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A universality law for sign correlations of eigenfunctions of differential operators

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Abstract. We establish a sign correlation universality law for sequences of functions $\{w_n\}_{n\in\mathbb{N}}$ satisfying a trigonometric asymptotic law. Our results are inspired by the classical WKB asymptotic approximation for Sturm–Liouville operators, and in particular we obtain non-trivial sign correlations for eigenfunctions of generic Schrödinger operators (including the harmonic oscillator), as well as Laguerre and Chebyshev polynomials. Given two distinct points $x, y \in \mathbb{R}$, we ask how often do $w_n(x)$ and $w_n(y)$ have the same sign. Asymptotically, one would expect this to be true half the time, but this turns out to not always be the case. Under certain natural assumptions, we prove that, for all $x \neq y$,

$$\frac{1}{3} \leq \lim_{N\to\infty} \frac{1}{N} \#\{0 \leq n < N : \operatorname{sgn}(w_n(x)) = \operatorname{sgn}(w_n(y))\} \leq \frac{2}{3},$$

and that these bounds are optimal, and can be attained. Our methods extend to other problems of similar flavor and we also discuss a number of open problems.

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1. Introduction

1.1. Setup. This paper is motivated by a simple and surprising property exhibited by the sequence of eigenfunctions for the eigenvalue problem of Sturm–Liouville differential operators. Consider, on the real line, the Schrödinger oper-

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ator associated to the potential V,

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x). \tag{1}$$

Here, $V\colon \mathbb{R} \to \mathbb{R}$ is some function satisfying $V(x) \to \infty$, as $|x| \to \infty$.

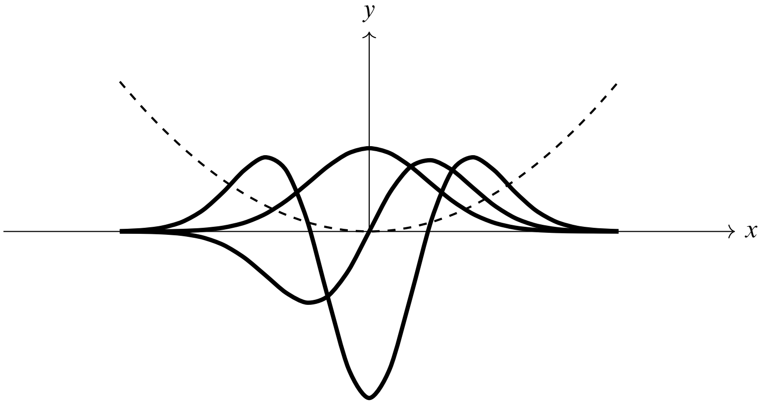

Figure 1. The potential $V(x) = x^2$ (dashed) and the first three eigenfunctions of the quantum harmonic oscillator.

The eigenvalue problem

$$H(w_n) = \lambda_n w_n \tag{2}$$

has been studied extensively, the simpler case $V(x) = x^2$ corresponding to the quantum harmonic oscillator whose eigenfunctions are given by the Hermite functions (see Figure 1). It is well understood that, as the eigenvalues become large, the second derivative dominates, and the eigenfunctions start to look locally like trigonometric functions. This phenomenon gives rise to the *WKB approximation*, named after Wentzel, Kramers, and Brillouin. The purpose of our paper is to establish a rather surprising universality statement for sign correlations of sequences of functions for which a kind of WKB approximation holds.

Our starting point is very simple to state: given two distinct points $x, y \in \mathbb{R}$, how often do $w_n(x)$ and $w_n(y)$ have the same sign? More precisely, we are interested in the *sign correlation limit*, defined as

$$\ell_{\{w_n\}}(x, y) := \lim_{N\to\infty} \frac{1}{N} \#\{0 \le n < N\colon \operatorname{sgn}(w_n(x)) = \operatorname{sgn}(w_n(y))\}. \tag{3}$$

One could be tempted to conjecture that, in the high frequency limit, the two points x, y decouple and the corresponding signs behave essentially like independent

Bernoulli random variables, thus exhibiting the same sign in roughly half of the cases. This seemingly natural conjecture turns out to be a good guess for the generic behavior of the system. However, earlier work of the authors [8] hinted at the possible existence of an exceptional set exhibiting a different kind of behavior, and motivated the present paper.

1.2. Main result. We state our first main result in general terms, but modelled by the WKB asymptotic trigonometric law. In §1.3 we apply our methods in order to obtain sign correlations for eigenfunctions of Schrödinger operators.

