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Spectral analysis of 2D outlier layout

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Abstract. Thompson's partition of a cyclic subnormal operator into normal and completely non-normal components is combined with a non-commutative calculus for hyponormal operators for separating outliers from the cloud, in rather general point distributions in the plane. The main result provides exact transformation formulas from the power moments of the prescribed point distribution into the moments of the uniform mass carried by the cloud. The proposed algorithm solely depends on the Hessenberg matrix associated to the original data. The robustness of the algorithm is reflected by the insensitivity of the output under trace class, or by a theorem of Voiculescu, under certain Hilbert–Schmidt class, additive perturbations of the Hessenberg matrix.

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1. Introduction

The Lebesgue decomposition of a positive measure supported on the real line or the circle and the spectral analysis of symmetric or unitary Hilbert space linear operators cannot be dissociated. We take for granted nowadays operator arguments and Hilbert space geometry features of scattering theory, dynamical systems or approximation schemes, all generally reflecting the fine structure of some underlying spectral measure.

The association of Lebesgue decomposition of measures in higher dimensions with spectral analysis is much less studied and developed. It naturally appeared in perturbation theory questions [40, 43, 28] and it also proved to be essential in some elaborate classification results concerning classes of non-normal operators [33, 20, 38]. In the present note we exploit such a classical, but not widely circulated chapter of modern operator theory with the specific aim at bringing forward an abstract tool with some statistical flavor, namely detecting "outliers" (that is the discrete part) from the "cloud" (the continuous, or better, 2D-absolutely continuous part) of a measure in the real plane. From the mathematical point of view the "subnormal dissection" of planar measures we propose is more convoluted, and hence more interesting. We are well aware of the ambiguity of terminology when mentioning outliers [18].

The case of two dimensions is special, not last because two real coordinates can be arranged into a single complex variable. Also the spectrum, broadly understood as the tangible numerical support of a linear transform, is contained in the complex plane, as well as its localized resolvents, usually interpreted as generalized Cauchy transforms of innate data. This immediately leads to the "black box" approach of restoring the whole from indirect measurements; we tacitly adopt such a perspective, familiar in more applied sciences.

The mathematical construct we propose can be summarized as follows. A compactly supported, positive measure μ in the complex plane is given. The closure of complex polynomials in the associated Lebesgue L^2-space is a Hilbert space whose inner product solely dependens on the *real* power moments of it. The multiplier S by the complex variable on this space is a cyclic subnormal operator whose spectrum contains the support of the generating measure. A landmark result of Thomson [38] decomposes this subnormal operator into orthogonal matrix blocks. One of these blocks, the normal component, gathers the point masses and some singular parts of μ. The other blocks are irreducible and fill with their spectrum some of the connected components of the complement of the support of μ. They provide a dissection of μ into mutual disjoint essential parts. In order to constructively identify these continuous components of μ we convert the moment data following a non-commutative calculus. Central to this step is Helton and Howe formula which relates traces of commutators of non-commutative polynomials in S and S^* to the principal function of S, a spectral invariant proposed by Pincus half a century ago [33, 12]. Trace class estimates in this situation go back to Berger and Shaw [5] and Voiculescu [41]. Finally, the conditioned moments are organized into an exponential transform which plays the role of "equilibrium potential" of the possibly thickened, continuous part of μ. The exponential transform is a superharmonic function decreasing on the complement from the value 1 at infinity, to the value 0, and behaving close to the boundary of these components as euclidean distance. See [15] for a recent survey of basic properties of the exponential transform. All in all, putting aside technical ingredients, we offer to the practitioner an algorithm for separating a simply connected cloud, or a union of simply connected clouds, from scattered outliers, which can be points, non-closed curves or even more complicated "thin" shapes; all in terms of moment data. The complex orthogonal polynomials and the associated Hessenberg matrix representing S naturally enter into computations. A different non-commutative perspective on outlier identification recently appeared in [4].

Omnipresent in our study, although not always explicit, are bounded point evaluations and reproducing kernels associated to some functional Hilbert spaces. For L^2 spaces of polynomials of a prescribed degrees, these are the well known Christoffel–Darboux kernels. In the classical one variable setting, the fine analysis and in particular asymptotics of these kernels provide the main separation tool between the 1D-absolute continuous part and singular part of a measure. In this direction an informative account is Nevai's eulogy of Geza Freud work [32]. The ultimate results, with far reaching consequences to approximation theory and beyond, are due to Totik and collaborators [39, 30]. For a novel application to ergodic theory see [21]. An operator point of view to the same topics, with unexpected applications to spectral theory, was recently advocated by Simon [36, 37, 35]. Not surprising, the multivariate analog of the Christoffel–Darboux kernel analysis is only nowadays slowly developing [9, 8, 24, 23, 7]. On this ground, potential benefits to the statistics of point distributions are emerging [25, 7, 26].

The present note remains at the purely theoretical level, including a simple toy example included in the last section. Computational and numerical aspects will be addressed in a forthcoming article, with particular attention to a comparison with some recent outlier separation methods proposed in [25, 7]. Having in mind such a sequel and its potential wider audience prompted us to balance the text and briefly comment a rather comprehensive bibliography.

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2. Preliminaries

We collect in the present section some known facts about the structure of two related classes of close to normal operators.

2.1. Thomson's partition of a measure. Let H be a separable, complex Hilbert space. A linear bounded operator $S \in \mathcal{L}(H)$ is called *subnormal* if there exists a larger Hilbert space $H \subset K$ and a normal operator $N \in \mathcal{L}(K)$, such that N leaves invariant H and $N|_H = S$. Subnormal operators are well studied, mainly with function theory tools, as the quintessential example is offered by multiplication by the complex variable on a Hilbert space of analytic functions. A subnormal operator S is called irreducible if it cannot be decomposed into a direct sum $S = S_1 \oplus S_2$ of non-trivial operators. Under a canonical minimality condition one proves that the spectrum of the normal extension is contained in the spectrum of the original operator:

$$\sigma(N) \subset \sigma(S),$$

the difference consisting in filling entire bounded "holes" of the complement, that is every $U \subset \mathbb{C} \setminus \sigma(N)$ bounded and connected satisfies either $U \cap \sigma(S) = \emptyset$ or $U \subset \sigma(S)$. The authoritative monograph [13] accurately exposes the basics of subnormal operator theory. In the sequel we freely use some standard terminology (essential spectrum, index, reproducing kernel, trace, ...) well explained there in a unifying context.

