t) := X(t)^* X(t), \quad t \in \mathbb{R}, \tag{1.1}$$

that are selfadjoint in $\mathfrak{H}$ and non-negative. The operator $A(t)$ acting in $\mathfrak{H}$ is generated by the closed quadratic form $\|X(t)u\|^2_{\mathfrak{H}_*}$, $u \in \operatorname{Dom} X_0$. Denote

$$A(0) = X_0^* X_0 =: A_0.$$

Put

$$\mathfrak{N} := \operatorname{Ker} A_0 = \operatorname{Ker} X_0, \quad \mathfrak{N}_* := \operatorname{Ker} X_0^*.$$

We assume that the point $\lambda_0 = 0$ is isolated in the spectrum of A_0 and

$$0 < n := \dim \mathfrak{N} < \infty, \quad n \leqslant n_* := \dim \mathfrak{N}_* \leqslant \infty.$$

By d_0 we denote the distance from the point zero to the rest of the spectrum of A_0 and by $F(t,s)$ we denote the spectral projection of the operator $A(t)$ for the interval $[0,s]$. Fix $\delta > 0$ such that $8\delta < d_0$. Next, we choose a number $t_0 > 0$ such that

$$t_0 \leqslant \delta^{1/2} \|X_1\|^{-1}_{\mathfrak{H}\to\mathfrak{H}_*}. \tag{1.2}$$

Then (see [9, Chapter 1, (1.3)]) $F(t,\delta) = F(t,3\delta)$ and $\operatorname{rank} F(t,\delta) = n$ for $|t| \leqslant t_0$. We often write $F(t)$ instead of $F(t,\delta)$. Let P and P_* be the orthogonal projections of $\mathfrak{H}$ onto $\mathfrak{N}$ and of $\mathfrak{H}_*$ onto $\mathfrak{N}_*$, respectively.

1.2. Operators Z and R.

Let

$$\mathfrak{D} := \operatorname{Dom} X_0 \cap \mathfrak{N}^\perp,$$

and let $u \in \mathfrak{H}_*$. Consider the following equation for the element $\psi \in \mathfrak{D}$ (cf. [9, Chapter 1, (1.7)]):

$$X_0^*(X_0\psi - u) = 0. \tag{1.3}$$

The equation is understood in the weak sense. In other words, $\psi \in \mathfrak{D}$ satisfies the identity

$$(X_0\psi, X_0\zeta)_{\mathfrak{H}_*} = (u, X_0\zeta)_{\mathfrak{H}_*}, \quad \text{for all } \zeta \in \mathfrak{D}.$$

Equation (1.3) has a unique solution ψ, and $\|X_0\psi\|_{\mathfrak{H}_*} \leqslant \|u\|_{\mathfrak{H}_*}$. Now, let $\omega \in \mathfrak{N}$ and $u = -X_1\omega$. The corresponding solution of equation (1.3) is denoted by $\psi(\omega)$. We define the bounded operator $Z\colon \mathfrak{H} \to \mathfrak{H}$ by the identities

$$Z\omega = \psi(\omega), \quad \omega \in \mathfrak{N}; \quad Zx = 0, \quad x \in \mathfrak{N}^\perp.$$

Note that

$$ZP = Z, \quad PZ = 0. \tag{1.4}$$

Now, we introduce an operator

$$R\colon \mathfrak{N} \longrightarrow \mathfrak{N}_*$$

(see [9, Chapter 1, Subsection 1.2]) as follows:

$$R\omega = X_0\psi(\omega) + X_1\omega \in \mathfrak{N}_*.$$

Another description of R is given by the formula

$$R = P_* X_1|_{\mathfrak{N}}.$$

1.3. The spectral germ. The selfadjoint operator

$$S := R^* R: \mathfrak{N} \longrightarrow \mathfrak{N}$$

is called the *spectral germ* of the operator family (1.1) at $t = 0$ (see [9, Chapter 1, Subsection 1.3]). This operator also can be written as

$$S = P X_1^* P_* X_1|_{\mathfrak{N}}.$$

So,

$$\|S\| \leqslant \|X_1\|^2. \tag{1.5}$$

The spectral germ S is called *nondegenerate,* if $\operatorname{Ker} S = \{0\}$ or, equivalently, $\operatorname{rank} R = n$.

In accordance with the analytic perturbation theory (see [31]), for $|t| \leqslant t_0$ there exist real-analytic functions $\lambda_l(t)$ and real-analytic $\mathfrak{H}$-valued functions $\phi_l(t)$ such that

$$A(t)\phi_l(t) = \lambda_l(t)\phi_l(t), \quad l = 1, \dots, n, \ |t| \leqslant t_0,$$

and $\phi_l(t)$, $l = 1, \dots, n$, form an orthonormal basis in the eigenspace $F(t)\mathfrak{H}$. For sufficiently small t_* ($\leqslant t_0$) and $|t| \leqslant t_*$, we have the following convergent power series expansions:

$$\lambda_l(t) = \gamma_l t^2 + \mu_l t^3 + \cdots, \qquad \gamma_l \geqslant 0, \ \mu_l \in \mathbb{R}, \ l = 1, \dots, n; \tag{1.6}$$

$$\phi_l(t) = \omega_l + t\phi_l^{(1)} + t^2\phi_l^{(2)} + \cdots, \quad l = 1, \dots, n.$$

The elements $\omega_l = \phi_l(0)$, $l = 1, \dots, n$, form an orthonormal basis in $\mathfrak{N}$.

In [9, Chapter 1, Subsection 1.6] it was shown that the numbers γ_l and the elements ω_l, $l = 1, \dots, n$, are eigenvalues and eigenvectors of the operator S:

$$S\omega_l = \gamma_l \omega_l, \quad l = 1, \dots, n. \tag{1.7}$$

The numbers γ_l and the vectors ω_l, $l = 1, \dots, n$, are called *threshold characteristics at the bottom of the spectrum* of the operator family $A(t)$.

1.4. Threshold approximations. We assume that

$$A(t) \geqslant c_* t^2 I, \quad |t| \leqslant t_0, \tag{1.8}$$

for some $c_* > 0$. This is equivalent to the following estimates for the eigenvalues $\lambda_l(t)$ of the operator $A(t)$: $\lambda_l(t) \geqslant c_* t^2$, $|t| \leqslant t_0$, $l = 1, \dots, n$. Taking (1.6) into account, we see that $\gamma_l \geqslant c_*$, $l = 1, \dots, n$. So, by (1.7), the germ S is nondegenerate:

$$S \geqslant c_* I_{\mathfrak{N}}. \tag{1.9}$$

As was shown in [9, Chapter 1, Theorem 4.1],

$$\|F(t) - P\| \leqslant C_1|t|, \quad |t| \leqslant t_0, \tag{1.10}$$

with

$$C_1 := \beta_1 \delta^{-1/2} \|X_1\|.$$

Besides (1.10), we need a more accurate approximation of the spectral projection obtained in [10, (2.10) and (2.15)]:

$$F(t) = P + tF_1 + F_2(t), \quad \|F_2(t)\| \leqslant C_2 t^2, \ |t| \leqslant t_0, \tag{1.11}$$

where

$$C_2 := \beta_2 \delta^{-1} \|X_1\|^2$$

and

$$F_1 = ZP + PZ^*. \tag{1.12}$$

From (1.4) and (1.12) it follows that

$$F_1 P = ZP. \tag{1.13}$$

In [9, Chapter 1, Theorem 5.2], it was proven that

$$\|(A(t) + \zeta I)^{-1} F(t) - (t^2 SP + \zeta I)^{-1} P\| \leqslant C_3 |t| (c_* t^2 + \zeta)^{-1}, \tag{1.14}$$

with $\zeta > 0$, $|t| \leqslant t_0$ and

$$C_3 := \beta_3 \delta^{-1/2} \|X_1\| (1 + c_*^{-1} \|X_1\|^2). \tag{1.15}$$

According to [13, Theorem 2.4], we have

$$\|A(t)^{1/2} F(t) - (t^2 S)^{1/2} P\| \leqslant C_4 t^2, \quad |t| \leqslant t_0; \tag{1.16}$$

with

$$C_4 := \beta_4 \delta^{-1/2} \|X_1\|^2 (1 + c_*^{-1/2} \|X_1\|). \tag{1.17}$$

Combining this with (1.5), we see that

$$\|A(t)^{1/2} F(t)\| \leqslant |t| \|S\|^{1/2} + C_4 t^2 \leqslant (\|X_1\| + C_4 t_0)|t|, \quad |t| \leqslant t_0. \tag{1.18}$$

We also need the following estimate for the operator $A(t)^{1/2} F_2(t)$ obtained in [12, (2.23)]:

$$\|A(t)^{1/2} F_2(t)\|_{\mathfrak{H} \to \mathfrak{H}} \leqslant C_5 t^2, \quad |t| \leqslant t_0, \tag{1.19}$$

where

$$C_5 := \beta_5 \delta^{-1/2} \|X_1\|^2.$$

1.5. Approximation of the operator $A(t)^{-1/2}F(t)$ for $t \neq 0$

Lemma 1.1. *For* $|t| \leqslant t_0$ *and* $t \neq 0$ *we have*

$$\|A(t)^{-1/2}F(t) - (t^2 S)^{-1/2}P\| \leqslant C_6. \tag{1.20}$$

The constant C_6 *is defined below in* (1.23) *and depends only on* δ, $\|X_1\|$, *and* c_*.

Proof. We have

$$A(t)^{-1/2}F(t) = \frac{1}{\pi}\int_0^\infty \zeta^{-1/2}(A(t)+\zeta I)^{-1}F(t)\,d\zeta, \quad t \neq 0. \tag{1.21}$$

(See, e.g., [49, Chapter III, Section 3, Subsection 4]). Similarly,

$$\begin{aligned}(t^2 S)^{-1/2}P &= \frac{1}{\pi}\int_0^\infty \zeta^{-1/2}(t^2 S + \zeta I_{\mathfrak{N}})^{-1}P\,d\zeta \\ &= \frac{1}{\pi}\int_0^\infty \zeta^{-1/2}(t^2 SP + \zeta I)^{-1}P\,d\zeta.\end{aligned} \tag{1.22}$$

Subtracting (1.22) from (1.21), using (1.14), and changing the variable

$$\tilde{\zeta} := (c_* t^2)^{-1}\zeta,$$

we obtain

$$\begin{aligned}\|A(t)^{-1/2}F(t) - (t^2 S)^{-1/2}P\| &\leqslant \frac{C_3}{\pi}\int_0^\infty \zeta^{-1/2}|t|(c_* t^2+\zeta)^{-1}\,d\zeta \\ &= \frac{C_3}{\pi}c_*^{-1/2}\int_0^\infty \tilde{\zeta}^{-1/2}(1+\tilde{\zeta})^{-1}\,d\tilde{\zeta} \\ &\leqslant \frac{C_3}{\pi}c_*^{-1/2}\Big(\int_0^1 \tilde{\zeta}^{-1/2}\,d\tilde{\zeta} + \int_1^\infty \tilde{\zeta}^{-3/2}\,d\tilde{\zeta}\Big) \\ &= 4\pi^{-1}c_*^{-1/2}C_3.\end{aligned}$$

We arrive at estimate (1.20) with the constant

$$C_6 := 4\pi^{-1}c_*^{-1/2}C_3. \tag{1.23}$$

□

2. Approximation of the operator $A(t)^{-1/2}\sin(\tau A(t)^{1/2})$

2.1. The principal term of approximation

Proposition 2.1. *For $|t| \leqslant t_0$ and $\tau \in \mathbb{R}$ we have*

$$\|(A(t)^{-1/2}\sin(\tau A(t)^{1/2}) - (t^2S)^{-1/2}\sin(\tau(t^2S)^{1/2}P))P\| \leqslant C_7(1+|\tau||t|). \tag{2.1}$$

The constant C_7 depends only on δ, $\|X_1\|$, and c_*.

Proof. For $t=0$ the operator under the norm sign in (2.1) is understood as a limit for $t \to 0$. Using the Taylor series expansion for the sine function, we see that this limit is equal to zero.

Now, let $t \neq 0$. We put

$$E(\tau) := e^{-i\tau A(t)^{1/2}}A(t)^{-1/2}F(t) - e^{-i\tau(t^2S)^{1/2}P}(t^2S)^{-1/2}P; \tag{2.2}$$

$$\begin{aligned}\Sigma(\tau) &:= e^{i\tau(t^2S)^{1/2}P}E(\tau)\\ &= e^{i\tau(t^2S)^{1/2}P}e^{-i\tau A(t)^{1/2}}A(t)^{-1/2}F(t) - (t^2S)^{-1/2}P.\end{aligned} \tag{2.3}$$

Then

$$\Sigma(0) = A(t)^{-1/2}F(t) - (t^2S)^{-1/2}P \tag{2.4}$$

and

$$\frac{d\Sigma(\tau)}{d\tau} = ie^{i\tau(t^2S)^{1/2}P}((t^2S)^{1/2}P - A(t)^{1/2}F(t))e^{-i\tau A(t)^{1/2}}A(t)^{-1/2}F(t). \tag{2.5}$$

By (1.8) and (1.16), the operator-valued function (2.5) satisfies the following estimate:

$$\left\|\frac{d\Sigma(\tau)}{d\tau}\right\| \leqslant C_4t^2\|A(t)^{-1/2}\| \leqslant C_4c_*^{-1/2}|t|, \quad |t| \leqslant t_0,\ t \neq 0. \tag{2.6}$$

Then, taking (1.20), (2.3), (2.4), and (2.6) into account, we see that

$$\|E(\tau)\| = \|\Sigma(\tau)\| \leqslant C_4c_*^{-1/2}|t||\tau| + \|\Sigma(0)\| \leqslant C_8(1+|\tau||t|), \tag{2.7}$$

with $|t| \leqslant t_0$, $t \neq 0$, and

$$C_8 := \max\{C_4c_*^{-1/2}; C_6\}. \tag{2.8}$$

(Cf. the proof of Theorem 2.5 from [13].) So,

$$\|A(t)^{-1/2}\sin(\tau A(t)^{1/2})F(t) - (t^2S)^{-1/2}\sin(\tau(t^2S)^{1/2}P)P\| \leqslant C_8(1+|\tau||t|). \tag{2.9}$$

By virtue of (1.8) and (1.10), from (2.9) we derive the inequality

$$\begin{aligned}&\|(A(t)^{-1/2}\sin(\tau A(t)^{1/2})-(t^2S)^{-1/2}\sin(\tau(t^2S)^{1/2}P))P\|\\&\quad\leqslant C_8(1+|\tau||t|)+\|A(t)^{-1/2}\sin(\tau A(t)^{1/2})(F(t)-P)\|\\&\quad\leqslant C_7(1+|\tau||t|),\end{aligned}\tag{2.10}$$

with $|t|\leqslant t_0$ and

$$C_7:=C_8+c_*^{-1/2}C_1.$$ □

2.2. Approximation in the "energy" norm. Now, we obtain another approximation for the operator $A(t)^{-1/2}\sin(\tau A(t)^{1/2})$ (in the "energy" norm).

Proposition 2.2. *For $\tau\in\mathbb{R}$ and $|t|\leqslant t_0$,*

$$\begin{aligned}&\|A(t)^{1/2}(A(t)^{-1/2}\sin(\tau A(t)^{1/2})-(I+tZ)(t^2S)^{-1/2}\sin(\tau(t^2S)^{1/2}P))P\|\\&\quad\leqslant C_9(|t|+|\tau|t^2).\end{aligned}\tag{2.11}$$

The constant C_9 depends only on δ, $\|X_1\|$, and c_*.

Proof. Note that

$$\begin{aligned}&A(t)^{1/2}e^{-i\tau A(t)^{1/2}}A(t)^{-1/2}P\\&\quad=A(t)^{1/2}e^{-i\tau A(t)^{1/2}}A(t)^{-1/2}F(t)P+e^{-i\tau A(t)^{1/2}}(P-F(t))P.\end{aligned}\tag{2.12}$$

By (1.10),

$$\|e^{-i\tau A(t)^{1/2}}(P-F(t))P\|\leqslant C_1|t|,\quad \tau\in\mathbb{R},\ |t|\leqslant t_0.\tag{2.13}$$

Next,

$$\begin{aligned}&A(t)^{1/2}e^{-i\tau A(t)^{1/2}}A(t)^{-1/2}F(t)P\\&\quad=A(t)^{1/2}F(t)e^{-i\tau(t^2S)^{1/2}P}(t^2S)^{-1/2}P+A(t)^{1/2}F(t)E(\tau)P,\end{aligned}\tag{2.14}$$

where $E(\tau)$ is given by (2.2). By (1.18) and (2.7), for $t\neq 0$

$$\|A(t)^{1/2}F(t)E(\tau)P\|\leqslant C_8(\|X_1\|+C_4t_0)(|t|+|\tau|t^2),\quad \tau\in\mathbb{R},\tag{2.15}$$

with $|t|\leqslant t_0$, $t\neq 0$. For $t=0$ the operator under the norm sign in (2.15) is understood as a limit for $t\to 0$. We have $e^{-i\tau A(t)^{1/2}}F(t)\to P$, as $t\to 0$. Next, by (1.9) and (1.16),

$$\begin{aligned}&\|A(t)^{1/2}F(t)e^{-i\tau(t^2S)^{1/2}P}(t^2S)^{-1/2}P-e^{-i\tau(t^2S)^{1/2}P}P\|\\&\quad=\|A(t)^{1/2}F(t)(t^2S)^{-1/2}P-P\|\leqslant c_*^{-1/2}C_4|t|,\quad \tau\in\mathbb{R},\end{aligned}$$

with $|t| \leqslant t_0$. Using these arguments, we see that the limit of the left-hand side of (2.15) as $t \to 0$ is equal to zero.

According to (1.11) and (1.13),

$$A(t)^{1/2}F(t)e^{-i\tau(t^2S)^{1/2}P}(t^2S)^{-1/2}P - A(t)^{1/2}(I+tZ)e^{-i\tau(t^2S)^{1/2}P}(t^2S)^{-1/2}P$$
$$= A(t)^{1/2}F_2(t)e^{-i\tau(t^2S)^{1/2}P}(t^2S)^{-1/2}P. \tag{2.16}$$

By (1.9) and (1.19),

$$\|A(t)^{1/2}F_2(t)e^{-i\tau(t^2S)^{1/2}P}(t^2S)^{-1/2}P\| \leqslant c_*^{-1/2}C_5|t|, \tag{2.17}$$

with $\tau \in \mathbb{R}$, $|t| \leqslant t_0$. Combining (2.12)–(2.17), we arrive at

$$\begin{aligned}&\|A(t)^{1/2}(e^{-i\tau A(t)^{1/2}}A(t)^{-1/2} - (I+tZ)e^{-i\tau(t^2S)^{1/2}P}(t^2S)^{-1/2}P)P\|\\ &\quad \leqslant C_9(|t| + |\tau|t^2),\end{aligned} \tag{2.18}$$

with $\tau \in \mathbb{R}$, $|t| \leqslant t_0$, and

$$C_9 := C_1 + c_*^{-1/2}C_5 + C_8(\|X_1\| + C_4t_0).$$

(Cf. the proof of Theorem 3.1 from [44].) □

2.3. Approximation of the operator $A(t)^{-1/2}\sin(\varepsilon^{-1}\tau A(t)^{1/2})P$. Now, we introduce a parameter $\varepsilon > 0$. We need to study the behavior of the operator $A(t)^{-1/2}\sin(\varepsilon^{-1}\tau A(t)^{1/2})P$ for small ε. Replace τ by $\varepsilon^{-1}\tau$ in (2.1):

$$\begin{aligned}&\|(A(t)^{-1/2}\sin(\varepsilon^{-1}\tau A(t)^{1/2}) - (t^2S)^{-1/2}\sin(\varepsilon^{-1}\tau(t^2S)^{1/2}P))P\|\\ &\quad \leqslant C_7(1+\varepsilon^{-1}|\tau||t|),\end{aligned}$$

with $|t| \leqslant t_0$, $\varepsilon > 0$, $\tau \in \mathbb{R}$. Multiplying this inequality by the "smoothing" factor $\varepsilon(t^2+\varepsilon^2)^{-1/2}$ and taking into account the inequalities $\varepsilon(t^2+\varepsilon^2)^{-1/2} \leqslant 1$ and $|\tau||t|(t^2+\varepsilon^2)^{-1/2} \leqslant |\tau|$, we obtain the following result.

Theorem 2.3. *For $\tau \in \mathbb{R}$, $\varepsilon > 0$, and $|t| \leqslant t_0$ we have*

$$\begin{aligned}&\|(A(t)^{-1/2}\sin(\varepsilon^{-1}\tau A(t)^{1/2}) - (t^2S)^{-1/2}\sin(\varepsilon^{-1}\tau(t^2S)^{1/2}P))\varepsilon(t^2+\varepsilon^2)^{-1/2}P\|\\ &\quad \leqslant C_7(1+|\tau|).\end{aligned}$$

Replacing τ by $\varepsilon^{-1}\tau$ in (2.11) and multiplying the operator by $\varepsilon^2(t^2+\varepsilon^2)^{-1}$, we arrive at the following statement.

Theorem 2.4. *For $\tau \in \mathbb{R}$, $\varepsilon > 0$, and $|t| \leqslant t_0$,*

$$\| A(t)^{1/2}(A(t)^{-1/2}\sin(\varepsilon^{-1}\tau A(t)^{1/2}) - (I + tZ)(t^2S)^{-1/2}\sin(\varepsilon^{-1}\tau(t^2S)^{1/2}P)) \cdot \varepsilon^2(t^2+\varepsilon^2)^{-1}P\| \leqslant C_9\varepsilon(1+|\tau|).$$

3. Approximation of the sandwiched operator sine

3.1. The operator family $A(t) = M^*\widehat{A}(t)M$. Now, we consider an operator family of the form $A(t) = M^*\widehat{A}(t)M$ (see [9, Chapter 1, Subsections 1.5 and 5.3]).

Let $\widehat{\mathfrak{H}}$ be yet another separable Hilbert space. Let $\widehat{X}(t) = \widehat{X}_0 + t\widehat{X}_1 : \widehat{\mathfrak{H}} \to \mathfrak{H}_*$ be a family of operators of the same form as $X(t)$, and suppose that $\widehat{X}(t)$ satisfies the assumptions of Subsection 1.1.

Let $M : \mathfrak{H} \to \widehat{\mathfrak{H}}$ be an isomorphism. Suppose that $M \operatorname{Dom} X_0 = \operatorname{Dom} \widehat{X}_0$; $X_0 = \widehat{X}_0 M$; $X_1 = \widehat{X}_1 M$. Then $X(t) = \widehat{X}(t)M$. Consider the family of operators

$$\widehat{A}(t) = \widehat{X}(t)^*\widehat{X}(t) : \widehat{\mathfrak{H}} \longrightarrow \widehat{\mathfrak{H}}. \tag{3.1}$$

Obviously,

$$A(t) = M^*\widehat{A}(t)M. \tag{3.2}$$

In what follows, all the objects corresponding to the family (3.1) are supplied with the upper mark "ˆ". Note that $\widehat{\mathfrak{N}} = M\mathfrak{N}$, $\hat{n} = n$, $\widehat{\mathfrak{N}}_* = \mathfrak{N}_*$, $\hat{n}_* = n_*$, and $\widehat{P}_* = P_*$.

We denote

$$Q := (MM^*)^{-1} = (M^*)^{-1}M^{-1} : \widehat{\mathfrak{H}} \longrightarrow \widehat{\mathfrak{H}}. \tag{3.3}$$

Let $Q_{\widehat{\mathfrak{N}}}$ be the block of Q in the subspace $\widehat{\mathfrak{N}}$: $Q_{\widehat{\mathfrak{N}}} = \widehat{P}Q|_{\widehat{\mathfrak{N}}} : \widehat{\mathfrak{N}} \to \widehat{\mathfrak{N}}$. Obviously, $Q_{\widehat{\mathfrak{N}}}$ is an isomorphism in $\widehat{\mathfrak{N}}$. Let

$$M_0 := (Q_{\widehat{\mathfrak{N}}})^{-1/2} : \widehat{\mathfrak{N}} \longrightarrow \widehat{\mathfrak{N}}.$$

As was shown in [43, Proposition 1.2], the orthogonal projection P of the space $\mathfrak{H}$ onto $\mathfrak{N}$ and the orthogonal projection $\widehat{P}$ of the space $\widehat{\mathfrak{H}}$ onto $\widehat{\mathfrak{N}}$ satisfy the following relation: $P = M^{-1}(Q_{\widehat{\mathfrak{N}}})^{-1}\widehat{P}(M^*)^{-1}$. Hence,

$$PM^* = M^{-1}(Q_{\widehat{\mathfrak{N}}})^{-1}\widehat{P} = M^{-1}M_0^2\widehat{P}. \tag{3.4}$$

According to [9, Chapter 1, Subsection 1.5], the spectral germs S and $\widehat{S}$ satisfy $S = PM^*\widehat{S}M|_{\mathfrak{N}}$.

For the operator family (3.1) we introduce the operator $\widehat{Z}_Q$ acting in $\widehat{\mathfrak{H}}$ and taking an element $\hat{u} \in \widehat{\mathfrak{H}}$ to the solution $\hat{\varphi}_Q$ of the problem

$$\widehat{X}_0^*(\widehat{X}_0\hat{\varphi}_Q + \widehat{X}_1\hat{\omega}) = 0, \quad Q\hat{\varphi}_Q \perp \widehat{\mathfrak{N}}, \tag{3.5}$$

where $\hat{\omega} := \widehat{P}\hat{u}$. Equation (3.5) is understood in the weak sense. As was shown in [10, Lemma 6.1], the operator Z for $A(t)$ and the operator $\widehat{Z}_Q$ satisfy

$$\widehat{Z}_Q = MZM^{-1}\widehat{P}. \tag{3.6}$$

3.2. The principal term of approximation for the sandwiched operator $A(t)^{-1/2}\sin(\tau A(t)^{1/2})$. In this subsection, we find an approximation for the operator $A(t)^{-1/2}\sin(\tau A(t)^{1/2})$, where $A(t)$ is given by (3.2), in terms of the germ $\widehat{S}$ of $\widehat{A}(t)$ and the isomorphism M. It is convenient to border the operator $A(t)^{-1/2}\sin(\tau A(t)^{1/2})$ by appropriate factors.

Proposition 3.1. *Suppose that the assumptions of Subsection 3.1 are satisfied. Then for $\tau \in \mathbb{R}$ and $|t| \leqslant t_0$ we have*

$$\begin{aligned} &\|MA(t)^{-1/2}\sin(\tau A(t)^{1/2})M^{-1}\widehat{P} \\ &\quad - M_0(t^2M_0\widehat{S}M_0)^{-1/2}\sin(\tau(t^2M_0\widehat{S}M_0)^{1/2})M_0^{-1}\widehat{P}\|_{\widehat{\mathfrak{H}}\to\widehat{\mathfrak{H}}} \\ &\leqslant C_7\|M\|\|M^{-1}\|(1+|\tau||t|). \end{aligned} \tag{3.7}$$

Here t_0 is defined according to (1.2), and C_7 is the constant from (2.10) depending only on δ, $\|X_1\|$, and c_*.

Proof. Estimate (3.7) follows from Proposition 2.1 by recalculation. In [13, Proposition 3.3], it was shown that

$$M\cos(\tau(t^2S)^{1/2}P)PM^* = M_0\cos(\tau(t^2M_0\widehat{S}M_0)^{1/2})M_0\widehat{P}. \tag{3.8}$$

Obviously,

$$(t^2S)^{-1/2}\sin(\tau(t^2S)^{1/2}P)P = \int_0^\tau \cos(\tilde{\tau}(t^2S)^{1/2}P)P\,d\tilde{\tau}. \tag{3.9}$$

Similarly,

$$(t^2M_0\widehat{S}M_0)^{-1/2}\sin(\tau(t^2M_0\widehat{S}M_0)^{1/2})M_0\widehat{P} = \int_0^\tau \cos(\tilde{\tau}(t^2M_0\widehat{S}M_0)^{1/2})M_0\widehat{P}\,d\tilde{\tau}. \tag{3.10}$$

Integrating (3.8) over τ and taking (3.9) and (3.10) into account, we conclude that

$$
\begin{aligned}
&M(t^2S)^{-1/2}\sin(\tau(t^2S)^{1/2}P)PM^*\\
&\quad= M_0(t^2M_0\widehat{S}M_0)^{-1/2}\sin(\tau(t^2M_0\widehat{S}M_0)^{1/2})M_0\widehat{P}.
\end{aligned}
\tag{3.11}
$$

Next, since $M_0 = (Q_{\widehat{\mathfrak{H}}})^{-1/2}$, using (3.4), we obtain $PM^*M_0^{-2}\widehat{P} = M^{-1}\widehat{P}$. So, by (3.11),

$$
\begin{aligned}
&M(t^2S)^{-1/2}\sin(\tau(t^2S)^{1/2}P)M^{-1}\widehat{P}\\
&\quad= M_0(t^2M_0\widehat{S}M_0)^{-1/2}\sin(\tau(t^2M_0\widehat{S}M_0)^{1/2})M_0^{-1}\widehat{P}.
\end{aligned}
\tag{3.12}
$$

Thus,

$$
\begin{aligned}
&MA(t)^{-1/2}\sin(\tau A(t)^{1/2})M^{-1}\widehat{P}\\
&\qquad - M_0(t^2M_0\widehat{S}M_0)^{-1/2}\sin(\tau(t^2M_0\widehat{S}M_0)^{1/2})M_0^{-1}\widehat{P}\\
&\quad= M(A(t)^{-1/2}\sin(\tau A(t)^{1/2})P - (t^2S)^{-1/2}\sin(\tau(t^2S)^{1/2}P)P)M^{-1}\widehat{P}.
\end{aligned}
\tag{3.13}
$$

Using Proposition 2.1 and (3.13), we arrive at inequality (3.7). □

3.3. Approximation with the corrector

Proposition 3.2. *Under the assumptions of Subsection 3.1, for $\tau \in \mathbb{R}$ and $|t| \leqslant t_0$ we have*

$$
\begin{aligned}
&\|\widehat{A}(t)^{1/2}(MA(t)^{-1/2}\sin(\tau A(t)^{1/2})M^{-1}\widehat{P}\\
&\qquad - (I + t\widehat{Z}_Q)M_0(t^2M_0\widehat{S}M_0)^{-1/2}\sin(\tau(t^2M_0\widehat{S}M_0)^{1/2})M_0^{-1}\widehat{P})\|_{\widehat{\mathfrak{H}}\to\widehat{\mathfrak{H}}}\\
&\quad\leqslant C_9\|M^{-1}\|(|t| + |\tau|t^2).
\end{aligned}
\tag{3.14}
$$

The constant C_9 is the same as in (2.18) and depends only on δ, $\|X_1\|$, and c_*.