A sequence $\{a_n\} \subseteq [0,1]$ is said to be *equidistributed in* $[0,1]$ if, for any subinterval $[c,d] \subseteq [0,1]$,

$$\lim_{N\to\infty} \frac{1}{N} \#\{0 \le n < N \colon a_n \in [c,d]\} = d - c.$$

A sequence $\{a_n\} \subseteq \mathbb{R}$ is said to be *equidistributed modulo 1* if the sequence of the fractional parts $\{a_n - \lfloor a_n \rfloor\}$ is equidistributed in $[0,1]$. Our first main result applies to a sequence of functions obeying a certain asymptotic behavior which is inspired by the WKB approximation, and is satisfied by several classical objects (see the examples in §4). Regarding notation, $o_n(1)$ will denote a quantity that tends to 0, as $n \to \infty$. We will also write $a_n = O(b_n)$, or $|a_n| \lesssim |b_n|$, if there exists a constant $C < \infty$ (independent of n) such that $|a_n| \le C|b_n|$, for every n.

Theorem 1 (main result). *Given $D \subseteq \mathbb{R}$, let $w_n \colon D \to \mathbb{R}$ be a sequence of functions satisfying*

$$w_n(x) = (1 + o_n(1))\phi(x,n) \cos(2\pi(\mu_n \varphi(x) - \theta)), \tag{4}$$

for every $x \in D$ and some $\{\mu_n\} \subset \mathbb{R}$, $\theta \in \mathbb{R}$, and function $\phi \colon D \times \mathbb{N} \to \mathbb{R}$. Consider distinct points $x, y \in D$ such that $\varphi(x) \neq \pm\varphi(y)$ and $\phi(x,n)\phi(y,n) > 0$ for all n. If the sequences $\{p^{-1}\mu_n\varphi(x)\}$ and $\{q^{-1}\mu_n\varphi(y)\}$ are equidistributed modulo 1 *for any $p, q \in \mathbb{Z} \setminus \{0\}$, then the sign correlation limit* (3) *exists, and satisfies*

$$\frac{1}{3} \le \ell_{\{w_n\}}(x,y) \le \frac{2}{3}. \tag{5}$$

Moreover, these constants are optimal.

We believe this result to be rather surprising. In particular, it establishes the existence of correlations different from $\frac{1}{2}$. These correlations are, however, universally bounded away from both 0 and 1. Theorem 1 motivates a number of natural questions, see §1.4 below.

1.3. Sharper asymptotics. The sign correlation limit can be computed *exactly* in a number of situations of interest. We proceed to describe one such situation. Let $V \in L^1_{\text{loc}}(\mathbb{R})$ be a locally integrable potential such that $V(x) \to \infty$, as $|x| \to \infty$, and assume V to be bounded from below,

$$\operatorname{essinf}_{x\in\mathbb{R}} V(x) > -\infty. \tag{6}$$

We renormalize the Hamiltonian by $H_V = -\frac{1}{4\pi^2}\frac{d^2}{dx^2} + V(x)$ (adapted to our choice of normalization of the Fourier transform, see §2 below). Under these conditions, the operator H_V given by (1) is known to have compact resolvent. In particular, H_V has purely discrete spectrum and a complete set of eigenfunctions, see [13, Theorem XIII.67]. This means that there exists an orthogonal basis $\{w_n\}$ of $L^2(\mathbb{R})$ such that $H_V(w_n) = \lambda_n w_n$, where the eigenvalues $\{\lambda_n\}$ form a non-decreasing sequence satisfying $\lambda_n \to \infty$, as $n \to \infty$. In addition, we require V to be an even function. This implies that the basis $\{w_n\}$ naturally splits into even and odd functions, since the corresponding subspaces are H_V-invariant. In particular, we can reorder the basis elements in such a way that w_n is an even function if n is even, and an odd function if n is odd. After doing so, the sequence $\{\lambda_n\}$ may no longer be non-decreasing, however we still have that $\lambda_n \to \infty$, as $n \to \infty$. By uniqueness of solutions to the eigenvalue problem (2), we may further impose $\operatorname{sgn}(w_{2n}(0)) = \operatorname{sgn}(w'_{2n+1}(0)) = (-1)^n$. Here and in the rest of the paper a prime denotes differentiation with respect to the variable x. We will also require both subsequences $\{\sqrt{\lambda_{2n}}x\}$ and $\{\sqrt{\lambda_{2n+1}}x\}$ to be equidistributed modulo 1, for every $x \neq 0$. Whether this should generically be the case is discussed in Problem (3) from §1.4 below. We are now ready to state our second main result.