A cyclic subnormal operator is necessarily of the following type. Let μ be a compactly supported, positive Borel measure on the complex plane $\mathbb{C}$. The multiplier $S_\mu = M_z$ acting on the closure $P^2(\mu)$ of polynomials in Lebesgue space $L^2(\mu)$ is obviously subnormal (the minimal normal extension is $N = M_z$ acting on $L^2(\mu)$) and has the constant function **1** as a cyclic vector. If the measure μ is not finitely atomic, then one can speak without ambiguity of the basis of $P^2(\mu)$ formed by the orthonormal polynomials

$$P_n(z) = \gamma_n z^n + \cdots \in \mathbb{C}[z], \quad n \geq 0,$$

where $\gamma_n > 0$ and

$$\langle P_k, P_j \rangle_{2,\mu} = \delta_{jk}, \quad j, k \geq 0.$$

The spectrum of N is equal to the closed support of μ, while the spectrum of S_μ contains $\mathrm{supp}(\mu)$ plus some "holes" of its complement.

Theorem 2.1 (Thomson). *Let μ be a positive Borel measure, compactly supported on $\mathbb{C}$. There exists a Borel partition $\Delta_0, \Delta_1, \ldots$ of the closed support of μ with the following properties*:

(1) $P^2(\mu) = L^2(\mu_0) \oplus P^2(\mu_1) \oplus P^2(\mu_2) \oplus \cdots$, *where* $\mu_j = \mu|_{\Delta_j}$, $j \geq 0$;

(2) *every operator* S_{μ_j}, $j \geq 1$, *is irreducible with spectral picture*

$$\sigma(S_{\mu_j}) \setminus \sigma_{\mathrm{ess}}(S_{\mu_j}) = G_j,$$

simply connected, and

$$\mathrm{supp}\, \mu_j \subset \overline{G_j}, \quad j \geq 1;$$

(3) *if* $\mu_0 = 0$, *then any element* $f \in P^2(\mu)$ *which vanishes* $[\mu]$*-a.e. on* $G = \bigcup_j G_j$ *is identically zero.*

The proof appeared in [38], even for L^p spaces, $1 \le p < \infty$. Note that only one summand in the decomposition of $P^2(\mu)$ is a full Lebesgue L^2-space. The immediate relevance of Thomson's Theorem for the general theory of subnormal operators was already discussed in [13] (published almost simultaneously with the original article). We confine to comment briefly the statement.

The operator $S_{\mu_0} = M_z \in \mathcal{L}(L^2(\mu_0))$ is the normal component of S_μ. If S_μ is not normal, then at least one of the summands S_{μ_j}, $j \ge 1$, is non-trivial. There can be finitely many, or at most countably many such irreducible summands. The spectral picture described in part 2) of the theorem means that the adjoint of every operator S_{μ_j} admits a continuum of eigenvalues of multiplicity one, labelled by the simply connected open set G_j:

$$\lambda \in G_j \implies [\ker(S_{\mu_j} - \lambda) = 0, \dim\ker(S^*_{\mu_j} - \bar{\lambda}) = 1],$$

and the range of $S_{\mu_j} - \lambda$ is closed. Moreover, the corresponding eigenvectors span $P^2(\mu_j)$, to the extent that this functional Hilbert space possesses a reproducing kernel. To be more specific, for every $\lambda \in G_j$ the corresponding point evaluation is bounded as a linear functional on $P^2(\mu_j)$, and hence on $P^2(\mu)$:

$$\Lambda^{\mu_j}(\lambda) := \inf\{\|p\|^2_{\mu_j}; p \in \mathbb{C}[z], \ p(\lambda) = 1\} > 0.$$

Above $\Lambda^\tau(\lambda)$ is the *Christoffel function* associated to the measure τ and point λ. This means that the support of μ_j must disconnect the simply connected open set G_j from the interior points of its complement. We will depict some examples in a subsequent section.

An array of function theoretic results, some providing answers to long standing open questions, stream from Thompson's partition of a measure and the structure of a cyclic subnormal operator, cf. Chapter VIII of [13]. An elaborate analysis of the limiting values of elements of $P^2(\mu)$ on the boundary of each chamber G_j is pursued in [2], with applications to the structure of invariant subspaces of the operator blocks S_j.

2.2. Hyponormal operators. A general subnormal operator $S \in \mathcal{L}(H)$ satisfies the commutator inequality

$$[S^*, S] \ge 0.$$

Indeed, denoting by $N \in \mathcal{L}(K)$ the normal extension of S and $P\colon N \to H$ the orthogonal projection, one finds for an arbitrary vector $x \in H$:

$$\langle [S^*, S]x, x\rangle = \|Sx\|^2 - \|S^*x\|^2 = \|Nx\|^2 - \|PN^*x\|^2 \ge \|Nx\|^2 - \|N^*x\|^2 = 0.$$

A linear bounded operator $T \in \mathcal{L}(H)$ subject to the constraint $[T^*, T] \geq 0$ is called *hyponormal*. Simple examples of weighted shifts or singular integral transforms show that not all hyponormal operators are subnormal. As a matter of fact, the spectral analysis of hyponormal operators is quite different than the complex function theory approach to subnormal operators. See for details [29].

Of particular interest are hyponormal operators with trace-class self-commutators. They are modeled by singular integrals (with Cauchy kernel singularity) on Lebesgue space on the real line. Early on, starting from such functional models Pincus [33] has identified a spectral invariant called the *principal function*. A decade of astounding discoveries led to the following trace formula, which can also be taken as a definition of the principal function. Let $T \in \mathcal{L}(H)$ be a hyponormal operator with trace-class self-commutator. There exists a function $g_T \in L^1_{\mathrm{comp}}(\mathbb{C}, \mathrm{d}A)$ with compact support, satisfying

$$\operatorname{Tr}[p(T, T^*), q(T, T^*)] = \frac{1}{\pi} \int_{\mathbb{C}} \Big(\frac{\partial p}{\partial \bar{z}} \frac{\partial q}{\partial z} - \frac{\partial q}{\partial \bar{z}} \frac{\partial p}{\partial z} \Big) g_T(z) \, \mathrm{d}A(z). \tag{1}$$

Above $p, q \in \mathbb{C}[z, \bar{z}]$ are polynomials, and the order of factors T, T^* in the functional calculus $p(T, T^*)$ does not affect the trace, due to relation (6). Originally, Carey and Pincus have defined g_T via a multiplicative commutator determinant, in its turn inspired by the perturbation determinant in the 1D theory of the phase shift. The trace formula above appeared in the works of Helton and Howe [20]. Proofs and historical details can be found in the monograph [29]. Important for our note is the observation that, for an irreducible hyponormal, but not normal operator T, the support of the principal function coincides with its spectrum:

$$\operatorname{supp} g_T = \sigma(T). \tag{2}$$

For a proof see p. 243 of [29].