Proof. Estimate (3.14) follows from Proposition 2.2 by recalculation. According to (3.6) and (3.12),

$$
\begin{aligned}
&t\widehat{Z}_QM_0(t^2M_0\widehat{S}M_0)^{-1/2}\sin(\tau(t^2M_0\widehat{S}M_0)^{1/2})M_0^{-1}\widehat{P}\\
&\quad= tMZM^{-1}M_0(t^2M_0\widehat{S}M_0)^{-1/2}\sin(\tau(t^2M_0\widehat{S}M_0)^{1/2})M_0^{-1}\widehat{P}\\
&\quad= tMZ(t^2S)^{-1/2}\sin(\tau(t^2S)^{1/2})PM^{-1}\widehat{P}.
\end{aligned}
\tag{3.15}
$$

Combining (3.15) with (3.2) and (3.13), we obtain

$$\begin{aligned}
&\|\widehat{A}(t)^{1/2}(MA(t)^{-1/2}\sin(\tau A(t)^{1/2})M^{-1}\widehat{P} \\
&\qquad - (I + t\widehat{Z}_Q)M_0(t^2M_0\widehat{S}M_0)^{-1/2}\sin(\tau(t^2M_0\widehat{S}M_0)^{1/2})M_0^{-1}\widehat{P})\|_{\widehat{\mathfrak{H}}\to\widehat{\mathfrak{H}}} \\
&\quad = \|A(t)^{1/2}(A(t)^{-1/2}\sin(\tau A(t)^{1/2})P \\
&\qquad\qquad - (I + tZ)(t^2S)^{-1/2}\sin(\tau(t^2S)^{1/2})P)M^{-1}\widehat{P}\|_{\widehat{\mathfrak{H}}\to\mathfrak{H}}.
\end{aligned}$$

Together with Proposition 2.2, this implies (3.14). □

3.4. Approximation of the sandwiched operator $A(t)^{-1/2}\sin(\varepsilon^{-1}\tau A(t)^{1/2})$. Writing down (3.7) and (3.14) with τ replaced by $\varepsilon^{-1}\tau$ and multiplying the corresponding inequalities by the "smoothing factors," we arrive at the following result.

Theorem 3.3. *Under the assumptions of Subsection 3.1, for $\tau \in \mathbb{R}$, $\varepsilon > 0$, and $|t| \leqslant t_0$*

$$\begin{aligned}
&\|(MA(t)^{-1/2}\sin(\varepsilon^{-1}\tau A(t)^{1/2})M^{-1}\widehat{P} \\
&\quad - M_0(t^2M_0\widehat{S}M_0)^{-1/2}\sin(\varepsilon^{-1}\tau(t^2M_0\widehat{S}M_0)^{1/2})M_0^{-1}\widehat{P}) \\
&\qquad \cdot \varepsilon(t^2+\varepsilon^2)^{-1/2}\widehat{P}\|_{\widehat{\mathfrak{H}}\to\widehat{\mathfrak{H}}} \\
&\quad \leqslant C_7\|M\|\|M^{-1}\|(1+|\tau|),
\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}
&\|\widehat{A}(t)^{1/2}(MA(t)^{-1/2}\sin(\varepsilon^{-1}\tau A(t)^{1/2})M^{-1}\widehat{P} \\
&\qquad - (I + t\widehat{Z}_Q)M_0(t^2M_0\widehat{S}M_0)^{-1/2} \\
&\qquad\qquad \cdot \sin(\varepsilon^{-1}\tau(t^2M_0\widehat{S}M_0)^{1/2})M_0^{-1}\widehat{P}) \\
&\quad \cdot \varepsilon^2(t^2+\varepsilon^2)^{-1}\widehat{P}\|_{\widehat{\mathfrak{H}}\to\widehat{\mathfrak{H}}} \\
&\quad \leqslant C_9\|M^{-1}\|\varepsilon(1+|\tau|).
\end{aligned} \tag{3.17}$$

The number t_0 is subject to (1.2), the constants C_7 and C_9 are the same as in (2.10) and (2.18).

Chapter II
Periodic differential operators in $L_2(\mathbb{R}^d;\mathbb{C}^n)$

In the present chapter, we describe the class of matrix second order differential operators admitting a factorization of the form $\mathcal{A} = \mathcal{X}^*\mathcal{X}$, where $\mathcal{X}$ is a homogeneous first order DO. This class was introduced and studied in [9, Chapter 2].

4. Factorized second order operators

4.1. Lattices Γ and $\widetilde{\Gamma}$. Let Γ be a lattice in $\mathbb{R}^d$ generated by the basis $\mathbf{a}_1, \dots, \mathbf{a}_d$:

$$\Gamma := \Big\{\mathbf{a} \in \mathbb{R}^d : \mathbf{a} = \sum_{j=1}^{d} \nu_j \mathbf{a}_j ,\ \nu_j \in \mathbb{Z}\Big\},$$

and let Ω be the elementary cell of the lattice Γ:

$$\Omega := \Big\{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} = \sum_{j=1}^{d} \zeta_j \mathbf{a}_j ,\ -\frac{1}{2} < \zeta_j < \frac{1}{2}\Big\}.$$

The basis $\mathbf{b}_1, \dots, \mathbf{b}_d$ dual to $\mathbf{a}_1, \dots, \mathbf{a}_d$ is defined by the relations

$$\langle \mathbf{b}_l , \mathbf{a}_j \rangle = 2\pi \delta_{lj} .$$

This basis generates the lattice $\widetilde{\Gamma}$ dual to Γ:

$$\widetilde{\Gamma} := \Big\{\mathbf{b} \in \mathbb{R}^d : \mathbf{b} = \sum_{j=1}^{d} \mu_j \mathbf{b}_j ,\ \mu_j \in \mathbb{Z}\Big\}.$$

Let $\widetilde{\Omega}$ be the first Brillouin zone of the lattice $\widetilde{\Gamma}$:

$$\widetilde{\Omega} := \{\mathbf{k} \in \mathbb{R}^d : |\mathbf{k}| < |\mathbf{k} - \mathbf{b}|,\ 0 \neq \mathbf{b} \in \widetilde{\Gamma}\}. \tag{4.1}$$

Let $|\Omega|$ be the Lebesgue measure of the cell Ω:

$$|\Omega| = \operatorname{meas} \Omega ,$$

and let

$$|\widetilde{\Omega}| = \operatorname{meas} \widetilde{\Omega} .$$

We put

$$2r_1 := \operatorname{diam} \Omega .$$

The maximal radius of the ball containing in $\operatorname{clos} \widetilde{\Omega}$ is denoted by r_0. Note that

$$2r_0 = \min_{0 \neq \mathbf{b} \in \widetilde{\Gamma}} |\mathbf{b}|. \tag{4.2}$$

With the lattice Γ, we associate the discrete Fourier transformation

$$v(\mathbf{x}) = |\Omega|^{-1/2} \sum_{\mathbf{b} \in \widetilde{\Gamma}} \hat{v}_{\mathbf{b}} e^{i \langle \mathbf{b}, \mathbf{x} \rangle}, \quad \mathbf{x} \in \Omega , \tag{4.3}$$

which is a unitary mapping of $l_2(\widetilde{\Gamma})$ onto $L_2(\Omega)$:

$$\int_{\Omega} |v(\mathbf{x})|^2 \, d\mathbf{x} = \sum_{\mathbf{b}\in\widetilde{\Gamma}} |\hat{v}_{\mathbf{b}}|^2. \tag{4.4}$$

Below by $\widetilde{H}^1(\Omega;\mathbb{C}^n)$ *we denote the subspace of functions from* $H^1(\Omega;\mathbb{C}^n)$ *whose* Γ*-periodic extension to* $\mathbb{R}^d$ *belongs to* $H^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{C}^n)$. We have

$$\|(\mathbf{D}+\mathbf{k})\mathbf{u}\|^2_{L_2(\Omega)} = \sum_{\mathbf{b}\in\widetilde{\Gamma}} |\mathbf{b}+\mathbf{k}|^2 |\hat{\mathbf{u}}_{\mathbf{b}}|^2, \quad \mathbf{u}\in\widetilde{H}^1(\Omega;\mathbb{C}^n),\ \mathbf{k}\in\mathbb{R}^d, \tag{4.5}$$

and convergence of the series in the right-hand side of (4.5) is equivalent to the relation $\mathbf{u}\in\widetilde{H}^1(\Omega;\mathbb{C}^n)$. From (4.1), (4.4), and (4.5) it follows that

$$\|(\mathbf{D}+\mathbf{k})\mathbf{u}\|^2_{L_2(\Omega)} \geqslant \sum_{\mathbf{b}\in\widetilde{\Gamma}} |\mathbf{k}|^2 |\hat{\mathbf{u}}_{\mathbf{b}}|^2 = |\mathbf{k}|^2 \|\mathbf{u}\|^2_{L_2(\Omega)}, \quad \mathbf{u}\in\widetilde{H}^1(\Omega;\mathbb{C}^n),\ \mathbf{k}\in\widetilde{\Omega}. \tag{4.6}$$

If $\psi(\mathbf{x})$ is a Γ-periodic measurable matrix-valued function in $\mathbb{R}^d$, we put

$$\bar{\psi} := |\Omega|^{-1} \int_{\Omega} \psi(\mathbf{x})\, d\mathbf{x}$$

and

$$\underline{\psi} := \left(|\Omega|^{-1} \int_{\Omega} \psi(\mathbf{x})^{-1}\, d\mathbf{x} \right)^{-1}.$$

Here, in the definition of $\bar{\psi}$ it is assumed that $\psi\in L_{1,\mathrm{loc}}(\mathbb{R}^d)$, and in the definition of $\underline{\psi}$ it is assumed that the matrix $\psi(\mathbf{x})$ is square and nondegenerate, and $\psi^{-1}\in L_{1,\mathrm{loc}}(\mathbb{R}^d)$.

4.2. The Gelfand transformation. Initially, the Gelfand transformation $\mathcal{U}$ is defined on the functions of the Schwartz class by the formula

$$\tilde{\mathbf{v}}(\mathbf{k},\mathbf{x}) = (\mathcal{U}\mathbf{v})(\mathbf{k},\mathbf{x}) = |\widetilde{\Omega}|^{-1/2} \sum_{\mathbf{a}\in\Gamma} e^{-i\langle \mathbf{k},\mathbf{x}+\mathbf{a}\rangle} \mathbf{v}(\mathbf{x}+\mathbf{a}),$$

with $\mathbf{v}\in\mathcal{S}(\mathbb{R}^d;\mathbb{C}^n)$, $\mathbf{x}\in\Omega$, $\mathbf{k}\in\widetilde{\Omega}$. Since

$$\int_{\widetilde{\Omega}} \int_{\Omega} |\tilde{\mathbf{v}}(\mathbf{k},\mathbf{x})|^2 \, d\mathbf{x}\, d\mathbf{k} = \int_{\mathbb{R}^d} |\mathbf{v}(\mathbf{x})|^2\, d\mathbf{x},$$

the transformation $\mathcal{U}$ extends by continuity up to a unitary mapping

$$\mathcal{U}\colon L_2(\mathbb{R}^d;\mathbb{C}^n) \longrightarrow \int_{\widetilde{\Omega}} \oplus L_2(\Omega;\mathbb{C}^n)\, d\mathbf{k}.$$

The relation $\mathbf{v} \in H^1(\mathbb{R}^d;\mathbb{C}^n)$ is equivalent to $\tilde{\mathbf{v}}(\mathbf{k},\cdot) \in \widetilde{H}^1(\Omega;\mathbb{C}^n)$ for a.e. $\mathbf{k} \in \widetilde{\Omega}$ and

$$\int_{\widetilde{\Omega}} \int_{\Omega} (|(\mathbf{D}+\mathbf{k})\tilde{\mathbf{v}}(\mathbf{k},\mathbf{x})|^2 + |\tilde{\mathbf{v}}(\mathbf{k},\mathbf{x})|^2)\, d\mathbf{x}\, d\mathbf{k} < \infty.$$

Under the Gelfand transformation, the operator of multiplication by a bounded periodic function in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ turns into multiplication by the same function on the fibers of the direct integral. The operator $\mathbf{D}$ applied to $\mathbf{v} \in H^1(\mathbb{R}^d;\mathbb{C}^n)$ turns into the operator $\mathbf{D}+\mathbf{k}$ applied to $\tilde{\mathbf{v}}(\mathbf{k},\cdot) \in \widetilde{H}^1(\Omega;\mathbb{C}^n)$.

4.3. Factorized second order operators. Let $b(\mathbf{D})$ be a matrix first order DO of the form $\sum_{j=1}^d b_j D_j$, where b_j, $j=1,\dots,d$, are constant matrices of size $m\times n$ (in general, with complex entries). *We always assume that $m \geqslant n$.* Suppose that the symbol $b(\boldsymbol{\xi}) = \sum_{j=1}^d b_j \xi_j$, $\boldsymbol{\xi} \in \mathbb{R}^d$, of the operator $b(\mathbf{D})$ has maximal rank: $\operatorname{rank} b(\boldsymbol{\xi}) = n$ for $0 \neq \boldsymbol{\xi} \in \mathbb{R}^d$. This condition is equivalent to the existence of constants $\alpha_0, \alpha_1 > 0$ such that

$$\alpha_0 \mathbf{1}_n \leqslant b(\boldsymbol{\theta})^* b(\boldsymbol{\theta}) \leqslant \alpha_1 \mathbf{1}_n, \quad \boldsymbol{\theta} \in \mathbb{S}^{d-1},\ 0 < \alpha_0 \leqslant \alpha_1 < \infty. \tag{4.7}$$

From (4.7) it follows that

$$|b_j| \leqslant \alpha_1^{1/2}, \quad j = 1,\dots,d. \tag{4.8}$$

Let Γ-periodic Hermitian $(m\times m)$-matrix-valued function $g(\mathbf{x})$ be positive definite and bounded together with the inverse matrix:

$$g(\mathbf{x}) > 0; \quad g, g^{-1} \in L_\infty(\mathbb{R}^d). \tag{4.9}$$

Suppose that $f(\mathbf{x})$ is a Γ-periodic $(n\times n)$-matrix-valued function such that $f, f^{-1} \in L_\infty(\mathbb{R}^d)$. In $L_2(\mathbb{R}^d;\mathbb{C}^n)$, consider DO $\mathcal{A}$ formally given by the differential expression

$$\mathcal{A} = f(\mathbf{x})^* b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D}) f(\mathbf{x}). \tag{4.10}$$

The precise definition of the operator $\mathcal{A}$ is given in terms of the quadratic form

$$\mathfrak{a}[\mathbf{u},\mathbf{u}] := (g b(\mathbf{D})(f\mathbf{u}), b(\mathbf{D})(f\mathbf{u}))_{L_2(\mathbb{R}^d)},$$

with

$$\mathbf{u} \in \operatorname{Dom} \mathfrak{a} := \{\mathbf{u} \in L_2(\mathbb{R}^d;\mathbb{C}^n) \colon f\mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n)\}.$$

Using the Fourier transformation and assumptions (4.7) and (4.9), it is easily seen that

$$\alpha_0 \|g^{-1}\|_{L_\infty}^{-1} \|\mathbf{D}(f\mathbf{u})\|^2_{L_2(\mathbb{R}^d)} \leqslant \mathfrak{a}[\mathbf{u},\mathbf{u}] \leqslant \alpha_1 \|g\|_{L_\infty} \|\mathbf{D}(f\mathbf{u})\|^2_{L_2(\mathbb{R}^d)}, \tag{4.11}$$

with $\mathbf{u} \in \operatorname{Dom} \mathfrak{a}$. Thus, the form $\mathfrak{a}[\cdot,\cdot]$ is closed and non-negative.

The operator $\mathcal{A}$ admits a factorization of the form $\mathcal{A} = \mathcal{X}^*\mathcal{X}$, where

$$\mathcal{X} := g(\mathbf{x})^{1/2} b(\mathbf{D}) f(\mathbf{x}) \colon L_2(\mathbb{R}^d;\mathbb{C}^n) \longrightarrow L_2(\mathbb{R}^d;\mathbb{C}^m), \quad \operatorname{Dom}\mathcal{X} = \operatorname{Dom}\mathfrak{a}.$$

5. Direct integral decomposition for the operator $\mathcal{A}$

5.1. The forms $\mathfrak{a}(\mathbf{k})$ and the operators $\mathcal{A}(\mathbf{k})$. We put

$$\mathfrak{H} := L_2(\Omega;\mathbb{C}^n), \quad \mathfrak{H}_* := L_2(\Omega;\mathbb{C}^m), \tag{5.1}$$

and consider the closed operator

$$\mathcal{X}(\mathbf{k}) \colon \mathfrak{H} \longrightarrow \mathfrak{H}_*, \quad \mathbf{k} \in \mathbb{R}^d,$$

defined on the domain

$$\operatorname{Dom}\mathcal{X}(\mathbf{k}) = \{\mathbf{u} \in \mathfrak{H} \colon f\mathbf{u} \in \widetilde{H}^1(\Omega;\mathbb{C}^n)\} =: \mathfrak{d}$$

by the expression

$$\mathcal{X}(\mathbf{k}) = g(\mathbf{x})^{1/2} b(\mathbf{D}+\mathbf{k}) f(\mathbf{x}).$$

The selfadjoint operator

$$\mathcal{A}(\mathbf{k}) := \mathcal{X}(\mathbf{k})^*\mathcal{X}(\mathbf{k})$$

in $L_2(\Omega;\mathbb{C}^n)$ is formally given by the differential expression

$$\mathcal{A}(\mathbf{k}) = f(\mathbf{x})^* b(\mathbf{D}+\mathbf{k})^* g(\mathbf{x}) b(\mathbf{D}+\mathbf{k}) f(\mathbf{x}) \tag{5.2}$$

with the periodic boundary conditions. The precise definition of the operator $\mathcal{A}(\mathbf{k})$ is given in terms of the closed quadratic form

$$\mathfrak{a}(\mathbf{k})[\mathbf{u},\mathbf{u}] := \|\mathcal{X}(\mathbf{k})\mathbf{u}\|^2_{\mathfrak{H}_*}, \quad \mathbf{u} \in \mathfrak{d}.$$

Using the discrete Fourier transformation (4.3) and assumptions (4.7) and (4.9), it is easily seen that

$$\begin{aligned}\alpha_0 \|g^{-1}\|^{-1}_{L_\infty} \|(\mathbf{D}+\mathbf{k})(f\mathbf{u})\|^2_{L_2(\Omega)} &\leqslant \mathfrak{a}(\mathbf{k})[\mathbf{u},\mathbf{u}] \\ &\leqslant \alpha_1 \|g\|_{L_\infty} \|(\mathbf{D}+\mathbf{k})(f\mathbf{u})\|^2_{L_2(\Omega)},\end{aligned} \tag{5.3}$$

with $\mathbf{u} \in \mathfrak{d}$. So, by the compactness of the embedding $\widetilde{H}^1(\Omega;\mathbb{C}^n) \hookrightarrow L_2(\Omega;\mathbb{C}^n)$, the spectrum of $\mathcal{A}(\mathbf{k})$ is discrete and the resolvent is compact.

By (4.6) and the lower estimate (5.3),

$$\mathcal{A}(\mathbf{k}) \geqslant c_* |\mathbf{k}|^2 I, \quad \mathbf{k} \in \operatorname{clos} \widetilde{\Omega}, \tag{5.4}$$

where

$$c_* := \alpha_0 \|g^{-1}\|_{L_\infty}^{-1} \|f^{-1}\|_{L_\infty}^{-2}.$$

We put

$$\mathfrak{N} := \operatorname{Ker} \mathcal{A}(0) = \operatorname{Ker} \mathcal{X}(0). \tag{5.5}$$

Then

$$\mathfrak{N} = \{\mathbf{u} \in L_2(\Omega; \mathbb{C}^n) \colon f\mathbf{u} = \mathbf{c} \in \mathbb{C}^n\}. \tag{5.6}$$

From (4.2) and (4.5) with $\mathbf{k} = 0$ it follows that

$$\|\mathbf{D}\mathbf{v}\|^2_{L_2(\Omega)} \geqslant 4r_0^2 \|\mathbf{v}\|^2_{L_2(\Omega)}, \quad \mathbf{v} = f\mathbf{u} \in \widetilde{H}^1(\Omega; \mathbb{C}^n), \int\limits_{\Omega} \mathbf{v}(\mathbf{x})\, d\mathbf{x} = 0.$$

Combining this with the lower estimate (5.3) for $\mathbf{k} = 0$, we see that the distance d_0 from the point zero to the rest of the spectrum of $\mathcal{A}(0)$ satisfies

$$d_0 \geqslant 4c_* r_0^2. \tag{5.7}$$

5.2. Direct integral decomposition for $\mathcal{A}$. Using the Gelfand transformation, we decompose the operator $\mathcal{A}$ into the direct integral of the operators $\mathcal{A}(\mathbf{k})$:

$$\mathcal{U}\mathcal{A}\mathcal{U}^{-1} = \int\limits_{\widetilde{\Omega}} \oplus \mathcal{A}(\mathbf{k})\, d\mathbf{k}. \tag{5.8}$$

This means the following. If $\mathbf{v} \in \operatorname{Dom} \mathfrak{a}$, then

$$\tilde{\mathbf{v}}(\mathbf{k}, \cdot) = (\mathcal{U}\mathbf{v})(\mathbf{k}, \cdot) \in \mathfrak{d} \quad \text{for a.e. } \mathbf{k} \in \widetilde{\Omega}, \tag{5.9}$$

$$\mathfrak{a}[\mathbf{v}, \mathbf{v}] = \int\limits_{\widetilde{\Omega}} \mathfrak{a}(\mathbf{k})[\tilde{\mathbf{v}}(\mathbf{k}, \cdot), \tilde{\mathbf{v}}(\mathbf{k}, \cdot)]\, d\mathbf{k}. \tag{5.10}$$

Conversely, if $\tilde{\mathbf{v}} \in \int_{\widetilde{\Omega}} \oplus L_2(\Omega; \mathbb{C}^n)\, d\mathbf{k}$ satisfies (5.9) and the integral in (5.10) is finite, then $\mathbf{v} \in \operatorname{Dom} \mathfrak{a}$ and (5.10) holds.

5.3. Incorporation of the operators $\mathcal{A}(\mathbf{k})$ into the abstract scheme. For $d > 1$ the operators $\mathcal{A}(\mathbf{k})$ depend on the multidimensional parameter $\mathbf{k}$. According to [9, Chapter 2], we consider the onedimensional parameter

$$t := |\mathbf{k}|.$$

We will apply the scheme of Chapter I. Herewith, all our considerations will depend on the additional parameter

$$\boldsymbol{\theta} = \mathbf{k}/|\mathbf{k}| \in \mathbb{S}^{d-1},$$

and we need to make our estimates uniform with respect to $\boldsymbol{\theta}$.

The spaces $\mathfrak{H}$ and $\mathfrak{H}_*$ are defined by (5.1). Let

$$X(t) = X(t,\boldsymbol{\theta}) := \mathcal{X}(t\boldsymbol{\theta}).$$

Then $X(t,\boldsymbol{\theta}) = X_0 + tX_1(\boldsymbol{\theta})$, where $X_0 = g(\mathbf{x})^{1/2}b(\mathbf{D})f(\mathbf{x})$, $\operatorname{Dom} X_0 = \mathfrak{d}$, and $X_1(\boldsymbol{\theta})$ is a bounded operator of multiplication by the matrix-valued function $g(\mathbf{x})^{1/2}b(\boldsymbol{\theta})f(\mathbf{x})$. We put

$$A(t) = A(t,\boldsymbol{\theta}) := \mathcal{A}(t\boldsymbol{\theta}).$$

Then $A(t,\boldsymbol{\theta}) = X(t,\boldsymbol{\theta})^* X(t,\boldsymbol{\theta})$. According to (5.5) and (5.6), $\mathfrak{N} = \operatorname{Ker} X_0 = \operatorname{Ker} \mathcal{A}(0)$, $\dim \mathfrak{N} = n$. The distance d_0 from the point zero to the rest of the spectrum of $\mathcal{A}(0)$ satisfied estimate (5.7). As was shown in [9, Chapter 2, Section 3], the condition $n \leqslant n_* = \dim \operatorname{Ker} X_0^*$ is also fulfilled. Thus, all the assumptions of Section 1 are valid.

In Subsection 1.1, it was required to choose the number $\delta < d_0/8$. Taking (5.4) and (5.7) into account, we put

$$\delta := c_* r_0^2/4 = (r_0/2)^2 \alpha_0 \|g^{-1}\|_{L_\infty}^{-1} \|f^{-1}\|_{L_\infty}^{-2}. \tag{5.11}$$

Next, by (4.7), the operator $X_1(\boldsymbol{\theta}) = g(\mathbf{x})^{1/2}b(\boldsymbol{\theta})f(\mathbf{x})$ satisfies

$$\|X_1(\boldsymbol{\theta})\| \leqslant \alpha_1^{1/2} \|g\|_{L_\infty}^{1/2} \|f\|_{L_\infty}. \tag{5.12}$$

This allows us to take the number

$$\begin{aligned} t_0 &:= \delta^{1/2} \alpha_1^{-1/2} \|g\|_{L_\infty}^{-1/2} \|f\|_{L_\infty}^{-1} \\ &= (r_0/2) \alpha_0^{1/2} \alpha_1^{-1/2} \|g\|_{L_\infty}^{-1/2} \|g^{-1}\|_{L_\infty}^{-1/2} \|f\|_{L_\infty}^{-1} \|f^{-1}\|_{L_\infty}^{-1} \end{aligned} \tag{5.13}$$

in the role of t_0 (see (1.2)). Obviously, $t_0 \leqslant r_0/2$, and the ball $|\mathbf{k}| \leqslant t_0$ lies in $\widetilde{\Omega}$. It is important that c_*, δ, and t_0 (see (5.4), (5.11), and (5.13)) do not depend on the parameter $\boldsymbol{\theta}$.

From (5.4) it follows that the spectral germ $S(\boldsymbol{\theta})$ (which now depends on $\boldsymbol{\theta}$) is nondegenerate:

$$S(\boldsymbol{\theta}) \geqslant c_* I_{\mathfrak{N}}. \tag{5.14}$$

It is important that the spectral germ is nondegenerate uniformly in $\boldsymbol{\theta}$.

6. The operator $\widehat{\mathcal{A}}$. The effective matrix. The effective operator

6.1. The operator $\widehat{\mathcal{A}}$. In the case where $f = \mathbf{1}_n$, we agree to mark all the objects by the upper hat "^". We have $\widehat{\mathfrak{H}} = \mathfrak{H} = L_2(\Omega;\mathbb{C}^n)$. For the operator

$$\widehat{\mathcal{A}} = b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D}), \tag{6.1}$$

the family

$$\widehat{\mathcal{A}}(\mathbf{k}) = b(\mathbf{D}+\mathbf{k})^* g(\mathbf{x}) b(\mathbf{D}+\mathbf{k}) \tag{6.2}$$

is denoted by $\widehat{A}(t;\boldsymbol{\theta})$. If $f = \mathbf{1}_n$, the kernel (5.6) takes the form

$$\widehat{\mathfrak{N}} = \{\mathbf{u} \in L_2(\Omega;\mathbb{C}^n) \colon \mathbf{u} = \mathbf{c} \in \mathbb{C}^n\}. \tag{6.3}$$

Let $\widehat{P}$ be the orthogonal projection of $\mathfrak{H}$ onto the subspace $\widehat{\mathfrak{N}}$. Then $\widehat{P}$ is the operator of averaging over the cell:

$$\widehat{P}\mathbf{u} = |\Omega|^{-1} \int_{\Omega} \mathbf{u}(\mathbf{x})\, d\mathbf{x}, \quad \mathbf{u} \in L_2(\Omega;\mathbb{C}^n). \tag{6.4}$$

From (5.4) with $f = \mathbf{1}_n$ it follows that

$$\widehat{\mathcal{A}}(\mathbf{k}) = \widehat{A}(t,\boldsymbol{\theta}) \geqslant \hat{c}_* t^2 I, \quad \mathbf{k} = t\boldsymbol{\theta} \in \operatorname{clos}\widetilde{\Omega}, \tag{6.5}$$

where

$$\hat{c}_* := \alpha_0 \|g^{-1}\|^{-1}_{L_\infty}.$$

6.2. The effective matrix. In accordance with [9, Chapter 3, Section 1], the spectral germ $\widehat{S}(\boldsymbol{\theta})$ of the operator family $\widehat{A}(t,\boldsymbol{\theta})$ acting in $\widehat{\mathfrak{N}}$ can be represented as

$$\widehat{S}(\boldsymbol{\theta}) = b(\boldsymbol{\theta})^* g^0 b(\boldsymbol{\theta}), \quad \boldsymbol{\theta} \in \mathbb{S}^{d-1}, \tag{6.6}$$

where $b(\boldsymbol{\theta})$ is the symbol of the operator $b(\mathbf{D})$ and g^0 is the so-called *effective matrix*. The constant positive $(m \times m)$-matrix g^0 is defined as follows. Assume that a Γ-periodic $(n \times m)$-matrix-valued function $\Lambda \in \widetilde{H}^1(\Omega)$ is the weak solution of the problem

$$b(\mathbf{D})^* g(\mathbf{x})(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m) = 0, \quad \int_{\Omega} \Lambda(\mathbf{x})\, d\mathbf{x} = 0. \tag{6.7}$$

Denote

$$\tilde{g}(\mathbf{x}) := g(\mathbf{x})(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m). \tag{6.8}$$

Then the effective matrix g^0 is given by

$$g^0 = |\Omega|^{-1} \int_{\Omega} \tilde{g}(\mathbf{x})\, d\mathbf{x}. \tag{6.9}$$

It turns out that the matrix g^0 is positive definite. In the case where $f = \mathbf{1}_n$, estimate (5.14) takes the form

$$\widehat{S}(\boldsymbol{\theta}) \geqslant \hat{c}_* I_{\widehat{\mathfrak{N}}}. \tag{6.10}$$

From (6.7) it is easy to derive that

$$\|b(\mathbf{D})\Lambda\|_{L_2(\Omega)} \leqslant |\Omega|^{1/2} m^{1/2} \|g\|_{L_\infty}^{1/2} \|g^{-1}\|_{L_\infty}^{1/2}. \tag{6.11}$$

We also need the following inequalities obtained in [11, (6.28) and Subsection 7.3]:

$$\|\Lambda\|_{L_2(\Omega)} \leqslant |\Omega|^{1/2} M_1, \quad M_1 := m^{1/2}(2r_0)^{-1}\alpha_0^{-1/2} \|g\|_{L_\infty}^{1/2} \|g^{-1}\|_{L_\infty}^{1/2}; \tag{6.12}$$

$$\|\mathbf{D}\Lambda\|_{L_2(\Omega)} \leqslant |\Omega|^{1/2} M_2, \quad M_2 := m^{1/2}\alpha_0^{-1/2} \|g\|_{L_\infty}^{1/2} \|g^{-1}\|_{L_\infty}^{1/2}. \tag{6.13}$$

6.3. The effective operator $\widehat{\mathcal{A}}^0$. By (6.6) and the homogeneity of the symbol $b(\mathbf{k})$, we have

$$\widehat{S}(\mathbf{k}) := t^2 \widehat{S}(\boldsymbol{\theta}) = b(\mathbf{k})^* g^0 b(\mathbf{k}), \quad \mathbf{k} \in \mathbb{R}^d,\ t = |\mathbf{k}|,\ \boldsymbol{\theta} = \mathbf{k}/|\mathbf{k}|. \tag{6.14}$$

The matrix $\widehat{S}(\mathbf{k})$ is the symbol of the differential operator

$$\widehat{\mathcal{A}}^0 = b(\mathbf{D})^* g^0 b(\mathbf{D}) \tag{6.15}$$

acting in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ on the domain $H^2(\mathbb{R}^d;\mathbb{C}^n)$ and called the *effective operator* for the operator $\widehat{\mathcal{A}}$.