Theorem 2 (sharper asymptotics). *Let $V \in L^1_{\text{loc}}(\mathbb{R})$ be an even potential, bounded from below in the sense of (6), and such that $V(x) \to \infty$, as $|x| \to \infty$. For each $n \in \mathbb{N}$, assume that for the associated eigenvalue problem $H_V(w_n) = \lambda_n w_n$, the following assertions hold*:

(H1) *the function w_n is even if n is even, and odd if n is odd*;

(H2) *the sequences $\{\sqrt{\lambda_{2n}}x\}$ and $\{\sqrt{\lambda_{2n+1}}x\}$ are equidistributed modulo 1, for any $x \in \mathbb{R} \setminus \{0\}$; and*

(H3) $\operatorname{sgn}(w_{2n}(0)) = \operatorname{sgn}(w'_{2n+1}(0)) = (-1)^n$.

Then the asymptotic

$$w_n(x) = (1 + o_n(1))\Big(w_n(0)^2 + \frac{w'_n(0)^2}{4\pi^2\lambda_n}\Big)^{1/2} \cos\Big(2\pi\Big(\sqrt{\lambda_n}x - \frac{n}{4}\Big)\Big) \tag{7}$$

holds uniformly on compact subsets of the real line. If x, y are distinct real numbers such that $\frac{x}{y} = \frac{p}{q}$ for some nonzero coprime integers p, q, then the sign correlation limit (3) *is given by*

$$\ell_{\{w_n\}}(x, y) = \begin{cases} \frac{1}{2} + \frac{1}{2pq} & \textit{if } p \equiv q \equiv 1 \mod 4, \textit{ or } p \equiv q \equiv 3 \mod 4, \\ \frac{1}{2} & \textit{otherwise.} \end{cases} \tag{8}$$

If $\frac{x}{y}$ is irrational, then $\ell_{\{w_n\}}(x, y) = \frac{1}{2}$.

The asymptotic (7) is exactly the one given via WKB approximation. The quadratic case $V(x) = x^2$, where the WKB approximation coincides with the classical asymptotic for Hermite polynomials, falls under the scope of Theorem 2 and is described in more detail in §4 (together with higher dimensional extensions, provided by the Laguerre polynomials).

1.4. Further remarks and open problems. (1) Can Theorems 1 and 2 be extended to sign correlations of three or more points? What can be said about the density with which a specific sign configuration, say $(+, -, +, -, -)$, can occur? Some of these may be universally bounded away from 0 and 1, while others may not be. In principle, our approach provides a framework for obtaining such bounds since each such question is reduced to a finite computation. However, the increase in complexity is substantial, which is why we have not been able to further explore this question. We believe it to be a promising avenue for future research.

(2) Is it possible to characterize the class of potentials V such that our result applies to eigenfunctions of the Schrödinger operator H_V? The WKB approximation seems to be a valuable tool, however, it is not clear to us whether a suitable theory on the equidistribution of the eigenvalues of differential operators exists. On the other hand, the asymptotic growth of $\{\lambda_n\}$, as $n \to \infty$, has been studied extensively (see [7]). Bohr's asymptotic formula

$$\#\{\lambda_n \leq \lambda\} \sim \frac{1}{\pi} \int_0^\infty \sqrt{(\lambda - V(x))_+}\, \mathrm{d}x,$$

gives some information about the possible equidistribution of $\{\sqrt{\lambda_n}\}$, but this question seems more subtle.

(3) As we shall see, these questions are connected to classical problems on the asymptotic behavior of geodesics on the d-dimensional torus $\mathbb{T}^d$. It is natural to expect that several of the new developments regarding strong forms of linear flow

rigidity [1, 2, 3, 4, 9, 15] can be used to make more precise statements in some special cases. We also note that at least for some classical families of orthogonal polynomials it should be possible to obtain more precise quantitative information – see §4 below for further details.

2. Useful lemmata

We start with a general result that will serve as a first step towards computing the sign correlation limit of a sequence of functions over a fixed finite set of points $\boldsymbol{a} = (a_1, \ldots, a_d) \in \mathbb{R}^d$.

Lemma 1. *Given* $\boldsymbol{a} \in \mathbb{R}^d$*, assume that* $\lambda \boldsymbol{a} \in \mathbb{Z}^d$*, for some* $\lambda > 0$*. Let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous* 1*-periodic function. Let* $\{\mu_n\} \subset \mathbb{R}$ *be a sequence such that* $\{\frac{\mu_n}{\lambda}\}$ *is equidistributed modulo* 1*. Let* $\boldsymbol{s} \in \{-1, 1\}^d$*. Then*

$$\lim_{N\to\infty} \frac{1}{N} \#\{0 \le n < N : (\operatorname{sgn}[f(\mu_n a_1)], \ldots, \operatorname{sgn}[f(\mu_n a_d)]) = \pm \boldsymbol{s}\} = \int_0^1 \Psi(\lambda t \boldsymbol{a}) \, \mathrm{d}t,$$

where the function $\Psi \colon \mathbb{R}^d \to \{0, 1\}$ *is defined as follows: given* $\boldsymbol{u} \in \mathbb{R}^d$*, then* $\Psi(\boldsymbol{u}) = 1$ *if* $(\operatorname{sgn}[f(u_1)], \ldots, \operatorname{sgn}[f(u_d)]) = \pm \boldsymbol{s}$*, and* $\Psi(\boldsymbol{u}) = 0$ *otherwise.*