To give a sense of the principal function, for a point λ disjoint of the essential spectrum of T one finds the Fredholm index formula:

$$g_T(\lambda) = -\operatorname{ind}(T - \lambda) = \dim \ker(T^* - \bar{\lambda}) - \dim \ker(T - \lambda). \tag{3}$$

The principal function enjoys a series of functoriality properties which are inherited from its curvature type definition (1).

Another relevant theorem due to Carey and Pincus [12] asserts that the principal function of a subnormal operator is integer valued.

One step further, we narrow our focus to hyponormal operators with rank-one self-commutator. This class of operators represents in many respects the two-dimensional analogue of rank-one perturbations of self-adjoint operators. In particular, the phase-shift of a rank-one self-adjoint perturbation is replaced in this context by the principal function. Without aiming at completeness, we extract below a couple of observations from the rich phenomenology of hyponormal operators with rank-one self-commutator. Details covering technical aspects touched below and additional bibliographical comments can be found in the recent lecture notes [15].

The foundation stone is Pincus' bijective correspondence between "shade functions" $g \in L^1_{\text{comp}}(\mathbb{C}, \mathrm{d}A)$, $0 \leq g \leq 1$, and irreducible hyponormal operators $T \in \mathcal{L}(H)$ with rank one self-commutator:

$$[T^*, T] = \xi\langle\cdot, \xi\rangle, \quad \xi \neq 0.$$

The explicit formula linking the two classes is

$$\begin{aligned} &\det[(T-w)(T^*-\bar{z})(T-w)^{-1}(T^*-\bar{z})^{-1}] \\ &= 1 - \langle (T^*-\bar{z})^{-1}\xi, (T^*-\bar{w})^{-1}\xi\rangle \\ &= \exp\Big(\frac{-1}{\pi}\int_{\mathbb{C}} \frac{g(\zeta)\,\mathrm{d}A(\zeta)}{(\zeta-w)(\bar{\zeta}-\bar{z})}\Big). \end{aligned} \tag{4}$$

Above z, w are originally outside the spectrum of T, equal to the support of g, but the second identity can be extended to the whole $\mathbb{C}^2$. As a matter of fact this is the multiplicative analog, and equivalent, of Helton and Howe additive formula, identifying g with the principal function g_T of T.

The only irreducible subnormal operator with rank-one self-commutator is the unilateral shift $S = S_z$, acting on the Hardy space of the unit disk $\mathbb{D}$. The self-commutator $[S^*, S]$ is the projection $\mathbf{1} \otimes \mathbf{1}$ on the constant functions. The principal function of S coincides with the characteristic function of the disk, and the above formulas read as

$$\begin{aligned} 1 - \frac{1}{w\bar{z}} &= 1 - \langle (S^*-\bar{z})^{-1}\mathbf{1}, (S^*-\bar{w})^{-1}\mathbf{1}\rangle \\ &= \exp\Big(\frac{-1}{\pi}\int_{\mathbb{D}} \frac{\mathrm{d}A(\zeta)}{(\zeta-w)(\bar{\zeta}-\bar{z})}\Big), \quad |z|, |w| > 1. \end{aligned}$$

We will soon return to this basic example.

3. Main result

We combine Thompson's Theorem and the theory of the principal function with the specific aim at finding the power moments of the characteristic function of the support of the non-normal component in Thompson's decomposition of the multiplier by the complex variable.

Theorem 3.1. *Let $\mu = \mu_0 + \mu_1 + \mu_2 + \ldots$ be Thompson's partition of a positive measure μ with compact support on $\mathbb{C}$. Let $S = S_\mu$ be the corresponding subnormal operator with irreducible parts having spectrum $\sigma(S_{\mu_j}) = \overline{G_j}$, $j \geq 1$.*

For every pair of non-negative integers k, ℓ the following trace exists and equals a moment integral:

$$\mathrm{Tr}[(S^*)^{k+1}, S^{\ell+1}] = \frac{(k+1)(\ell+1)}{\pi} \sum_j \int_{\overline{G_j}} \bar{z}^k z^\ell \, \mathrm{d}A(z). \tag{5}$$

Proof. The subnormal operator $S = S_\mu$ is hyponormal and cyclic. A theorem due to Berger and Shaw [5] asserts that the self-commutator of S is trace class:

$$\mathrm{Tr}[S^*, S] < \infty. \tag{6}$$

Due to non-negativity $\mathrm{Tr}[S^*, S] = 0$ if and only if S is normal.

Consider an irreducible component $S_j = S_{\mu_j}$, $j \geq 1$, of S. According to the index formula (3), $g_{S_j}(\lambda) = 1$ for every $\lambda \in G_j$. Moreover, Thompson's Theorem yields $\sigma(S_j) = \overline{G_j}$, and on the other hand $\sigma(S_j) = \operatorname{supp} g_{S_j}$. A theorem due to Carey and Pincus asserts that the principal function of a subnormal operator is integer valued [12]. Hence

$$g_{S_j} = \chi_{\overline{G_j}}.$$

Thus, for a fixed integer $j \geq 1$ Helton and Howe formula reads

$$\mathrm{Tr}[(S_j^*)^{k+1}, S_j^{\ell+1}] = \frac{(k+1)(\ell+1)}{\pi} \int_{\overline{G_j}} \bar{z}^k z^\ell \, \mathrm{d}A(z), \quad k, \ell \geq 0.$$

Since S is a direct sum of these irreducible operators and S_0 does not contribute to the trace of commutators, the statement is proved. □

A different proof of Berger-Shaw Theorem is due to Voiculescu [41], where the modulus of quasi-triangularity and other perturbation theory concepts enter into discussion.

By exploiting another feature of the principal function, namely Berger's cyclic multiplicity inequality

$$g_T \leq \mathrm{mult}(T),$$

see [6], we derive the following notable property of the open chambers G_j.

Corollary 3.2. *In the conditions of Thompson's Theorem, the area measure of every intersection $\overline{G_j} \cap \overline{G_k}$, $j \neq k$, is zero.*

Proof. Indeed,

$$g_S = g_{S_1} + g_{S_2} + \cdots \le 1$$

and

$$g_S \le 1, \quad \mathrm{d}A\text{-a.e.}$$

While $g_{S_j} = \chi_{\overline{G_j}}$, $j \ge 1$. □

4. Shape reconstruction

In this section we focus on the special nature of the power moments derived from the commutator formula (5) by sketching some known reconstruction algorithms. The specific nature of the generating measure of the new moments (absolutely continuous, with a bounded weight with respect to Lebesgue 2D measure) simplifies and enhances the reconstruction and approximation procedures.