Let $\widehat{\mathcal{A}}^0(\mathbf{k})$ be the operator family in $L_2(\Omega;\mathbb{C}^n)$ corresponding to the effective operator $\widehat{\mathcal{A}}^0$. Then $\widehat{\mathcal{A}}^0(\mathbf{k}) = b(\mathbf{D}+\mathbf{k})^* g^0 b(\mathbf{D}+\mathbf{k})$ with periodic boundary conditions: $\operatorname{Dom} \widehat{\mathcal{A}}^0(\mathbf{k}) = \widetilde{H}^2(\Omega;\mathbb{C}^n)$. So, by (6.4) and (6.14),

$$\widehat{S}(\mathbf{k})\widehat{P} = \widehat{\mathcal{A}}^0(\mathbf{k})\widehat{P}. \tag{6.16}$$

From estimate (6.10) for the symbol of the operator $\widehat{\mathcal{A}}^0(\mathbf{k})$ it follows that

$$\widehat{\mathcal{A}}^0(\mathbf{k}) \geqslant \hat{c}_* |\mathbf{k}|^2 I, \quad \mathbf{k} \in \widetilde{\Omega}. \tag{6.17}$$

6.4. Properties of the effective matrix. The effective matrix g^0 satisfies the estimates known in homogenization theory as the Voigt–Reuss bracketing (see, e.g., [9, Chapter 3, Theorem 1.5]).

Proposition 6.1. *Let g^0 be the effective matrix (6.9). Then*

$$\underline{g} \leqslant g^0 \leqslant \bar{g}. \tag{6.18}$$

If $m = n$, then $g^0 = \underline{g}$.

From inequalities (6.18) it follows that

$$|g^0| \leqslant \|g\|_{L_\infty}, \quad |(g^0)^{-1}| \leqslant \|g^{-1}\|_{L_\infty}. \tag{6.19}$$

Now, we distinguish the cases where one of the inequalities in (6.18) becomes an identity. See [9, Chapter 3, Propositions 1.6 and 1.7].

Proposition 6.2. *The equality $g^0 = \bar{g}$ is equivalent to the relations*

$$b(\mathbf{D})^* \mathbf{g}_k(\mathbf{x}) = 0, \quad k = 1, \dots, m, \tag{6.20}$$

where $\mathbf{g}_k(\mathbf{x})$, $k = 1, \dots, m$, are the columns of the matrix-valued function $g(\mathbf{x})$.

Proposition 6.3. *The identity $g^0 = \underline{g}$ is equivalent to the relations*

$$\mathbf{l}_k(\mathbf{x}) = \mathbf{l}_k^0 + b(\mathbf{D})\mathbf{w}_k, \quad \mathbf{l}_k^0 \in \mathbb{C}^m,\ \mathbf{w}_k \in \widetilde{H}^1(\Omega; \mathbb{C}^m),\ k = 1, \dots, m, \tag{6.21}$$

where $\mathbf{l}_k(\mathbf{x})$, $k = 1, \dots, m$, are the columns of the matrix-valued function $g(\mathbf{x})^{-1}$.

7. Approximation of the sandwiched operator $\mathcal{A}(\mathbf{k})^{-1/2} \sin(\varepsilon^{-1}\tau \mathcal{A}(\mathbf{k})^{1/2})$

Now, we consider the operator $\mathcal{A}(\mathbf{k})^{-1/2} \sin(\varepsilon^{-1}\tau \mathcal{A}(\mathbf{k})^{1/2})$ in the general case where $f \neq \mathbf{1}_n$. Recall that $\mathcal{A}(\mathbf{k})$ is the operator (5.2). Then

$$\mathcal{A}(\mathbf{k}) = f(\mathbf{x})^* \widehat{\mathcal{A}}(\mathbf{k}) f(\mathbf{x}). \tag{7.1}$$

7.1. Incorporation of $\mathcal{A}(\mathbf{k})$ in the framework of Section 3. As was shown in Subsection 5.3, the operator $\mathcal{A}(\mathbf{k})$ satisfies the assumptions of Section 1. Now the assumptions of Subsection 3.1 are valid with $\mathfrak{H} = \widehat{\mathfrak{H}} = L_2(\Omega; \mathbb{C}^n)$ and $\mathfrak{H}_* = L_2(\Omega; \mathbb{C}^m)$. The role of $\widehat{A}(t)$ is played by $\widehat{A}(t, \boldsymbol{\theta}) = \widehat{\mathcal{A}}(t\boldsymbol{\theta})$, and the role of $A(t)$ is played by $A(t, \boldsymbol{\theta}) = \mathcal{A}(t\boldsymbol{\theta})$. An isomorphism M is the operator of multiplication by the function $f(\mathbf{x})$. Relation (3.2) corresponds to the identity (7.1).

Next, the operator Q (see (3.3)) is the operator of multiplication by the matrix-valued function

$$Q(\mathbf{x}) := (f(\mathbf{x})f(\mathbf{x})^*)^{-1}. \tag{7.2}$$

The block $Q_{\widehat{\mathfrak{N}}}$ of Q in the subspace $\widehat{\mathfrak{N}}$ (see (6.3)) is the operator of multiplication by the constant matrix

$$\overline{Q} = (\underline{ff^*})^{-1} = |\Omega|^{-1} \int\limits_{\Omega} (f(\mathbf{x})f(\mathbf{x})^*)^{-1}\, d\mathbf{x}.$$

The operator

$$M_0 := (Q_{\widehat{\mathfrak{N}}})^{-1/2}$$

acts in $\widehat{\mathfrak{N}}$ as multiplication by the matrix

$$f_0 := (\overline{Q})^{-1/2} = (\underline{ff^*})^{1/2}.$$

Obviously,

$$|f_0| \leqslant \|f\|_{L_\infty}, \quad |f_0^{-1}| \leqslant \|f^{-1}\|_{L_\infty}. \tag{7.3}$$

Now, we specify the operators from (3.16) and (3.17). By (6.14),

$$t^2 M_0 \widehat{S}(\boldsymbol{\theta}) M_0 = f_0 b(\mathbf{k})^* g^0 b(\mathbf{k}) f_0, \quad t = |\mathbf{k}|,\ \boldsymbol{\theta} = \mathbf{k}/|\mathbf{k}|. \tag{7.4}$$

Let $\mathcal{A}^0$ be the following operator in $L_2(\mathbb{R}^d;\mathbb{C}^n)$:

$$\mathcal{A}^0 = f_0 b(\mathbf{D})^* g^0 b(\mathbf{D}) f_0, \quad \operatorname{Dom}\mathcal{A}^0 = H^2(\mathbb{R}^d;\mathbb{C}^n). \tag{7.5}$$

Let $\mathcal{A}^0(\mathbf{k})$ be the corresponding operator family in $L_2(\Omega;\mathbb{C}^n)$ given by the expression

$$\mathcal{A}^0(\mathbf{k}) = f_0 b(\mathbf{D}+\mathbf{k})^* g^0 b(\mathbf{D}+\mathbf{k}) f_0 \tag{7.6}$$

with the periodic boundary conditions. By (6.16), (6.17), (7.3), and the identity $c_* = \hat{c}_* \|f^{-1}\|^{-2}_{L_\infty}$, the symbol of the operator $\mathcal{A}^0$ satisfies the estimate

$$f_0 b(\mathbf{k})^* g^0 b(\mathbf{k}) f_0 \geqslant c_* |\mathbf{k}|^2 \mathbf{1}_n, \quad \mathbf{k} \in \mathbb{R}^d. \tag{7.7}$$

Hence, using the Fourier series representation for the operator $\mathcal{A}^0(\mathbf{k})$ and (4.5), we deduce that

$$\mathcal{A}^0(\mathbf{k}) \geqslant c_* |\mathbf{k}|^2 I, \quad \mathbf{k} \in \operatorname{clos} \widetilde{\Omega}. \tag{7.8}$$

By (6.4), (7.4), and (7.6), we obtain $t^2 M_0 \widehat{S}(\boldsymbol{\theta}) M_0 \widehat{P} = \mathcal{A}^0(\mathbf{k})\widehat{P}$, whence

$$\begin{aligned} &M_0 (t^2 M_0 \widehat{S}(\boldsymbol{\theta}) M_0)^{-1/2} \sin(\varepsilon^{-1}\tau (t^2 M_0 \widehat{S}(\boldsymbol{\theta}) M_0)^{1/2}) M_0^{-1} \widehat{P} \\ &\quad = f_0 \mathcal{A}^0(\mathbf{k})^{-1/2} \sin(\varepsilon^{-1}\tau \mathcal{A}^0(\mathbf{k})^{1/2}) f_0^{-1} \widehat{P}. \end{aligned} \tag{7.9}$$

In accordance with [11, Section 5], the role of $\widehat{Z}_Q$ is played by the operator

$$\widehat{Z}_Q(\boldsymbol{\theta}) = \Lambda_Q b(\boldsymbol{\theta})\widehat{P}. \tag{7.10}$$

Here Λ_Q is the operator of multiplication by the Γ-periodic $(n\times m)$-matrix-valued solution $\Lambda_Q(\mathbf{x})$ of the problem

$$b(\mathbf{D})^* g(\mathbf{x})(b(\mathbf{D})\Lambda_Q(\mathbf{x}) + \mathbf{1}_m) = 0, \quad \int_\Omega Q(\mathbf{x})\Lambda_Q(\mathbf{x})\, d\mathbf{x} = 0.$$

Note that

$$\Lambda_Q(\mathbf{x}) = \Lambda(\mathbf{x}) + \Lambda_Q^0, \quad \Lambda_Q^0 := -(\overline{Q})^{-1}(\overline{Q\Lambda}), \tag{7.11}$$

where Λ is the Γ-periodic solution of problem (6.7). From (7.10) it follows that $t\widehat{Z}_Q(\boldsymbol{\theta})\widehat{P} = \Lambda_Q b(\mathbf{k})\widehat{P} = \Lambda_Q b(\mathbf{D}+\mathbf{k})\widehat{P}$.

7.2. Estimates in the case where $|\mathbf{k}| \leqslant t_0$. Consider the operator

$$\mathcal{H}_0 = -\Delta$$

acting in $L_2(\mathbb{R}^d;\mathbb{C}^n)$. Under the Gelfand transformation, this operator is decomposed into the direct integral of the operators $\mathcal{H}_0(\mathbf{k})$ acting in $L_2(\Omega;\mathbb{C}^n)$ and given by the differential expression $|\mathbf{D}+\mathbf{k}|^2$ with the periodic boundary conditions. Denote

$$\mathcal{R}(\mathbf{k},\varepsilon) := \varepsilon^2(\mathcal{H}_0(\mathbf{k}) + \varepsilon^2 I)^{-1}. \tag{7.12}$$

Obviously,

$$\mathcal{R}(\mathbf{k},\varepsilon)\widehat{P} = \varepsilon^2(t^2+\varepsilon^2)^{-1}\widehat{P}, \quad |\mathbf{k}| = t. \tag{7.13}$$

In order to approximate the operator $f\mathcal{A}(\mathbf{k})^{-1/2}\sin(\varepsilon^{-1}\tau\mathcal{A}(\mathbf{k})^{1/2})f^{-1}$, we apply Theorem 3.3. We only need to specify the constants in estimates. The constants c_*, δ, and t_0 are defined by (5.4), (5.11), and (5.13). Using estimate (5.12), we choose the following values of constants from (1.10), (1.11), and (1.15):

$$\begin{aligned}
C_1 &:= \beta_1\delta^{-1/2}\alpha_1^{1/2}\|g\|_{L_\infty}^{1/2}\|f\|_{L_\infty},\\
C_2 &:= \beta_2\delta^{-1}\alpha_1\|g\|_{L_\infty}\|f\|_{L_\infty}^2,\\
C_3 &:= \beta_3\delta^{-1/2}\alpha_1^{1/2}\|g\|_{L_\infty}^{1/2}\|f\|_{L_\infty}(1 + c_*^{-1}\alpha_1\|g\|_{L_\infty}\|f\|_{L_\infty}^2).
\end{aligned}$$

Similarly, we choose the constants from (1.17) and (1.19)

$$\begin{aligned}
C_4 &:= \beta_4\delta^{-1/2}\alpha_1\|g\|_{L_\infty}\|f\|_{L_\infty}^2(1 + c_*^{-1/2}\alpha_1^{1/2}\|g\|_{L_\infty}^{1/2}\|f\|_{L_\infty}),\\
C_5 &:= \beta_5\delta^{-1/2}\alpha_1\|g\|_{L_\infty}\|f\|_{L_\infty}^2.
\end{aligned}$$

Using these C_1, C_3, C_4, and C_5, we fix the constants from (1.23), (2.8), (2.10), and (2.18):

$$\begin{aligned} C_6 &:= 4\pi^{-1} c_*^{-1/2} C_3, \\ C_7 &:= C_8 + c_*^{-1/2} C_1, \\ C_8 &:= \max\{C_4 c_*^{-1/2}; C_6\}, \\ C_9 &:= C_1 + c_*^{-1/2} C_5 + C_8(\alpha_1^{1/2} \|g\|_{L_\infty}^{1/2} \|f\|_{L_\infty} + C_4 t_0). \end{aligned}$$

By Theorem 3.3, taking (7.9), (7.10), and (7.13) into account, we have

$$\begin{aligned} &\|(f\mathcal{A}(\mathbf{k})^{-1/2} \sin(\varepsilon^{-1}\tau\mathcal{A}(\mathbf{k})^{1/2}) f^{-1} - f_0 \mathcal{A}^0(\mathbf{k})^{-1/2} \sin(\varepsilon^{-1}\tau\mathcal{A}^0(\mathbf{k})^{1/2}) f_0^{-1}) \\ &\quad \cdot \mathcal{R}(\mathbf{k},\varepsilon)^{1/2} \widehat{P}\|_{L_2(\Omega)\to L_2(\Omega)} \\ &\quad \leqslant C_7 \|f\|_{L_\infty} \|f^{-1}\|_{L_\infty} (1+|\tau|), \end{aligned} \tag{7.14}$$

for $\tau \in \mathbb{R}$, $\varepsilon > 0$, $|\mathbf{k}| \leqslant t_0$, and

$$\begin{aligned} &\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2} (f\mathcal{A}(\mathbf{k})^{-1/2} \sin(\varepsilon^{-1}\tau\mathcal{A}(\mathbf{k})^{1/2}) f^{-1} \\ &\qquad - (I + \Lambda_Q b(\mathbf{D}+\mathbf{k})) f_0 \mathcal{A}^0(\mathbf{k})^{-1/2} \sin(\varepsilon^{-1}\tau\mathcal{A}^0(\mathbf{k})^{1/2}) f_0^{-1}) \\ &\quad \cdot \mathcal{R}(\mathbf{k},\varepsilon) \widehat{P}\|_{L_2(\Omega)\to L_2(\Omega)} \\ &\quad \leqslant C_9 \|f^{-1}\|_{L_\infty} \varepsilon (1+|\tau|), \end{aligned} \tag{7.15}$$

for $\tau \in \mathbb{R}$, $\varepsilon > 0$, $|\mathbf{k}| \leqslant t_0$.

Using (7.11), we show that Λ_Q can be replaced by Λ in (7.15). Only the constant in the estimate will change under such replacement. Indeed, due to the presence of the projection $\widehat{P}$, taking (4.7), (6.2), (7.3), and (7.13), and the inequality $|\sin x|/|x| \leqslant 1$ into account, we have

$$\begin{aligned} &\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2} \Lambda_Q^0 b(\mathbf{D}+\mathbf{k}) f_0 \mathcal{A}^0(\mathbf{k})^{-1/2} \\ &\qquad \cdot \sin(\varepsilon^{-1}\tau\mathcal{A}^0(\mathbf{k})^{1/2}) f_0^{-1} \mathcal{R}(\mathbf{k},\varepsilon) \widehat{P}\|_{L_2(\Omega)\to L_2(\Omega)} \\ &\quad \leqslant \|g\|_{L_\infty}^{1/2} \|b(\mathbf{k}) \Lambda_Q^0 b(\mathbf{k}) f_0 \mathcal{A}^0(\mathbf{k})^{-1/2} \\ &\qquad\quad \cdot \sin(\varepsilon^{-1}\tau\mathcal{A}^0(\mathbf{k})^{1/2}) f_0^{-1} \mathcal{R}(\mathbf{k},\varepsilon) \widehat{P}\|_{L_2(\Omega)\to L_2(\Omega)} \\ &\quad \leqslant \alpha_1 \|g\|_{L_\infty}^{1/2} |\Lambda_Q^0| |\mathbf{k}|^2 \|f\|_{L_\infty} \|f^{-1}\|_{L_\infty} |\tau| \varepsilon (|\mathbf{k}|^2 + \varepsilon^2)^{-1}, \quad \varepsilon > 0, \end{aligned} \tag{7.16}$$

with $\tau \in \mathbb{R}$, $\mathbf{k} \in \operatorname{clos} \widetilde{\Omega}$. Next, according to [11, Section 7],

$$|\Lambda_Q^0| \leqslant m^{1/2} (2r_0)^{-1} \alpha_0^{-1/2} \|g\|_{L_\infty}^{1/2} \|g^{-1}\|_{L_\infty}^{1/2} \|f\|_{L_\infty}^2 \|f^{-1}\|_{L_\infty}^2. \tag{7.17}$$

Combining (7.11) and (7.15)–(7.17), we arrive at the estimate

$$\begin{aligned}\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}(f\mathcal{A}(\mathbf{k})^{-1/2}\sin(\varepsilon^{-1}\tau\mathcal{A}(\mathbf{k})^{1/2})f^{-1}\\ -(I+\Lambda b(\mathbf{D}+\mathbf{k}))f_0\mathcal{A}^0(\mathbf{k})^{-1/2}\sin(\varepsilon^{-1}\tau\mathcal{A}^0(\mathbf{k})^{1/2})f_0^{-1})\\ \cdot\mathcal{R}(\mathbf{k},\varepsilon)\widehat{P}\|_{L_2(\Omega)\to L_2(\Omega)}\\ \leqslant C_{10}\varepsilon(1+|\tau|),\end{aligned} \tag{7.18}$$

with $\varepsilon>0$, $\tau\in\mathbb{R}$, $\mathbf{k}\in\operatorname{clos}\widetilde{\Omega}$, $|\mathbf{k}|\leqslant t_0$, and

$$C_{10}:=C_9\|f^{-1}\|_{L_\infty}+m^{1/2}(2r_0)^{-1}\alpha_0^{-1/2}\alpha_1\|g\|_{L_\infty}\|g^{-1}\|_{L_\infty}^{1/2}\|f\|_{L_\infty}^3\|f^{-1}\|_{L_\infty}^3.$$

7.3. Approximations for $|\mathbf{k}|>t_0$. By (5.4) and (7.8),

$$\|\mathcal{A}(\mathbf{k})^{-1/2}\|_{L_2(\Omega)\to L_2(\Omega)}\leqslant c_*^{-1/2}t_0^{-1},\quad \|\mathcal{A}^0(\mathbf{k})^{-1/2}\|_{L_2(\Omega)\to L_2(\Omega)}\leqslant c_*^{-1/2}t_0^{-1}, \tag{7.19}$$

for $\mathbf{k}\in\operatorname{clos}\widetilde{\Omega}$, $|\mathbf{k}|>t_0$. By (7.13),

$$\|\mathcal{R}(\mathbf{k},\varepsilon)\widehat{P}\|_{L_2(\Omega)\to L_2(\Omega)}\leqslant 1,\quad \mathbf{k}\in\operatorname{clos}\widetilde{\Omega}. \tag{7.20}$$

Combining (7.3) and (7.19) and (7.20), we obtain

$$\begin{aligned}\|(f\mathcal{A}(\mathbf{k})^{-1/2}\sin(\varepsilon^{-1}\tau\mathcal{A}(\mathbf{k})^{1/2})f^{-1}\\ -f_0\mathcal{A}^0(\mathbf{k})^{-1/2}\sin(\varepsilon^{-1}\tau\mathcal{A}^0(\mathbf{k})^{1/2})f_0^{-1})\\ \cdot\mathcal{R}(\mathbf{k},\varepsilon)^{1/2}\widehat{P}\|_{L_2(\Omega)\to L_2(\Omega)}\\ \leqslant 2c_*^{-1/2}t_0^{-1}\|f\|_{L_\infty}\|f^{-1}\|_{L_\infty},\end{aligned} \tag{7.21}$$

with $\varepsilon>0$, $\tau\in\mathbb{R}$, $\mathbf{k}\in\operatorname{clos}\widetilde{\Omega}$, $|\mathbf{k}|>t_0$. Bringing together (7.14) and (7.21), we conclude that

$$\begin{aligned}\|(f\mathcal{A}(\mathbf{k})^{-1/2}\sin(\varepsilon^{-1}\tau\mathcal{A}(\mathbf{k})^{1/2})f^{-1}-f_0\mathcal{A}^0(\mathbf{k})^{-1/2}\sin(\varepsilon^{-1}\tau\mathcal{A}^0(\mathbf{k})^{1/2})f_0^{-1})\\ \cdot\mathcal{R}(\mathbf{k},\varepsilon)^{1/2}\widehat{P}\|_{L_2(\Omega)\to L_2(\Omega)}\\ \leqslant\max\{C_7;2c_*^{-1/2}t_0^{-1}\}\|f\|_{L_\infty}\|f^{-1}\|_{L_\infty}(1+|\tau|),\end{aligned} \tag{7.22}$$

for $\varepsilon>0$, $\tau\in\mathbb{R}$, $\mathbf{k}\in\operatorname{clos}\widetilde{\Omega}$.

Now, we proceed to estimation of the operator under the norm sign in (7.18) for $|\mathbf{k}|>t_0$. By (7.13) and the elementary inequality $t^2+\varepsilon^2\geqslant 2\varepsilon t_0$, $t>t_0$, we have

$$\|\mathcal{R}(\mathbf{k},\varepsilon)\widehat{P}\|_{L_2(\Omega)\to L_2(\Omega)}\leqslant(2t_0)^{-1}\varepsilon,\quad \varepsilon>0,\ \mathbf{k}\in\operatorname{clos}\widetilde{\Omega},\ |\mathbf{k}|>t_0. \tag{7.23}$$

By (7.1) and (7.23),

$$\begin{aligned}
&\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2} f \mathcal{A}(\mathbf{k})^{-1/2} \sin(\varepsilon^{-1}\tau\mathcal{A}(\mathbf{k})^{1/2}) f^{-1}\mathcal{R}(\mathbf{k},\varepsilon)\widehat{P}\|_{L_2(\Omega)\to L_2(\Omega)} \\
&= \|\sin(\varepsilon^{-1}\tau\mathcal{A}(\mathbf{k})^{1/2}) f^{-1}\mathcal{R}(\mathbf{k},\varepsilon)\widehat{P}\|_{L_2(\Omega)\to L_2(\Omega)} \\
&\leqslant \varepsilon(2t_0)^{-1}\|f^{-1}\|_{L_\infty},
\end{aligned} \tag{7.24}$$

with $\varepsilon > 0$, $\tau \in \mathbb{R}$, $\mathbf{k} \in \operatorname{clos}\widetilde{\Omega}$, $|\mathbf{k}| > t_0$.

From (6.19), (7.3), (7.6), and (7.23) it follows that

$$\begin{aligned}
&\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2} f_0\mathcal{A}^0(\mathbf{k})^{-1/2} \sin(\varepsilon^{-1}\tau\mathcal{A}^0(\mathbf{k})^{1/2}) f_0^{-1}\mathcal{R}(\mathbf{k},\varepsilon)\widehat{P}\|_{L_2(\Omega)\to L_2(\Omega)} \\
&\quad\leqslant \varepsilon(2t_0)^{-1}\|g^{1/2}b(\mathbf{D}+\mathbf{k}) f_0 \sin(\varepsilon^{-1}\tau\mathcal{A}^0(\mathbf{k})^{1/2})\mathcal{A}^0(\mathbf{k})^{-1/2} f_0^{-1}\widehat{P}\|_{L_2(\Omega)\to L_2(\Omega)} \\
&\quad\leqslant \varepsilon(2t_0)^{-1}\|g\|_{L_\infty}^{1/2}\|g^{-1}\|_{L_\infty}^{1/2} \\
&\qquad\qquad \cdot \|\mathcal{A}^0(\mathbf{k})^{1/2}\sin(\varepsilon^{-1}\tau\mathcal{A}^0(\mathbf{k})^{1/2})\mathcal{A}^0(\mathbf{k})^{-1/2} f_0^{-1}\widehat{P}\|_{L_2(\Omega)\to L_2(\Omega)} \\
&\quad\leqslant \varepsilon(2t_0)^{-1}\|g\|_{L_\infty}^{1/2}\|g^{-1}\|_{L_\infty}^{1/2}\|f^{-1}\|_{L_\infty},
\end{aligned} \tag{7.25}$$

for $\varepsilon > 0$, $\tau \in \mathbb{R}$, $\mathbf{k} \in \operatorname{clos}\widetilde{\Omega}$, $|\mathbf{k}| > t_0$.

Next,

$$\begin{aligned}
&\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\Lambda b(\mathbf{D}+\mathbf{k}) f_0\mathcal{A}^0(\mathbf{k})^{-1/2} \sin(\varepsilon^{-1}\tau\mathcal{A}^0(\mathbf{k})^{1/2}) f_0^{-1}\mathcal{R}(\mathbf{k},\varepsilon)\widehat{P} \\
&\quad = (\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\Lambda\widehat{P}_m) b(\mathbf{D}+\mathbf{k}) f_0\mathcal{A}^0(\mathbf{k})^{-1/2} \sin(\varepsilon^{-1}\tau\mathcal{A}^0(\mathbf{k})^{1/2}) f_0^{-1}\mathcal{R}(\mathbf{k},\varepsilon)\widehat{P},
\end{aligned}$$

where $\widehat{P}_m$ is the orthogonal projection of the space $\mathfrak{H}_* = L_2(\Omega;\mathbb{C}^m)$ onto the subspace of constants. According to [12, (6.22)],

$$\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\Lambda\widehat{P}_m\|_{L_2(\Omega)\to L_2(\Omega)} \leqslant C_\Lambda, \quad \mathbf{k}\in\widetilde{\Omega}, \tag{7.26}$$

where the constant C_Λ depends only on m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and the parameters of the lattice Γ.

By (6.19), (7.3), (7.6), (7.23), and (7.26),

$$\begin{aligned}
&\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\Lambda b(\mathbf{D}+\mathbf{k}) f_0\mathcal{A}^0(\mathbf{k})^{-1/2} \sin(\varepsilon^{-1}\tau\mathcal{A}^0(\mathbf{k})^{1/2}) f_0^{-1}\mathcal{R}(\mathbf{k},\varepsilon)\widehat{P}\|_{L_2(\Omega)\to L_2(\Omega)} \\
&\quad\leqslant C_\Lambda\|g^{-1}\|_{L_\infty}^{1/2}\|f^{-1}\|_{L_\infty}(2t_0)^{-1}\varepsilon,
\end{aligned} \tag{7.27}$$

for $\varepsilon > 0$, $\tau \in \mathbb{R}$, $\mathbf{k} \in \operatorname{clos}\widetilde{\Omega}$, and $|\mathbf{k}| > t_0$.