Proof. Consider the function $g(t) := \Psi(\lambda t \boldsymbol{a})$, which satisfies $g(t + 1) = g(t)$, for every $t \in \mathbb{R}$. By construction, we have that

$$\begin{aligned} &\{0 \le n < N : (\operatorname{sgn}[f(\mu_n a_1)], \ldots, \operatorname{sgn}[f(\mu_n a_d)]) = \pm \boldsymbol{s}\} \\ &\quad = \Big\{0 \le n < N : g\Big(\frac{\mu_n}{\lambda}\Big) = 1\Big\}. \end{aligned}$$

Since the function g is 1-periodic and the sequence $\{\frac{\mu_n}{\lambda}\}$ is equidistributed modulo 1, we have that, as $N \to \infty$,

$$\begin{aligned} &\frac{1}{N} \#\{0 \le n < N : (\operatorname{sgn}[f(\mu_n a_1)], \ldots, \operatorname{sgn}[f(\mu_n a_d)]) = \pm \boldsymbol{s}\} \\ &\quad \longrightarrow |\{t \in [0, 1] : g(t) = 1\}| = \int_0^1 g(t) \, \mathrm{d}t. \end{aligned}$$

The last identity follows from the fact that the function g takes values in $\{0, 1\}$. This concludes the proof of the lemma. □

Only the case $d = 2$ of Lemma 1 will be relevant to our applications. For the remainder of the section, we will discuss integrals of the function

$$\Phi(x, y) := \operatorname{sgn}(\cos(2\pi x)\cos(2\pi y)) \tag{9}$$

over rays of the two-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, which will play a key role in the proof of our main theorems. We remark that the Haar measure on $\mathbb{T}^2$ coincides with the Lebesgue measure on the fundamental domain $[0, 1]^2$. We further note that, given a ray $\gamma\colon \mathbb{R} \to \mathbb{T}^2$ defined by $\gamma(t) = (At-\alpha, Bt-\beta)$ for some $A, B \neq 0$, then

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \Phi(\gamma(t))\,\mathrm{d}t = \lim_{T\to\infty} \frac{1}{T}\int_0^T \Phi(\tilde{\gamma}(t))\,\mathrm{d}t, \tag{10}$$

where $\tilde{\gamma}\colon \mathbb{R} \to \mathbb{T}^2$ is in turn given by $\tilde{\gamma}(t) = (t, at + b)$, with $a = B/A$ and $b = (B/A)\alpha - \beta$. The following lemma is well known, with suitable modifications and vast generalizations appearing in [5, 6, 12]. For the sake of completeness, we provide a short proof.

Lemma 2. *Given $a, b \in \mathbb{R}$, let $\gamma(t) = (t, at + b)$ be the corresponding line in $\mathbb{R}^2$. Then the limit*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \Phi(\gamma(t))\,\mathrm{d}t \tag{11}$$

exists. Moreover, if the limit is nonzero, then the coefficient a is a rational number.

Proof. Since the function Φ is 1-periodic in the variables x and y, the problem reduces to a standard question in equidistribution theory on the 2-dimensional torus $\mathbb{T}^2$. If a is irrational, then the line $t \mapsto (t, at + b)$ is densely wound and equidistributes over $\mathbb{T}^2$, and the averaged integral in (11) converges to the average value of Φ,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \Phi(\gamma(t))\,\mathrm{d}t = \int_{\mathbb{T}^2} \Phi(x, y)\,\mathrm{d}x\,\mathrm{d}y = 0,$$

see [6, §2.3]. If a is rational, then the line $t \mapsto (t, at + b)$ gives rise to a closed geodesic on $\mathbb{T}^2$, and the existence of the limit (11) follows from periodicity. □

The next lemma further analyzes the case of rational slope $a = p/q \in \mathbb{Q}$. It is of quantitative flavor, and relies on the explicit form of the function Φ. We achieve this by resorting to Fourier series, and will normalize the Fourier coefficients of an integrable function $f\colon [0,1] \to \mathbb{C}$ in the following way:

$$\hat{f}(n) = \int_0^1 f(x) e^{-2\pi i n x} \, \mathrm{d}x.$$

Lemma 3. *Let $A, B \in \mathbb{R}$ be nonzero real numbers, such that $A/B = p/q$ for some coprime $p, q \in \mathbb{Z}$. Let $\alpha, \beta \in \mathbb{R}$ and let $\gamma(t) = (At - \alpha, Bt - \beta)$ be the corresponding ray on $\mathbb{T}^2$. If either p or q are even, then*