4.1. In our specific situation of a positive measure μ carrying its Thompson's partition, we face, returning to the notation of the preceding section, a characteristic function

$$g = g_{S_\mu} = \sum_j \chi_{\overline{G_j}}.$$

Its moments appear in relation (5),

$$a_{k\ell} = \int_{\mathbb{C}} \zeta^k \bar{\zeta}^\ell g(\zeta)\,\mathrm{d}A(\zeta), \quad k, l \ge 0,$$

and will be organized in the exponential of a formal generating series

$$E_g(w,z) = \exp\Big[\frac{-1}{\pi} \sum_{k,\ell=0}^{\infty} \frac{a_{k\ell}}{w^{k+1}\bar{z}^{\ell+1}}\Big].$$

This is of course the power expansion at infinity of the double Cauchy integral appearing in (4). In general we define

$$E_g(w,z) = \exp\Big(\frac{-1}{\pi} \int_{\mathbb{C}} \frac{g(\zeta)\,\mathrm{d}A(\zeta)}{(\zeta - w)(\bar{\zeta} - \bar{z})}\Big), \quad z, w \in \mathbb{C},\ z \ne w.$$

We recall a few of the properties of the *exponential transform* E_g:

(a) the function E_g can be extended by continuity to $\mathbb{C}^2$ by assuming the value $E_g(z,z) = 0$ whenever $\int_{\mathbb{C}} \frac{g(\zeta)\,\mathrm{d}A(\zeta)}{|\zeta - z|^2} = \infty$;

(b) the function $E_g(w,z)$ is analytic in $w \in \mathbb{C} \setminus \operatorname{supp}(g)$ and antianalytic in $z \in \mathbb{C} \setminus \operatorname{supp}(g)$;

(c) the kernel $1 - E_g(w,z)$ is positive semi-definite in $\mathbb{C}^2$;

(d) the behavior at infinity contains as a first term the Cauchy transform of g:

$$E_g(w,z) = \frac{1}{\bar{z}}\Big[\frac{-1}{\pi}\int_{\mathbb{C}} \frac{g(\zeta)\,\mathrm{d}A(\zeta)}{\zeta - w}\Big] + O\Big(\frac{1}{|z|^2}\Big), \quad |z| \to \infty.$$

The case of a characteristic function $g = \chi_\Omega$ of a bounded domain Ω is particularly relevant for our note. In this case we simply write E_Ω instead of E_g, and we record the following properties (all proved and well commented in [15]).

(i) The equation
$$\frac{\partial E_\Omega(w,z)}{\partial \bar{w}} = \frac{E_\Omega(w,z)}{\bar{w} - \bar{z}}$$
holds for $z \in \Omega$ and $w \in \mathbb{C} \setminus \Omega$.

(ii) The function $E_\Omega(w,z)$ extends analytically/antianalytically from $(\mathbb{C} \setminus \overline{\Omega})^2$ across real analytic arcs of the boundary of Ω.

(iii) Assume $\partial\Omega$ is piecewise smooth. Then $z \mapsto E_\Omega(z,z)$ is a superharmonic function on the complement of Ω, with value 1 at infinity, vanishing on $\overline{\Omega}$ and satisfying
$$E_\Omega(z,z) \approx \operatorname{dist}(z, \partial\Omega)$$
for $z \in \mathbb{C} \setminus \overline{\Omega}$ close to $\partial\Omega$.

A Riemann-Hilbert factorization also characterizes E_Ω, see [15]. The feature which turns the exponential transform into a suitable shape reconstruction from moments tool is its rationality on a class of domains which approximate in Hausdorff distance any planar domain. More specifically, a bounded open set U of the complex plane is called a *quadrature domain* for analytic functions if there is a distribution of finite support τ in U (combination of point masses and their derivatives), such that

$$\int_U f\,\mathrm{d}A = \tau(f), \quad f \in L^1_a(U, \mathrm{d}A),$$

where $L^1_a(U, \mathrm{d}A)$ stands for the space of analytic functions in U which are integrable. The *order* of a quadrature domain is the number of nodes, counting multiplicity, in the above cubature formula. A quadrature domain has always a real algebraic boundary, even irreducible if U is connected. The simplest example is of course a disk. Any simply connected quadrature domain is a conformal image of the disk by a rational function. The term was coined by Aharonov and Shapiro for specific function theory purposes, but soon it was realized that quadrature domains are relevant in fluid mechanics, geophysics, integrable systems and operator theory. An informative and accessible survey is [17].

Quadrature domains can be exactly reconstructed from moments via formal algebraic manipulations of the exponential transform. Namely, let d be a fixed integer and let $(a_{k\ell})_{k,\ell=0}^{d}$, be a hermitian matrix of potential moments of a "shade function" $g(z), 0 \leq g \leq 1$. Consider the truncated exponential transform

$$F(w,z) = \exp\left[\frac{-1}{\pi}\sum_{k,\ell=0}^{d}\frac{a_{k\ell}}{w^{k+1}\bar{z}^{\ell+1}}\right] = 1 - \sum_{m,n=0}^{\infty}\frac{b_{mn}}{w^{m+1}\bar{z}^{n+1}}.$$

A necessary and sufficient condition that $(a_{k\ell})_{k,\ell=0}^{d}$ represent the moments of a quadrature domain of order d is

$$\det(b_{mn})_{m,n=0}^{d} = 0,$$

or equivalently the existence of a monic polynomial $P(z)$ of degree d and a rational function of the form

$$R_d(w,z) = 1 - \frac{\sum_{m,n=0}^{d-1} c_{mn} w^m \bar{z}^n}{P(w)\overline{P(z)}},$$

such that, at infinity

$$F(w,z) - R_d(w,z) = O\Big(\frac{1}{w^{d+1}\bar{z}^{d}}, \frac{1}{w^{d}\bar{z}^{d+1}}\Big).$$

The reader will recognize above a typical 2D Padé approximation scheme. Moreover, for any shade function g, the exponential transform E_g coincides with E_Ω, where Ω is a quadrature domain if and only if

$$E_g(w,z) = 1 - \frac{\sum_{m,n=0}^{d-1} c_{mn} w^m \bar{z}^n}{P(w)\overline{P(z)}}, \quad |z|, |w| \gg 1.$$

In this case the zeros of P coincide with the quadrature nodes, while the numerator is the irreducible defining polynomial of the boundary of Ω:

$$\partial\Omega \subset \Big\{z \in \mathbb{C}: \sum_{m,n=0}^{d-1} c_{mn} z^m \bar{z}^n = |P(z)|^2\Big\}.$$

The above Padé approximation procedure was proposed for the reconstruction of planar shapes in [14] and it is also fully explained in [15].