Combining (7.18), (7.24), (7.25), and (7.27), we conclude that

$$\begin{aligned}\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}&(f\mathcal{A}(\mathbf{k})^{-1/2}\sin(\varepsilon^{-1}\tau\mathcal{A}(\mathbf{k})^{1/2})f^{-1}\\ &\qquad -(I+\Lambda b(\mathbf{D}+\mathbf{k}))f_0\mathcal{A}^0(\mathbf{k})^{-1/2}\sin(\varepsilon^{-1}\tau\mathcal{A}^0(\mathbf{k})^{1/2})f_0^{-1})\\ &\cdot\mathcal{R}(\mathbf{k},\varepsilon)\widehat{P}\|_{L_2(\Omega)\to L_2(\Omega)}\\ &\leqslant C_{11}\varepsilon(1+|\tau|),\end{aligned}\tag{7.28}$$

with $\varepsilon>0$, $\tau\in\mathbb{R}$, $\mathbf{k}\in\operatorname{clos}\widetilde{\Omega}$. Here

$$C_{11}:=\max\{C_{10};(2t_0)^{-1}\|f^{-1}\|_{L_\infty}(1+\|g\|_{L_\infty}^{1/2}\|g^{-1}\|_{L_\infty}^{1/2}+C_\Lambda\|g^{-1}\|_{L_\infty}^{1/2})\}.$$

7.4. Removal of the operator $\widehat{P}$. Now, we show that, in the operator under the norm sign in (7.22) the projection $\widehat{P}$ can be replaced by the identity operator. After such replacement, only the constant in the estimate will be different. To show this, we estimate the norm of the operator $\mathcal{R}(\mathbf{k},\varepsilon)^{1/2}(I-\widehat{P})$ by using the discrete Fourier transform:

$$\|\mathcal{R}(\mathbf{k},\varepsilon)^{1/2}(I-\widehat{P})\|_{L_2(\Omega)\to L_2(\Omega)}=\max_{0\neq\mathbf{b}\in\widetilde{\Gamma}}\varepsilon(|\mathbf{b}+\mathbf{k}|^2+\varepsilon^2)^{-1/2}\leqslant\varepsilon r_0^{-1},\tag{7.29}$$

with $\varepsilon>0$, $\mathbf{k}\in\operatorname{clos}\widetilde{\Omega}$. Next, applying the spectral theorem and the elementary inequality $|\sin x|/|x|\leqslant 1$, $x\in\mathbb{R}$, we conclude that

$$\|\mathcal{A}(\mathbf{k})^{-1/2}\sin(\varepsilon^{-1}\tau\mathcal{A}(\mathbf{k})^{1/2})\|_{L_2(\Omega)\to L_2(\Omega)}\leqslant\varepsilon^{-1}|\tau|.\tag{7.30}$$

Similarly,

$$\|\mathcal{A}^0(\mathbf{k})^{-1/2}\sin(\varepsilon^{-1}\tau\mathcal{A}^0(\mathbf{k})^{1/2})\|_{L_2(\Omega)\to L_2(\Omega)}\leqslant\varepsilon^{-1}|\tau|.\tag{7.31}$$

Bringing together (7.3) and (7.29)–(7.31), we arrive at the estimate

$$\begin{aligned}\|(f\mathcal{A}(\mathbf{k})^{-1/2}&\sin(\varepsilon^{-1}\tau\mathcal{A}(\mathbf{k})^{1/2})f^{-1}-f_0\mathcal{A}^0(\mathbf{k})^{-1/2}\sin(\varepsilon^{-1}\tau\mathcal{A}^0(\mathbf{k})^{1/2})f_0^{-1})\\ &\cdot\mathcal{R}(\mathbf{k},\varepsilon)^{1/2}(I-\widehat{P})\|_{L_2(\Omega)\to L_2(\Omega)}\\ &\leqslant 2r_0^{-1}\|f\|_{L_\infty}\|f^{-1}\|_{L_\infty}|\tau|.\end{aligned}$$

Combining this with (7.22), we see that

$$\begin{aligned}\|(f\mathcal{A}(\mathbf{k})^{-1/2}&\sin(\varepsilon^{-1}\tau\mathcal{A}(\mathbf{k})^{1/2})f^{-1}-f_0\mathcal{A}^0(\mathbf{k})^{-1/2}\sin(\varepsilon^{-1}\tau\mathcal{A}^0(\mathbf{k})^{1/2})f_0^{-1})\\ &\cdot\mathcal{R}(\mathbf{k},\varepsilon)^{1/2}\|_{L_2(\Omega)\to L_2(\Omega)}\\ &\leqslant C_{12}(1+|\tau|),\end{aligned}\tag{7.32}$$

with $\varepsilon > 0$, $\tau \in \mathbb{R}$, $\mathbf{k} \in \operatorname{clos} \widetilde{\Omega}$, and

$$C_{12} := (2r_0^{-1} + \max\{C_7; 2c_*^{-1/2} t_0^{-1}\}) \|f\|_{L_\infty} \|f^{-1}\|_{L_\infty}.$$

Now, we show that the operator $\widehat{P}$ in the principal terms of approximation (7.28) can be removed. Let us estimate the operator $\mathcal{R}(\mathbf{k}, \varepsilon)(I - \widehat{P})$ using the discrete Fourier transform:

$$\|\mathcal{R}(\mathbf{k}, \varepsilon)(I - \widehat{P})\|_{L_2(\Omega) \to L_2(\Omega)} = \max_{0 \neq \mathbf{b} \in \widetilde{\Gamma}} \varepsilon^2 (|\mathbf{b} + \mathbf{k}|^2 + \varepsilon^2)^{-1} \leqslant \varepsilon r_0^{-1}, \tag{7.33}$$

for $\varepsilon > 0$, $\mathbf{k} \in \operatorname{clos} \widetilde{\Omega}$. By (7.1) and (7.33),

$$\begin{aligned} &\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2} f \mathcal{A}(\mathbf{k})^{-1/2} \sin(\varepsilon^{-1} \tau \mathcal{A}(\mathbf{k})^{1/2}) f^{-1} \mathcal{R}(\mathbf{k}, \varepsilon)(I - \widehat{P})\|_{L_2(\Omega) \to L_2(\Omega)} \\ &\quad = \|\sin(\varepsilon^{-1} \tau \mathcal{A}(\mathbf{k})^{1/2}) f^{-1} \mathcal{R}(\mathbf{k}, \varepsilon)(I - \widehat{P})\|_{L_2(\Omega) \to L_2(\Omega)} \\ &\quad \leqslant \|f^{-1}\|_{L_\infty} \varepsilon r_0^{-1}, \end{aligned} \tag{7.34}$$

with $\varepsilon > 0$, $\tau \in \mathbb{R}$, $\mathbf{k} \in \operatorname{clos} \widetilde{\Omega}$. Next, by (6.2), (6.19), (7.3), (7.6), and (7.33),

$$\begin{aligned} &\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2} f_0 \mathcal{A}^0(\mathbf{k})^{-1/2} \sin(\varepsilon^{-1} \tau \mathcal{A}^0(\mathbf{k})^{1/2}) f_0^{-1} \mathcal{R}(\mathbf{k}, \varepsilon)(I - \widehat{P})\|_{L_2(\Omega) \to L_2(\Omega)} \\ &\quad \leqslant \|g\|_{L_\infty}^{1/2} \|g^{-1}\|_{L_\infty}^{1/2} \|f^{-1}\|_{L_\infty} \varepsilon r_0^{-1}, \end{aligned} \tag{7.35}$$

with $\varepsilon > 0$, $\tau \in \mathbb{R}$, $\mathbf{k} \in \operatorname{clos} \widetilde{\Omega}$. Combining (7.28), (7.34), and (7.35), we get

$$\begin{aligned} &\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2} (f \mathcal{A}(\mathbf{k})^{-1/2} \sin(\varepsilon^{-1} \tau \mathcal{A}(\mathbf{k})^{1/2}) f^{-1} \\ &\qquad - (I + \Lambda b(\mathbf{D} + \mathbf{k}) \widehat{P}) f_0 \mathcal{A}^0(\mathbf{k})^{-1/2} \sin(\varepsilon^{-1} \tau \mathcal{A}^0(\mathbf{k})^{1/2}) f_0^{-1}) \\ &\quad \cdot \mathcal{R}(\mathbf{k}, \varepsilon)\|_{L_2(\Omega) \to L_2(\Omega)} \\ &\quad \leqslant C_{13} \varepsilon (1 + |\tau|), \end{aligned} \tag{7.36}$$

with $\varepsilon > 0$, $\tau \in \mathbb{R}$, $\mathbf{k} \in \operatorname{clos} \widetilde{\Omega}$, and

$$C_{13} := C_{11} + r_0^{-1} \|f^{-1}\|_{L_\infty} (1 + \|g\|_{L_\infty}^{1/2} \|g^{-1}\|_{L_\infty}^{1/2}).$$

8. Approximation of the sandwiched operator $\mathcal{A}^{-1/2} \sin(\varepsilon^{-1} \tau \mathcal{A}^{1/2})$

8.1. Let $\mathcal{A}$ and $\mathcal{A}^0$ be the operators (4.10) and (7.5), respectively, acting in $L_2(\mathbb{R}^d; \mathbb{C}^n)$. Recall the notation $\mathcal{H}_0 = -\Delta$ and put

$$\mathcal{R}(\varepsilon) := \varepsilon^2 (\mathcal{H}_0 + \varepsilon^2 I)^{-1}.$$

Using the Gelfand transformation, we decompose this operator into the direct integral of the operators (7.12):

$$\mathcal{R}(\varepsilon) = \mathcal{U}^{-1}\Big(\int_{\widetilde{\Omega}} \oplus \mathcal{R}(\mathbf{k}, \varepsilon)\, d\mathbf{k} \Big)\mathcal{U}. \tag{8.1}$$

In $L_2(\mathbb{R}^d; \mathbb{C}^n)$, we introduce the operator

$$\Pi := \mathcal{U}^{-1}[\widehat{P}]\mathcal{U}.$$

Here $[\widehat{P}]$ is the projection in $\int_{\widetilde{\Omega}} \oplus L_2(\Omega; \mathbb{C}^n)\, d\mathbf{k}$ acting on fibers as the operator $\widehat{P}$ (see (6.4)). As was shown in [11, (6.8)], Π is the pseudodifferential operator in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ with the symbol $\chi_{\widetilde{\Omega}}(\boldsymbol{\xi})$, where $\chi_{\widetilde{\Omega}}$ is the characteristic function of the set $\widetilde{\Omega}$. That is

$$(\Pi \mathbf{u})(\mathbf{x}) = (2\pi)^{-d/2} \int_{\widetilde{\Omega}} e^{i\langle \mathbf{x}, \boldsymbol{\xi}\rangle} \hat{\mathbf{u}}(\boldsymbol{\xi})\, d\boldsymbol{\xi}.$$

Here $\hat{\mathbf{u}}(\boldsymbol{\xi})$ is the Fourier image of the function $\mathbf{u} \in L_2(\mathbb{R}^d; \mathbb{C}^n)$.

Theorem 8.1. *Under the assumptions of Subsection 8.1, for $\varepsilon > 0$ and $\tau \in \mathbb{R}$ we have*

$$\begin{aligned} &\| (f\mathcal{A}^{-1/2} \sin(\varepsilon^{-1}\tau \mathcal{A}^{1/2}) f^{-1} \\ &\quad - f_0(\mathcal{A}^0)^{-1/2} \sin(\varepsilon^{-1}\tau (\mathcal{A}^0)^{1/2}) f_0^{-1}) \\ &\qquad \cdot \mathcal{R}(\varepsilon)^{1/2} \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \\ &\quad \leqslant C_{12}(1 + |\tau|), \end{aligned} \tag{8.2}$$

and

$$\begin{aligned} &\| \widehat{\mathcal{A}}^{1/2} (f\mathcal{A}^{-1/2} \sin(\varepsilon^{-1}\tau \mathcal{A}^{1/2}) f^{-1} \\ &\qquad - (I + \Lambda b(\mathbf{D})\Pi) f_0(\mathcal{A}^0)^{-1/2} \sin(\varepsilon^{-1}\tau (\mathcal{A}^0)^{1/2}) f_0^{-1}) \\ &\quad \cdot \mathcal{R}(\varepsilon) \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \\ &\quad \leqslant C_{13}\varepsilon(1 + |\tau|). \end{aligned} \tag{8.3}$$

The constants C_{12} and C_{13} depend only on m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|f\|_{L_\infty}$, $\|f^{-1}\|_{L_\infty}$, and the parameters of the lattice Γ.

Proof. By (5.8), the similar identity for $\mathcal{A}^0$, and (8.1), from (7.32) we deduce estimate (8.2).

From (7.36) via the Gelfand transform we derive inequality (8.3). □

8.2. Removal of the operator Π in the corrector for $d \leqslant 4$. Now, we show that the operator Π in estimate (8.3) can be removed for $d \leqslant 4$.

Theorem 8.2. *Under the assumptions of Subsection 8.1, let $d \leqslant 4$. Then, for $0 < \varepsilon \leqslant 1$ and $\tau \in \mathbb{R}$,*

$$\begin{aligned} &\| \widehat{\mathcal{A}}^{1/2} (f \mathcal{A}^{-1/2} \sin(\varepsilon^{-1} \tau \mathcal{A}^{1/2}) f^{-1} \\ &\qquad - (I + \Lambda b(\mathbf{D})) f_0 (\mathcal{A}^0)^{-1/2} \sin(\varepsilon^{-1} \tau (\mathcal{A}^0)^{1/2}) f_0^{-1}) \\ &\quad \cdot \mathcal{R}(\varepsilon) \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \\ &\leqslant C_{14} \varepsilon (1 + |\tau|). \end{aligned} \tag{8.4}$$

The constant C_{14} depends only on m, n, d, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|f\|_{L_\infty}$, $\|f^{-1}\|_{L_\infty}$, and the parameters of the lattice Γ.

To prove Theorem 8.2, we need the following result, see [44, Proposition 9.3].

Proposition 8.3. *Let $l = 1$ for $d = 1$, $l > 1$ for $d = 2$, and $l = d/2$ for $d \geqslant 3$. Then the operator $\widehat{\mathcal{A}}^{1/2}[\Lambda]$ is a continuous mapping of $H^l(\mathbb{R}^d; \mathbb{C}^m)$ to $L_2(\mathbb{R}^d; \mathbb{C}^n)$, and*

$$\| \widehat{\mathcal{A}}^{1/2} [\Lambda] \|_{H^l(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant \mathfrak{C}_d. \tag{8.5}$$

Here the constant $\mathfrak{C}_d$ depends only on m, n, d, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and the parameters of the lattice Γ; for $d = 2$ it depends also on l.

Proof of Theorem 8.2. Taking into account that the matrix-valued function (7.4) is the symbol of the operator $\mathcal{A}^0$ and the function $\chi_{\widetilde{\Omega}}(\boldsymbol{\xi})$ is the symbol of Π, using (4.7), (7.3), and (7.7) we have

$$\begin{aligned} &\| b(\mathbf{D})(I - \Pi) f_0 (\mathcal{A}^0)^{-1/2} \sin(\varepsilon^{-1} \tau (\mathcal{A}^0)^{1/2}) f_0^{-1} \mathcal{R}(\varepsilon) \|_{L_2(\mathbb{R}^d) \to H^2(\mathbb{R}^d)} \\ &\quad \leqslant \sup_{\boldsymbol{\xi} \in \mathbb{R}^d} (1 + |\boldsymbol{\xi}|^2) |b(\boldsymbol{\xi})| (1 - \chi_{\widetilde{\Omega}}(\boldsymbol{\xi})) |f_0| \\ &\qquad\qquad \cdot |(f_0 b(\boldsymbol{\xi})^* g^0 b(\boldsymbol{\xi}) f_0)^{-1/2}| |f_0^{-1}| \varepsilon^2 (|\boldsymbol{\xi}|^2 + \varepsilon^2)^{-1} \\ &\quad \leqslant \sup_{|\boldsymbol{\xi}| \geqslant r_0} (1 + |\boldsymbol{\xi}|^2) \alpha_1^{1/2} |\boldsymbol{\xi}| \|f\|_{L_\infty} c_*^{-1/2} |\boldsymbol{\xi}|^{-1} \|f^{-1}\|_{L_\infty} \varepsilon^2 (|\boldsymbol{\xi}|^2 + \varepsilon^2)^{-1} \\ &\quad \leqslant \alpha_1^{1/2} c_*^{-1/2} \|f\|_{L_\infty} \|f^{-1}\|_{L_\infty} \varepsilon^2 \sup_{|\boldsymbol{\xi}| \geqslant r_0} (1 + |\boldsymbol{\xi}|^2) |\boldsymbol{\xi}|^{-2} \\ &\quad \leqslant \alpha_1^{1/2} c_*^{-1/2} \|f\|_{L_\infty} \|f^{-1}\|_{L_\infty} (r_0^{-2} + 1) \varepsilon^2. \end{aligned} \tag{8.6}$$

For $d \leqslant 4$, we can take $l \leqslant 2$ in Proposition 8.3. So, combining (8.5) and (8.6), we have

$$\begin{aligned} &\| \widehat{\mathcal{A}}^{1/2} [\Lambda] b(\mathbf{D})(I - \Pi) f_0 (\mathcal{A}^0)^{-1/2} \sin(\varepsilon^{-1} \tau (\mathcal{A}^0)^{1/2}) f_0^{-1} \mathcal{R}(\varepsilon) \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \\ &\quad \leqslant \varepsilon^2 C'_{14}, \end{aligned}$$

where

$$C'_{14} := \alpha_1^{1/2} c_*^{-1/2} (r_0^{-2} + 1)\mathfrak{C}_d \|f\|_{L_\infty} \|f^{-1}\|_{L_\infty}.$$

Combining this with (8.3), we arrive at estimate (8.4) with

$$C_{14} := C_{13} + C'_{14}. \qquad \square$$

8.3. On the possibility of removal of the operator Π from the corrector. Sufficient conditions on Λ. It is possible to eliminate the operator Π for $d \geqslant 5$ by imposing the following assumption on the matrix-valued function Λ.

Condition 8.4. The operator $[\Lambda]$ is continuous from $H^2(\mathbb{R}^d; \mathbb{C}^m)$ to $H^1(\mathbb{R}^d; \mathbb{C}^n)$.

Actually, it is sufficient to impose the following condition to remove Π for $d \geqslant 5$.

Condition 8.5. Assume that the periodic solution Λ of problem (6.7) belongs to $L_d(\Omega)$.

Proposition 8.6. *For $d \geqslant 3$, Condition 8.5 implies Condition 8.4.*

To prove Proposition 8.6 we need the following statement.

Lemma 8.7. *Let $d \geqslant 3$. Assume that Condition 8.5 is satisfied. Then the operator $g^{1/2} b(\mathbf{D})[\Lambda]$ is a continuous mapping of $H^2(\mathbb{R}^d; \mathbb{C}^m)$ to $L_2(\mathbb{R}^d; \mathbb{C}^m)$ and*

$$\|g^{1/2} b(\mathbf{D})[\Lambda]\|_{H^2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant \mathfrak{C}_\Lambda. \tag{8.7}$$

The constant $\mathfrak{C}_\Lambda$ depends only on d, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|\Lambda\|_{L_d(\Omega)}$, and the parameters of the lattice Γ.

Proof. The proof is quite similar to the proof of Proposition 8.8 from [45].

Let $\mathbf{v}_j(\mathbf{x})$, $j = 1, \dots, m$, be the columns of the matrix $\Lambda(\mathbf{x})$. In other words, $\mathbf{v}_j$ is the Γ-periodic solution of the problem

$$b(\mathbf{D})^* g(\mathbf{x})(b(\mathbf{D})\mathbf{v}_j(\mathbf{x}) + \mathbf{e}_j) = 0, \quad \int_\Omega \mathbf{v}_j(\mathbf{x})\, d\mathbf{x} = 0. \tag{8.8}$$

Here $\{\mathbf{e}_j\}_{j=1}^m$ is the standard orthonormal basis in $\mathbb{C}^m$. Let $u \in H^2(\mathbb{R}^d)$. Then

$$g^{1/2} b(\mathbf{D})(\mathbf{v}_j u) = g^{1/2}(b(\mathbf{D})\mathbf{v}_j)u + \sum_{l=1}^d g^{1/2} b_l (D_l u)\mathbf{v}_j. \tag{8.9}$$

We estimate the second term on the right-hand side of (8.9):

$$\Big\| \sum_{l=1}^{d} g^{1/2} b_l (D_l u) \mathbf{v}_j \Big\|_{L_2(\mathbb{R}^d)} \leqslant \|g\|^{1/2}_{L_\infty} \alpha_1^{1/2} d^{1/2} \Big(\int_{\mathbb{R}^d} |\mathbf{D}u|^2 |\mathbf{v}_j|^2 \, d\mathbf{x} \Big)^{1/2}. \quad (8.10)$$

Next,

$$\int_{\mathbb{R}^d} |\mathbf{D}u|^2 |\mathbf{v}_j|^2 \, d\mathbf{x} = \sum_{\mathbf{a}\in\Gamma} \int_{\Omega+\mathbf{a}} |\mathbf{D}u|^2 |\mathbf{v}_j|^2 \, d\mathbf{x}. \quad (8.11)$$

By the Hölder inequality with indices $s = d/2$ and $s' = d/(d-2)$,

$$\int_{\Omega+\mathbf{a}} |\mathbf{D}u|^2 |\mathbf{v}_j|^2 \, d\mathbf{x} \leqslant \Big(\int_{\Omega} |\mathbf{v}_j|^d \, d\mathbf{x} \Big)^{2/d} \Big(\int_{\Omega+\mathbf{a}} |\mathbf{D}u|^{2d/(d-2)} \, d\mathbf{x} \Big)^{(d-2)/d}. \quad (8.12)$$

By the continuous embedding $H^1(\Omega) \hookrightarrow L_{2d/(d-2)}(\Omega)$,

$$\Big(\int_{\Omega+\mathbf{a}} |\mathbf{D}u|^{2d/(d-2)} \, d\mathbf{x} \Big)^{(d-2)/2d} \leqslant C_\Omega \|\mathbf{D}u\|_{H^1(\Omega+\mathbf{a})}. \quad (8.13)$$

The embedding constant C_Ω depends only on d and Ω (i.e., on the lattice Γ). From (8.11)–(8.13) it follows that

$$\int_{\mathbb{R}^d} |\mathbf{D}u|^2 |\mathbf{v}_j|^2 \, d\mathbf{x} \leqslant C_\Omega^2 \|\mathbf{v}_j\|^2_{L_d(\Omega)} \|u\|^2_{H^2(\mathbb{R}^d)}. \quad (8.14)$$

Using (8.10), from (8.14) we derive the estimate

$$\Big\| \sum_{l=1}^{d} g^{1/2} b_l (D_l u) \mathbf{v}_j \Big\|_{L_2(\mathbb{R}^d)} \leqslant \|g\|^{1/2}_{L_\infty} \alpha_1^{1/2} d^{1/2} C_\Omega \|\mathbf{v}_j\|_{L_d(\Omega)} \|u\|_{H^2(\mathbb{R}^d)}. \quad (8.15)$$

Next, equation (8.8) implies that

$$\int_{\mathbb{R}^d} \Big(\langle g(\mathbf{x}) b(\mathbf{D}) \mathbf{v}_j, b(\mathbf{D})\mathbf{w} \rangle + \sum_{l=1}^{d} \langle b_l^* g(\mathbf{x}) \mathbf{e}_j, D_l \mathbf{w} \rangle \Big) \, d\mathbf{x} = 0 \quad (8.16)$$

for any $\mathbf{w} \in H^1(\mathbb{R}^d; \mathbb{C}^n)$ such that $\mathbf{w}(\mathbf{x}) = 0$ for $|\mathbf{x}| > R$ (with some $R > 0$).

Let $u \in C_0^\infty(\mathbb{R}^d)$. We put $\mathbf{w}(\mathbf{x}) = |u(\mathbf{x})|^2 \mathbf{v}_j(\mathbf{x})$. Then

$$b(\mathbf{D})\mathbf{w} = |u|^2 b(\mathbf{D}) \mathbf{v}_j + \sum_{l=1}^{d} b_l (D_l |u|^2) \mathbf{v}_j.$$

Substituting this expression into (8.16), we obtain

$$\int_{\mathbb{R}^d} \Big(\Big\langle g(\mathbf{x}) b(\mathbf{D}) \mathbf{v}_j, |u|^2 b(\mathbf{D}) \mathbf{v}_j + \sum_{l=1}^{d} b_l (D_l |u|^2) \mathbf{v}_j \Big\rangle + \sum_{l=1}^{d} \langle b_l^* g(\mathbf{x}) \mathbf{e}_j, D_l(|u|^2 \mathbf{v}_j) \rangle \Big)\, d\mathbf{x} = 0.$$

Hence,

$$J_0 := \int_{\mathbb{R}^d} |g^{1/2} b(\mathbf{D}) \mathbf{v}_j|^2 |u|^2 \, d\mathbf{x} = J_1 + J_2, \tag{8.17}$$

where

$$J_1 = - \int_{\mathbb{R}^d} \Big\langle g^{1/2} b(\mathbf{D}) \mathbf{v}_j, \sum_{l=1}^{d} g^{1/2} b_l (D_l |u|^2) \mathbf{v}_j \Big\rangle \, d\mathbf{x},$$

$$J_2 = - \int_{\mathbb{R}^d} \sum_{l=1}^{d} \langle b_l^* g(\mathbf{x}) \mathbf{e}_j, D_l(|u|^2 \mathbf{v}_j) \rangle \, d\mathbf{x}$$

$$= - \int_{\mathbb{R}^d} \sum_{l=1}^{d} \langle b_l^* g(\mathbf{x}) \mathbf{e}_j, D_l(\mathbf{v}_j u) u^* + \mathbf{v}_j u (D_l u^*) \rangle \, d\mathbf{x}.$$

By (4.8),

$$|J_1| \leqslant \|g\|_{L_\infty}^{1/2} \alpha_1^{1/2} d^{1/2} \int_{\mathbb{R}^d} 2 |g^{1/2} b(\mathbf{D}) \mathbf{v}_j| |u| |\mathbf{D} u| |\mathbf{v}_j| \, d\mathbf{x}$$

$$\leqslant \frac{1}{2} \int_{\mathbb{R}^d} |g^{1/2} b(\mathbf{D}) \mathbf{v}_j|^2 |u|^2 \, d\mathbf{x} + 2 \|g\|_{L_\infty} \alpha_1 d \int_{\mathbb{R}^d} |\mathbf{D} u|^2 |\mathbf{v}_j|^2 \, d\mathbf{x}.$$

Combining this with (8.14), we see that

$$|J_1| \leqslant \frac{1}{2} J_0 + 2 \|g\|_{L_\infty} \alpha_1 d C_\Omega^2 \|\mathbf{v}_j\|_{L_d(\Omega)}^2 \|u\|_{H^2(\mathbb{R}^d)}^2. \tag{8.18}$$

Now we proceed to estimating the term J_2. By (4.8),

$$\int_{\mathbb{R}^d} |b_l^* g(\mathbf{x}) \mathbf{e}_j|^2 |u|^2 \, d\mathbf{x} \leqslant \alpha_1 \|g\|_{L_\infty}^2 \|u\|_{L_2(\mathbb{R}^d)}^2.$$

Then

$$|J_2| \leqslant \sum_{l=1}^{d} \|ub_l^* g \mathbf{e}_j\|_{L_2(\mathbb{R}^d)} (\|D_l(\mathbf{v}_j u)\|_{L_2(\mathbb{R}^d)} + \|\mathbf{v}_j (D_l u^*)\|_{L_2(\mathbb{R}^d)})$$
$$\leqslant \mu \|\mathbf{D}(\mathbf{v}_j u)\|^2_{L_2(\mathbb{R}^d)} + (4^{-1} + (4\mu)^{-1}) d\alpha_1 \|g\|^2_{L_\infty} \|u\|^2_{L_2(\mathbb{R}^d)}$$
$$+ \int_{\mathbb{R}^d} |\mathbf{v}_j|^2 |\mathbf{D}u^*|^2 \, d\mathbf{x}$$

for any $\mu > 0$. By (8.14),

$$|J_2| \leqslant \mu \|\mathbf{D}(\mathbf{v}_j u)\|^2_{L_2(\mathbb{R}^d)} + ((4^{-1} + (4\mu)^{-1}) d\alpha_1 \|g\|^2_{L_\infty} + C_\Omega^2 \|\mathbf{v}_j\|^2_{L_d(\Omega)}) \|u\|^2_{H^2(\mathbb{R}^d)}. \tag{8.19}$$

Now, relations (8.17), (8.18), and (8.19) imply that

$$\frac{1}{2} J_0 \leqslant \mu \|\mathbf{D}(\mathbf{v}_j u)\|^2_{L_2(\mathbb{R}^d)} + \Big((2\|g\|_{L_\infty} \alpha_1 d + 1) C_\Omega^2 \|\mathbf{v}_j\|^2_{L_d(\Omega)} + \Big(\frac{1}{4} + \frac{1}{4\mu}\Big) d\alpha_1 \|g\|^2_{L_\infty} \Big) \|u\|^2_{H^2(\mathbb{R}^d)}. \tag{8.20}$$

Comparing (8.9), (8.15), (8.17), and (8.20), we obtain

$$\|g^{1/2} b(\mathbf{D})(\mathbf{v}_j u)\|^2_{L_2(\mathbb{R}^d)}$$
$$\leqslant 2J_0 + 2\|g\|_{L_\infty} \alpha_1 d C_\Omega^2 \|\mathbf{v}_j\|^2_{L_d(\Omega)} \|u\|^2_{H^2(\mathbb{R}^d)}$$
$$\leqslant 4\mu \|\mathbf{D}(\mathbf{v}_j u)\|^2_{L_2(\mathbb{R}^d)}$$
$$+ ((10\|g\|_{L_\infty} \alpha_1 d + 4) C_\Omega^2 \|\mathbf{v}_j\|^2_{L_d(\Omega)} + (1 + \mu^{-1}) d\alpha_1 \|g\|^2_{L_\infty}) \|u\|^2_{H^2(\mathbb{R}^d)}. \tag{8.21}$$

By (4.11) (with $f = \mathbf{1}_n$),

$$4\mu \|\mathbf{D}(\mathbf{v}_j u)\|^2_{L_2(\mathbb{R}^d)} \leqslant 4\mu \alpha_0^{-1} \|g^{-1}\|_{L_\infty} \|g^{1/2} b(\mathbf{D})(\mathbf{v}_j u)\|^2_{L_2(\mathbb{R}^d)}$$
$$= \frac{1}{2} \|g^{1/2} b(\mathbf{D})(\mathbf{v}_j u)\|^2_{L_2(\mathbb{R}^d)},$$

for $\mu = \frac{1}{8} \alpha_0 \|g^{-1}\|^{-1}_{L_\infty}$. Together with (8.21) this implies

$$\|g^{1/2} b(\mathbf{D})(\mathbf{v}_j u)\|^2_{L_2(\mathbb{R}^d)} \leqslant \mathcal{C}_j^2 \|u\|^2_{H^2(\mathbb{R}^d)},$$

where

$$\mathcal{C}_j^2 := (20\|g\|_{L_\infty} \alpha_1 d + 8) C_\Omega^2 \|\mathbf{v}_j\|^2_{L_d(\Omega)} + (2 + 16\alpha_0^{-1} \|g^{-1}\|_{L_\infty}) d\alpha_1 \|g\|^2_{L_\infty}.$$

Thus,

$$\|g^{1/2}b(\mathbf{D})[\mathbf{v}_j]\|_{H^2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant \mathfrak{C}_j, \quad j = 1, \dots, m,$$

whence

$$\|g^{1/2}b(\mathbf{D})[\Lambda]\|_{H^2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant \Big(\sum_{j=1}^{m} \mathfrak{C}_j^2\Big)^{1/2} =: \mathfrak{C}_\Lambda;$$

i.e., (8.7) is true. □

Proof of Proposition 8.6. Let $u \in H^2(\mathbb{R}^d)$. Similarly to (8.11)–(8.14),

$$\|\mathbf{v}_j u\|^2_{L_2(\mathbb{R}^d)} \leqslant C_\Omega^2 \|\mathbf{v}_j\|^2_{L_d(\Omega)} \|u\|^2_{H^1(\mathbb{R}^d)}.$$

Here $\mathbf{v}_j(\mathbf{x})$, $j = 1, \dots, m$, are the columns of the matrix $\Lambda(\mathbf{x})$. Thus,

$$\|[\Lambda]u\|^2_{L_2(\mathbb{R}^d)} \leqslant C_\Omega^2 \sum_{j=1}^{m} \|\mathbf{v}_j\|^2_{L_d(\Omega)} \|u\|^2_{H^1(\mathbb{R}^d)}. \tag{8.22}$$

By (4.11) with $f = \mathbf{1}_n$, and Lemma 8.7,

$$\begin{aligned} \|\mathbf{D}[\Lambda]u\|^2_{L_2(\mathbb{R}^d)} &\leqslant \alpha_0^{-1} \|g^{-1}\|_{L_\infty} \|g^{1/2}b(\mathbf{D})[\Lambda]u\|^2_{L_2(\mathbb{R}^d)} \\ &\leqslant \alpha_0^{-1}\|g^{-1}\|_{L_\infty} \mathfrak{C}_\Lambda^2 \|u\|^2_{H^2(\mathbb{R}^d)}. \end{aligned} \tag{8.23}$$

Combining (8.22) and (8.23), we obtain

$$\|[\Lambda]u\|^2_{H^1(\mathbb{R}^d)} \leqslant \Big(C_\Omega^2 \sum_{j=1}^{m} \|\mathbf{v}_j\|^2_{L_d(\Omega)} + \alpha_0^{-1}\|g^{-1}\|_{L_\infty}\mathfrak{C}_\Lambda^2\Big) \|u\|^2_{H^2(\mathbb{R}^d)},$$

for $u \in H^2(\mathbb{R}^d)$. □

Theorem 8.8. *Let $d \geqslant 5$. Under Condition 8.4, for $0 < \varepsilon \leqslant 1$ and $\tau \in \mathbb{R}$,*

$$\begin{aligned} &\|\widehat{\mathcal{A}}^{1/2}(f\mathcal{A}^{-1/2}\sin(\varepsilon^{-1}\tau\mathcal{A}^{1/2})f^{-1} \\ &\qquad - (I + \Lambda b(\mathbf{D})) f_0 (\mathcal{A}^0)^{-1/2}\sin(\varepsilon^{-1}\tau(\mathcal{A}^0)^{1/2}) f_0^{-1}) \\ &\quad \cdot \mathcal{R}(\varepsilon)\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \\ &\leqslant C_{15}\varepsilon(1 + |\tau|), \end{aligned} \tag{8.24}$$

where the constant C_{15} depends only on m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|f\|_{L_\infty}$, $\|f^{-1}\|_{L_\infty}$, the parameters of the lattice Γ, and the norm $\|[\Lambda]\|_{H^2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)}$.