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \Phi(\gamma(t)) \, \mathrm{d}t = 0.$$

If both p and q are odd, then

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \Phi(\gamma(t)) \, \mathrm{d}t = (-1)^{\frac{p+q}{2}+1} \frac{8}{\pi^2 pq} \sum_{\ell=0}^{\infty} \frac{\cos(2\pi(2\ell+1)(p\beta - q\alpha))}{(2\ell+1)^2}.$$

In particular, in this case, we have that

$$\left| \lim_{T \to \infty} \frac{1}{T} \int_0^T \Phi(\gamma(t)) \ \mathrm{d}t \right| \le \frac{1}{|pq|} \quad (p, q \text{ odd})$$

where equality is attained if and only if $p\beta - q\alpha$ is an integer.

Proof. By periodicity, recall (10), we have that

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \Phi(\gamma(t)) \ \mathrm{d}t = \int_0^1 \Phi(pt - \alpha, qt - \beta) \, \mathrm{d}t.$$

Expanding the function Φ in Fourier series,

$$\Phi(x, y) = \frac{4}{\pi^2} \sum_{\substack{n,m \in \mathbb{Z} \\ m,n \neq 0}} \frac{\sin(\frac{\pi n}{2}) \sin(\frac{\pi m}{2})}{mn} e^{2\pi i (mx + ny)},$$

we obtain that

$$\begin{aligned}
&\int_0^1 \Phi(pt-\alpha, qt-\beta)\,\mathrm{d}t \\
&\quad = \frac{4}{\pi^2} \sum_{\substack{n,m\in\mathbb{Z}\\ m,n\neq 0}} \frac{\sin(\frac{\pi n}{2})\sin(\frac{\pi m}{2})}{mn} \int_0^1 e^{2\pi i(mp+nq)t} e^{-2\pi i(m\alpha+n\beta)}\,\mathrm{d}t \\
&\quad = \frac{8}{\pi^2 pq} \sum_{k=1}^{\infty} \frac{\sin(\frac{\pi kp}{2})\sin(\frac{\pi kq}{2})}{k^2} \cos(2\pi k(p\beta-q\alpha)).
\end{aligned}$$

This quantity vanishes if either p or q are even. On the other hand, if both p and q are odd, then

$$\int_0^1 \Phi(pt-\alpha, qt-\beta)\,\mathrm{d}t = (-1)^{\frac{p+q}{2}+1} \frac{8}{\pi^2 pq} \sum_{\ell=0}^{\infty} \frac{\cos(2\pi(2\ell+1)(p\beta-q\alpha))}{(2\ell+1)^2}.$$

Since $\sum_{\ell=0}^{\infty} \frac{1}{(2\ell+1)^2} = \frac{\pi^2}{8}$, the triangle inequality implies

$$\left| \int_0^1 \Phi(pt-\alpha, qt-\beta)\,\mathrm{d}t \right| \le \frac{1}{|pq|},$$

where equality is attained if and only if $p\beta - q\alpha \in \mathbb{Z}$. □

3. Proofs of Theorems 1 and 2

Proof of Theorem 1. Let $x \neq y \in D$ be given, satisfying $\varphi(x) \neq \pm\varphi(y)$. No generality is lost in assuming that $\varphi(x)/\varphi(y) = p/q$ for some coprime $p, q \in \mathbb{Z}$, and that $p/\varphi(x) = q/\varphi(y) > 0$, for otherwise Lemma 2 would imply that $\ell_{\{w_n\}}(x, y) = \frac{1}{2}$, and there is nothing to prove. Now, since the function (4) satisfies $\phi(x, n)\phi(y, n) > 0$ for all n, we have that

$$\ell_{\{w_n\}}(x, y) = \ell_{\{u_n\}}(x, y), \tag{12}$$

where $u_n(x) := \cos(2\pi(\mu_n\varphi(x) - \theta))$. We focus on the latter limit, and prepare to apply Lemma 1 with $\boldsymbol{a} = (\varphi(x), \varphi(y))$, $\boldsymbol{s} = (1, 1)$, $\lambda = p/\varphi(x) = q/\varphi(y)$, and $f(z) = \cos(2\pi(z - \theta))$. Note that our equidistribution assumption implies that the sequence $\{\frac{\mu_n}{\lambda}\} = \{p^{-1}\mu_n\varphi(x)\}$ is equidistributed modulo 1, and so all the hypotheses of Lemma 1 are satisfied. The conclusion is that

$$\ell_{\{u_n\}}(x, y) = \int_0^1 \Psi(pt - \theta, qt - \theta)\,\mathrm{d}t, \tag{13}$$

where the function Ψ is related to Φ from (9) via $\Phi = 2\Psi - 1$. It then follows from (12) and (13) that

$$\ell_{\{w_n\}}(x, y) = \frac{1}{2} + \frac{1}{2}\int_0^1 \Phi(pt - \theta, qt - \theta)\,\mathrm{d}t. \tag{14}$$