4.2. One can approximate unions of simply connected domains with smooth boundary without going through the costly non-linear exponential transform. This time one can invoke simply the Christoffel–Darboux kernel and its behavior inside, outside or on the boundary of the set. The necessary estimates for this approach appear in [16], while a refinement of the method, for non-simply connected sets (an "archipelago of islands with lakes") appears in [34]. We indicate only one typical result. Specifically, if Ω is such an archipelago, and $P_n(z)$ denote the associated complex orthogonal polynomials, then Christoffel's function

$$\Lambda_n^{\Omega}(\lambda) = \inf\{\|p\|_{2,\Omega}^2,\ p \in \mathbb{C}[z],\ \deg(p) \le n,\ p(\lambda) = 1\},$$

satisfies

$$\sqrt{\pi}\,\mathrm{dist}(z, \partial\Omega) \le \sqrt{\Lambda_n^{\Omega}(z)} \le C\,\mathrm{dist}(z, \partial\Omega)$$

for $z \in \Omega$, close to $\partial\Omega$ and C is a positive constant. Moreover,

$$\Lambda_n^{\Omega}(\lambda) = O\Big(\frac{1}{n}\Big), \quad \lambda \in \partial\Omega.$$

In the exterior of $\overline{\Omega}$, Christoffel's function decays exponentially to zero. On analytic boundaries one has sharp estimates, complemented by some computational analysis and graphical experiments, see again [16, 34].

4.3. Finally, if it is a priori given that the boundary of an open set Ω is given by a single real polynomial q of degree d, and Ω is its sublevel set, then Stokes Theorem (of geometric measure theory) allows one to identify q from all power moments of Ω of degree less than or equal to $3d$. The additional assumption that Ω is convex drops this bound to $2d$. For details see [27]. The same idea is generalized there to the reconstruction of algebraic domains carrying an exponential weight.

5. Exclusion of outliers

In this section we provide a few instances of Function Theory flavor which support the Algorithm we propose. A more constructive matrix analysis approach reformulates our approximation scheme.

5.1. Function theory examples. We start this section with a few examples, validating, or showing the limits, of the ideal separation of "outliers" from the "cloud" we propose. Thompson's decomposition of a positive Boreal measure in 2D strongly uncouples (in orthogonal Hilbert space directions) the normal part of the multiplier by the complex variable from its pure subnormal part. This will be the first step in our scheme. On simple examples, that is a cloud of uniform area mass versus finitely many atoms, the partition we propose will do as expected. However, on more sophisticated measures, the decomposition might be counterintuitive.

Example 5.1. Two measures supported by the unit circle $\mathbb{T}$ produce the same essential spectrum, yet their Thompson's partition is very different. Specifically, let $\mathrm{d}\,\theta$ denote arc length, and

$$\mu = \sum_{k=1}^{\infty} \frac{1}{2^k} \delta_{z_k},$$

where (z_k) is an everywhere dense sequence on $\mathbb{T}$.

The operator $S_{\mathrm{d}\,\theta}$ is pure subnormal; this is the unilateral shift acting on Hardy space of the disk. The spectrum is the closure of the unit disk,

$$\sigma(S_{\mathrm{d}\,\theta}) = \overline{\mathbb{D}},$$

the essential spectrum is the boundary

$$\sigma_{\mathrm{ess}}(S_{\mathrm{d}\,\theta}) = \mathbb{T},$$

and the principal function is the characteristic function of the disk. The associated exponential transform is

$$E_{\mathbb{D}}(w, z) = \exp\Big[\frac{-1}{\pi} \int_{\mathbb{D}} \frac{\mathrm{d}\,A(\zeta)}{(\zeta - w)(\bar{\zeta} - \bar{z})}\Big] = 1 - \frac{1}{w\bar{z}}, \quad |z|, |w| > 1,$$

and indeed $z = 0$ is the quadrature node of $\mathbb{D}$ and $1 - z\bar{z} = 0$ is the equation defining the boundary.

On the other hand, S_μ is a diagonalizable operator, hence normal, with spectrum filling a continuum. In particular its essential spectrum coincides with the spectrum

$$\sigma(S_\mu) = \sigma_{\mathrm{ess}}(S_\mu) = \mathbb{T}.$$

As a matter of fact, μ can be any singular measure with respect to arc length, and closed support equal to the full circle. Indeed, the celebrated Szegö's Theorem asserts that $P^2(\mu) = L^2(\mu)$ for every measure μ supported by the unit circle and singular with respect to Lebesgue measure, see for instance Corollary III.12.9 in [13].

Example 5.2. We remain in the closed unit disk, and consider after Kriete [22] a measure μ which inside the disk is rotationally invariant

$$\mu|_{\mathbb{D}} = G(r)r \,\mathrm{d}\theta \,\mathrm{d}\theta$$

while on the boundary has the form

$$\mu|_{\mathbb{T}} = w \,\mathrm{d}\theta + \nu_s,$$

where G, w are positive, continuous weights and ν_s is singular with respect to arc length. In general, $L^2(\nu_s)$ is a direct summand of $P^2(\mu)$, but whether $L^2(w \,\mathrm{d}\theta)$ is also a direct summand of $P^2(\mu)$ is a challenging question, touching delicate harmonic analysis chapters; a sufficient condition depending on the growth of the function G and the behavior of the boundary weight w. Without entering into details, we mention that there exists an example (attributed by Kriete to Vol'berg) with $G(r) = 1,\ r \in [0, 1)$, and weight $w(\theta)$ necessarily satisfying

$$\int_{-\pi}^{\pi} \log w(\theta) \,\mathrm{d}\theta = -\infty,$$

so that

$$P^2(\chi_{\mathbb{D}} \,\mathrm{d}A + w \,\mathrm{d}\theta) = L^2_a(\mathbb{D}) \oplus L^2(w \,\mathrm{d}\theta).$$

Above $L^2_a(\mathbb{D}, \mathrm{d}A) = P^2(\chi_{\mathbb{D}} \,\mathrm{d}A)$ is the Bergman space of the unit disk, that is the space of analytic functions in Ω which are square summable on Ω.

In such a case, our splitting schema will separate the area measure in the disk from the boundary component $w \,\mathrm{d}\theta$.

Speaking about the Bergman space of a simply connected domain Ω, classical results of Carleman (1923), then later Markushevich and Farrell (1934) prove that

$$L^2_a(\Omega, \mathrm{d}A) = P^2(\chi_\Omega \,\mathrm{d}A),$$

for a Jordan domain, respectively for a Carathéodory domain. In this situation, Thompson's partition has obviously only one chamber, equal to Ω, and no normal part. By definition, Ω is a *Carathéodory domain* if its boundary equals the boundary of the unbounded connected component of the complement of its closure. For example, a Carathéodory domain does not have internal slits. The early survey [31] is invaluable for information on the contributions of the Russian and Armenian schools to this very topics.