Proof. Under Condition 8.4, by (4.7), (6.1), and (8.6), we have

$$\begin{aligned}
&\|\widehat{\mathcal{A}}^{1/2}[\Lambda]b(\mathbf{D})(I-\Pi)f_0(\mathcal{A}^0)^{-1/2}\sin(\varepsilon^{-1}\tau(\mathcal{A}^0)^{1/2})f_0^{-1}\mathcal{R}(\varepsilon)\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\\
&\quad\leqslant \|g\|_{L_\infty}^{1/2}\alpha_1^{1/2}\|\mathbf{D}[\Lambda]\|_{H^2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\alpha_1^{1/2}c_*^{-1/2}\|f\|_{L_\infty}\|f^{-1}\|_{L_\infty}(r_0^{-2}+1)\varepsilon^2\\
&\quad\leqslant C_{15}'\varepsilon^2,
\end{aligned}$$

where

$$C_{15}' := \alpha_1 c_*^{-1/2}\|g\|_{L_\infty}^{1/2}\|f\|_{L_\infty}\|f^{-1}\|_{L_\infty}\|[\Lambda]\|_{H^2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)}(r_0^{-2}+1).$$

Combining this with (8.3), we arrive at estimate (8.24) with the constant

$$C_{15} := C_{13} + C_{15}'. \qquad \square$$

For $d \geqslant 5$, removal of the operator Π in the corrector also can be achieved by increasing the degree of the operator $\mathcal{R}(\varepsilon)$. In the application to homogenization of the hyperbolic Cauchy problem, this corresponds to more restrictive assumptions on the regularity of the initial data.

The proof of the following result is quite similar to that of Theorem 8.2.

Proposition 8.9. *Let $d \geqslant 5$. Then for $\tau \in \mathbb{R}$, $0 < \varepsilon \leqslant 1$, we have*

$$\begin{aligned}
&\|\widehat{\mathcal{A}}^{1/2}(f\mathcal{A}^{-1/2}\sin(\varepsilon^{-1}\tau\mathcal{A}^{1/2})f^{-1}\\
&\qquad - (I+\Lambda b(\mathbf{D}))f_0(\mathcal{A}^0)^{-1/2}\sin(\varepsilon^{-1}\tau(\mathcal{A}^0)^{1/2})f_0^{-1})\\
&\quad\cdot\mathcal{R}(\varepsilon)^{d/4}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\\
&\quad\leqslant C_{16}\varepsilon(1+|\tau|).
\end{aligned}$$

The constant C_{16} depends only on m, n, d, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|f\|_{L_\infty}$, $\|f^{-1}\|_{L_\infty}$, and the parameters of the lattice Γ.

Chapter III
Homogenization problem for hyperbolic systems

9. Approximation of the sandwiched operator $\mathcal{A}_\varepsilon^{-1/2}\sin(\tau\mathcal{A}_\varepsilon^{1/2})$

For a Γ-periodic measurable function $\psi(\mathbf{x})$ in $\mathbb{R}^d$ we denote

$$\psi^\varepsilon(\mathbf{x}) := \psi(\varepsilon^{-1}\mathbf{x}), \quad \varepsilon > 0.$$

Let $[\psi^\varepsilon]$ be the operator of multiplication by the function $\psi^\varepsilon(\mathbf{x})$. *Our main object* is the operator $\mathcal{A}_\varepsilon$, $\varepsilon > 0$, acting in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ and formally given by the differential expression

$$\mathcal{A}_\varepsilon = f^\varepsilon(\mathbf{x})^* b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}) f^\varepsilon(\mathbf{x}). \tag{9.1}$$

Denote

$$\widehat{\mathcal{A}}_\varepsilon = b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}). \tag{9.2}$$

The precise definitions of these operators are given in terms of the corresponding quadratic forms. The coefficients of the operators (9.1) and (9.2) oscillate rapidly as $\varepsilon \to 0$.

Our goal is to approximate the sandwiched operator $\mathcal{A}_\varepsilon^{-1/2}\sin(\tau\mathcal{A}_\varepsilon^{1/2})$. The results are applied to homogenization of the solutions of the Cauchy problem for hyperbolic systems.

9.1. The principal term of approximation. Let T_ε be the *unitary scaling transformation* in $L_2(\mathbb{R}^d;\mathbb{C}^n)$:

$$(T_\varepsilon \mathbf{u})(\mathbf{x}) := \varepsilon^{d/2}\mathbf{u}(\varepsilon\mathbf{x}), \quad \varepsilon > 0.$$

Then $\mathcal{A}_\varepsilon = \varepsilon^{-2}T_\varepsilon^* \mathcal{A} T_\varepsilon$. Thus,

$$\mathcal{A}_\varepsilon^{-1/2}\sin(\tau\mathcal{A}_\varepsilon^{1/2}) = \varepsilon T_\varepsilon^* \mathcal{A}^{-1/2}\sin(\varepsilon^{-1}\tau\mathcal{A}^{1/2})T_\varepsilon.$$

The operator $\mathcal{A}^0$ satisfies a similar identity. Next,

$$(\mathcal{H}_0 + I)^{-1/2} = \varepsilon T_\varepsilon^*(\mathcal{H}_0 + \varepsilon^2 I)^{-1/2}T_\varepsilon = T_\varepsilon^* \mathcal{R}(\varepsilon)^{1/2}T_\varepsilon.$$

Note that for any s the operator $(\mathcal{H}_0 + I)^{s/2}$ is an isometric isomorphism of the Sobolev space $H^s(\mathbb{R}^d;\mathbb{C}^n)$ onto $L_2(\mathbb{R}^d;\mathbb{C}^n)$. Indeed, for $\mathbf{u} \in H^s(\mathbb{R}^d;\mathbb{C}^n)$

$$\|(\mathcal{H}_0 + I)^{s/2}\mathbf{u}\|^2_{L_2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} (|\boldsymbol{\xi}|^2 + 1)^s |\hat{\mathbf{u}}(\boldsymbol{\xi})|^2\, d\boldsymbol{\xi} = \|\mathbf{u}\|^2_{H^s(\mathbb{R}^d)}. \tag{9.3}$$

Using these arguments, from (8.2) we deduce the following result.

Theorem 9.1. *Let $\mathcal{A}_\varepsilon$ be the operator* (9.1) *and let $\mathcal{A}^0$ be the operator* (7.5)*. Then, for $\varepsilon > 0$ and $\tau \in \mathbb{R}$,*

$$\begin{aligned} &\| f^\varepsilon \mathcal{A}_\varepsilon^{-1/2}\sin(\tau\mathcal{A}_\varepsilon^{1/2})(f^\varepsilon)^{-1} - f_0(\mathcal{A}^0)^{-1/2}\sin(\tau(\mathcal{A}^0)^{1/2})f_0^{-1}\|_{H^1(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \\ &\quad \leqslant C_{12}\varepsilon(1+|\tau|). \end{aligned} \tag{9.4}$$

The constant C_{12} is controlled in terms of r_0, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|f\|_{L_\infty}$, and $\|f^{-1}\|_{L_\infty}$.

By (7.3) and the elementary inequality $|\sin x|/|x| \leqslant 1$, $x \in \mathbb{R}$,

$$\begin{aligned}\| f^\varepsilon \mathcal{A}_\varepsilon^{-1/2} \sin(\tau \mathcal{A}_\varepsilon^{1/2})(f^\varepsilon)^{-1} - f_0(\mathcal{A}^0)^{-1/2} \sin(\tau (\mathcal{A}^0)^{1/2}) f_0^{-1} \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \\ \leqslant 2 \| f \|_{L_\infty} \| f^{-1} \|_{L_\infty} |\tau|.\end{aligned} \tag{9.5}$$

Interpolating between (9.5) and (9.4), we obtain the following result.

Theorem 9.2. *Under the assumptions of Theorem 9.1, for* $0 \leqslant s \leqslant 1$, $\tau \in \mathbb{R}$, *and* $\varepsilon > 0$,

$$\begin{aligned}\| f^\varepsilon \mathcal{A}_\varepsilon^{-1/2} \sin(\tau \mathcal{A}_\varepsilon^{1/2})(f^\varepsilon)^{-1} - f_0(\mathcal{A}^0)^{-1/2} \sin(\tau (\mathcal{A}^0)^{1/2}) f_0^{-1} \|_{H^s(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \\ \leqslant \mathfrak{C}_1(s)(1 + |\tau|)\varepsilon^s,\end{aligned}$$

where

$$\mathfrak{C}_1(s) := (2 \| f \|_{L_\infty} \| f^{-1} \|_{L_\infty})^{1-s} C_{12}^s.$$

9.2. Approximation with corrector. Now, we obtain an approximation with the correction term taken into account. We put

$$\Pi_\varepsilon := T_\varepsilon^* \Pi T_\varepsilon.$$

Then Π_ε is the pseudodifferential operator in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ with the symbol $\chi_{\widetilde{\Omega}/\varepsilon}(\boldsymbol{\xi})$, i.e.,

$$(\Pi_\varepsilon \mathbf{u})(\mathbf{x}) = (2\pi)^{-d/2} \int_{\widetilde{\Omega}/\varepsilon} e^{i\langle \mathbf{x}, \boldsymbol{\xi} \rangle} \hat{\mathbf{u}}(\boldsymbol{\xi})\, d\boldsymbol{\xi}. \tag{9.6}$$

Obviously, $\Pi_\varepsilon \mathbf{D}^\sigma \mathbf{u} = \mathbf{D}^\sigma \Pi_\varepsilon \mathbf{u}$ for $\mathbf{u} \in H^\kappa(\mathbb{R}^d; \mathbb{C}^n)$ and any multiindex σ of length $|\sigma| \leqslant \kappa$. Note that

$$\| \Pi_\varepsilon \|_{H^\kappa(\mathbb{R}^d) \to H^\kappa(\mathbb{R}^d)} \leqslant 1, \quad \kappa \in \mathbb{Z}_+.$$

The following results were obtained in [37, Proposition 1.4] and [12, Subsection 10.2].

Proposition 9.3. *For any function* $\mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^n)$ *we have*

$$\| \Pi_\varepsilon \mathbf{u} - \mathbf{u} \|_{L_2(\mathbb{R}^d)} \leqslant \varepsilon r_0^{-1} \| \mathbf{D} \mathbf{u} \|_{L_2(\mathbb{R}^d)}, \quad \varepsilon > 0.$$

Proposition 9.4. *Let* $\Phi(\mathbf{x})$ *be a* Γ*-periodic function in* $\mathbb{R}^d$ *such that* $\Phi \in L_2(\Omega)$. *Then the operator* $[\Phi^\varepsilon]\Pi_\varepsilon$ *is bounded in* $L_2(\mathbb{R}^d; \mathbb{C}^n)$, *and*

$$\| [\Phi^\varepsilon] \Pi_\varepsilon \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant |\Omega|^{-1/2} \| \Phi \|_{L_2(\Omega)}, \quad \varepsilon > 0.$$

Theorem 9.5. *Let $\Lambda(\mathbf{x})$ be the Γ-periodic solution of problem* (6.7)*. Let Π_ε be the operator* (9.6)*. Then, under the assumptions of Theorem 9.1, for $\varepsilon > 0$ and $\tau \in \mathbb{R}$*

$$\begin{aligned} &\| f^\varepsilon \mathcal{A}_\varepsilon^{-1/2} \sin(\tau \mathcal{A}_\varepsilon^{1/2})(f^\varepsilon)^{-1} \\ &\qquad - (I + \varepsilon \Lambda^\varepsilon b(\mathbf{D})\Pi_\varepsilon) f_0 (\mathcal{A}^0)^{-1/2} \sin(\tau (\mathcal{A}^0)^{1/2}) f_0^{-1} \|_{H^2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \\ &\quad \leqslant C_{17}\varepsilon(1+|\tau|), \end{aligned} \tag{9.7}$$

where the constant C_{17} depends only on m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|f\|_{L_\infty}$, $\|f^{-1}\|_{L_\infty}$, and the parameters of the lattice Γ.

Proof. By the scaling transformation, (8.3) implies that

$$\begin{aligned} &\|\widehat{\mathcal{A}}_\varepsilon^{1/2}(f^\varepsilon \mathcal{A}_\varepsilon^{-1/2} \sin(\tau \mathcal{A}_\varepsilon^{1/2})(f^\varepsilon)^{-1} \\ &\qquad - (I + \varepsilon \Lambda^\varepsilon b(\mathbf{D})\Pi_\varepsilon) f_0 (\mathcal{A}^0)^{-1/2} \sin(\tau (\mathcal{A}^0)^{1/2}) f_0^{-1}) \\ &\quad \cdot (\mathcal{H}_0 + I)^{-1}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \\ &\quad \leqslant C_{13}\varepsilon(1+|\tau|). \end{aligned} \tag{9.8}$$

Note that, by (4.7), (4.9), and (9.2),

$$\hat{c}_* \|\mathbf{D}\mathbf{u}\|^2_{L_2(\mathbb{R}^d)} \leqslant \|\widehat{\mathcal{A}}_\varepsilon^{1/2}\mathbf{u}\|^2_{L_2(\mathbb{R}^d)}, \quad \mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n), \tag{9.9}$$

where the constant $\hat{c}_*$ is defined by (6.5). From (9.8) and (9.9) it follows that

$$\begin{aligned} &\|\mathbf{D}(f^\varepsilon \mathcal{A}_\varepsilon^{-1/2} \sin(\tau \mathcal{A}_\varepsilon^{1/2})(f^\varepsilon)^{-1} \\ &\qquad - (I + \varepsilon \Lambda^\varepsilon b(\mathbf{D})\Pi_\varepsilon) f_0 (\mathcal{A}^0)^{-1/2} \sin(\tau (\mathcal{A}^0)^{1/2}) f_0^{-1}) \\ &\quad \cdot (\mathcal{H}_0 + I)^{-1}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \\ &\quad \leqslant \hat{c}_*^{-1/2} C_{13}\varepsilon(1+|\tau|). \end{aligned} \tag{9.10}$$

Now, we estimate the $(L_2 \to L_2)$-norm of the correction term. Let $\Pi_\varepsilon^{(m)}$ be the pseudodifferential operator in $L_2(\mathbb{R}^d;\mathbb{C}^m)$ with the symbol $\chi_{\widetilde{\Omega}/\varepsilon}(\boldsymbol{\xi})$. By Proposition 9.4 and (6.12),

$$\|\Lambda^\varepsilon \Pi_\varepsilon^{(m)}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant M_1. \tag{9.11}$$

Using (6.19), (7.3), (7.5), and (9.11),

$$\begin{aligned} &\|\varepsilon \Lambda^\varepsilon b(\mathbf{D})\Pi_\varepsilon f_0 (\mathcal{A}^0)^{-1/2} \sin(\tau (\mathcal{A}^0)^{1/2}) f_0^{-1} (\mathcal{H}_0 + I)^{-1}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \\ &\quad \leqslant \varepsilon \|\Lambda^\varepsilon \Pi_\varepsilon^{(m)}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \\ &\qquad\quad \cdot \|b(\mathbf{D}) f_0 (\mathcal{A}^0)^{-1/2} \sin(\tau (\mathcal{A}^0)^{1/2}) f_0^{-1} (\mathcal{H}_0 + I)^{-1}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \\ &\quad \leqslant \varepsilon M_1 \|g^{-1}\|^{1/2}_{L_\infty} \|\sin(\tau (\mathcal{A}^0)^{1/2}) f_0^{-1} (\mathcal{H}_0 + I)^{-1}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \\ &\quad \leqslant \varepsilon M_1 \|g^{-1}\|^{1/2}_{L_\infty} \|f^{-1}\|_{L_\infty}. \end{aligned} \tag{9.12}$$

Combining (9.3), (9.4), (9.10), and (9.12), we arrive at estimate (9.7) with the constant $C_{17} := \hat{c}_*^{-1/2} C_{13} + C_{12} + M_1 \|g^{-1}\|_{L_\infty}^{1/2} \|f^{-1}\|_{L_\infty}$. $\square$

By interpolation, from Theorem 9.5 we derive the following result.

Theorem 9.6. *Under the assumptions of Theorem 9.5, for* $0 \leqslant s \leqslant 1$, $\tau \in \mathbb{R}$, *and* $0 < \varepsilon \leqslant 1$

$$\begin{aligned} &\| f^\varepsilon \mathcal{A}_\varepsilon^{-1/2} \sin(\tau \mathcal{A}_\varepsilon^{1/2}) (f^\varepsilon)^{-1} \\ &\quad - (I + \varepsilon \Lambda^\varepsilon b(\mathbf{D}) \Pi_\varepsilon) f_0 (\mathcal{A}^0)^{-1/2} \sin(\tau (\mathcal{A}^0)^{1/2}) f_0^{-1} \|_{H^{s+1}(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \\ &\leqslant \mathfrak{C}_2(s)(1 + |\tau|)\varepsilon^s. \end{aligned} \tag{9.13}$$

Here the constant $\mathfrak{C}_2(s)$ *depends only on* s, m, α_0, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|f\|_{L_\infty}$, $\|f^{-1}\|_{L_\infty}$, *and the parameters of the lattice* Γ.

Proof. Let us estimate the left-hand side of (9.13) for $s = 0$. By (4.7), (9.1), and the elementary inequality $|\sin x|/|x| \leqslant 1$, $x \in \mathbb{R}$,

$$\begin{aligned} &\| f^\varepsilon \mathcal{A}_\varepsilon^{-1/2} \sin(\tau \mathcal{A}_\varepsilon^{1/2}) (f^\varepsilon)^{-1} \|_{H^1(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \\ &\quad \leqslant \|f\|_{L_\infty} \|f^{-1}\|_{L_\infty} |\tau| + \|\mathbf{D} f^\varepsilon \mathcal{A}_\varepsilon^{-1/2} \sin(\tau \mathcal{A}_\varepsilon^{1/2}) (f^\varepsilon)^{-1}\|_{H^1(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \\ &\quad \leqslant \|f\|_{L_\infty} \|f^{-1}\|_{L_\infty} |\tau| + \alpha_0^{-1/2} \|g^{-1}\|_{L_\infty}^{1/2} \|f^{-1}\|_{L_\infty}. \end{aligned} \tag{9.14}$$

Similarly, by (4.7), (7.3), and (7.5),

$$\begin{aligned} &\| f_0 (\mathcal{A}^0)^{-1/2} \sin(\tau (\mathcal{A}^0)^{1/2}) f_0^{-1} \|_{H^1(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \\ &\quad \leqslant \|f\|_{L_\infty} \|f^{-1}\|_{L_\infty} |\tau| + \alpha_0^{-1/2} \|g^{-1}\|_{L_\infty}^{1/2} \|f^{-1}\|_{L_\infty}. \end{aligned} \tag{9.15}$$

From (6.19), (7.3), and (9.11) it follows that

$$\begin{aligned} &\|\varepsilon \Lambda^\varepsilon b(\mathbf{D}) \Pi_\varepsilon f_0 (\mathcal{A}^0)^{-1/2} \sin(\tau (\mathcal{A}^0)^{1/2}) f_0^{-1} \|_{H^1(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \\ &\quad \leqslant \varepsilon M_1 \|g^{-1}\|_{L_\infty}^{1/2} \|f^{-1}\|_{L_\infty} \\ &\qquad + \varepsilon \|\mathbf{D} \Lambda^\varepsilon b(\mathbf{D}) \Pi_\varepsilon f_0 (\mathcal{A}^0)^{-1/2} \sin(\tau (\mathcal{A}^0)^{1/2}) f_0^{-1} \|_{H^1(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \\ &\quad \leqslant \varepsilon M_1 \|g^{-1}\|_{L_\infty}^{1/2} \|f^{-1}\|_{L_\infty} \\ &\qquad + \|(\mathbf{D}\Lambda)^\varepsilon b(\mathbf{D}) \Pi_\varepsilon f_0 (\mathcal{A}^0)^{-1/2} \sin(\tau (\mathcal{A}^0)^{1/2}) f_0^{-1} \|_{H^1(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \\ &\qquad + \varepsilon \|\Lambda^\varepsilon \mathbf{D} b(\mathbf{D}) \Pi_\varepsilon f_0 (\mathcal{A}^0)^{-1/2} \sin(\tau (\mathcal{A}^0)^{1/2}) f_0^{-1} \|_{H^1(\mathbb{R}^d) \to L_2(\mathbb{R}^d)}. \end{aligned} \tag{9.16}$$

By Proposition 9.4, (6.13), (6.19), (7.3), and (7.5),

$$\begin{aligned} &\|(\mathbf{D}\Lambda)^\varepsilon b(\mathbf{D}) \Pi_\varepsilon f_0 (\mathcal{A}^0)^{-1/2} \sin(\tau (\mathcal{A}^0)^{1/2}) f_0^{-1} \|_{H^1(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \\ &\quad \leqslant M_2 \|g^{-1}\|_{L_\infty}^{1/2} \|f^{-1}\|_{L_\infty}. \end{aligned} \tag{9.17}$$

Next, according to (6.19), (7.5), and (9.11),

$$\begin{aligned}&\varepsilon\|\Lambda^\varepsilon \mathbf{D} b(\mathbf{D})\Pi_\varepsilon f_0(\mathcal{A}^0)^{-1/2}\sin(\tau(\mathcal{A}^0)^{1/2})f_0^{-1}\|_{H^1(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\\&\quad\leqslant \varepsilon M_1\|g^{-1}\|_{L_\infty}^{1/2}\|\mathbf{D}\sin(\tau(\mathcal{A}^0)^{1/2})f_0^{-1}\|_{H^1(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}.\end{aligned}\tag{9.18}$$

Since the operator $\mathcal{A}^0$ with constant coefficients commutes with the differentiation $\mathbf{D}$, we have

$$\|\mathbf{D}\sin(\tau(\mathcal{A}^0)^{1/2})\|_{H^1(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\leqslant 1.$$

Together with (7.3) and (9.16)–(9.18) this yields

$$\begin{aligned}&\|\varepsilon\Lambda^\varepsilon b(\mathbf{D})\Pi_\varepsilon f_0(\mathcal{A}^0)^{-1/2}\sin(\tau(\mathcal{A}^0)^{1/2})f_0^{-1}\|_{H^1(\mathbb{R}^d)\to H^1(\mathbb{R}^d)}\\&\quad\leqslant (2\varepsilon M_1+M_2)\|g^{-1}\|_{L_\infty}^{1/2}\|f^{-1}\|_{L_\infty}.\end{aligned}\tag{9.19}$$

Relations (9.14), (9.15), and (9.19) imply that

$$\begin{aligned}&\|f^\varepsilon\mathcal{A}_\varepsilon^{-1/2}\sin(\tau\mathcal{A}_\varepsilon^{1/2})(f^\varepsilon)^{-1}\\&\quad-(I+\varepsilon\Lambda^\varepsilon b(\mathbf{D})\Pi_\varepsilon)f_0(\mathcal{A}^0)^{-1/2}\sin(\tau(\mathcal{A}^0)^{1/2})f_0^{-1}\|_{H^1(\mathbb{R}^d)\to H^1(\mathbb{R}^d)}\\&\quad\leqslant C_{18}(1+|\tau|),\end{aligned}\tag{9.20}$$

with $\tau\in\mathbb{R}$, $0<\varepsilon\leqslant 1$, and

$$C_{18}:=\max\{2\|f\|_{L_\infty}\|f^{-1}\|_{L_\infty};(2\alpha_0^{-1/2}+2M_1+M_2)\|g^{-1}\|_{L_\infty}^{1/2}\|f^{-1}\|_{L_\infty}\}.$$

Interpolating between (9.20) and (9.7), we deduce estimate (9.13) with

$$\mathfrak{C}_2(s):=C_{18}^{1-s}C_{17}^{s}.$$

□

9.3. The case where $d\leqslant 4$. Now we apply Theorem 8.2. By the scaling transformation, (8.4) implies that

$$\begin{aligned}&\|\widehat{\mathcal{A}}_\varepsilon^{1/2}(f^\varepsilon\mathcal{A}_\varepsilon^{-1/2}\sin(\tau\mathcal{A}_\varepsilon^{1/2})(f^\varepsilon)^{-1}\\&\quad-(I+\varepsilon\Lambda^\varepsilon b(\mathbf{D}))f_0(\mathcal{A}^0)^{-1/2}\sin(\tau(\mathcal{A}^0)^{1/2})f_0^{-1})\\&\quad\cdot(\mathcal{H}_0+I)^{-1}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\\&\quad\leqslant C_{14}\varepsilon(1+|\tau|),\end{aligned}\tag{9.21}$$

for $0<\varepsilon\leqslant 1$, $\tau\in\mathbb{R}$. Combining this with (9.9), we obtain

$$\begin{aligned}&\|\mathbf{D}(f^\varepsilon\mathcal{A}_\varepsilon^{-1/2}\sin(\tau\mathcal{A}_\varepsilon^{1/2})(f^\varepsilon)^{-1}\\&\quad-(I+\varepsilon\Lambda^\varepsilon b(\mathbf{D}))f_0(\mathcal{A}^0)^{-1/2}\sin(\tau(\mathcal{A}^0)^{1/2})f_0^{-1})\\&\quad\cdot(\mathcal{H}_0+I)^{-1}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\\&\quad\leqslant \hat{c}_*^{-1/2}C_{14}\varepsilon(1+|\tau|),\end{aligned}\tag{9.22}$$

for $0<\varepsilon\leqslant 1$, $\tau\in\mathbb{R}$.

Let us estimate the $(L_2 \to L_2)$-norm of the corrector. By the scaling transformation,

$$\begin{aligned} &\varepsilon \|\Lambda^\varepsilon b(\mathbf{D}) f_0 (\mathcal{A}^0)^{-1/2} \sin(\tau (\mathcal{A}^0)^{1/2}) f_0^{-1} (\mathcal{H}_0 + I)^{-1} \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \\ &\quad = \varepsilon \|\Lambda b(\mathbf{D}) f_0 (\mathcal{A}^0)^{-1/2} \sin(\varepsilon^{-1} \tau (\mathcal{A}^0)^{1/2}) f_0^{-1} \mathcal{R}(\varepsilon) \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)}. \end{aligned} \tag{9.23}$$

The $(H^s \to L_2)$-norm of the operator $[\Lambda]$ was estimated in [44, Proposition 11.3].

Proposition 9.7. *Let $s = 0$ for $d = 1$, $s > 0$ for $d = 2$, $s = d/2 - 1$ for $d \geqslant 3$. Then the operator $[\Lambda]$ is a continuous mapping of $H^s(\mathbb{R}^d; \mathbb{C}^m)$ to $L_2(\mathbb{R}^d; \mathbb{C}^m)$, and*

$$\|[\Lambda]\|_{H^s(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant \mathfrak{C}_d,$$

there the constant $\mathfrak{C}_d$ depends only on d, m, n, α_0, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and the parameters of the lattice Γ; in the case $d = 2$ it depends also on s.

Now we consider only the case $d \leqslant 4$. So, by Proposition 9.7,

$$\|[\Lambda]\|_{H^1(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant \mathfrak{C}_d, \quad d \leqslant 4. \tag{9.24}$$

Thus, we need to estimate the operator $b(\mathbf{D}) f_0 (\mathcal{A}^0)^{-1/2} \sin(\varepsilon^{-1} \tau (\mathcal{A}^0)^{1/2}) f_0^{-1} \mathcal{R}(\varepsilon)$ in the $(L_2 \to H^1)$-norm. By (6.19), (7.3), and (7.5), for any d we have

$$\begin{aligned} &\|b(\mathbf{D}) f_0 (\mathcal{A}^0)^{-1/2} \sin(\varepsilon^{-1} \tau (\mathcal{A}^0)^{1/2}) f_0^{-1} \mathcal{R}(\varepsilon) \|_{L_2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \\ &\quad \leqslant \|b(\mathbf{D}) f_0 (\mathcal{A}^0)^{-1/2} \sin(\varepsilon^{-1} \tau (\mathcal{A}^0)^{1/2}) f_0^{-1} \mathcal{R}(\varepsilon) \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \\ &\qquad + \|\mathbf{D} b(\mathbf{D}) f_0 (\mathcal{A}^0)^{-1/2} \sin(\varepsilon^{-1} \tau (\mathcal{A}^0)^{1/2}) f_0^{-1} \mathcal{R}(\varepsilon) \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \\ &\quad \leqslant 2 \|g^{-1}\|_{L_\infty}^{1/2} \|f^{-1}\|_{L_\infty}, \quad \tau \in \mathbb{R}, \quad 0 < \varepsilon \leqslant 1. \end{aligned} \tag{9.25}$$

The following result is a direct consequence of (9.4) and (9.22)–(9.25).