The latter integral was computed in the course of the proof of Lemma 3, and is non-zero only if both p, q are odd. In that case, applying Lemma 3 with $A = \varphi(x)$, $B = \varphi(y)$, and $\alpha = \beta = \theta$, yields

$$\left|\int_0^1 \Phi(pt - \theta, qt - \theta)\,\mathrm{d}t\right| \leq \frac{1}{|pq|}. \tag{15}$$

To finish the argument, note that p, q both being odd, and $\varphi(x) \neq \pm\varphi(y)$, jointly force the inequality $\frac{1}{|pq|} \leq \frac{1}{3}$. Estimates (14) and (15) then imply (5), which is the first desired conclusion. To verify the claimed optimality, recall the cases of equality in Lemma 3 and consider the particular case when $\varphi(x) = 3\varphi(y)$ and $\theta(\varphi(x) - \varphi(y)) \in \mathbb{Z}$. □

Proof of Theorem 2. Let us briefly recall the proof of the asymptotic (7). Start by noting that two linearly independent solutions of the associated homogeneous equation $w_n'' + 4\pi^2\lambda_n w_n = 0$ are given by

$$w_{n,1}(x) := \cos(2\pi\sqrt{\lambda_n}x) \text{ and } w_{n,2}(x) := \sin(2\pi\sqrt{\lambda_n}x),$$

and have constant Wronskian

$$W(w_n^{(1)}, w_n^{(2)}) := \det\begin{pmatrix} w_{n,1} & w_{n,2} \\ w_{n,1}' & w_{n,2}' \end{pmatrix} = 2\pi\sqrt{\lambda_n}.$$

The general solution to the eigenvalue problem (2) is then given by

$$a\cos(2\pi\sqrt{\lambda_n}x) + b\sin(2\pi\sqrt{\lambda_n}x) + \frac{2\pi}{\sqrt{\lambda_n}}\int_0^x \sin(2\pi\sqrt{\lambda_n}(x - t))V(t)w_n(t)\,\mathrm{d}t, \tag{16}$$

for some $a, b \in \mathbb{R}$, as can be easily checked by direct differentiation. Evaluating (16) and its derivative at zero while appealing to hypotheses (H1) and (H3), we then have that

$$w_n(x) = \sqrt{w_n(0)^2 + \frac{w_n'(0)^2}{4\pi^2\lambda_n}} \cos\Big(2\pi\Big(\sqrt{\lambda_n}x - \frac{n}{4}\Big)\Big) + \frac{2\pi}{\sqrt{\lambda_n}} \int_0^x \sin(2\pi\sqrt{\lambda_n}(x-t))V(t)w_n(t)\,\mathrm{d}t. \tag{17}$$

Define $M_n(x) := \max\{|w_n(y)| : y \in [0, x]\}$. Applying the integral form of Grönwall's inequality [17, Theorem 1.10] to (17), we deduce

$$M_n(x) \le \Big(w_n(0)^2 + \frac{w_n'(0)^2}{4\pi^2\lambda_n}\Big)^{1/2} + o_n(1)M_n(x),$$

and therefore

$$M_n(x) \lesssim \Big(w_n(0)^2 + \frac{w_n'(0)^2}{4\pi^2\lambda_n}\Big)^{1/2},$$

from where asymptotic (7) follows at once.