Asking the same completeness of polynomials in Bergman space beyond Carathéodory domains turns out to be very challenging, and interesting. It is not our aim to enter into inherent technical details. We confine to record a recent outstanding observation of Akeroyd [1] stating that there exists a slit Γ joining an internal point to a boundary point of the unit disk, so that $L^2_a(\mathbb{D} \setminus \Gamma, \mathrm{d}A) = P^2(\chi_{\mathbb{D}\setminus\Gamma}\, \mathrm{d}A)$. In general, this is not the case, the relative geometry of the curve Γ inside $\mathbb{D}$ altering the completeness of polynomials . See [10] for an authoritative text, containing definitive results.

Example 5.3. To get closer to the aim of this note, we have to mention the weighted Bergman space case also. Let Ω be a simply connected domain and $w\colon \Omega \to (0,\infty)$ a continuous weight. We denote by $L^2_a(\Omega, w\,\mathrm{d}A)$ the space of analytic functions in Ω which are square summable with respect to the measure $w\,\mathrm{d}A$ on Ω. Early theorems of Hedberg [19] assert that

$$L^2_a(\Omega, w\,\mathrm{d}A) = P^2(\chi_\Omega w\,\mathrm{d}A),$$

if either $w(z) = |f(z)|$, where f is an analytic function in Ω subject to the growth condition

$$\int_\Omega (|f|^{-\delta} + |f|^{1+\delta})\,\mathrm{d}A < \infty,$$

for some $\delta > 0$, or

$$L^2_a(\mathbb{D}, (w \circ \phi)\,\mathrm{d}A) = P^2(\chi_{\mathbb{D}}(w \circ \phi)\,\mathrm{d}A),$$

where $\phi\colon \mathbb{D} \to \Omega$ is a conformal mapping. For instance a positive weight w making the pull-back on the disk $w \circ \phi$ rotationally invariant is appropriate for polynomial density in the weighted Bergman space. In other terms w is constant on the level sets of the inner Green function of Ω. See [11] for complementary results and history of this completeness problem.

5.2. The algorithm. After this lengthy and still sketchy preparation for a possible delimitation of the mathematical meaning of a "cloud," we are ready to put forward the announced algorithm.

Algorithm for separation of outliers from a cloud. Let $\mu = \chi_\Omega w\,\mathrm{d}A + \nu$ be a positive measure, where Ω is a bounded simply connected domain, w is a positive continuous weight on Ω and ν be a positive measure of compact support. We impose Thompson's decomposition

$$P^2(\mu) = P^2(\chi_\Omega w\,\mathrm{d}A) \oplus L^2(\nu). \tag{7}$$

The given data are the power moments

$$s_{k\ell} = \int_{\mathbb{C}} z^k \bar{z}^\ell \, \mathrm{d}\mu(z), \quad k, \ell \geq 0,$$

all assumed finite.

(1) Compute the associated complex orthogonal polynomials $P_k(z),\ k \geq 0$, and the corresponding truncated Christoffel–Darboux kernel

$$K_d^\mu(w, z) = \sum_{j=0}^{d} P_j(w) \bar{P_j}(z), \quad d \geq 0.$$

(2) For every pair of integers $k, \ell \geq 0$ compute the trace of commutator

$$\begin{aligned}
&\mathrm{Tr}[(S^*)^{k+1}, S^{\ell+1}] \\
&\quad = \sum_{j=0}^{\infty} \lim_{d\to\infty} \int_C \Big(z^{\ell+1} \bar{z}^{k+1} |p_j(z)|^2 - z^{\ell+1} \overline{p_j(z)} \\
&\qquad\qquad \int_C K_d^\mu(z, \zeta) \bar{\zeta}^{k+1} p_j(\zeta) \mathrm{d}\mu(\zeta) \Big) \mathrm{d}\mu(z) \\
&\quad = \sum_{j=0}^{\infty} \lim_{d\to\infty} \int_{\mathbb{C}\times\mathbb{C}} K_d^\mu(z, \zeta) z^{\ell+1} \overline{p_j(z)} \\
&\qquad\qquad [\bar{z}^{k+1} p_j(z) - \bar{\zeta}^{k+1} p_j(\zeta)] \mathrm{d}\mu(\zeta) \mathrm{d}\mu(z).
\end{aligned}$$

(3) The moments of the "cloud" carrying uniform mass equal to one are

$$a_{k\ell} = \int_\Omega z^k \bar{z}^\ell \, \mathrm{d}A(z) = \frac{\pi}{(k+1)(\ell+1)} \mathrm{Tr}[(S^*)^{k+1}, S^{\ell+1}], \quad k, \ell \geq 0.$$

(4) Use one of the reconstruction of shapes from moments procedures (outlined in Section 4) to approximate, or find exactly Ω.

The limits above, that is the trace of the commutator, exist by Berger and Shaw Theorem. The fact that Thompson's decomposition has only one open chamber follows from the fact that polynomials are integrable with respect to the measure $\chi_\Omega w \, \mathrm{d}A$ and the weight w is continuous and positive. Indeed, under these assumptions, every point of Ω (assumed connected) is a bounded point evaluation for $P^2(\chi_\Omega w \, \mathrm{d}A)$.

Notice that step (2) solely depends on the moments of the measure μ. The hypotheses in the algorithm can obviously be relaxed, allowing for instance a union of disjoint simply connected domains and an array of singular measures ν (with respect to area) whose support does not disconnect the plane. However, in such a context, the necessary decomposition (7) is not always easy to establish.

We provide a criterion assuring the validity of (7).

Proposition 5.4. *Let $\mu = \chi_\Omega w \, \mathrm{d}A + \nu$ be a positive measure, where Ω is a bounded simply connected domain with smooth boundary and w is a positive continuous weight on Ω. Suppose ν is a positive measure of compact support, so that the complement of $\overline{\Omega} \cup \operatorname{supp}(\nu)$ is path connected in the complex plane and $P^2(\nu) = L^2(\nu)$.*

If $\nu(\overline{\Omega}) = 0$, then decomposition (7) holds true.

Proof. Let $\rho\colon \mathbb{C} \to \mathbb{R}$ be a defining function for the boundary of Ω:

$$\Omega = \{z \in \mathbb{C}, \rho(z) < 0\}, \quad \nabla_w \rho \neq 0,\ w \in \partial\Omega.$$

For $\epsilon > 0$ sufficiently small the domain

$$\Omega_\epsilon = \{z \in \mathbb{C},\ \rho(z) < \epsilon\}$$

is still simply connected with smooth boundary and $\partial\Omega_\epsilon$ is a deformation retract of $\overline{\Omega_\epsilon} \setminus \Omega$. Thus the complement of the compact set $K_\epsilon = \overline{\Omega} \cup [\operatorname{supp}(\nu) \setminus \Omega_\epsilon]$ is still path connected. Denote by χ_ϵ the characteristic function of K_ϵ.