Theorem 9.8. *Let $d \leqslant 4$. Under the assumptions of Theorem 9.5,*

$$\begin{aligned} &\|f^\varepsilon \mathcal{A}_\varepsilon^{-1/2} \sin(\tau \mathcal{A}_\varepsilon^{1/2}) (f^\varepsilon)^{-1} \\ &\qquad - (I + \varepsilon \Lambda^\varepsilon b(\mathbf{D})) f_0 (\mathcal{A}^0)^{-1/2} \sin(\tau (\mathcal{A}^0)^{1/2}) f_0^{-1} \|_{H^2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \\ &\quad \leqslant C_{19} \varepsilon (1 + |\tau|), \end{aligned} \tag{9.26}$$

with $0 < \varepsilon \leqslant 1$, $\tau \in \mathbb{R}$. The constant

$$C_{19} := \hat{c}_*^{-1/2} C_{14} + C_{12} + 2 \mathfrak{C}_d \|g^{-1}\|_{L_\infty}^{1/2} \|f^{-1}\|_{L_\infty}$$

depends only on d, m, n, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|f\|_{L_\infty}$, $\|f^{-1}\|_{L_\infty}$, and the parameters of the lattice Γ.

9.4. Removal of Π_ε from the corrector for $d \geqslant 5$. The following result can be deduced from Theorem 8.8.

Theorem 9.9. *Let $d \geqslant 5$. Let Condition 8.4 be satisfied. Then, under the assumptions of Theorem 9.5, for $0 < \varepsilon \leqslant 1$ and $\tau \in \mathbb{R}$ we have*

$$\begin{aligned} &\| f^\varepsilon \mathcal{A}_\varepsilon^{-1/2} \sin(\tau \mathcal{A}_\varepsilon^{1/2})(f^\varepsilon)^{-1} \\ &\quad - (I + \varepsilon \Lambda^\varepsilon b(\mathbf{D})) f_0 (\mathcal{A}^0)^{-1/2} \sin(\tau (\mathcal{A}^0)^{1/2}) f_0^{-1} \|_{H^2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \\ &\leqslant C_{20} \varepsilon (1 + |\tau|), \end{aligned} \tag{9.27}$$

where the constant C_{20} depends only on m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|f\|_{L_\infty}$, $\|f^{-1}\|_{L_\infty}$, the parameters of the lattice Γ, and the norm $\|[\Lambda]\|_{H^2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)}$.

Proof. The proof is similar to that of Theorem 9.5. Combining (6.19), (7.3), (7.5), (8.24), (9.4), and (9.9), we arrive at the estimate (9.27) with

$$C_{20} := \hat{c}_*^{-1/2} C_{15} + C_{12} + \|[\Lambda]\|_{H^2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \|g^{-1}\|_{L_\infty}^{1/2} \|f^{-1}\|_{L_\infty}. \qquad \square$$

Theorem 9.10. *Let $d \geqslant 5$. Under the assumptions of Theorem 9.5, for $0 < \varepsilon \leqslant 1$ and $\tau \in \mathbb{R}$ we have*

$$\begin{aligned} &\| f^\varepsilon \mathcal{A}_\varepsilon^{-1/2} \sin(\tau \mathcal{A}_\varepsilon^{1/2})(f^\varepsilon)^{-1} \\ &\quad - (I + \varepsilon \Lambda^\varepsilon b(\mathbf{D})) f_0 (\mathcal{A}^0)^{-1/2} \sin(\tau (\mathcal{A}^0)^{1/2}) f_0^{-1} \|_{H^{d/2}(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \\ &\leqslant C_{21} \varepsilon (1 + |\tau|). \end{aligned} \tag{9.28}$$

The constant C_{21} depends only on d, m, n, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|f\|_{L_\infty}$, $\|f^{-1}\|_{L_\infty}$, and the parameters of the lattice Γ.

Proof. By the scaling transformation, from Proposition 8.9 it follows that

$$\begin{aligned} &\| \widehat{\mathcal{A}}_\varepsilon^{1/2} (f^\varepsilon \mathcal{A}_\varepsilon^{-1/2} \sin(\tau \mathcal{A}_\varepsilon^{1/2})(f^\varepsilon)^{-1} \\ &\quad - (I + \varepsilon \Lambda^\varepsilon b(\mathbf{D})) f_0 (\mathcal{A}^0)^{-1/2} \sin(\tau (\mathcal{A}^0)^{1/2}) f_0^{-1}) \\ &\quad \cdot (\mathcal{H}_0 + I)^{-d/4} \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \\ &\leqslant C_{16} \varepsilon (1 + |\tau|), \end{aligned}$$

with $0 < \varepsilon \leqslant 1$, $\tau \in \mathbb{R}$. By (9.9),

$$\begin{aligned} &\| \mathbf{D} (f^\varepsilon \mathcal{A}_\varepsilon^{-1/2} \sin(\tau \mathcal{A}_\varepsilon^{1/2})(f^\varepsilon)^{-1} \\ &\quad - (I + \varepsilon \Lambda^\varepsilon b(\mathbf{D})) f_0 (\mathcal{A}^0)^{-1/2} \sin(\tau (\mathcal{A}^0)^{1/2}) f_0^{-1}) \\ &\quad \cdot (\mathcal{H}_0 + I)^{-d/4} \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \\ &\leqslant \hat{c}_*^{-1/2} C_{16} \varepsilon (1 + |\tau|), \end{aligned} \tag{9.29}$$

with $0 < \varepsilon \leqslant 1$, $\tau \in \mathbb{R}$. By Proposition 9.7, and (6.19), (7.3), and (7.5),

$$\begin{aligned}
&\|\Lambda^\varepsilon b(\mathbf{D}) f_0(\mathcal{A}^0)^{-1/2}\sin(\tau(\mathcal{A}^0)^{1/2}) f_0^{-1}(\mathcal{H}_0+I)^{-d/4}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\\
&\quad\leqslant \mathfrak{C}_d \|b(\mathbf{D}) f_0(\mathcal{A}^0)^{-1/2}\sin(\tau(\mathcal{A}^0)^{1/2}) f_0^{-1}(\mathcal{H}_0+I)^{-d/4}\|_{L_2(\mathbb{R}^d)\to H^{d/2-1}(\mathbb{R}^d)}\\
&\quad\leqslant \mathfrak{C}_d \|g^{-1}\|_{L_\infty}^{1/2}\|f^{-1}\|_{L_\infty}\|(\mathcal{H}_0+I)^{-d/4}\|_{L_2(\mathbb{R}^d)\to H^{d/2-1}(\mathbb{R}^d)}\\
&\quad\leqslant \mathfrak{C}_d \|g^{-1}\|_{L_\infty}^{1/2}\|f^{-1}\|_{L_\infty}.
\end{aligned} \tag{9.30}$$

Combining (9.4), (9.29), and (9.30), we arrive at estimate (9.28) with the constant

$$C_{21} := \hat{c}_*^{-1/2} C_{16} + C_{12} + \mathfrak{C}_d \|g^{-1}\|_{L_\infty}^{1/2}\|f^{-1}\|_{L_\infty}. \qquad \square$$

9.5. Removal of Π_ε. Interpolational results. To obtain the analogue of Theorem 9.6 with Π_ε replaced by I we need the continuity of the operator

$$\varepsilon[\Lambda^\varepsilon] b(\mathbf{D}) f_0(\mathcal{A}^0)^{-1/2}\sin(\tau(\mathcal{A}^0)^{1/2}) f_0^{-1}$$

in $H^1(\mathbb{R}^d;\mathbb{C}^n)$, i.e., we need the boundedness of $\|[(\mathbf{D}\Lambda)^\varepsilon]\|_{H^1(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}$ and $\|[\Lambda^\varepsilon]\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}$. Due to Parseval's theorem, $\|[\Lambda^\varepsilon]\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} < \infty$ holds if and only if the matrix-valued function Λ is subject to the following condition.

Condition 9.11. Assume that the Γ-periodic solution $\Lambda(\mathbf{x})$ of problem (6.7) is bounded, i.e., $\Lambda \in L_\infty(\mathbb{R}^d)$.

Under Condition 9.11, the operator $[(\mathbf{D}\Lambda)^\varepsilon]$ is bounded from H^1 to L_2 due to the following result obtained in [37, Corollary 2.4].

Lemma 9.12. *Under Condition 9.11, for any function $u \in H^1(\mathbb{R}^d)$ and $\varepsilon > 0$ we have*

$$\int_{\mathbb{R}^d} |(\mathbf{D}\Lambda)^\varepsilon(\mathbf{x})|^2 |u(\mathbf{x})|^2\, d\mathbf{x} \leqslant \mathfrak{c}_1 \|u\|^2_{L_2(\mathbb{R}^d)} + \mathfrak{c}_2\varepsilon^2 \|\Lambda\|^2_{L_\infty}\|\mathbf{D}u\|^2_{L_2(\mathbb{R}^d)}.$$

The constants $\mathfrak{c}_1$ and $\mathfrak{c}_2$ depend on m, d, α_0, α_1, $\|g\|_{L_\infty}$, and $\|g^{-1}\|_{L_\infty}$.

Some cases where Condition 9.11 is fulfilled automatically were distinguished in [12, Lemma 8.7].

Proposition 9.13. *Suppose that at least one of the following assumptions is satisfied*:

1°) $d \leqslant 2$;

2°) *the dimension* $d \geqslant 1$ *is arbitrary and the operator* $\mathcal{A}_\varepsilon$ *has the form* $\mathcal{A}_\varepsilon = \mathbf{D}^* g^\varepsilon(\mathbf{x})\mathbf{D}$, *where* $g(\mathbf{x})$ *is symmetric matrix with real entries*;

3°) *the dimension* d *is arbitrary and* $g^0 = \underline{g}$, *i.e., relations* (6.21) *are true.*

Then Condition 9.11 *is fulfilled.*

Surely, if $\Lambda \in L_\infty$, then Condition 8.5 holds automatically. Then, by Proposition 8.6, for $d \geqslant 5$, the assumptions of Theorem 9.9 are satisfied.

We are going to check that under Condition 9.11 the analog of Theorem 9.6 is valid without any smoothing operator in the corrector. To do this, we estimate the $(H^1 \to H^1)$-norm of the operators under the norm sign in (9.26) (or (9.27)). By (6.19), (7.3), (7.5), and Lemma 9.12, we obtain

$$\begin{aligned} &\|\varepsilon\Lambda^\varepsilon b(\mathbf{D}) f_0 (\mathcal{A}^0)^{-1/2} \sin(\tau(\mathcal{A}^0)^{1/2}) f_0^{-1}\|_{H^1(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \\ &\quad \leqslant 2\varepsilon \|\Lambda\|_{L_\infty} \|g^{-1}\|^{1/2}_{L_\infty} \|f^{-1}\|_{L_\infty} + \|g^{-1}\|^{1/2}_{L_\infty} \|f^{-1}\|_{L_\infty} (\mathfrak{c}_1 + \mathfrak{c}_2 \|\Lambda\|^2_{L_\infty})^{1/2}, \end{aligned} \tag{9.31}$$

with $0 < \varepsilon \leqslant 1$, $\tau \in \mathbb{R}$. Combining (9.14), (9.15), and (9.31), we deduce that

$$\begin{aligned} &\| f^\varepsilon \mathcal{A}_\varepsilon^{-1/2} \sin(\tau \mathcal{A}_\varepsilon^{1/2}) (f^\varepsilon)^{-1} \\ &\qquad - (I + \varepsilon\Lambda^\varepsilon b(\mathbf{D})) f_0 (\mathcal{A}^0)^{-1/2} \sin(\tau(\mathcal{A}^0)^{1/2}) f_0^{-1}\|_{H^1(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \\ &\quad \leqslant C_{22}(1 + |\tau|), \quad 0 < \varepsilon \leqslant 1, \quad \tau \in \mathbb{R}, \end{aligned} \tag{9.32}$$

where

$$\begin{aligned} C_{22} := \|f^{-1}\|_{L_\infty} \max\{&2\|f\|_{L_\infty}; \\ &\|g^{-1}\|^{1/2}_{L_\infty} (2\alpha_0^{-1/2} + 2\|\Lambda\|_{L_\infty} + (\mathfrak{c}_1 + \mathfrak{c}_2\|\Lambda\|^2_{L_\infty})^{1/2})\}. \end{aligned}$$

Interpolating between (9.32) and (9.26) for $d \leqslant 4$ and between (9.32) and (9.27) for $d \geqslant 5$, we arrive at the following result.

Theorem 9.14. *Suppose that the assumptions of Theorem* 9.1 *are satisfied and Condition* 9.11 *holds. Then, for* $0 \leqslant s \leqslant 1$ *and* $\tau \in \mathbb{R}$, $0 < \varepsilon \leqslant 1$

$$\begin{aligned} &\| f^\varepsilon \mathcal{A}_\varepsilon^{-1/2} \sin(\tau \mathcal{A}_\varepsilon^{1/2}) (f^\varepsilon)^{-1} \\ &\qquad - (I + \varepsilon\Lambda^\varepsilon b(\mathbf{D})) f_0 (\mathcal{A}^0)^{-1/2} \sin(\tau(\mathcal{A}^0)^{1/2}) f_0^{-1}\|_{H^{s+1}(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \\ &\qquad \leqslant \mathfrak{C}_3(s)(1 + |\tau|)\varepsilon^s, \end{aligned}$$

where

$$\mathfrak{C}_3(s) := \begin{cases} C_{22}^{1-s} C_{19}^s & \text{for } d \leqslant 4, \\ C_{22}^{1-s} C_{20}^s & \text{for } d \geqslant 5. \end{cases}$$

9.6. The case where the corrector is equal to zero. Assume that $g^0 = \bar{g}$, i.e., relations (6.20) are valid. Then the Γ-periodic solution of problem (6.7) is equal to zero: $\Lambda(\mathbf{x}) = 0$, and Theorem 9.6 implies the following result.

Proposition 9.15. *Suppose that relations* (6.20) *hold. Then under the assumptions of Theorem* 9.1, *for* $0 \leqslant s \leqslant 1$ *and* $\tau \in \mathbb{R}$, $0 < \varepsilon \leqslant 1$ *we have*

$$\| f^\varepsilon \mathcal{A}_\varepsilon^{-1/2} \sin(\tau \mathcal{A}_\varepsilon^{1/2})(f^\varepsilon)^{-1} - f_0 (\mathcal{A}^0)^{-1/2} \sin(\tau (\mathcal{A}^0)^{1/2}) f_0^{-1} \|_{H^{s+1}(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leqslant \mathfrak{C}_2(s)(1 + |\tau|)\varepsilon^s.$$

10. Homogenization of hyperbolic systems with periodic coefficients

10.1. The statement of the problem. Homogenization for the solutions of hyperbolic systems. Our goal is to apply the results of Section 9 to homogenization for the solutions of the problem

$$\begin{cases} Q^\varepsilon(\mathbf{x}) \dfrac{\partial^2 \mathbf{u}_\varepsilon(\mathbf{x}, \tau)}{\partial \tau^2} = -b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}) \mathbf{u}_\varepsilon(\mathbf{x}, \tau) + Q^\varepsilon(\mathbf{x}) \mathbf{F}(\mathbf{x}, \tau), \\ \mathbf{u}_\varepsilon(\mathbf{x}, 0) = 0, \quad \dfrac{\partial \mathbf{u}_\varepsilon(\mathbf{x}, 0)}{\partial \tau} = \boldsymbol{\psi}(\mathbf{x}), \end{cases} \tag{10.1}$$

where $\boldsymbol{\psi} \in L_2(\mathbb{R}^d; \mathbb{C}^n)$, $\mathbf{F} \in L_{1,\mathrm{loc}}(\mathbb{R}; L_2(\mathbb{R}^d; \mathbb{C}^n))$, and $Q(\mathbf{x})$ is a Γ-periodic $(n \times n)$-matrix-valued function (7.2). Substituting

$$\mathbf{z}_\varepsilon(\cdot, \tau) := (f^\varepsilon)^{-1} \mathbf{u}_\varepsilon(\cdot, \tau)$$

into (10.1), we rewrite problem (10.1) as

$$\begin{cases} \dfrac{\partial^2 \mathbf{z}_\varepsilon(\mathbf{x}, \tau)}{\partial \tau^2} = -f^\varepsilon(\mathbf{x})^* b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}) f^\varepsilon(\mathbf{x}) \mathbf{z}_\varepsilon(\mathbf{x}, \tau) + f^\varepsilon(\mathbf{x})^{-1} \mathbf{F}(\mathbf{x}, \tau), \\ \mathbf{z}_\varepsilon(\mathbf{x}, 0) = 0, \quad \dfrac{\partial \mathbf{z}_\varepsilon(\mathbf{x}, 0)}{\partial \tau} = f^\varepsilon(\mathbf{x})^{-1} \boldsymbol{\psi}(\mathbf{x}). \end{cases}$$

Then

$$\mathbf{z}_\varepsilon(\cdot, \tau) = \mathcal{A}_\varepsilon^{-1/2} \sin(\tau \mathcal{A}_\varepsilon^{1/2})(f^\varepsilon)^{-1} \boldsymbol{\psi} + \int_0^\tau \mathcal{A}_\varepsilon^{-1/2} \sin((\tau - \tilde{\tau}) \mathcal{A}_\varepsilon^{1/2})(f^\varepsilon)^{-1} \mathbf{F}(\cdot, \tilde{\tau}) \, d\tilde{\tau} \tag{10.2}$$

and

$$\mathbf{u}_\varepsilon(\cdot, \tau) = f^\varepsilon \mathcal{A}_\varepsilon^{-1/2} \sin(\tau \mathcal{A}_\varepsilon^{1/2})(f^\varepsilon)^{-1} \boldsymbol{\psi} + \int_0^\tau f^\varepsilon \mathcal{A}_\varepsilon^{-1/2} \sin((\tau - \tilde{\tau}) \mathcal{A}_\varepsilon^{1/2})(f^\varepsilon)^{-1} \mathbf{F}(\cdot, \tilde{\tau}) \, d\tilde{\tau}. \tag{10.3}$$

Let $\mathbf{u}_0(\mathbf{x},\tau)$ be the solution of the effective problem

$$\begin{cases} \overline{Q}\dfrac{\partial^2 \mathbf{u}_0(\mathbf{x},\tau)}{\partial \tau^2} = -b(\mathbf{D})^* g^0 b(\mathbf{D})\mathbf{u}_0(\mathbf{x},\tau) + \overline{Q}\mathbf{F}(\mathbf{x},\tau), \\ \mathbf{u}_0(\mathbf{x},0) = 0, \quad \dfrac{\partial \mathbf{u}_0(\mathbf{x},0)}{\partial \tau} = \boldsymbol{\psi}(\mathbf{x}), \end{cases} \tag{10.4}$$

where

$$\overline{Q} = |\Omega|^{-1} \int_\Omega Q(\mathbf{x})\, d\mathbf{x}.$$

Similarly to (10.2) and (10.3), we obtain

$$\begin{aligned} \mathbf{u}_0(\cdot,\tau) = {} & f_0 (\mathcal{A}^0)^{-1/2} \sin(\tau(\mathcal{A}^0)^{1/2}) f_0^{-1}\boldsymbol{\psi} \\ & + \int_0^\tau f_0 (\mathcal{A}^0)^{-1/2} \sin((\tau-\tilde{\tau})(\mathcal{A}^0)^{1/2}) f_0^{-1}\mathbf{F}(\cdot,\tilde{\tau})\, d\tilde{\tau}. \end{aligned} \tag{10.5}$$

Using Theorems 9.1, 9.5, and identities (10.3), (10.5), we arrive at the following result.

Theorem 10.1. *Let* $\mathbf{u}_\varepsilon$ *be the solution of problem* (10.1) *and let* $\mathbf{u}_0$ *be the solution of the effective problem* (10.4).

1°. *Let* $\boldsymbol{\psi} \in H^1(\mathbb{R}^d;\mathbb{C}^n)$ *and let* $\mathbf{F} \in L_{1,\mathrm{loc}}(\mathbb{R}; H^1(\mathbb{R}^d;\mathbb{C}^n))$. *Then for* $\tau \in \mathbb{R}$ *and* $\varepsilon > 0$ *we have*

$$\|\mathbf{u}_\varepsilon(\cdot,\tau) - \mathbf{u}_0(\cdot,\tau)\|_{L_2(\mathbb{R}^d)} \leqslant C_{12}\varepsilon(1+|\tau|)(\|\boldsymbol{\psi}\|_{H^1(\mathbb{R}^d)} + \|\mathbf{F}\|_{L_1((0,\tau);H^1(\mathbb{R}^d))}).$$

2°. *Let* $\boldsymbol{\psi} \in H^2(\mathbb{R}^d;\mathbb{C}^n)$ *and let* $\mathbf{F} \in L_{1,\mathrm{loc}}(\mathbb{R}; H^2(\mathbb{R}^d;\mathbb{C}^n))$. *Let* $\Lambda(\mathbf{x})$ *be the* Γ*-periodic solution of problem* (6.7). *Let* Π_ε *be the smoothing operator* (9.6). *By* $\mathbf{v}_\varepsilon$ *we denote the first order approximation*:

$$\mathbf{v}_\varepsilon(\mathbf{x},\tau) := \mathbf{u}_0(\mathbf{x},\tau) + \varepsilon\Lambda^\varepsilon b(\mathbf{D})\Pi_\varepsilon \mathbf{u}_0(\mathbf{x},\tau). \tag{10.6}$$

Then for $\tau \in \mathbb{R}$ *and* $\varepsilon > 0$ *we have*

$$\|\mathbf{u}_\varepsilon(\cdot,\tau) - \mathbf{v}_\varepsilon(\cdot,\tau)\|_{H^1(\mathbb{R}^d)} \leqslant C_{17}\varepsilon(1+|\tau|)(\|\boldsymbol{\psi}\|_{H^2(\mathbb{R}^d)} + \|\mathbf{F}\|_{L_1((0,\tau);H^2(\mathbb{R}^d))}). \tag{10.7}$$

Remark 10.2. If $d \leqslant 4$ (or $d \geqslant 5$ and Condition 8.4 is satisfied), then we can use Theorem 9.8 (respectively, Theorem 9.9), i.e., the estimate of the form (10.7) is valid with $\mathbf{v}_\varepsilon$ replaced by

$$\mathbf{v}_\varepsilon^{(0)}(\mathbf{x},\tau) := \mathbf{u}_0(\mathbf{x},\tau) + \varepsilon\Lambda^\varepsilon b(\mathbf{D})\mathbf{u}_0(\mathbf{x},\tau). \tag{10.8}$$

Theorem 9.10 implies the following statement.

Proposition 10.3. *Assume that* $d \geqslant 5$. *Let* $\boldsymbol{\psi} \in H^{d/2}(\mathbb{R}^d;\mathbb{C}^n)$ *and let* $\mathbf{F} \in L_{1,\mathrm{loc}}(\mathbb{R}; H^{d/2}(\mathbb{R}^d;\mathbb{C}^n))$. *Let* $\mathbf{u}_\varepsilon$ *and* $\mathbf{u}_0$ *be the solutions of problems* (10.1) *and* (10.4) *respectively. Let* $\mathbf{v}_\varepsilon^{(0)}$ *be given by* (10.8). *Then for* $\tau \in \mathbb{R}$ *and* $0 < \varepsilon \leqslant 1$ *we have*

$$\begin{aligned} &\|\mathbf{u}_\varepsilon(\cdot,\tau) - \mathbf{v}_\varepsilon^{(0)}(\cdot,\tau)\|_{H^1(\mathbb{R}^d)} \\ &\quad \leqslant C_{21}\varepsilon(1+|\tau|)(\|\boldsymbol{\psi}\|_{H^{d/2}(\mathbb{R}^d)} + \|\mathbf{F}\|_{L_1((0,\tau);H^{d/2}(\mathbb{R}^d))}). \end{aligned}$$

Applying Theorems 9.2 and 9.6, we arrive at the following result.

Theorem 10.4. *Let* $\mathbf{u}_\varepsilon$ *be the solution of problem* (10.1) *and let* $\mathbf{u}_0$ *be the solution of the effective problem* (10.4).

1°. *Let* $\boldsymbol{\psi} \in H^s(\mathbb{R}^d;\mathbb{C}^n)$ *and* $\mathbf{F} \in L_{1,\mathrm{loc}}(\mathbb{R}; H^s(\mathbb{R}^d;\mathbb{C}^n))$, $0 \leqslant s \leqslant 1$. *Then, for* $\tau \in \mathbb{R}$ *and* $\varepsilon > 0$

$$\begin{aligned} &\|\mathbf{u}_\varepsilon(\cdot,\tau) - \mathbf{u}_0(\cdot,\tau)\|_{L_2(\mathbb{R}^d)} \\ &\quad \leqslant \mathfrak{C}_1(s)(1+|\tau|)\varepsilon^s(\|\boldsymbol{\psi}\|_{H^s(\mathbb{R}^d)} + \|\mathbf{F}\|_{L_1((0,\tau);H^s(\mathbb{R}^d))}). \end{aligned}$$

Under the additional assumption that $\mathbf{F} \in L_1(\mathbb{R}_\pm; H^s(\mathbb{R}^d;\mathbb{C}^n))$, *for* $0<s\leqslant 1$, $|\tau| = \varepsilon^{-\alpha}$, $0 < \varepsilon \leqslant 1$, $0 < \alpha < s$

$$\begin{aligned} &\|\mathbf{u}_\varepsilon(\cdot,\pm\varepsilon^{-\alpha}) - \mathbf{u}_0(\cdot,\pm\varepsilon^{-\alpha})\|_{L_2(\mathbb{R}^d)} \\ &\quad \leqslant 2\mathfrak{C}_1(s)\varepsilon^{s-\alpha}(\|\boldsymbol{\psi}\|_{H^s(\mathbb{R}^d)} + \|\mathbf{F}\|_{L_1(\mathbb{R}_\pm;H^s(\mathbb{R}^d))}). \end{aligned}$$

2°. *Let* $\boldsymbol{\psi} \in H^{1+s}(\mathbb{R}^d;\mathbb{C}^n)$ *and* $\mathbf{F} \in L_{1,\mathrm{loc}}(\mathbb{R}; H^{1+s}(\mathbb{R}^d;\mathbb{C}^n))$, $0 \leqslant s \leqslant 1$. *Let* $\mathbf{v}_\varepsilon$ *be given by* (10.6). *Then, for* $\tau \in \mathbb{R}$ *and* $0 < \varepsilon \leqslant 1$

$$\begin{aligned} &\|\mathbf{u}_\varepsilon(\cdot,\tau) - \mathbf{v}_\varepsilon(\cdot,\tau)\|_{H^1(\mathbb{R}^d)} \\ &\quad \leqslant \mathfrak{C}_2(s)(1+|\tau|)\varepsilon^s(\|\boldsymbol{\psi}\|_{H^{1+s}(\mathbb{R}^d)} + \|\mathbf{F}\|_{L_1((0,\tau);H^{1+s}(\mathbb{R}^d))}). \end{aligned}$$

Under the additional assumption that $\mathbf{F} \in L_1(\mathbb{R}_\pm; H^{1+s}(\mathbb{R}^d;\mathbb{C}^n))$, *where* $0 < s \leqslant 1$, *for* $\tau = \pm\varepsilon^{-\alpha}$, $0 < \varepsilon \leqslant 1$, $0 < \alpha < s$

$$\begin{aligned} &\|\mathbf{u}_\varepsilon(\cdot,\pm\varepsilon^{-\alpha}) - \mathbf{v}_\varepsilon(\cdot,\pm\varepsilon^{-\alpha})\|_{H^1(\mathbb{R}^d)} \\ &\quad \leqslant 2\mathfrak{C}_2(s)\varepsilon^{s-\alpha}(\|\boldsymbol{\psi}\|_{H^{1+s}(\mathbb{R}^d)} + \|\mathbf{F}\|_{L_1(\mathbb{R}_\pm;H^{1+s}(\mathbb{R}^d))}). \end{aligned}$$

By the Banach–Steinhaus theorem, this result implies the following theorem.

Theorem 10.5. *Let* $\mathbf{u}_\varepsilon$ *be the solution of problem* (10.1)*, and let* $\mathbf{u}_0$ *be the solution of the effective problem* (10.4)*.*

1°. *Let* $\boldsymbol{\psi} \in L_2(\mathbb{R}^d;\mathbb{C}^n)$ *and* $\mathbf{F} \in L_{1,\mathrm{loc}}(\mathbb{R}; L_2(\mathbb{R}^d;\mathbb{C}^n))$*. Then, for* $\tau \in \mathbb{R}$

$$\lim_{\varepsilon\to 0} \|\mathbf{u}_\varepsilon(\cdot,\tau) - \mathbf{u}_0(\cdot,\tau)\|_{L_2(\mathbb{R}^d)} = 0.$$

2°. *Let* $\boldsymbol{\psi} \in H^1(\mathbb{R}^d;\mathbb{C}^n)$ *and* $\mathbf{F} \in L_{1,\mathrm{loc}}(\mathbb{R}; H^1(\mathbb{R}^d;\mathbb{C}^n))$*. Let* $\mathbf{v}_\varepsilon$ *be given by* (10.6)*. Then, for* $\tau \in \mathbb{R}$

$$\lim_{\varepsilon\to 0} \|\mathbf{u}_\varepsilon(\cdot,\tau) - \mathbf{v}_\varepsilon(\cdot,\tau)\|_{H^1(\mathbb{R}^d)} = 0.$$

Applying Theorem 9.14, we make the following observation.

Remark 10.6. For $0 < \varepsilon \leqslant 1$, under Condition 9.11, the analogs of Theorems 10.1, 10.4, and 10.5 are valid with the operator Π_ε replaced by the identity operator.