The rest of the proof follows similar steps to those of Theorem 1. Firstly, we can restrict attention to the case of rational x/y. Secondly,

$$\ell_{\{w_n\}}(x, y) = \ell_{\{v_n\}}(x, y),$$

where $v_n(x) := \cos(2\pi(\sqrt{\lambda_n}x - \frac{n}{4}))$. Thirdly, given the equidistribution assumption (H2), Lemma 1 again applies and reduces the computation to

$$\ell_{\{v_n\}}(x, y) = \int_0^1 \Big(\frac{\Psi_0 + \Psi_1}{2}\Big)(pt, qt)\,\mathrm{d}t.$$

Here, the functions Ψ_0, Ψ_1 are given by $\Phi_0 =: 2\Psi_0 - 1$ and $\Phi_1 =: 2\Psi_0 - 1$, where $\Phi_0 := \Phi$ was given in (9), and $\Phi_1(x, y) := \Phi\big(x - \frac{1}{4}, y - \frac{1}{4}\big)$. Consequently,

$$\ell_{\{w_n\}}(x, y) = \frac{1}{2} + \frac{1}{4}\int_0^1 \Phi(pt, qt)\,\mathrm{d}t + \frac{1}{4}\int_0^1 \Phi\Big(pt - \frac{1}{4}, qt - \frac{1}{4}\Big)\,\mathrm{d}t.$$

These integrals can be calculated with Lemma 3. Invoking it with $\alpha = \beta = 0$, and then with $\alpha = \beta = \frac{1}{4}$, yields

$$\ell_{\{w_n\}}(x, y) = \frac{1}{2} + \frac{1}{4pq}((-1)^{\frac{p+q}{2}+1} + (-1)^{p+1}).$$

This is readily seen to be equivalent to the result as stated in (8). ◻

4. Further examples: Hermite functions, Laguerre polynomials and sets of bounded remainder

4.1. Hermite functions. One could think of replacing hypothesis (4) from Theorem 1 by a less restrictive assumption of the form

$$w_n(x) = (1 + o_n(1))\phi(x,n)\cos(2\pi(\mu_n\varphi(x) - \theta_n)),$$

where $\{\theta_n\}$ is now a *sequence*. Without any further assumption, several steps of the preceding proofs break down completely. However, if some quantitative control on the speed with which the sequences $\{\mu_n\varphi(x)\}$ and $\{\mu_n\varphi(y)\}$ equidistribute modulo 1 is known, then we can allow for a certain degree of variability in the sequence $\{\theta_n\}$. It is not clear to us what the sharp version of such a statement would be, and we leave it for future research.

Cases in which the sequence $\{\theta_n\}$ changes rapidly with n, but does so in a structured manner, are also of interest. Such cases may be dealt with by partitioning $\{w_n\}$ into an appropriate number of subsequences, as we now illustrate. A particularly nice example which fits into this framework (and served as original inspiration for Theorem 2) is that of the Schrödinger operator on the real line,

$$H := -\frac{1}{4\pi^2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2.$$

The operator H is diagonalized by the Hermite functions,

$$\varphi_n(x) := H_n(\sqrt{2\pi}x)e^{-\pi x^2}.$$

Here, $\{H_n(x)\}$ denote the classical Hermite polynomials, which are orthogonal with respect to the standard Gaussian measure $e^{-\pi x^2}\,\mathrm{d}x$. As is well known,

$$H(\varphi_n) = \frac{2n+1}{2\pi}\varphi_n. \tag{18}$$

Moreover, the asymptotic from [16, Theorem 8.22.6 and Formula (8.22.8)],

$$H_n(\sqrt{2\pi}x)e^{-\pi x^2} = (1 + o_n(1))\frac{\Gamma(n+1)}{\Gamma\left(\frac{n}{2}+1\right)}\cos\left(2\pi\left(\sqrt{\frac{2n+1}{2\pi}}x - \frac{n}{4}\right)\right)$$

shows that the eigenfunctions $\{\varphi_n\}$ in (18) do not satisfy the assumptions of Theorem 1, but that the subsequences $\{\varphi_{2n}\}$ and $\{\varphi_{2n+1}\}$ do. We further note that the basis $\{\varphi_n\}$ diagonalizes the Fourier transform in the following sense: the elements of $\{\varphi_n\}$ are pairwise orthogonal, dense in $L^2(\mathbb{R})$, and

$$\hat{\varphi}_n(\xi) = \int_{-\infty}^{\infty} \varphi_n(x) e^{-2\pi i \xi x}\, \mathrm{d}x = (-i)^n \varphi_n(\xi).$$

A simple consequence of Theorem 2 (plus a short computation) is the following.