Let $h \in L^2(\nu)$. In view of Runge Theorem, the function $\chi_\epsilon h$ can be approximated in $L^2(\mu)$ by a sequence of polynomials. Since $\nu(\overline{\Omega}) = 0$, letting ϵ tend to zero we obtain a sequence of polynomials $q_n \in \mathbb{C}[z]$ satisfying

$$\lim_n \|h - q_n\|_{2,\nu} = 0,$$

and

$$\lim_n \|q_n\|_{2,\chi_\Omega w \, \mathrm{d}A} = 0.$$

Thus the sequence (q_n) is Cauchy in the space $P^2(\mu)$. Its limit is an element $H \in P^2(\mu)$ which is equal to the function h when restricted to the support of the measure ν and it is identically zero in the domain Ω. Moreover, the definition of H as an element of $L^2(\mu) = L^2(\chi_\Omega w \, \mathrm{d}A) \oplus L^2(\nu)$ is independent of the sequence of approximants (q_n).

In conclusion the space $L^2(\nu)$ is a direct summand of $P^2(\mu)$. □

Variations of the above proof are immediate. For instance allowing Ω to be a union of simply connected domains whose closures are mutually disjoint and ν to be a measure supported on a finite union of closed arcs or points, disjoint of $\overline{\Omega}$, so that the complement of $\overline{\Omega} \cup \mathrm{supp}(\nu)$ is connected.

5.3. Matrix analysis interpretation. An interpretation of the algorithm in terms of the associated Hessenberg matrix is at hand without additional complications. We provide one observation in this direction, with numerical matrix analysis flavor. To be more specific, write, in the conditions and notation adopted in the algorithm

$$z p_k(z) = \sum_{n=0}^{k+1} h_{nk} p_n(z), \quad k \geq 0.$$

The Hessenberg matrix $(h_{nk})_{n,k=0}^{\infty}$ has only the first sub-diagonal non-zero and it represents the subnormal operator $S = S_\mu$ with respect to the orthonormal basis $(p_k)_{k=0}^{\infty}$.

Fix an integer $j \geq 0$. Then

$$\langle S^* S p_j, p_j \rangle = \| S p_j \|^2 = \sum_{n=0}^{j+1} |h_{nj}|^2,$$

while

$$\langle S S^* p_j, p_j \rangle = \| S^* p_j \|^2 = \sum_{k=0}^{\infty} |\langle S^* p_j, p_k \rangle|^2 = \sum_{k=0}^{\infty} |h_{jk}|^2.$$

Hence

$$\mathrm{Tr}[S^*, S] = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (|h_{kj}|^2 - |h_{jk}|^2).$$

In view of Helton and Howe formula we infer:

Proposition 5.5. *Let (h_{kj}) denote Hessenberg's matrix associated to the complex orthogonal polynomials in $P^2(\mu)$. If the measure μ satisfies the conditions in the Algorithm, then the area of the "cloud" Ω can be recovered from the identity*

$$\mathrm{Area}(\Omega) = \pi \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (|h_{kj}|^2 - |h_{jk}|^2).$$

Similarly one can derive a formula for the center of mass of Ω:

$$\int_\Omega z \, \mathrm{d}A(z) = \frac{\pi}{2} \sum_{j=0}^{\infty} \sum_{\ell \leq k+1} h_{\ell k} (h_{kj} \overline{h_{\ell j}} - h_{j\ell} \overline{h_{jk}}).$$

The alert reader will recognize that the ordering of terms and summation in Proposition 5.5 is crucial. For instance, if

$$\sum_{j,k=0}^{\infty} |h_{jk}|^2 < \infty,$$

that is S is a Hilbert–Schmidt operator, then $\mathrm{Area}(\Omega) = 0$. Recall also that $\mathrm{Tr}[A, B] = 0$ for any trace-class operator A and bounded operator B. Consequently a trace class perturbation of the Hessenberg matrix S will not affect step (2) in the Algorithm, and hence the transformed moments $(a_{k\ell})$. Even more, a theorem of Voiculescu [42] implies the same invariance of traces of commutators (appearing in step (2)) by any Hilbert–Schmidt perturbation of S subject to $[S^*, S]$ being trace-class. We put together these observations in the following statement.

Proposition 5.6. *In the conditions of the Algorithm, let* $\widetilde{S} = S + K$ *be an additive perturbation of Hessenberg's matrix* S. *If* K *is trace-class, or Hilbert–Schmidt and* $\mathrm{Tr}\,|[\widetilde{S}^*, \widetilde{S}]| < \infty$, *then the output is unchanged*:

$$\mathrm{Tr}[(S^*)^{k+1}, S^{\ell+1}] = \mathrm{Tr}[(\widetilde{S}^*)^{k+1}, \widetilde{S}^{\ell+1}], \quad k, \ell \geq 0.$$

More details about an asymptotic triangularization with respect to the Hilbert–Schmidt class of the operator S^* (fulfilling the constraints of the Algorithm) can be found in [42].

6. Final comments

We illustrate the algorithm of this article by a simple toy example built on the Beta distribution. The cloud will be the unit disk and outliers will be distributed on two concentric circles of a larger radius. We pass to polar coordinates $z = re^{it}$ with $r \geq 0$ and $t \in [-\pi, \pi]$. Let α, β be positive constants defining the rotationally invariant weight

$$w(z) = r^{\alpha-1}(1-r)^{\beta-1}.$$

The moments are

$$\begin{aligned} u_{k\ell} &= \int_{\mathbb{D}} z^k \bar{z}^\ell w(z)\,\mathrm{d}A(z) \\ &= 2\pi\delta_{k\ell} \int_0^1 r^{2k+1} r^{\alpha-1}(1-r)^{\beta-1}\,\mathrm{d}\theta \end{aligned}$$

$$= 2\pi\delta_{k\ell}B(2k+\alpha+1,\beta)$$
$$= 2\pi\delta_{k\ell}\frac{\Gamma(2k+\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+2k+1)}, \quad k,\ell \geq 0.$$

See for instance [3] for the closed form of the Beta integral.