10.2. Approximation of the flux. Let $\mathbf{p}_\varepsilon(\mathbf{x},\tau)$ be the "flux"

$$\mathbf{p}_\varepsilon(\mathbf{x},\tau) := g^\varepsilon(\mathbf{x}) b(\mathbf{D}) \mathbf{u}_\varepsilon(\mathbf{x},\tau). \tag{10.9}$$

Theorem 10.7. *Suppose that the assumptions of Theorem* 10.1(2°) *are satisfied. Let* $\mathbf{p}_\varepsilon$ *be the "flux"* (10.9)*, and let* $\tilde{g}(\mathbf{x})$ *be the matrix-valued function* (6.8)*. Then for* $\tau \in \mathbb{R}$ *and* $\varepsilon > 0$ *we have*

$$\begin{aligned} &\|\mathbf{p}_\varepsilon(\cdot,\tau) - \tilde{g}^\varepsilon b(\mathbf{D}) \Pi_\varepsilon \mathbf{u}_0(\cdot,\tau)\|_{L_2(\mathbb{R}^d)} \\ &\quad \leqslant C_{23}\varepsilon(1+|\tau|)(\|\boldsymbol{\psi}\|_{H^2(\mathbb{R}^d)} + \|\mathbf{F}\|_{L_1((0,\tau);H^2(\mathbb{R}^d))}). \end{aligned} \tag{10.10}$$

The constant C_{23} *depends only on* m*,* d*,* α_0*,* α_1*,* $\|g\|_{L_\infty}$*,* $\|g^{-1}\|_{L_\infty}$*,* $\|f\|_{L_\infty}$*,* $\|f^{-1}\|_{L_\infty}$*, and the parameters of the lattice* Γ*.*

Proof. From (9.3), (9.8), (10.3), and (10.5), it follows that

$$\begin{aligned} &\|\widehat{\mathcal{A}}_\varepsilon^{1/2}(\mathbf{u}_\varepsilon(\cdot,\tau) - (I + \varepsilon\Lambda^\varepsilon b(\mathbf{D})\Pi_\varepsilon)\mathbf{u}_0(\cdot,\tau))\|_{L_2(\mathbb{R}^d)} \\ &\quad \leqslant C_{13}\varepsilon(1+|\tau|)(\|\boldsymbol{\psi}\|_{H^2(\mathbb{R}^d)} + \|\mathbf{F}\|_{L_1((0,\tau);H^2(\mathbb{R}^d))}). \end{aligned} \tag{10.11}$$

By (9.2) and Proposition 9.3,

$$\|\widehat{\mathcal{A}}_\varepsilon^{1/2}(\Pi_\varepsilon - I)\mathbf{u}_0(\cdot,\tau)\|_{L_2(\mathbb{R}^d)} \leqslant \varepsilon\alpha_1^{1/2} r_0^{-1} \|g\|_{L_\infty}^{1/2} \|\mathbf{D}^2 \mathbf{u}_0(\cdot,\tau)\|_{L_2(\mathbb{R}^d)}. \tag{10.12}$$

Using (7.3), (10.5), and the inequality $|\sin x|/|x| \leqslant 1$, $x \in \mathbb{R}$, we obtain

$$\begin{aligned} &\|\mathbf{D}^2\mathbf{u}_0(\cdot,\tau)\|_{L_2(\mathbb{R}^d)} \\ &\quad\leqslant \|\mathbf{u}_0(\cdot,\tau)\|_{H^2(\mathbb{R}^d)} \\ &\quad\leqslant |\tau|\|f\|_{L_\infty}\|f^{-1}\|_{L_\infty}(\|\boldsymbol{\psi}\|_{H^2(\mathbb{R}^d)} + \|\mathbf{F}\|_{L_1((0,\tau);H^2(\mathbb{R}^d))}). \end{aligned} \tag{10.13}$$

Combining (10.9) and (10.11)–(10.13), we arrive at

$$\begin{aligned} &\|\mathbf{p}_\varepsilon(\cdot,\tau) - g^\varepsilon b(\mathbf{D})(I + \varepsilon\Lambda^\varepsilon b(\mathbf{D}))\Pi_\varepsilon\mathbf{u}_0(\cdot,\tau)\|_{L_2(\mathbb{R}^d)} \\ &\quad\leqslant C_{24}\varepsilon(1+|\tau|)(\|\boldsymbol{\psi}\|_{H^2(\mathbb{R}^d)} + \|\mathbf{F}\|_{L_1((0,\tau);H^2(\mathbb{R}^d))}), \end{aligned} \tag{10.14}$$

where

$$C_{24} := C_{13}\|g\|^{1/2}_{L_\infty} + \alpha_1^{1/2} r_0^{-1}\|g\|_{L_\infty}\|f\|_{L_\infty}\|f^{-1}\|_{L_\infty}.$$

We have

$$\begin{aligned} &\varepsilon g^\varepsilon b(\mathbf{D})\Lambda^\varepsilon b(\mathbf{D})\Pi_\varepsilon\mathbf{u}_0(\cdot,\tau) \\ &\quad= g^\varepsilon(b(\mathbf{D})\Lambda)^\varepsilon b(\mathbf{D})\Pi_\varepsilon\mathbf{u}_0(\cdot,\tau) + \varepsilon g^\varepsilon\sum_{l=1}^d b_l\Lambda^\varepsilon\Pi_\varepsilon^{(m)}D_l b(\mathbf{D})\mathbf{u}_0(\cdot,\tau). \end{aligned} \tag{10.15}$$

By (4.7), (4.8), (9.11), and (10.13),

$$\begin{aligned} &\Big\|\varepsilon g^\varepsilon\sum_{l=1}^d b_l\Lambda^\varepsilon\Pi_\varepsilon^{(m)}D_l b(\mathbf{D})\mathbf{u}_0(\cdot,\tau)\Big\|_{L_2(\mathbb{R}^d)} \\ &\quad\leqslant \varepsilon\|g\|_{L_\infty}\alpha_1 d^{1/2}M_1\|\mathbf{D}^2\mathbf{u}_0(\cdot,\tau)\|_{L_2(\mathbb{R}^d)} \\ &\quad\leqslant \varepsilon|\tau|\alpha_1 d^{1/2}M_1\|g\|_{L_\infty}\|f\|_{L_\infty}\|f^{-1}\|_{L_\infty}(\|\boldsymbol{\psi}\|_{H^2(\mathbb{R}^d)} + \|\mathbf{F}\|_{L_1((0,\tau);H^2(\mathbb{R}^d))}). \end{aligned} \tag{10.16}$$

Now, relations (6.8) and (10.14)–(10.16) imply estimate (10.10) with the constant

$$C_{23} := C_{24} + \alpha_1 d^{1/2}M_1\|g\|_{L_\infty}\|f\|_{L_\infty}\|f^{-1}\|_{L_\infty}. \qquad \square$$

Lemma 10.8. *For $\varepsilon > 0$ and $\tau \in \mathbb{R}$ we have*

$$\begin{aligned} &\|g^\varepsilon b(\mathbf{D})f^\varepsilon\mathcal{A}_\varepsilon^{-1/2}\sin(\tau\mathcal{A}_\varepsilon^{1/2})(f^\varepsilon)^{-1} \\ &\quad - \tilde{g}^\varepsilon b(\mathbf{D})\Pi_\varepsilon f_0(\mathcal{A}^0)^{-1/2}\sin(\tau(\mathcal{A}^0)^{1/2})f_0^{-1}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \\ &\leqslant C_{25}. \end{aligned} \tag{10.17}$$

Here

$$C_{25} := (\|g\|^{1/2}_{L_\infty} + \|g\|_{L_\infty}\|g^{-1}\|^{1/2}_{L_\infty}(m^{1/2}\|g\|^{1/2}_{L_\infty}\|g^{-1}\|^{1/2}_{L_\infty} + 1))\|f^{-1}\|_{L_\infty}.$$

Proof. By (9.1),

$$\|g^\varepsilon b(\mathbf{D}) f^\varepsilon \mathcal{A}_\varepsilon^{-1/2}\sin(\tau\mathcal{A}_\varepsilon^{1/2})(f^\varepsilon)^{-1}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant \|g\|_{L_\infty}^{1/2}\|f^{-1}\|_{L_\infty}. \tag{10.18}$$

Next, by (6.19), (7.3), and (7.5),

$$\begin{aligned}&\|\tilde{g}^\varepsilon \Pi_\varepsilon^{(m)} b(\mathbf{D}) f_0(\mathcal{A}^0)^{-1/2}\sin(\tau(\mathcal{A}^0)^{1/2}) f_0^{-1}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\\ &\quad\leqslant \|\tilde{g}^\varepsilon \Pi_\varepsilon^{(m)}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\|g^{-1}\|_{L_\infty}^{1/2}\|f^{-1}\|_{L_\infty}.\end{aligned} \tag{10.19}$$

Using Proposition 9.4 and (6.8) and (6.11), we obtain

$$\begin{aligned}\|\tilde{g}^\varepsilon \Pi_\varepsilon^{(m)}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} &\leqslant \|g\|_{L_\infty}(|\Omega|^{-1/2}\|b(\mathbf{D})\Lambda\|_{L_2(\Omega)}+1)\\ &\leqslant \|g\|_{L_\infty}(m^{1/2}\|g\|_{L_\infty}^{1/2}\|g^{-1}\|_{L_\infty}^{1/2}+1).\end{aligned} \tag{10.20}$$

Combining (10.18)–(10.20), we arrive at estimate (10.17). □

Theorem 10.9. 1°. *Let* $\mathbf{u}_\varepsilon$ *and* $\mathbf{u}_0$ *be the solutions of problems* (10.1) *and* (10.4), *respectively, for* $\boldsymbol{\psi}\in H^s(\mathbb{R}^d;\mathbb{C}^n)$ *and* $\mathbf{F}\in L_{1,\mathrm{loc}}(\mathbb{R};H^s(\mathbb{R}^d;\mathbb{C}^n))$, *where* $0\leqslant s\leqslant 2$. *Let* $\mathbf{p}_\varepsilon$ *be given by* (10.9) *and let* $\tilde{g}(\mathbf{x})$ *be the matrix-valued function* (6.8). *Then for* $\tau\in\mathbb{R}$ *and* $\varepsilon>0$ *we have*

$$\begin{aligned}&\|\mathbf{p}_\varepsilon(\cdot,\tau)-\tilde{g}^\varepsilon b(\mathbf{D})\Pi_\varepsilon\mathbf{u}_0(\cdot,\tau)\|_{L_2(\mathbb{R}^d)}\\ &\quad\leqslant \mathfrak{C}_4(s)(1+|\tau|)^{s/2}\varepsilon^{s/2}(\|\boldsymbol{\psi}\|_{H^s(\mathbb{R}^d)}+\|\mathbf{F}\|_{L_1((0,\tau);H^s(\mathbb{R}^d))}).\end{aligned} \tag{10.21}$$

Here

$$\mathfrak{C}_4(s):=C_{25}^{1-s/2}C_{23}^{s/2}.$$

Under the additional assumption that $\mathbf{F}\in L_1(\mathbb{R}_\pm;H^s(\mathbb{R}^d;\mathbb{C}^n))$, *where* $0\leqslant s\leqslant 2$, *for* $|\tau|=\varepsilon^{-\alpha}$, $0<\varepsilon\leqslant 1$, $0<\alpha<1$, *we have*

$$\begin{aligned}&\|\mathbf{p}_\varepsilon(\cdot,\pm\varepsilon^{-\alpha})-\tilde{g}^\varepsilon b(\mathbf{D})\Pi_\varepsilon\mathbf{u}_0(\cdot,\pm\varepsilon^{-\alpha})\|_{L_2(\mathbb{R}^d)}\\ &\quad\leqslant 2^{s/2}\mathfrak{C}_4(s)\varepsilon^{s(1-\alpha)/2}(\|\boldsymbol{\psi}\|_{H^s(\mathbb{R}^d)}+\|\mathbf{F}\|_{L_1(\mathbb{R}_\pm;H^s(\mathbb{R}^d))}).\end{aligned} \tag{10.22}$$

2°. *If* $\boldsymbol{\psi}\in L_2(\mathbb{R}^d;\mathbb{C}^n)$ *and* $\mathbf{F}\in L_{1,\mathrm{loc}}(\mathbb{R};L_2(\mathbb{R}^d;\mathbb{C}^n))$, *then*

$$\lim_{\varepsilon\to 0}\|\mathbf{p}_\varepsilon(\cdot,\tau)-\tilde{g}^\varepsilon b(\mathbf{D})\Pi_\varepsilon\mathbf{u}_0(\cdot,\tau)\|_{L_2(\mathbb{R}^d)}=0,\quad \tau\in\mathbb{R}.$$

3°. *If* $\boldsymbol{\psi}\in L_2(\mathbb{R}^d;\mathbb{C}^n)$ *and* $\mathbf{F}\in L_1(\mathbb{R}_\pm;L_2(\mathbb{R}^d;\mathbb{C}^n))$, *then*

$$\lim_{\varepsilon\to 0}\|\mathbf{p}_\varepsilon(\cdot,\pm\varepsilon^{-\alpha})-\tilde{g}^\varepsilon b(\mathbf{D})\Pi_\varepsilon\mathbf{u}_0(\cdot,\pm\varepsilon^{-\alpha})\|_{L_2(\mathbb{R}^d)}=0,\quad 0<\varepsilon\leqslant 1,\ 0<\alpha<1.$$

Proof. 1°. Rewriting estimate (10.10) with $\mathbf{F} = 0$ in operator terms and interpolating with estimate (10.17), we conclude that

$$\begin{aligned}&\|g^\varepsilon b(\mathbf{D}) f^\varepsilon \mathcal{A}_\varepsilon^{-1/2} \sin(\tau \mathcal{A}_\varepsilon^{1/2})(f^\varepsilon)^{-1} \\ &\qquad - \tilde{g}^\varepsilon b(\mathbf{D})\Pi_\varepsilon f_0(\mathcal{A}^0)^{-1/2} \sin(\tau(\mathcal{A}^0)^{1/2}) f_0^{-1}\|_{H^s(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \\ &\quad\leqslant C_{25}^{1-s/2} C_{23}^{s/2}(1+|\tau|)^{s/2}\varepsilon^{s/2}.\end{aligned}$$

Thus, by (10.3) and (10.5), we derive estimate (10.21).

2°. The assertion follows from (10.21) by the Banach–Steinhaus theorem.

3°. The assertion is a consequence of (10.22) and the Banach–Steinhaus theorem. □

10.3. On the possibility to remove Π_ε from approximation of the flux

Theorem 10.10. *Under the assumptions of Theorem 10.7, let $d \leqslant 4$. Then for $\tau \in \mathbb{R}$ and $0 < \varepsilon \leqslant 1$*

$$\begin{aligned}&\|\mathbf{p}_\varepsilon(\cdot,\tau) - \tilde{g}^\varepsilon b(\mathbf{D})\mathbf{u}_0(\cdot,\tau)\|_{L_2(\mathbb{R}^d)} \\ &\quad\leqslant C_{26}\varepsilon(1+|\tau|)(\|\boldsymbol{\psi}\|_{H^2(\mathbb{R}^d)} + \|\mathbf{F}\|_{L_1((0,\tau);H^2(\mathbb{R}^d))}).\end{aligned} \tag{10.23}$$

The constant C_{26} depends only on m, n, d, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|f\|_{L_\infty}$, $\|f^{-1}\|_{L_\infty}$, and the parameters of the lattice Γ.

Proof. The proof repeats the proof of Theorem 10.7 with some simplifications. By (9.21), (10.3), and (10.5),

$$\begin{aligned}&\|\widehat{\mathcal{A}}_\varepsilon^{1/2}(\mathbf{u}_\varepsilon(\cdot,\tau) - (I + \varepsilon\Lambda^\varepsilon b(\mathbf{D}))\mathbf{u}_0(\cdot,\tau))\|_{L_2(\mathbb{R}^d)} \\ &\quad\leqslant C_{14}\varepsilon(1+|\tau|)(\|\boldsymbol{\psi}\|_{H^2(\mathbb{R}^d)} + \|\mathbf{F}\|_{L_1((0,\tau);H^2(\mathbb{R}^d))}).\end{aligned} \tag{10.24}$$

Then, according to (9.2) and (10.9),

$$\begin{aligned}&\|\mathbf{p}_\varepsilon(\cdot,\tau) - g^\varepsilon b(\mathbf{D})(I + \varepsilon\Lambda^\varepsilon b(\mathbf{D}))\mathbf{u}_0(\cdot,\tau)\|_{L_2(\mathbb{R}^d)} \\ &\quad\leqslant \|g\|_{L_\infty}^{1/2} C_{14}\varepsilon(1+|\tau|)(\|\boldsymbol{\psi}\|_{H^2(\mathbb{R}^d)} + \|\mathbf{F}\|_{L_1((0,\tau);H^2(\mathbb{R}^d))}).\end{aligned} \tag{10.25}$$

Similarly to (10.15),

$$\begin{aligned}&\varepsilon g^\varepsilon b(\mathbf{D})\Lambda^\varepsilon b(\mathbf{D})\mathbf{u}_0(\cdot,\tau) \\ &\quad= g^\varepsilon(b(\mathbf{D})\Lambda)^\varepsilon b(\mathbf{D})\mathbf{u}_0(\cdot,\tau) + \varepsilon g^\varepsilon \sum_{l=1}^d b_l \Lambda^\varepsilon D_l b(\mathbf{D})\mathbf{u}_0(\cdot,\tau).\end{aligned} \tag{10.26}$$

Let us estimate the second summand in the right-hand side. By (4.8),

$$\begin{aligned}&\Big\|\varepsilon g^\varepsilon \sum_{l=1}^d b_l \Lambda^\varepsilon D_l b(\mathbf{D})\mathbf{u}_0(\cdot,\tau)\Big\|_{L_2(\mathbb{R}^d)}\\&\quad\leqslant \varepsilon \|g\|_{L_\infty}(d\alpha_1)^{1/2}\|\Lambda^\varepsilon \mathbf{D} b(\mathbf{D})\mathbf{u}_0(\cdot,\tau)\|_{L_2(\mathbb{R}^d)}\\&\quad\leqslant \varepsilon \|g\|_{L_\infty}(d\alpha_1)^{1/2}\|[\Lambda]\|_{H^1(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\|\mathbf{D} b(\mathbf{D})\mathbf{u}_0(\cdot,\tau)\|_{H^1(\mathbb{R}^d)},\end{aligned} \tag{10.27}$$

for $0<\varepsilon\leqslant 1$. By (6.19), (7.3), (7.5), and (10.5),

$$\|\mathbf{D} b(\mathbf{D})\mathbf{u}_0(\cdot,\tau)\|_{H^1(\mathbb{R}^d)} \leqslant \|g^{-1}\|^{1/2}_{L_\infty}\|f^{-1}\|_{L_\infty}(\|\boldsymbol{\psi}\|_{H^2(\mathbb{R}^d)}+\|\mathbf{F}\|_{L_1((0,\tau);H^2(\mathbb{R}^d))}). \tag{10.28}$$

Combining (9.24), (10.27), and (10.28), we have

$$\begin{aligned}&\Big\|\varepsilon g^\varepsilon \sum_{l=1}^d b_l \Lambda^\varepsilon D_l b(\mathbf{D})\mathbf{u}_0(\cdot,\tau)\Big\|_{L_2(\mathbb{R}^d)}\\&\leqslant \varepsilon \|g\|_{L_\infty}(d\alpha_1)^{1/2}\mathfrak{C}_d\|g^{-1}\|^{1/2}_{L_\infty}\|f^{-1}\|_{L_\infty}(\|\boldsymbol{\psi}\|_{H^2(\mathbb{R}^d)}+\|\mathbf{F}\|_{L_1((0,\tau);H^2(\mathbb{R}^d))}),\end{aligned} \tag{10.29}$$

for $d\leqslant 4$. Now relations (6.8), (10.25), (10.26), and (10.29) imply estimate (10.23) with the constant

$$C_{26} := C_{14}\|g\|^{1/2}_{L_\infty} + (d\alpha_1)^{1/2}\mathfrak{C}_d\|g\|_{L_\infty}\|g^{-1}\|^{1/2}_{L_\infty}\|f^{-1}\|_{L_\infty}. \qquad \square$$

Let $d\geqslant 5$ and let Condition 8.4 be satisfied. Then, by the scaling transformation, the analog of (9.21) (with the constant C_{15} instead of C_{14}) follows from (8.24). We wish to remove Π_ε from approximation for the flux similarly to (10.24)–(10.29). According to [34, Subsection 1.6, Proposition 1], Condition 8.4 implies the boundedness of $[\Lambda]$ as an operator from $H^1(\mathbb{R}^d;\mathbb{C}^m)$ to $L_2(\mathbb{R}^d;\mathbb{C}^n)$ with the estimate $\|[\Lambda]\|_{H^1(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\leqslant C\|[\Lambda]\|_{H^2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)}$.

The following statement can be checked by analogy with the proof of Theorem 10.10.

Theorem 10.11. *Let $d\geqslant 5$. Let Condition 8.4 be satisfied. Then, under the assumptions of Theorem 10.7, for $0<\varepsilon\leqslant 1$ and $\tau\in\mathbb{R}$*

$$\begin{aligned}&\|\mathbf{p}_\varepsilon(\cdot,\tau)-\tilde g^\varepsilon b(\mathbf{D})\mathbf{u}_0(\cdot,\tau)\|_{L_2(\mathbb{R}^d)}\\&\quad\leqslant C_{27}\varepsilon(1+|\tau|)(\|\boldsymbol{\psi}\|_{H^2(\mathbb{R}^d)}+\|\mathbf{F}\|_{L_1((0,\tau);H^2(\mathbb{R}^d))}).\end{aligned}$$

The constant

$$C_{27} := C_{15}\|g\|^{1/2}_{L_\infty} + (d\alpha_1)^{1/2}\|g\|_{L_\infty}\|g^{-1}\|^{1/2}_{L_\infty}\|f^{-1}\|_{L_\infty}\|[\Lambda]\|_{H^1(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}$$

depends only on d, m, n, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|f\|_{L_\infty}$, $\|f^{-1}\|_{L_\infty}$, the parameters of the lattice Γ, and the norm $\|[\Lambda]\|_{H^2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)}$.

By analogy with (10.24)–(10.29), using Proposition 9.7, from Theorem 9.10 we derive the following result.

Theorem 10.12. *Let $d \geqslant 5$. Let $\mathbf{u}_\varepsilon$ and $\mathbf{u}_0$ be the solutions of problems* (10.1) *and* (10.4)*, respectively, where we suppose $\boldsymbol{\psi} \in H^{d/2}(\mathbb{R}^d;\mathbb{C}^n)$ and $\mathbf{F} \in L_1((0,\tau); H^{d/2}(\mathbb{R}^d;\mathbb{C}^n))$. Let $\mathbf{p}_\varepsilon$ be defined by* (10.9) *and let $\tilde{g}$ be the matrix-valued function* (6.8)*. Then, for $0 < \varepsilon \leqslant 1$ and $\tau \in \mathbb{R}$*

$$\|\mathbf{p}_\varepsilon(\cdot,\tau) - \tilde{g}^\varepsilon b(\mathbf{D})\mathbf{u}_0(\cdot,\tau)\|_{L_2(\mathbb{R}^d)} \leqslant C_{28}\varepsilon(1+|\tau|)(\|\boldsymbol{\psi}\|_{H^{d/2}(\mathbb{R}^d)} + \|\mathbf{F}\|_{L_1((0,\tau);H^{d/2}(\mathbb{R}^d))}).$$

The constant C_{28} depends only on d, m, n, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|f\|_{L_\infty}$, $\|f^{-1}\|_{L_\infty}$, and the parameters of the lattice Γ.

To obtain interpolational results without any smoothing operator, we need to prove the analog of Lemma 10.8 without Π_ε. I.e., we want to prove the $(L_2 \to L_2)$-boundedness of the operator

$$\tilde{g}^\varepsilon b(\mathbf{D}) f_0 (\mathcal{A}^0)^{-1/2} \sin(\tau(\mathcal{A}^0)^{1/2}) f_0^{-1}. \tag{10.30}$$

The following property of $\tilde{g}$ was obtained in [44, Proposition 9.6]. (The one-dimensional case will be considered in Subsection 10.4 below.)

Proposition 10.13. *Let $l > 1$ for $d = 2$, and $l = d/2$ for $d \geqslant 3$. The operator $[\tilde{g}]$ is a continuous mapping of $H^l(\mathbb{R}^d;\mathbb{C}^m)$ to $L_2(\mathbb{R}^d;\mathbb{C}^m)$, and*

$$\|[\tilde{g}]\|_{H^l(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant \mathfrak{C}'_d.$$

The constant $\mathfrak{C}'_d$ depends only d, m, n, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and the parameters of the lattice Γ; for $d = 2$ it depends also on l.

So, for $d \geqslant 2$, we can not expect the $(L_2 \to L_2)$-boundedness of the operator (10.30). The $(H^2 \to L_2)$-continuity of the operator (10.30) was used in Theorem 10.10 and, under Condition 8.4, in Theorem 10.11. (The $(H^2 \to L_2)$-boundedness of $[\tilde{g}]$ follows from [34, Subsection 1.3.2, Lemma 1].) So, without any additional conditions on Λ, using Proposition 10.13, we can obtain some interpolational results only for $d \leqslant 3$.

By (6.19), (7.3), (7.5), and Proposition 10.13,

$$\|\tilde{g}^\varepsilon b(\mathbf{D}) f_0 (\mathcal{A}^0)^{-1/2} \sin(\tau(\mathcal{A}^0)^{1/2}) f_0^{-1}\|_{H^l(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant \mathfrak{C}'_d \|g^{-1}\|^{1/2}_{L_\infty} \|f^{-1}\|_{L_\infty}. \tag{10.31}$$

(Here l is as in Proposition 10.13.)

Combining (10.18) and (10.31) and interpolating with (10.23), we obtain the following result.

Theorem 10.14. *Let* $2 \leqslant d \leqslant 3$*, and let* $1 < l < 2$ *for* $d = 2$ *and* $l = 3/2$ *for* $d = 3$*. Let* $0 \leqslant s \leqslant 1$*. Assume that* $\theta = l + (2-l)s$ *for* $d = 2$ *and* $\theta = 3/2 + s/2$ *for* $d = 3$*. Let* $\mathbf{u}_\varepsilon$ *and* $\mathbf{u}_0$ *be the solutions of problems* (10.1) *and* (10.4)*, respectively, where* $\boldsymbol{\psi} \in H^\theta(\mathbb{R}^d;\mathbb{C}^n)$ *and* $\mathbf{F} \in L_1((0,\tau);H^\theta(\mathbb{R}^d;\mathbb{C}^n))$*. Let* $\mathbf{p}_\varepsilon$ *be the flux* (10.9) *and let* $\tilde{g}$ *be the matrix-valued function* (6.8)*. Then for* $0 < \varepsilon \leqslant 1$ *and* $\tau \in \mathbb{R}$ *we have*

$$\|\mathbf{p}_\varepsilon(\cdot,\tau) - \tilde{g}^\varepsilon b(\mathbf{D})\mathbf{u}_0(\cdot,\tau)\|_{L_2(\mathbb{R}^d)} \leqslant \mathfrak{C}_5(s)\varepsilon^s(1+|\tau|)^s(\|\boldsymbol{\psi}\|_{H^\theta(\mathbb{R}^d)} + \|\mathbf{F}\|_{L_1((0,\tau);H^\theta(\mathbb{R}^d))}).$$

Here

$$\mathfrak{C}_5(s) := C_{27}^s(\|g\|_{L_\infty}^{1/2} + \mathfrak{C}'_d\|g^{-1}\|_{L_\infty}^{1/2})^{1-s}\|f^{-1}\|_{L_\infty}^{1-s}.$$

10.4. The special case. Suppose that $g^0 = \underline{g}$, i.e., relations (6.21) hold. For $d = 1$, identity $g^0 = \underline{g}$ is always true, see, e.g., [50, Chapter I, §2]. In accordance with [11, Remark 3.5], in this case the matrix-valued function (6.8) is constant and coincides with g^0, i.e., $\tilde{g}(\mathbf{x}) = g^0 = \underline{g}$. The following statement is a consequence of Theorem 10.9(1°.)

Proposition 10.15. *Assume that relations* (6.21) *hold. Let* $\mathbf{u}_\varepsilon$ *and* $\mathbf{u}_0$ *be the solutions of problems* (10.1) *and* (10.4)*, respectively, for* $\boldsymbol{\psi} \in H^s(\mathbb{R}^d;\mathbb{C}^n)$ *and* $\mathbf{F} \in L_{1,\mathrm{loc}}(\mathbb{R};H^s(\mathbb{R}^d;\mathbb{C}^n))$*, where* $0 \leqslant s \leqslant 2$*. Let* $\mathbf{p}_\varepsilon$ *be given by* (10.9)*. Then for* $\tau \in \mathbb{R}$ *and* $\varepsilon > 0$*,*

$$\|\mathbf{p}_\varepsilon(\cdot,\tau) - g^0 b(\mathbf{D})\mathbf{u}_0(\cdot,\tau)\|_{L_2(\mathbb{R}^d)} \leqslant \mathfrak{C}_6(s)(1+|\tau|)^{s/2}\varepsilon^{s/2}(\|\boldsymbol{\psi}\|_{H^s(\mathbb{R}^d)} + \|\mathbf{F}\|_{L_1((0,\tau);H^s(\mathbb{R}^d))}). \tag{10.32}$$

Here

$$\mathfrak{C}_6(s) := \mathfrak{C}_4(s) + 2^{1-s/2}r_0^{-s/2}\|g\|_{L_\infty}^{1/2}\|f^{-1}\|_{L_\infty}.$$

Proof. We wish to remove the operator Π_ε from the approximation (10.21). Obviously, $\|\Pi_\varepsilon - I\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant 2$. According to Proposition 9.3,

$$\|\Pi_\varepsilon - I\|_{H^2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant \|\Pi_\varepsilon - I\|_{H^1(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant \varepsilon r_0^{-1}.$$

Then, by interpolation, $\|\Pi_\varepsilon - I\|_{H^s(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant 2^{1-s/2}r_0^{-s/2}\varepsilon^{s/2}$, $0 \leqslant s \leqslant 2$. Combining this with (6.19), (7.3), (7.5), (10.5), and taking into account that the operator $\mathcal{A}^0$ with constant coefficients commutes with the smoothing operator Π_ε, we obtain

$$\|g^0 b(\mathbf{D})(\Pi_\varepsilon - I)\mathbf{u}_0(\cdot,\tau)\|_{L_2(\mathbb{R}^d)} \leqslant 2^{1-s/2}r_0^{-s/2}\|g\|_{L_\infty}^{1/2}\|f^{-1}\|_{L_\infty}\varepsilon^{s/2}(\|\boldsymbol{\psi}\|_{H^s(\mathbb{R}^d)} + \|\mathbf{F}\|_{L_1((0,\tau);H^s(\mathbb{R}^d))}). \tag{10.33}$$

Now, from identity $g^0 = \tilde{g}$, (10.21), and (10.33) we derive estimate (10.32). $\square$

11. Applications of the general results

The following examples were previously considered in [9, 13, 25, 26].