Proposition 3 (Sign correlations for Hermite functions). *Let $x, y \neq 0$ and $y/x \in \mathbb{Z}$. Then*

$$\ell_{\{\varphi_n\}}(x,y) = \begin{cases} \frac{1}{2} + \frac{x}{2y} & \text{if } \frac{y}{x} \equiv 1 \mod 4, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

4.2. Laguerre polynomials. An extension of the previous example to higher dimensions involves the so-called *Laguerre polynomials*. Let $\{L_n^\nu(x)\}$ be the (generalized) Laguerre polynomials with parameter $\nu > -1$, defined via

$$\int_0^\infty L_n^\nu(x) L_m^\nu(x) x^\nu e^{-x}\ \mathrm{d}x = \frac{\Gamma(n+\nu+1)}{n!} \delta(n-m). \tag{19}$$

It is well known, see [16, Formula (8.22.2)], that

$$L_n^\nu(2\pi x^2) e^{-\pi x^2} = (1 + o_n(1)) \frac{n^{\nu/2-1/4}}{\sqrt{\pi}(2\pi)^{\nu/2+1/4} x^{\nu+1/2}} \cdot \cos\left(2\pi\left(\sqrt{\frac{4n+2\nu+2}{2\pi}}x - \frac{2\nu+1}{8}\right)\right).$$

It is also known that the set of Laguerre functions

$$\mathbf{x} \in \mathbb{R}^d \longmapsto \Phi_n(\mathbf{x}) := L_n^\nu(2\pi|\mathbf{x}|^2) e^{-\pi|\mathbf{x}|^2},$$

with $\nu = d/2 - 1$, diagonalizes the operator $\mathbf{H} = -\frac{1}{4\pi^2}\Delta + |\mathbf{x}|^2$ over the space of radial functions in $\mathbb{R}^d$, and that

$$\mathbf{H}(\Phi_n) = \frac{(4n+2\nu+2)}{2\pi} \Phi_n.$$

We also note that $\{\Phi_n\}$ diagonalizes the Fourier transform over the space of square integrable radial functions in $\mathbb{R}^d$. Indeed,

$$\hat{\Phi}_n(\boldsymbol{\xi}) = \int_{\mathbb{R}^d} \Phi_n(\mathbf{x}) e^{-2\pi i \boldsymbol{\xi}\cdot\mathbf{x}}\, \mathrm{d}\mathbf{x} = (-1)^n \Phi_n(\boldsymbol{\xi}).$$

The following result is a direct application of Lemma 3 with $\alpha = \beta = \frac{2\nu+1}{8} = \frac{d-1}{8}$.

Proposition 4 (Sign correlations for Laguerre functions). *Let $r_1, r_2 > 0$ be radii such that $\frac{r_1}{r_2} = \frac{p}{q}$ for some coprime integers p and q. Then*

$$\ell_{\{\Phi_n\}}(r_1, r_2) = \begin{cases} \frac{1}{2} & \text{if } p \text{ or } q \text{ is even, or if } p, q \text{ and } \dfrac{(p-q)(d-1)}{2} \text{ are odd,} \\ \frac{1}{2} - \frac{1}{2pq}(-1)^{\frac{p+q}{2} + \frac{(p-q)(d-1)}{4}} & \text{otherwise.} \end{cases}$$

4.3. Sets of bounded remainder. In this final section, we describe a curious phenomenon that was discovered by accident. Consider the family of Chebyshev polynomials of the first kind on the interval $[-1, 1]$, denoted $\{T_n\}_{n \geq 0}$ and defined via $T_n(x) := \cos(n \arccos x)$. This turns out to be one of the extremal examples for Theorem 1 since

$$\#\left\{0 \leq n < N : \operatorname{sgn} T_n\left(\cos\left(\frac{2\pi}{10}\right)\right) = \operatorname{sgn} T_n\left(\cos\left(3\frac{2\pi}{10}\right)\right)\right\} = (1 + o_N(1))\frac{N}{3}.$$

Indeed, the arising quantities simplify to

$$T_n\left(\cos\left(\frac{2\pi}{10}\right)\right) = \cos\left(\frac{2\pi n}{10}\right) \quad \text{and} \quad T_n\left(\cos\left(3\frac{2\pi}{10}\right)\right) = \cos\left(3\frac{2\pi n}{10}\right),$$

and both sequences $\{\frac{2\pi n}{10}\}$ and $\{3\frac{2\pi n}{10}\}$ are equidistributed modulo 1. If we go one step further and try to understand the error term, we encounter the following surprising phenomenon. At least for $N \leq 10^4$, we used to Mathematica to verify that

$$\left|\#\left\{0 \leq n < N : \operatorname{sgn} T_n\left(\cos\left(\frac{2\pi}{10}\right)\right) = \operatorname{sgn} T_n\left(\cos\left(3\frac{2\pi}{10}\right)\right)\right\} - \frac{N}{3}\right| \leq 10.$$

This somewhat surprising behavior is porbably related to the fine structure of Kronecker sequences [10, 11, 14]; one cannot hope for such strong results in general. This is reminiscent of exciting new developments in the theory of continuous flows on the torus, Beck's [1, 2, 3, 4] superuniformity theory, that may have nontrivial implications. We also refer to a related paper by Grepstad & Larcher on sets with bounded remainder [9], which seems to provide further interesting directions of research.

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