Let $R, \rho > 1$ and $\theta, \sigma \in (0,\pi)$. For a continuous function ϕ in the plane we define the measure ν as

$$\int \phi \, \mathrm{d}\nu = \int_{-\theta}^{\theta} \phi(Re^{it}) \, \mathrm{d}t + \phi(\rho e^{i\sigma}) + \phi(\rho e^{-i\sigma}).$$

The moments are

$$v_{k\ell} = 2R^{k+\ell}\frac{\sin(k-\ell)\theta}{k-\ell} + 2\rho^{k+l}\cos(k-\ell)\sigma, \quad k,\ell \geq 0.$$

In case $k = \ell$ we define

$$\left.\frac{\sin(k-\ell)\theta}{k-\ell}\right|_{k=\ell} = \theta.$$

The space $P^2(\chi_{\mathbb{D}} w \, \mathrm{d}A)$ has the monomials as an orthogonal basis and each complex number of the open unit disk is a bounded point evaluation. On the other hand, the support of the measure ν is disjoint of the closed unit disk and it does not disconnect the plane. Thus, the measure $\mu = \chi_{\mathbb{D}} w \, \mathrm{d}A + \nu$ fulfills the conditions in the statement of the Algorithm. The moments of μ are

$$s_{k\ell} = 2\pi\delta_{k\ell}B(2k+\alpha+1,\beta) + 2R^{k+\ell}\frac{\sin(k-\ell)\theta}{k-\ell} + 2\rho^{k+l}\cos(k-\ell)\sigma, \quad k,\ell \geq 0.$$

These moments depend on six independent parameters $\alpha, \beta, R, \rho, \theta, \sigma$.

Normally at this point we should run the algorithm, that is compute successively traces of commutators of the multiplier S_μ and its adjoint. Since we know from the beginning Thompson's decomposition of the measure μ we infer on theoretical grounds that the moments of the cloud (endowed with Lebesgue measure) are

$$a_{k\ell} = \int_{\mathbb{D}} z^k \bar{z}^\ell \, \mathrm{d}A(z) = \delta_{k\ell}\frac{\pi}{k+1}, \quad k,\ell \geq 0.$$

Note that these moments are independent of the six parameters. As a matter of fact we only need the first three moments a_{00}, a_{01}, a_{11} to determine indeed that the cloud is the unit disk.

To be more precise, with the notation and conventions introduced in Section 4.1, $a_{10} = \overline{a_{01}} = 0$ and the truncated formal exponential transform yields

$$\begin{aligned}\exp\Big[-\frac{1}{z\bar{w}} - \frac{1}{2z^2\bar{w}^2}\Big] &= \Big(1 - \frac{1}{z\bar{w}} + \frac{1}{2z^2\bar{w}^2}\Big)\Big(1 - \frac{1}{2z^2\bar{w}^2}\Big) + O\Big(\frac{1}{w^3}, \frac{1}{\bar{z}^3}\Big)\\ &= 1 - \frac{1}{z\bar{w}} + \frac{0}{z\bar{w}^2} + \frac{1}{z^2\bar{w}} + \frac{0}{z^2\bar{w}^2} + O\Big(\frac{1}{w^3}, \frac{1}{\bar{z}^3}\Big)\\ &= \frac{z\bar{w} - 1}{z\bar{w}} + O\Big(\frac{1}{w^3}, \frac{1}{\bar{z}^3}\Big).\end{aligned}$$

We find $b_{00} = 1, b_{01} = b_{10} = b_{11} = 0$. From the vanishing determinant $b_{00}b_{11} - b_{10}b_{01} = 0$ we deduce that the cloud is a quadrature domain of order 1 with algebraic boundary given by the equation

$$1 - z\bar{z} = 0$$

and the only quadrature node at $z = 0$.

Without adding cumbersome formulas to the initial data above, we note that replacing the unit disk by

$$|z - c|^2 \le M^2, \quad c \in \mathbb{C},\ M > 0,$$

leads at the final step of identification to the following formal computations. The initial moments are

$$\begin{aligned}a_{00} &= \pi M^2,\\ a_{01} &= \int\limits_{|z-c|\le M} z\,\mathrm{d}A(z) = \pi M c = \overline{a_{10}},\\ a_{11} &= \int\limits_{|z-c|\le M} |z|^2\,\mathrm{d}A(z) = 2\pi\int\limits_0^M (|c|^2 + r^2) r\,\mathrm{d}\theta = \pi M^2|c|^2 + \pi\frac{M^4}{2}.\end{aligned}$$

The truncated exponential transforms is

$$\begin{aligned}&\exp\Big[-\frac{M^2}{z\bar{w}} - \frac{M^2\bar{c}}{z\bar{w}^2} - \frac{M^2 c}{z^2\bar{w}} - \frac{M^2|c|^2 + \frac{M^4}{2}}{z^2\bar{w}^2}\Big]\\ &\quad = 1 - \frac{M^2}{z\bar{w}} - \frac{M^2\bar{c}}{z\bar{w}^2} - \frac{M^2 c}{z^2\bar{w}} - \frac{M^2|c|^2}{z^2\bar{w}^2} + O\Big(\frac{1}{w^3}, \frac{1}{\bar{z}^3}\Big).\end{aligned}$$

We infer

$$b_{00} = M^2, \quad b_{10} = M^2 c, \quad b_{01} = M^2\bar{c}, \quad b_{11} = M^2|c|^2.$$

The vanishing determinant $b_{00}b_{11} - b_{10}b_{01} = 0$ and the linear dependence of the columns of the matrix $(b_{k\ell})_{k,\ell=0}^{1}$ identify the monic factor $P(z) = z - c$ in the denominator $P(z)\overline{P(w)}$ of the rational approximant of the full exponential transform. Finally, as in the one dimensional diagonal Padé approximation scheme, one finds

$$\begin{aligned}(z-c)(\bar{w}-\bar{c})\Big[1 - \frac{M^2}{z\bar{w}} - \frac{M^2\bar{c}}{z\bar{w}^2} - \frac{M^2 c}{z^2\bar{w}} - \frac{M^2|c|^2}{z^2\bar{w}^2}\Big] \\ = (z-c)(0\bar{w}-\bar{c}) - M^2 + O\Big(\frac{1}{z^2}, \frac{1}{\bar{w}^2}\Big).\end{aligned}$$

We deduce from here that the equation of the generating shape possessing moments $a_{00}, a_{10}, a_{01}, a_{11}$ is necessarily the disk of equation $|z - c|^2 \le M^2$.

We refer to [14] for details about the exact reconstruction algorithm of quadrature domains and to [15] for the associated Padé approximation scheme.

In the above toy example the Hessenberg matrix S_μ associated to the original measure depending on all parameters has more structure: it is a weighted shift perturbed by some simple Toeplitz matrices. This observation would simplify the direct computations of the traces of commutators of monomials in S_μ and S_μ^*. For a general measure the multiplier S_μ is not a structured matrix, hence some approximation procedure is required to be incorporated and combined with the trace formulas. The qualitative aspects of this process, complemented by some real data experiments, will be addressed in a forthcoming article.

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