11.1. The acoustics equation. In $L_2(\mathbb{R}^d)$, we consider the operator

$$\widehat{\mathcal{A}} = \mathbf{D}^* g(\mathbf{x})\mathbf{D} = -\operatorname{div} g(\mathbf{x})\nabla, \tag{11.1}$$

where $g(\mathbf{x})$ is a periodic symmetric matrix with real entries. Assume that $g(\mathbf{x}) > 0$, $g, g^{-1} \in L_\infty$. The operator $\widehat{\mathcal{A}}$ describes a periodic acoustical medium. The operator (11.1) is a particular case of the operator (6.1). Now we have $n = 1$, $m = d$, $b(\mathbf{D}) = \mathbf{D}$, $\alpha_0 = \alpha_1 = 1$. Consider the operator $\widehat{\mathcal{A}}_\varepsilon = \mathbf{D}^* g^\varepsilon(\mathbf{x})\mathbf{D}$, whose coefficients oscillate rapidly for small ε.

Let us write down the effective operator. In the case under consideration, the Γ-periodic solution of problem (6.7) is a row:

$$\Lambda(\mathbf{x}) = i\Phi(\mathbf{x}), \quad \Phi(\mathbf{x}) = (\Phi_1(\mathbf{x}), \dots, \Phi_d(\mathbf{x})),$$

where $\Phi_j \in \widetilde{H}^1(\Omega)$ is the solution of the problem

$$\operatorname{div} g(\mathbf{x})(\nabla\Phi_j(\mathbf{x}) + \mathbf{e}_j) = 0, \quad \int_\Omega \Phi_j(\mathbf{x})\, d\mathbf{x} = 0.$$

Here $\mathbf{e}_j$, $j = 1, \dots, d$, is the standard orthonormal basis in $\mathbb{R}^d$. Clearly, the functions $\Phi_j(\mathbf{x})$ are real-valued, and the entries of $\Lambda(\mathbf{x})$ are purely imaginary. By (6.8), the columns of the $(d \times d)$-matrix-valued function $\tilde{g}(\mathbf{x})$ are the vector-valued functions $g(\mathbf{x})(\nabla\Phi_j(\mathbf{x}) + \mathbf{e}_j)$, $j = 1, \dots, d$. The effective matrix is defined according to (6.9):

$$g^0 = |\Omega|^{-1} \int_\Omega \tilde{g}(\mathbf{x})\, d\mathbf{x}.$$

Clearly, $\tilde{g}(\mathbf{x})$ and g^0 have real entries. If $d = 1$, then $m = n = 1$, whence $g^0 = \underline{g}$.

Let $Q(\mathbf{x})$ be a Γ-periodic function on $\mathbb{R}^d$ such that $Q(\mathbf{x}) > 0$, $Q, Q^{-1} \in L_\infty$. The function $Q(\mathbf{x})$ describes the density of the medium.

Consider the Cauchy problem for the acoustics equation in the medium with rapidly oscillating characteristics:

$$\begin{cases} Q^\varepsilon(\mathbf{x})\dfrac{\partial^2 u_\varepsilon(\mathbf{x}, \tau)}{\partial\tau^2} = -\operatorname{div} g^\varepsilon(\mathbf{x})\nabla u_\varepsilon(\mathbf{x}, \tau), \quad \mathbf{x} \in \mathbb{R}^d,\ \tau \in \mathbb{R}, \\ u_\varepsilon(\mathbf{x}, 0) = 0, \quad \dfrac{\partial u_\varepsilon(\mathbf{x}, 0)}{\partial\tau} = \psi(\mathbf{x}), \end{cases} \tag{11.2}$$

where $\psi \in L_2(\mathbb{R}^d)$ is a given function. (For simplicity, we consider the homogeneous equation.) Then the homogenized problem takes the form

$$\begin{cases} \overline{Q}\dfrac{\partial^2 u_0(\mathbf{x},\tau)}{\partial \tau^2} = -\operatorname{div} g^0 \nabla u_0(\mathbf{x},\tau), \quad \mathbf{x} \in \mathbb{R}^d,\ \tau \in \mathbb{R}, \\ u_0(\mathbf{x},0) = 0, \quad \dfrac{\partial u_0(\mathbf{x},0)}{\partial \tau} = \psi(\mathbf{x}). \end{cases} \tag{11.3}$$

According to [33, Chapter III, Theorem 13.1], $\Lambda \in L_\infty$ and the norm $\|\Lambda\|_{L_\infty}$ does not exceed a constant depending on d, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and Ω. Applying Theorems 10.4 and 10.9(1°) and taking into account Remark 10.6, we arrive at the following result.

Proposition 11.1. *Under the assumptions of Subsection 11.1, let u_ε be the solution of problem (11.2) and let u_0 be the solution of the effective problem (11.3).*

1°. *Let $\psi \in H^s(\mathbb{R}^d)$ for some $0 \leqslant s \leqslant 1$. Then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have*

$$\|u_\varepsilon(\cdot,\tau) - u_0(\cdot,\tau)\|_{L_2(\mathbb{R}^d)} \leqslant \mathfrak{C}_6(s)(1+|\tau|)\varepsilon^s \|\psi\|_{H^s(\mathbb{R}^d)}.$$

2°. *Let $\psi \in H^{s+1}(\mathbb{R}^d)$ for some $0 \leqslant s \leqslant 1$. Then for $\tau \in \mathbb{R}$ and $0 < \varepsilon \leqslant 1$ we have*

$$\|u_\varepsilon(\cdot,\tau) - u_0(\cdot,\tau) - \varepsilon \Phi^\varepsilon \nabla u_0(\cdot,\tau)\|_{H^1(\mathbb{R}^d)} \leqslant \mathfrak{C}_7(s)(1+|\tau|)\varepsilon^s \|\psi\|_{H^{1+s}(\mathbb{R}^d)}.$$

3°. *Let $\psi \in H^s(\mathbb{R}^d)$ for some $0 \leqslant s \leqslant 2$. Let Π_ε be defined by (9.6). Then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have*

$$\|g^\varepsilon \nabla u_\varepsilon(\cdot,\tau) - \tilde{g}^\varepsilon \Pi_\varepsilon \nabla u_0(\cdot,\tau)\|_{L_2(\mathbb{R}^d)} \leqslant \mathfrak{C}_8(s)(1+|\tau|)^{s/2}\varepsilon^{s/2} \|\psi\|_{H^s(\mathbb{R}^d)}.$$

The constants $\mathfrak{C}_6(s)$, $\mathfrak{C}_7(s)$, and $\mathfrak{C}_8(s)$ depend only on s, d, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and parameters of the lattice Γ.

11.2. The operator of elasticity theory. Let $d \geqslant 2$. We represent the operator of elasticity theory in the form used in [9, Chapter 5, §2]. Let ζ be an orthogonal second rank tensor in $\mathbb{R}^d$; in the standard orthonormal basis in $\mathbb{R}^d$, it can be represented by a matrix $\zeta = \{\zeta_{jl}\}_{j,l=1}^d$. We shall consider symmetric tensors ζ, which will be identified with vectors $\zeta_* \in \mathbb{C}^m$, $2m = d(d+1)$, by the following rule. The vector ζ_* is formed by all components ζ_{jl}, $j \leqslant l$, and the pairs (j,l) are put in order in some fixed way. Let χ be an $(m \times m)$-matrix, $\chi = \operatorname{diag}\{\chi_{(j,l)}\}$, where $\chi_{(j,l)} = 1$ for $j = l$ and $\chi_{(j,l)} = 2$ for $j < l$. Then $|\boldsymbol{\zeta}|^2 = \langle \chi\zeta_*, \zeta_* \rangle_{\mathbb{C}^m}$.

Let $\mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^d)$ be the *displacement vector*. Then the *deformation tensor* is has the entries $e_{jl}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_l} + \frac{\partial u_l}{\partial x_j}\right)$. The corresponding vector is denoted by

$e_*(\mathbf{u})$. The relation $b(\mathbf{D})\mathbf{u} = -ie_*(\mathbf{u})$ determines an $(m\times d)$-matrix homogeneous DO $b(\mathbf{D})$ uniquely; the symbol of this DO is a matrix with real entries. For instance, with an appropriate ordering, we have

$$b(\boldsymbol{\xi}) = \begin{pmatrix} \xi_1 & 0 \\ \frac{\xi_2}{2} & \frac{\xi_1}{2} \\ 0 & \xi_2 \end{pmatrix}, \qquad d = 2;$$

$$b(\boldsymbol{\xi}) = \begin{pmatrix} \xi_1 & 0 & 0 \\ \frac{\xi_2}{2} & \frac{\xi_1}{2} & 0 \\ 0 & \xi_2 & 0 \\ 0 & \frac{\xi_3}{2} & \frac{\xi_2}{2} \\ 0 & 0 & \xi_3 \\ \frac{\xi_3}{2} & 0 & \frac{\xi_1}{2} \end{pmatrix}, \quad d = 3.$$

Let $\sigma(\mathbf{u})$ be the *stress tensor*, and let $\sigma_*(\mathbf{u})$ be the corresponding vector. The Hooke law can be represented by the relation $\sigma_*(\mathbf{u}) = g(\mathbf{x})e_*(\mathbf{u})$, where $g(\mathbf{x})$ is an $(m \times m)$ matrix (which gives a "concise" description of the Hooke tensor). This matrix characterizes the parameters of the elastic (in general, anisotropic) medium. We assume that $g(\mathbf{x})$ is Γ-periodic and such that $g(\mathbf{x}) > 0$, and $g, g^{-1} \in L_\infty$.

The energy of elastic deformations is given by the quadratic form

$$\begin{aligned} \mathfrak{w}[\mathbf{u},\mathbf{u}] &= \frac{1}{2}\int_{\mathbb{R}^d} \langle \sigma_*(\mathbf{u}), e_*(\mathbf{u})\rangle_{\mathbb{C}^m}\, d\mathbf{x} \\ &= \frac{1}{2}\int_{\mathbb{R}^d} \langle g(\mathbf{x})b(\mathbf{D})\mathbf{u}, b(\mathbf{D})\mathbf{u}\rangle_{\mathbb{C}^m}\, d\mathbf{x}, \end{aligned} \tag{11.4}$$

for $\mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^d)$. The operator $\mathcal{W}$ generated by this form is the operator of elasticity theory. Thus, the operator $2\mathcal{W} = b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D}) = \widehat{\mathcal{A}}$ is of the form (6.1) with $n = d$ and $m = d(d+1)/2$.

In the case of an *isotropic* medium, the expression for the form (11.4) simplifies significantly and depends only on two functional *Lamé parameters* $\lambda(\mathbf{x})$, $\mu(\mathbf{x})$:

$$\mathfrak{w}[\mathbf{u},\mathbf{u}] = \int_{\mathbb{R}^d} \Big(\mu(\mathbf{x})|e(\mathbf{u})|^2 + \frac{\lambda(\mathbf{x})}{2}|\operatorname{div}\mathbf{u}|^2\Big)\, d\mathbf{x}.$$

The parameter μ is the *shear modulus*. The modulus $\lambda(\mathbf{x})$ may be negative. Often, another parameter $\kappa(\mathbf{x}) = \lambda(\mathbf{x}) + 2\mu(\mathbf{x})/d$ is introduced instead of $\lambda(\mathbf{x})$; κ is called the *modulus of volume compression*. In the isotropic case, the conditions that ensure the positive definiteness of the matrix $g(\mathbf{x})$ are $\mu(\mathbf{x}) \geqslant \mu_0 > 0$, $\kappa(\mathbf{x}) \geqslant \kappa_0 > 0$. We write down the "isotropic" matrices g for $d = 2$ and $d = 3$:

$$g = \begin{pmatrix} \kappa+\mu & 0 & \kappa-\mu \\ 0 & 4\mu & 0 \\ \kappa-\mu & 0 & \kappa+\mu \end{pmatrix}, \qquad d = 2;$$

$$g = \frac{1}{3}\begin{pmatrix} 3\kappa+4\mu & 0 & 3\kappa-2\mu & 0 & 3\kappa-2\mu & 0 \\ 0 & 12\mu & 0 & 0 & 0 & 0 \\ 3\kappa-2\mu & 0 & 3\kappa+4\mu & 0 & 3\kappa-2\mu & 0 \\ 0 & 0 & 0 & 12\mu & 0 & 0 \\ 3\kappa-2\mu & 0 & 3\kappa-2\mu & 0 & 3\kappa+4\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 12\mu \end{pmatrix}, \quad d = 3.$$

Consider the operator $\mathcal{W}_\varepsilon = \frac{1}{2}\widehat{\mathcal{A}}_\varepsilon$ with rapidly oscillating coefficients. The effective matrix g^0 and the effective operator $\mathcal{W}^0 = \frac{1}{2}\widehat{\mathcal{A}}^0$ are defined by the general rules (see (6.8), (6.9), and (6.15)).

Let $Q(\mathbf{x})$ be a Γ-periodic $(d \times d)$-matrix-valued function such that $Q(\mathbf{x}) > 0$, $Q, Q^{-1} \in L_\infty$. Usually, $Q(\mathbf{x})$ is a scalar-valued function describing the density of the medium. We assume that $Q(\mathbf{x})$ is a matrix-valued function in order to take possible anisotropy into account.

Consider the following Cauchy problem for the system of elasticity theory:

$$\begin{cases} Q^\varepsilon(\mathbf{x})\dfrac{\partial^2 \mathbf{u}_\varepsilon(\mathbf{x},\tau)}{\partial \tau^2} = -\mathcal{W}_\varepsilon \mathbf{u}_\varepsilon(\mathbf{x},\tau), \quad \mathbf{x} \in \mathbb{R}^d,\ \tau \in \mathbb{R}, \\ \mathbf{u}_\varepsilon(\mathbf{x},0) = 0, \quad \dfrac{\partial \mathbf{u}_\varepsilon(\mathbf{x},0)}{\partial \tau} = \boldsymbol{\psi}(\mathbf{x}), \end{cases} \tag{11.5}$$

where $\boldsymbol{\psi} \in L_2(\mathbb{R}^d;\mathbb{C}^d)$ is a given function. The homogenized problem takes the form

$$\begin{cases} \overline{Q}\dfrac{\partial^2 \mathbf{u}_0(\mathbf{x},\tau)}{\partial \tau^2} = -\mathcal{W}^0 \mathbf{u}_0(\mathbf{x},\tau), \quad \mathbf{x} \in \mathbb{R}^d,\ \tau \in \mathbb{R}, \\ \mathbf{u}_0(\mathbf{x},0) = 0, \quad \dfrac{\partial \mathbf{u}_0(\mathbf{x},0)}{\partial \tau} = \boldsymbol{\psi}(\mathbf{x}). \end{cases}$$

Theorems 10.4 and 10.9 can be applied to problem (11.5). If $d = 2$, then Condition 9.11 is satisfied according to Proposition 9.13. So, we can use Theorem 9.14. If $d = 3$, then Theorem 9.8 is applicable.

11.3. The model equation of electrodynamics. We cannot include the general Maxwell operator in the scheme developed above; we have to assume that the magnetic permeability is unit. In $L_2(\mathbb{R}^3;\mathbb{C}^3)$, we consider the model operator $\mathcal{L}$ formally given by the expression $\mathcal{L} = \operatorname{curl}\eta(\mathbf{x})^{-1}\operatorname{curl} - \nabla\nu(\mathbf{x})\operatorname{div}$. Here the *dielectric permittivity* $\eta(\mathbf{x})$ is Γ-periodic (3×3)-matrix-valued function in $\mathbb{R}^3$ with real entries such that $\eta(\mathbf{x}) > 0$; $\eta, \eta^{-1} \in L_\infty$; $\nu(\mathbf{x})$ is real-valued Γ-periodic function in $\mathbb{R}^3$ such that $\nu(\mathbf{x}) > 0$; $\nu, \nu^{-1} \in L_\infty$. The precise definition of $\mathcal{L}$ is given via the closed positive form

$$\int_{\mathbb{R}^3} \left(\langle \eta(\mathbf{x})^{-1}\operatorname{curl}\mathbf{u}, \operatorname{curl}\mathbf{u}\rangle + \nu(\mathbf{x})|\operatorname{div}\mathbf{u}|^2\right) d\mathbf{x}, \quad \mathbf{u} \in H^1(\mathbb{R}^3;\mathbb{C}^3).$$

The operator $\mathcal{L}$ can be written as $\widehat{\mathcal{A}} = b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D})$ with $n = 3$, $m = 4$, and

$$b(\mathbf{D}) = \begin{pmatrix} -i\operatorname{curl} \\ -i\operatorname{div} \end{pmatrix}, \quad g(\mathbf{x}) = \begin{pmatrix} \eta(\mathbf{x})^{-1} & 0 \\ 0 & \nu(\mathbf{x}) \end{pmatrix}. \tag{11.6}$$

The corresponding symbol of $b(\mathbf{D})$ is

$$b(\boldsymbol{\xi}) = \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \\ \xi_1 & \xi_2 & \xi_3 \end{pmatrix}.$$

According to [9, §7.2] the effective matrix has the form

$$g^0 = \begin{pmatrix} (\eta^0)^{-1} & 0 \\ 0 & \underline{\nu} \end{pmatrix},$$

where η^0 is the effective matrix for the scalar elliptic operator $-\operatorname{div}\eta\nabla = \mathbf{D}^*\eta\mathbf{D}$. The effective operator is given by

$$\mathcal{L}^0 = \operatorname{curl}(\eta^0)^{-1}\operatorname{curl} - \nabla\underline{\nu}\operatorname{div}.$$

Let $\mathbf{v}_j \in \widetilde{H}^1(\Omega;\mathbb{C}^3)$ be the Γ-periodic solution of the problem

$$b(\mathbf{D})^* g(\mathbf{x})(b(\mathbf{D})\mathbf{v}_j(\mathbf{x}) + \mathbf{e}_j) = 0, \quad \int_\Omega \mathbf{v}_j(\mathbf{x})\, d\mathbf{x} = 0,$$

$j = 1,2,3,4$. Here $\mathbf{e}_j$, $j = 1,2,3,4$, is the standard orthonormal basis in $\mathbb{C}^4$. As was shown in [11, §14], the solutions $\mathbf{v}_j$, $j = 1,2,3$, can be determined as follows. Let $\widetilde{\Phi}_j(\mathbf{x})$ be the Γ-periodic solution of the problem

$$\operatorname{div}\eta(\mathbf{x})(\nabla\widetilde{\Phi}_j(\mathbf{x}) + \mathbf{c}_j) = 0, \quad \int_\Omega \widetilde{\Phi}_j(\mathbf{x})\, d\mathbf{x} = 0,$$

with $j = 1,2,3$, where $\mathbf{c}_j = (\eta^0)^{-1}\tilde{\mathbf{e}}_j$, and $\tilde{\mathbf{e}}_j$, $j = 1,2,3$, is the standard orthonormal basis in $\mathbb{C}^3$. Let $\mathbf{q}_j$ be the Γ-periodic solution of the problem

$$\Delta \mathbf{q}_j = \eta(\nabla\widetilde{\Phi}_j + \mathbf{c}_j) - \tilde{\mathbf{e}}_j, \quad \int\limits_{\Omega} \mathbf{q}_j(\mathbf{x})\, d\mathbf{x} = 0.$$

Then $\mathbf{v}_j = i \operatorname{curl} \mathbf{q}_j$, $j = 1,2,3$.

Next, we have $\mathbf{v}_4 = i\nabla\phi$, where ϕ is the Γ-periodic solution of the problem

$$\Delta\phi = \underline{\nu}(\nu(\mathbf{x}))^{-1} - 1, \quad \int\limits_{\Omega} \phi(\mathbf{x})\, d\mathbf{x} = 0.$$

The matrix $\Lambda(\mathbf{x})$ is the (3×4)-matrix with the columns $i \operatorname{curl} \mathbf{q}_1$, $i \operatorname{curl} \mathbf{q}_2$, $i \operatorname{curl} \mathbf{q}_3$, $i\nabla\phi$. By $\Psi(\mathbf{x})$ we denote the (3×3)-matrix-valued function with the columns $\operatorname{curl} \mathbf{q}_1$, $\operatorname{curl} \mathbf{q}_2$, $\operatorname{curl} \mathbf{q}_3$ (then $\Psi(\mathbf{x})$ has real entries). We put $\mathbf{w} = \nabla\phi$. Then $\Lambda(\mathbf{x})b(\mathbf{D}) = \Psi(\mathbf{x}) \operatorname{curl} + \mathbf{w}(\mathbf{x}) \operatorname{div}$.

The application of Theorems 9.1 and 9.8 gives the following result.

Theorem 11.2. *Under the assumptions of Subsection 11.3, denote*

$$\mathcal{L}_\varepsilon := \operatorname{curl}(\eta^\varepsilon)^{-1} \operatorname{curl} - \nabla\nu^\varepsilon \operatorname{div}.$$

Then, for $\tau \in \mathbb{R}$

$$\begin{aligned}\|\mathcal{L}_\varepsilon^{-1/2} \sin(\tau\mathcal{L}_\varepsilon^{1/2}) - (\mathcal{L}^0)^{-1/2} \sin(\tau(\mathcal{L}^0)^{1/2})\|_{H^1(\mathbb{R}^3)\to L_2(\mathbb{R}^3)} \\ \leqslant C_{12}\varepsilon(1+|\tau|),\end{aligned} \tag{11.7}$$

with $\varepsilon > 0$, and

$$\begin{aligned}&\|\mathcal{L}_\varepsilon^{-1/2} \sin(\tau\mathcal{L}_\varepsilon^{1/2}) \\ &\quad - (I + \varepsilon\Psi^\varepsilon \operatorname{curl} + \varepsilon\mathbf{w}^\varepsilon \operatorname{div})(\mathcal{L}^0)^{-1/2} \sin(\tau(\mathcal{L}^0)^{1/2})\|_{H^2(\mathbb{R}^3)\to H^1(\mathbb{R}^3)} \\ &\leqslant C_{19}\varepsilon(1+|\tau|),\end{aligned} \tag{11.8}$$

with $0 < \varepsilon \leqslant 1$. The constants C_{12} and C_{19} depend only on $\|\eta\|_{L_\infty}$, $\|\eta^{-1}\|_{L_\infty}$, $\|\nu\|_{L_\infty}$, $\|\nu^{-1}\|_{L_\infty}$, and the parameters of the lattice Γ.

Also, we can apply (interpolational) Theorems 9.2 and 9.6. But in this case the correction term contains the smoothing operator Π_ε (see (9.6)). We omit the details.

It turns out that the operators $\mathcal{L}_\varepsilon$ and $\mathcal{L}^0$ split in the Weyl decomposition $L_2(\mathbb{R}^3;\mathbb{C}^3) = J \oplus G$ simultaneously. Here the "solenoidal" subspace J consists of vector functions $\mathbf{u} \in L_2(\mathbb{R}^3;\mathbb{C}^3)$ for which $\operatorname{div}\mathbf{u} = 0$ (in the sense of distributions) and the "potential" subspace is $G := \{\mathbf{u} = \nabla\phi : \phi \in H^1_{\mathrm{loc}}(\mathbb{R}^3), \nabla\phi \in L_2(\mathbb{R}^3;\mathbb{C}^3)\}$. The Weyl decomposition reduces the operators $\mathcal{L}_\varepsilon$ and $\mathcal{L}^0$, i.e., $\mathcal{L}_\varepsilon = \mathcal{L}_{\varepsilon,J} \oplus \mathcal{L}_{\varepsilon,G}$ and $\mathcal{L}^0 = \mathcal{L}^0_J \oplus \mathcal{L}^0_G$. The part $\mathcal{L}_{\varepsilon,J}$ acting in the "solenoidal" subspace J is formally defined by the differential expression $\operatorname{curl}\eta^\varepsilon(\mathbf{x})^{-1}\operatorname{curl}$, while the part $\mathcal{L}_{\varepsilon,G}$ acting in the "potential" subspace G corresponds to the expression $-\nabla\nu^\varepsilon(\mathbf{x})\nabla$. The parts $\mathcal{L}^0_J$ and $\mathcal{L}^0_G$ can be written in the same way. The Weyl decomposition allows us to apply Theorem 11.2 to homogenization of the Cauchy problem for the model hyperbolic equation appearing in electrodynamics:

$$\begin{cases} \partial_\tau^2 \mathbf{u}_\varepsilon = -\operatorname{curl}\eta^\varepsilon(\mathbf{x})^{-1}\operatorname{curl}\mathbf{u}_\varepsilon, \quad \operatorname{div}\mathbf{u}_\varepsilon = 0, \\ \mathbf{u}_\varepsilon(\mathbf{x},0) = 0, \quad \partial_\tau \mathbf{u}_\varepsilon(\mathbf{x},0) = \boldsymbol{\psi}(\mathbf{x}). \end{cases} \tag{11.9}$$

The effective problem takes the form

$$\begin{cases} \partial_\tau^2 \mathbf{u}_0 = -\operatorname{curl}(\eta^0)^{-1}\operatorname{curl}\mathbf{u}_0, \quad \operatorname{div}\mathbf{u}_0 = 0, \\ \mathbf{u}_0(\mathbf{x},0) = 0, \quad \partial_\tau \mathbf{u}_0(\mathbf{x},0) = \boldsymbol{\psi}(\mathbf{x}). \end{cases} \tag{11.10}$$

Let $\mathcal{P}$ be the orthogonal projection of $L_2(\mathbb{R}^3;\mathbb{C}^3)$ onto J. Then (see [9, Subsection 2.4 of Chapter 7]) the operator $\mathcal{P}$ (restricted to $H^s(\mathbb{R}^3;\mathbb{C}^3)$) is also the orthogonal projection of the space $H^s(\mathbb{R}^3;\mathbb{C}^3)$ onto the subspace $J \cap H^s(\mathbb{R}^3;\mathbb{C}^3)$ for all $s > 0$.

Restricting the operators under the norm sign in (11.7) and (11.8) to the subspaces $J \cap H^1(\mathbb{R}^3;\mathbb{C}^3)$ and $J \cap H^2(\mathbb{R}^3;\mathbb{C}^3)$, respectively, and multiplying by $\mathcal{P}$ from the left, we see that Theorem 11.2 implies the following result.

Theorem 11.3. *Under the assumptions of Subsection 11.3, let $\mathbf{u}_\varepsilon$ and $\mathbf{u}_0$ be the solutions of problems (11.9) and (11.10), respectively.*

1°. *Let $\boldsymbol{\psi} \in J \cap H^1(\mathbb{R}^3;\mathbb{C}^3)$. Then for $\varepsilon > 0$ and $\tau \in \mathbb{R}$ we have*

$$\|\mathbf{u}_\varepsilon(\cdot,\tau) - \mathbf{u}_0(\cdot,\tau)\|_{L_2(\mathbb{R}^3)} \leqslant C_{12}\varepsilon(1+|\tau|)\|\boldsymbol{\psi}\|_{H^1(\mathbb{R}^3)}.$$

2°. *Let $\boldsymbol{\psi} \in J \cap H^2(\mathbb{R}^3;\mathbb{C}^3)$. Then for $0 < \varepsilon \leqslant 1$ and $\tau \in \mathbb{R}$ we have*

$$\|\mathbf{u}_\varepsilon(\cdot,\tau) - \mathbf{u}_0(\cdot,\tau) - \varepsilon\Psi^\varepsilon\operatorname{curl}\mathbf{u}_0(\cdot,\tau)\|_{H^1(\mathbb{R}^3)} \leqslant C_{19}\varepsilon(1+|\tau|)\|\boldsymbol{\psi}\|_{H^2(\mathbb{R}^3)}.$$

According to (11.6), the role of the flux for problem (11.9) is played by the vector-valued function

$$\mathbf{p}_\varepsilon = g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon = -i\begin{pmatrix} (\eta^\varepsilon)^{-1}\operatorname{curl}\mathbf{u}_\varepsilon \\ \nu^\varepsilon\operatorname{div}\mathbf{u}_\varepsilon \end{pmatrix} = -i\begin{pmatrix} (\eta^\varepsilon)^{-1}\operatorname{curl}\mathbf{u}_\varepsilon \\ 0 \end{pmatrix}.$$

To approximate the flux, we apply Theorem 10.10. The matrix $\tilde{g} = g(\mathbf{1} + b(\mathbf{D})\Lambda)$ has a block-diagonal structure, see [11, Subsection 14.3]): the upper left (3×3) block is represented by the matrix with the columns $\nabla\widetilde{\Phi}_j(\mathbf{x}) + \mathbf{c}_j$, $j = 1, 2, 3$. We denote this block by $a(\mathbf{x})$. The element at the right lower corner is equal to $\underline{\nu}$. The other elements are zero. Then, by (11.6) and (11.10),

$$\tilde{g}^\varepsilon b(\mathbf{D})\mathbf{u}_0 = -i \begin{pmatrix} a^\varepsilon \operatorname{curl} \mathbf{u}_0 \\ 0 \end{pmatrix}.$$

We arrive at the following statement.

Theorem 11.4. *Under the assumptions of Theorem 11.3, let* $\boldsymbol{\psi} \in J \cap H^2(\mathbb{R}^3; \mathbb{C}^3)$. *Then for* $0 < \varepsilon \leqslant 1$ *and* $\tau \in \mathbb{R}$ *we have*

$$\|(\eta^\varepsilon)^{-1} \operatorname{curl} \mathbf{u}_\varepsilon(\cdot, \tau) - a^\varepsilon \operatorname{curl} \mathbf{u}_0(\cdot, \tau)\|_{L_2(\mathbb{R}^3)} \leqslant C_{26}\varepsilon(1 + |\tau|)\|\boldsymbol{\psi}\|_{H^2(\mathbb{R}^3)}.$$

The constant C_{26} *depends only on* $\|\eta\|_{L_\infty}$, $\|\eta^{-1}\|_{L_\infty}$, $\|\nu\|_{L_\infty}$, $\|\nu^{-1}\|_{L_\infty}$, *and the parameters of the lattice* Γ.

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